OPTIMAL LQ-FEEDBACK CONTROL FOR A CLASS OF FIRST-ORDER HYPERBOLIC DISTRIBUTED PARAMETER SYSTEMS

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Abstract. The Linear-Quadratic (LQ) optimal control problem is studied for a class of first-order hyperbolic partial differential equation models by using a nonlinear infinite-dimensional (distributed parameter) Hilbert state-space description. First the dynamical properties of the linearized model around some equilibrium profile are studied. Next the LQ-feedback operator is computed by using the corresponding operator Riccati algebraic equation whose solution is obtained *via* a related matrix Riccati differential equation in the space variable. Then the latter is applied to the nonlinear model, and the resulting closed-loop system dynamical performances are analyzed.

1. Introduction

Phenomena in many chemical processes include transport processes that are described by partial differential equations (PDE's): see *e.g.* [12,15,18]. When diffusive transport is negligible and convective transport is dominant, processes are described by first-order hyperbolic PDE's. The class of such processes includes plug flow reactors.

The dynamical analysis and (optimal) control of distributed parameter systems modeled by hyperbolic PDE's are important questions that have received a lot of attention in the system and control literature: see *e.g.* [7,8,20] for the study of such questions from the applied mathematics point of view, and *e.g.* [12,16,24] for applications to process control. In most practical situations the original infinite-dimensional systems described by PDE's are spatially discretized to obtain finite-dimensional approximate models, and then finite-dimensional controllers are designed and implemented. However, the latter approach neglects the distributed nature of the original system; this may lead to inappropriate control design that neglects the dynamical properties of the infinite-dimensional system and may impose some limitations on the controller performances.

In [4,5], the Linear-Quadratic (LQ) problem is studied for a nonisothermal plug flow reactor model that represents a particular process described by first-order hyperbolic PDE's. In this paper, the LQ-optimal control

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problem is studied for a more general class of first-order hyperbolic PDE models by using a nonlinear infinite-dimensional Hilbert state-space description. Section 2 describes both the nonlinear model and linearized model of this class of systems. Section 3 focuses on dynamical properties (exponential stability, reachability and observability) of the linearized model. The optimal control design is the object of Section 4. An LQ-optimal feedback is computed by using the corresponding operator Riccati algebraic equation, whose solution is obtained *via* a related matrix Riccati differential equation. Then, in Section 5, this LQ-optimal feedback is applied to the nonlinear model, and the resulting closed-loop system dynamical performances are analyzed. Criteria are given that guarantee that the closed-loop system is asymptotically stable. Moreover optimality is proved with respect to a modified cost criterion.

2. MODELS DESCRIPTION

2.1. **Nonlinear model**

Let us consider the following semilinear first-order hyperbolic PDE system in one spatial dimension:

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + f(x, u) \tag{2.1}$$

$$y = h(x) \tag{2.2}$$

subject to the boundary condition: for all $t \geq 0$:

$$x(0, t) = b \tag{2.3}$$

and the initial condition: for almost all $z \in [0, 1]$:

$$x(z, 0) = x_0(z) \tag{2.4}$$

where $z \in [0, 1]$ (without loss of generality) and $t \in [0, \infty)$ denote the spatial variable and the time variable respectively, $x(t) := [x_1(\cdot, t), \ldots x_n(\cdot, t)]^T \in H := L^2(0, 1)^n$, $u(t) := [u_1(\cdot, t), \ldots u_m(\cdot, t)]^T \in U := L^2(0, 1)^m$, and $y(t) := [y_1(\cdot, t), \ldots y_p(\cdot, t)]^T \in Y := L^2(0, 1)^p$ denote the vectors of state (input, output, respectively) variables at time t, the functions $f := [f_1 \ldots f_n] : H \times U \longrightarrow H$ and $h := [h_1 \ldots h_p] : H \longrightarrow Y$ are continuous vector functions, $b \in \mathbb{R}^n$ is a constant (column) vector and x_0 is in H.

Let us assume that the system (2.1)–(2.2) has at least one equilibrium profile $(x_e, u_e) \in H \times U$, *i.e.* (x_e, u_e) is a solution of the following ordinary differential equation

$$\frac{\mathrm{d}x_e}{\mathrm{d}z} = f(x_e(z), u_e(z)), \quad x_e(0) = b.$$

Remark 2.1. (a) In this paper, it will be assumed that the hyperbolic PDE (2.1), together with the boundary condition (2.3) and the initial condition (2.4) is well posed, *i.e.* that, with any fixed input function u, the state trajectories exist such that, for all $t \geq 0$, $x(t)$ is in H. See *e.g.* [8], Chapter 3, for more detail concerning the existence of solutions to semilinear and quasilinear systems; see also [22,23] for the analysis of the state trajectories of such models, by means of general mathematical descriptions involving abstract differential equations on Banach or Hilbert spaces. In addition, specific existence conditions for the state trajectories will be established in Section 5, for such hyperbolic PDE system under LQ-optimal state feedback.

(b) If the function f is assumed to be Lipschitz continuous on its domain, then the system (2.1)–(2.2) has a unique equilibrium profile x_e, corresponding to a fixed input function u_e. This applies in particular to the nonisothermal plug flow reactor studied in [3–5].

In what follows, although we do not necessarily linearize the system (2.1)–(2.2) around the trivial equilibrium $(0, 0)$, we shall keep for simplicity x and u as notations for the new variables $x - x_e$ and $u - u_e$, respectively.

2.2. **Linearized model**

The linearization of the system (2.1)–(2.2) around an equilibrium profile leads to the following linear time-invariant infinite-dimensional state-space system on the Hilbert space H:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \in H \\ y(t) = Cx(t). \end{cases} \tag{2.5}$$

Here A is the linear (unbounded) operator defined on its domain:

$$D(A) = \left\{ x \in H : x \text{ is a.c.}, \ \frac{\mathrm{d}x}{\mathrm{d}z} \in H \text{ and } x(0) = 0 \right\} \tag{2.6}$$

(where a.c. means that the function x is absolutely continuous) by

$$A = \begin{bmatrix} -\frac{\mathrm{d}.}{\mathrm{d}z} - \alpha_{1,1}I & -\alpha_{1,2}I & \dots & -\alpha_{1,n}I \\ -\alpha_{2,1}I & -\frac{\mathrm{d}.}{\mathrm{d}z} - \alpha_{2,2}I & \dots & -\alpha_{2,n}I \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n,1}I & -\alpha_{n,2}I & \dots & -\frac{\mathrm{d}.}{\mathrm{d}z} - \alpha_{n,n}I \end{bmatrix} \tag{2.7}$$

where I denotes the identity operator on $L^2(0,1)$. The operators $B \in \mathcal{L}(U,H)$ and $C \in \mathcal{L}(H,Y)$ are the bounded linear operators defined by

$$Bu = \begin{bmatrix} \beta_{1,1}I & \beta_{1,2}I & \dots & \beta_{1m}I \\ \beta_{2,1}I & \beta_{2,2}I & \dots & \beta_{2,m}I \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n,1}I & \beta_{n,2}I & \dots & \beta_{n,m}I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \tag{2.8}$$

and

$$Cx = \begin{bmatrix} w_{1,1}I & w_{1,2}I & \dots & w_{1,n}I \\ w_{2,1}I & w_{2,2}I & \dots & w_{2,n}I \\ \vdots & \vdots & \ddots & \vdots \\ w_{p,1}I & w_{n,2}I & \dots & w_{p,n}I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.9}$$

Denoting by $\underline{k}$ the set of positive integers $\{1, \dots, k\}$, the functions $\alpha_{i,j}(z)$, $i,j \in \underline{n}$, $\beta_{i,j}(z)$, $i \in \underline{n}$, $j \in \underline{m}$ and $w_{i,j}(z)$, $i \in \underline{p}$, $j \in \underline{n}$ correspond to the Jacobians of the nonlinear functions f and h of the model equations evaluated at the chosen equilibrium profile and are given by

$$\alpha_{i,j} = -\frac{\partial f_i}{\partial x_j}(x_e, u_e), \ \beta_{i,j} = \frac{\partial f_i}{\partial u_j}(x_e, u_e), \ w_{i,j} = \frac{\partial h_i}{\partial x_j}(x_e).$$

Remark 2.2. It is assumed that all the functions $\alpha_{i,j}$, $\beta_{i,j}$ and $w_{i,j}$ are in $L^\infty(0,1)$. These conditions guarantee in particular that the linear operators B and C defined above are bounded on U and H respectively.

Let us denote by $\mathcal{M}, \mathcal{N}$ and $\mathcal{C}$ the following space-varying matrices, that will be useful in the rest of this paper:

$$\mathcal{M} := -\left[\alpha_{i,j}\right]_{i,j \in \underline{n}}, \ \mathcal{N} := \left[\beta_{i,j}\right]_{i \in \underline{n} \,;\, j \in \underline{m}}, \ \mathcal{C} := \left[w_{i,j}\right]_{i \in \underline{p} \,;\, j \in \underline{n}}. \tag{2.10}$$

3. Dynamical properties of the linearized model

This section focuses on dynamical properties, *viz.* exponential stability, reachability and observability of the linearized model (2.5)–(2.9). The fundamental observation leading to these properties is the strong relation between the infinite-dimensional linear system $\Sigma(A, B, C)$ (see *e.g.* [13]) and the finite-dimensional linear space-varying system $\Sigma(\mathcal{M}, \mathcal{N}, \mathcal{C})$ (see *e.g.* [10]) given by (2.10). This relation is observed thanks to the fact that the solution of a related operator Lyapunov equation can be built from the solution of the corresponding matrix Lyapunov equation: see Section 3.2. First let us define the controllability matrix of a matrix pair $(\mathcal{M}, \mathcal{N})$ (see [25], p. 66):

$$Q_c(z) := [P_0(z) \vdots P_1(z) \vdots \ldots \vdots P_{n-1}(z)] \tag{3.1}$$

where

$$P_{k+1}(z) = -\mathcal{M}(z)P_k(z) + \frac{\mathrm{d}P_k}{\mathrm{d}z}(z), \quad P_0(z) = \mathcal{N}(z). \tag{3.2}$$

The observability matrix of a matrix pair $(\mathcal{C}, \mathcal{M})$ is defined similarly (see [25], p. 66):

$$Q_o(z) := [S_0(z) \vdots S_1(z) \vdots \ldots \vdots S_{n-1}(z)] \tag{3.3}$$

where

$$S_{k+1}(z) = -\mathcal{M}^T(z)S_k(z) + \frac{\mathrm{d}S_k}{\mathrm{d}z}(z), \quad S_0(z) = \mathcal{C}^T(z). \tag{3.4}$$

3.1. **Exponential stability**

The first focus concerns the exponential stability of the C_0-semigroup generated by the operator A. In order to prove the exponential stability of this C_0-semigroup, the complete observability (see [25]) of some related matrix pair is needed.

Lemma 3.1. *The following matrix Lyapunov differential equation (MLDE)*

$$-\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}z} = \mathcal{P}\mathcal{M} + \mathcal{M}^T\mathcal{P} + I_n, \quad \mathcal{P}(1) = 0 \tag{3.5}$$

admits a positive definite solution, viz. *the observability gramian $\mathcal{P}$, on $[0, 1)$.*

Proof. Observe that the corresponding observability matrix of the pair $(I_n, \mathcal{M})$ (see Eqs. (3.3)–(3.4)) is full rank, whence by using [25], Theorem 5, p. 69, the matrix pair $(I_n, \mathcal{M})$ is completely observable on the interval $[0, 1)$. Consequently, the MLDE (3.5) has a positive definite solution on the interval $[0, 1)$ (see *e.g.* [10]). □

By observing that the solution of the corresponding operator Lyapunov algebraic equation (OLAE) can be built from the solution of the MLDE (3.5), Lemma 3.1 leads to the following exponential stability result.

Theorem 3.1. *The operator A defined by (2.6)–(2.7) is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded linear operators on H.*

Proof. In view of [13], Theorem 5.3.1, p. 217, it suffices to prove that there exists a positive definite operator $P \in \mathcal{L}(H)$ satisfying the following OLAE

$$PA + A^*P + I = 0 \ \text{ with } \ P(D(A)) \subset D(A^*). \tag{3.6}$$

In view of the structure of the operator A, it seems natural to look for a solution of the form $P = [\mathcal{P}_{i,j}(z) I]_{i,j\in\underline{n}}$, where I is the identity operator and every $\mathcal{P}_{i,j}$ is a real valued function defined on $[0, 1]$ and is the (i, j)-entry of a symmetric matrix function $z \in [0, 1] \mapsto \mathcal{P}(z) \in \mathbb{R}^{n\times n}$. The notation $[\mathcal{P}_{i,j}(z) I]_{i,j\in\underline{n}}$ is adopted here in order

to avoid any confusion between a matrix with scalar valued entries and an (multiplication) operator matrix. By a straightforward calculation, the following identities can be shown to hold:

$$PA = PA_0 + PM \quad \text{and} \quad A^*P = A_0^*P + M^*P, \tag{3.7}$$

where the operator A_0 is the unbounded linear diagonal operator defined on its domain

$$D(A_0) := \left\{ x \in H : x \text{ is a.c.}, \frac{\mathrm{d}x}{\mathrm{d}z} \in H \text{ and } x(0) = 0 \right\}$$

by

$$A_0 := \begin{bmatrix} -\frac{\mathrm{d}.}{\mathrm{d}z} & 0 & \dots & 0 \\ 0 & -\frac{\mathrm{d}.}{\mathrm{d}z} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\frac{\mathrm{d}.}{\mathrm{d}z} \end{bmatrix}, \tag{3.8}$$

and the operator M is given by $M = [\mathcal{M}_{i,j}(z) \, I]_{i,j\in\underline{n}}$. On the other hand, it can be shown that

$$PA_0 + A_0^*P = \frac{\mathrm{d}P}{\mathrm{d}z}. \tag{3.9}$$

Therefore, if the matrix $\mathcal{P} := [\mathcal{P}_{i,j}(z)]_{i,j\in\underline{n}}$ is the positive definite solution of the MLDE (3.5), then the operator $P := [\mathcal{P}_{i,j}(z)I]_{i,j\in\underline{n}}$ is a solution of the OLAE (3.6), since the final condition $\mathcal{P}(1) = 0$ implies that $P(D(A)) \subset D(A^*)$. Moreover this solution is positive definite. Indeed, for any $x \in H$, one has

$$\langle Px, x \rangle_H = \int_0^1 (x^T(z)\mathcal{P}(z)x(z))\mathrm{d}z.$$

Hence one can conclude that P is also positive definite since the matrix function $\mathcal{P}$ is positive definite on $[0,1]$ by Lemma 3.1. $\qquad\square$

The following result is an obvious consequence of Theorem 3.1 (see *e.g.* [13]).

Corollary 3.1. *The operator pair (A, B) is exponentially stabilizable and the operator pair (C, A) is exponentially detectable.*

3.2. **Reachability and observability**

The second question concerns the reachability of (A, B), where A is given by (2.6)–(2.7) and B is given by (2.8). Noting that the fact that the reachability operator Lyapunov algebraic equation can be related to some matrix Lyapunov differential equation leads to the following result:

Theorem 3.2. *Let A and B be the operators defined by (2.6)–(2.7) and (2.8), respectively. Let $\mathcal{M}$ and $\mathcal{N}$ be the matrices defined by (2.10). If the matrix pair $(\mathcal{M}, \mathcal{N})$ is totally controllable on $[0, 1]$ then the operator pair (A, B) is reachable.*

Proof. First recall that the operator A generates an exponentially stable C_0-semigroup. Then by [13], Theorem 4.1.23, p. 160, the extended reachability gramian, denoted by L_B, is the unique self-adjoint solution of the following OLAE:

$$L_B(D(A^*)) \subset D(A), \quad \text{and} \quad [AL_B + L_BA^* + BB^*]x = 0, \; x \in D(A^*). \tag{3.10}$$

In order to prove the reachability of (A, B) and in view of [13], Theorem 4.1.22, p. 160, it is enough to show that the solution L_B is positive definite. Note that by a straightforward calculation it can be shown that if the matrix $\mathcal{L}_B := (l_{i,j}(z))_{1 \leq i,j \leq n}$ is the positive definite solution of the MLDE

$$\frac{\mathrm{d}\mathcal{L}_B}{\mathrm{d}z} = \mathcal{M}\mathcal{L}_B + \mathcal{L}_B\mathcal{M}^T + \mathcal{N}\mathcal{N}^T, \ \mathcal{L}_B(0) = 0, \tag{3.11}$$

whose existence is guaranteed by the total controllability of the matrix pair $(\mathcal{M}, \mathcal{N})$, then the operator $\mathrm{L}_B := (l_{i,j}(z)I)_{1 \leq i,j \leq n}$ is the solution of the OLAE (3.10). Moreover L_B is positive definite since the matrix function $\mathcal{L}_B$ is positive definite on $[0, 1]$. □

In view of [25], Theorem 3, p. 69, the following corollary is a straightforward consequence of Theorem 3.2:

Corollary 3.2. *If the controllability matrix $Q_c(z)$, defined by equations (3.1)–(3.2), is full rank, i.e. $rank(Q_c(z)) = n$ for all $z \in [0, 1]$, then the operator pair (A, B) is reachable.*

By using duality and arguments similar to those used for the reachability analysis, it can be shown that the observability follows from the total observability of some related space-varying system. This is summarized in the following theorem and corollary:

Theorem 3.3. *Let A and C be the operators defined by (2.6)–(2.7) and (2.9), respectively. Let $\mathcal{M}$ and $\mathcal{C}$ be the matrices defined by (2.10). If the matrix pair $(\mathcal{C}, \mathcal{M})$ is completely observable on $[0, 1]$ then the operator pair (C, A) is observable.*

Corollary 3.3. *If the observability matrix $Q_o(z)$, defined by equation (3.3)–(3.4) is full rank, i.e. $rank(Q_o(z)) = n$ for some $z \in [0, 1]$, then the operator pair (C, A) is observable.*

4. LQ-OPTIMAL CONTROL DESIGN

This section is devoted to the design of an optimal LQ-feedback (see *e.g.* [11,13]) for the linearized model given by (2.5)–(2.9). More precisely the aim is to find a control law which minimizes the following cost criterion:

$$\Lambda(x_0; u) = \int_0^\infty \left(\langle y(s), \ y(s) \rangle + \langle u(s), \ u(s) \rangle \right) \mathrm{d}s. \tag{4.1}$$

First note that the fact that A generates an exponentially stable C_0-semigroup (see Th. 3.1) implies that (A, B) is exponentially stabilizable and (C, A) is exponentially detectable.

It is well known (see *e.g.* [13]) that, under those conditions, the solution of this problem can be obtained by finding the nonnegative self-adjoint operator $Q_o \in \mathcal{L}(H)$ that solves the operator Riccati algebraic equation (ORAE), *viz.*

$$[A^*Q_o + Q_oA + C^*C - Q_oBB^*Q_o]x = 0, \tag{4.2}$$

for all $x \in D(A)$, where $Q_o(D(A)) \subset D(A^*)$.

In order to solve this equation, the following lemma, which can be easily proven by noting that the matrices $\mathcal{C}^T\mathcal{C}$ and $\mathcal{N}\mathcal{N}^T$ are positive semi-definite (see [1], Cor. 6.7.36), is useful.

Lemma 4.1. *Let us consider the following matrix Riccati differential equation (MRDE):*

$$-\frac{\mathrm{d}\Phi}{\mathrm{d}z} = \mathcal{M}^T\Phi + \Phi\mathcal{M} + \mathcal{C}^T\mathcal{C} - \Phi\mathcal{N}\mathcal{N}^T\Phi, \ \ \Phi(1) = 0. \tag{4.3}$$

Then the latter has a unique positive semi-definite solution on $[0, 1]$.

Now we are in a position to state the following result:

Theorem 4.1. *Let us consider the linearized model* (2.5)–(2.9). *Let* $\mathcal{N} \in \mathbb{R}^{n \times m}$ *be given by* (2.10) *and* $\Phi \in \mathbb{R}^{n \times n}$ *be the solution of the MRDE* (4.3). *Consider*

$$\Psi(z) =: -\mathcal{N}^T(z)\Phi(z) \in \mathbb{R}^{m \times n}. \tag{4.4}$$

Then $Q_0 := [\Phi_{i,j}\, I]_{i,j \in \underline{n}}$ *is the unique self-adjoint positive semi-definite solution of the operator Riccati differential equation* (4.2). *Hence the operator*

$$K_0 := [\Psi_{i,j}\, I]_{i \in \underline{m},\, j \in \underline{n}} \tag{4.5}$$

is the optimal LQ-feedback and the LQ-optimal control law is given by

$$u_{opt}(z,t) = \Psi(z)x(z,t). \tag{4.6}$$

Proof. Let us prove that Q_o is the unique self-adjoint positive semi-definite solution of the ORAE (4.2). First observe that

$$A^*Q_0 = A_0^*\Phi + M^*\Phi \quad \text{and} \quad Q_0A = \Phi A_0 + \Phi M,$$

where A_0 is given by (3.8). By direct calculation, it can be easily shown that

$$A_0^*\Phi + \Phi A_0 = \left[\frac{\mathrm{d}\Phi_{i,j}}{\mathrm{d}z}\, I\right]_{i,j \in \underline{n}}.$$

On the other hand $C^*C = [(\mathcal{C}^T\mathcal{C})_{i,j}\, I]_{i,j \in \underline{n}}$ and $Q_0BB^*Q_0 = [(\Phi\mathcal{N}\mathcal{N}^T\Phi)_{i,j}\, I]_{i,j \in \underline{n}}$. Therefore if the matrix Φ is a solution of the MRDE (4.3), then $Q_o = \Phi I$ is a solution of the ORAE (4.2) since the initial condition $\Phi(1) = 0$ implies that $Q_o(D(A)) \subset D(A^*)$. Moreover the fact that the MRDE (4.3) has a unique positive semi-definite solution Φ on $[0, 1]$ implies that the operator $Q_o = \Phi I$ is the unique self-adjoint positive semi-definite solution of the ORAE (4.2). $\square$

Remark 4.1. In view of Theorem 3.2 and Corollary 3.2, if the matrix pair $(\mathcal{M}, \mathcal{N})$ is totally controllable, or equivalently the controllability matrix Q_c is full rank on the interval $[0, 1]$, then the LQ-optimal feedback K_0 can also be uniquely determined by spectral factorization (see *e.g.* [9]). This alternative method was successfully applied to a nonisothermal plug flow reactor in [4,5].

5. Nonlinear closed-loop system performances analysis

This section focuses on the performances of the resulting closed-loop nonlinear system based on the implementation of the optimal LQ-feedback designed in the previous section. Here, we are interested in two questions. The first one is the asymptotic stability. The second one is the optimality of the designed LQ-control for the nonlinear system since its optimality is proved only on the linearized model with respect to a chosen quadratic criterion. Without loss of generality let us assume that the optimal LQ-feedback is computed for the linearized model around the zero equilibrium profile.

5.1. Stability analysis

In this subsection, we are interested in the asymptotic stability of the closed-loop nonlinear model. The latter can be described by applying the LQ-optimal state feedback K_0, given by (4.5), to the nonlinear model (2.1)–(2.2):

$$\begin{cases} \dot{x}(t) = A_0x(t) + N_0(x(t)) \\ x(0) = x_0 \in D(A_0) \end{cases} \tag{5.1}$$

where the operator A_0 is the linear (unbounded) diagonal operator given by equation (3.8) and the nonlinear operator N_0 is defined on H by

$$N_0(x) := f(x, K_0 x). \tag{5.2}$$

First we assume that the operator $\mathcal{N}_0$ is defined everywhere. The analysis of the following results is based on concepts and results developed *e.g.* in [9,14,19,21].

Definition 5.1. Let $\mathcal{A}$ be an operator on a Banach space X with domain $D(\mathcal{A})$.

(i) $\mathcal{A}$ is said to be *dissipative* if, for all x, $x' \in D(\mathcal{A})$ and for all $\lambda > 0$,

$$\|x - x'\| \leq \|(x - x') - \lambda(\mathcal{A}x - \mathcal{A}x')\|,$$

or equivalently, for all x, $x' \in D(\mathcal{A})$, there exists a bounded linear functional f on X such that $f(x - x') = \|x - x'\|^2 = \|f\|^2$ and $f(\mathcal{A}x - \mathcal{A}x') \leq 0$.

In addition $\mathcal{A}$ is said to be *strictly dissipative* if the conditions above hold with strict inequalities, for all x, $x' \in D(\mathcal{A})$ such that $x \neq x'$.

(ii) $\mathcal{A}$ is said to be *(strictly) m-dissipative* if it is (strictly) dissipative and $\mathcal{R}(I - \mathcal{A}) = X$, where $\mathcal{R}(T)$ denotes the range of an operator T.

Theorem 5.1. *Consider the closed-loop nonlinear system* (5.1)–(5.2). *If the operator N_0 is strictly dissipative on the Hilbert space H, then the operator $A_0 + N_0$ is the generator of a unique nonlinear contraction semigroup $\Gamma(t)$ on H. Moreover, for any $x_0 \in H$,*

$$x(t, x_0) := \Gamma(t)x_0 \to 0 \ as \ t \to \infty$$

i.e. *the zero state is an asymptotically stable equilibrium state.*

Proof. Observe that each diagonal entry of the operator A_0 generates an exponentially stable C_0-semigroup of contraction (see [13], Ex. 2.2.4, p. 34), whence it is m-dissipative. It follows that the operator A_0 is m-dissipative.

On the other hand $(I - \lambda A_0)^{-1}$ is compact. Indeed, consider the operator

$$\mathcal{A}_d := -\frac{\mathrm{d}.}{\mathrm{d}z}$$

defined on the domain

$$D(\mathcal{A}_d) = \left\{ x \in L^2(0,1) : x \text{ is absolutely continuous, } \frac{\mathrm{d}x}{\mathrm{d}z} \in L^2(0,1) \text{ and } x(0) = 0 \right\}.$$

A straightforward computation shows that, for any $\lambda > 0$ and for any $g \in L^2(0,1)$,

$$((I - \lambda\mathcal{A}_d)^{-1}g)(z) = \frac{1}{\lambda} \exp\left(-\frac{z}{\lambda}\right) \int_0^z \exp\left(\frac{\eta}{\lambda}\right) g(\eta)\mathrm{d}\eta.$$

Using this identity, it can be shown that $(I - \lambda\mathcal{A}_d)^{-1}$ is a Hilbert-Schmidt operator, whence it is compact. It follows that the operator $(I - \lambda A_0)^{-1}$ is also compact, as a diagonal matrix of compact operators. Then, by using the strict dissipativity of N_0 the conclusion follows directly from [2], Theorem 4.3.1, pp. 59–60. $\qquad \square$

Theorem 5.2. *Consider the closed-loop nonlinear system* (5.1)–(5.2). *Assume that the operator N_0 is Lipschitz continuous with Lipschitz constant l_0. Then, if $l_0 < \mathrm{e}^{-1}$, the operator $A_0 + N_0$ is the generator of a unique nonlinear contraction semigroup $\Gamma(t)$ on H. Moreover, for any $x_0 \in H$,*

$$x(t, x_0) := \Gamma(t)x_0 \to 0 \ as \ t \to \infty$$

i.e. *the zero state is an asymptotically stable equilibrium state.*

Proof. Let us denote by ω_0 the growth bound of the C_0-semigroup $(\mathrm{e}^{A_0 t})_{t \geq 0}$. By using [13], Theorem 2.1.6(e), p. 18, and its proof, for all $\omega > \omega_0$, there exists a t_ω such that

$$\frac{\log \|\mathrm{e}^{A_0 t}\|}{t} < \omega, \quad \text{for } t \geq t_\omega, \text{ or equivalently } \|\mathrm{e}^{A_0 t}\| \leq \mathrm{e}^\omega, \text{ for } t \geq t_\omega.$$

Note also that $\|\mathrm{e}^{A_0 t}\|$ is bounded on every finite subinterval of $[0, \infty)$, in particular in the interval $[0, t_\omega]$, *i.e.* there exists $M_{0,\omega} > 0$ such that

$$\|\mathrm{e}^{A_0 t}\| \leq M_{0,\omega}, \quad \forall 0 \leq t \leq t_\omega.$$

Then $\forall \omega \geq \omega_0$, there exists a constant $M_\omega = \begin{cases} M_{0,\omega}, & \text{if } \omega > 0 \\ \mathrm{e}^{-\omega t_\omega} M_{0,\omega} & \text{if } \omega < 0 \end{cases}$, such that

$$\|\mathrm{e}^{A_0 t}\| \leq M_\omega \mathrm{e}^{\omega t}, \text{ for all } t \geq 0. \tag{5.3}$$

On the other hand, it can be shown that $\|\mathrm{e}^{A_0 t}\| = 0$, for all $t \geq 1$ and $\|\mathrm{e}^{A_0 t}\| = 1$, for all $t < 1$. Then (a) the growth bound of the C_0-semigroup $(\mathrm{e}^{A_0 t})_{t \geq 0}$ is equal to $-\infty$, (b) the constant t_ω defined above is equal to 1 and (c) the constant $M_{0,\omega}$ is equal to 1. Consequently, for all $\omega < 0$

$$\|\mathrm{e}^{A_0 t}\| \leq M_\omega \mathrm{e}^{\omega t}, \text{ for all } t \geq 0, \quad \text{where } M_\omega = \mathrm{e}^{-\omega}. \tag{5.4}$$

By using [6], Corollary 13, p. 126, the asymptotic stability follows from [6], Comment 14, p. 126, since $\sup_{\omega < 0}\{-\omega \mathrm{e}^\omega\} = \mathrm{e}^{-1}$. □

In the previous results, we have assumed that the nonlinear operator N_0 is defined everywhere. Now we are interested in the asymptotic stability of the system (5.1) when the operator N_0 is defined on a convex closed subset $F \subset H$. First the following technical concept is recalled:

Definition 5.2. Let $\mathcal{A}$ be a dissipative operator. Let X_0 be a subset of X. $\mathcal{A}$ is said to be in $Q(X_0)$ if $\overline{D(\mathcal{A})} \subset X_0$ and $X_0 \subset \mathcal{R}(I - \lambda \mathcal{A})$ for all $\lambda > 0$.

On the other hand, the following result, whose proof can be found in [2,6], is needed.

Lemma 5.1 ([6], Th. 16). *Let F be a closed convex subset of X. Consider a linear closed dissipative operator A_0 such that $(I - \lambda A_0)^{-1}$ is compact for some $\lambda > 0$. Consider a Lipschitz continuous nonlinear operator N_0 on F. Assume that $\mathcal{A} = A_0 + N_0$ is strictly dissipative and the restriction of A_0 to $D(A_0) \cap F$ is in $Q(F)$ and that the condition*

$$\liminf_{\lambda \to 0^+} \lambda^{-1} d(F, x + \lambda N_0(x)) = 0, \text{ for } x \in \overline{D(\mathcal{A})},$$

holds, where $d(F, p)$ denotes the distance from $p \in X$ to F. Let $\Gamma(t)$ be the contraction semigroup generated by $\mathcal{A}$. Assume that $\overline{x}$ is an equilibrium point of $\mathcal{A}$. Then for any $x_0 \in \overline{D(\mathcal{A})}$,

$$x(t, x_0) := \Gamma(t)x_0 \to \overline{x} \text{ as } t \to \infty.$$

Now we are in a position to state the following result:

Theorem 5.3. *Consider the closed-loop nonlinear model (5.1)–(5.2). Assume that the operator N_0 is Lipschitz continuous on F and the operator $\mathcal{A} := A_0 + N_0$ is strictly dissipative. If $F \subset \mathcal{R}(I - \lambda A_0)$ for all $\lambda > 0$ and if the condition*

$$\liminf_{\lambda \to 0^+} \lambda^{-1} d(F, x + \lambda N_0(x)) = 0, \text{ for } x \in \overline{D(\mathcal{A})},$$

holds, then, for any $x_0 \in \overline{D(\mathcal{A})}$,

$$x(t, x_0) := \Gamma(t)x_0 \to 0 \text{ as } t \to \infty$$

i.e. *the zero state is an asymptotically stable equilibrium state.*

Proof. This result is a consequence of Lemma 5.1. Indeed, the compactness and the m-dissipativity of A_0 are shown in the proof of Theorem 5.1. On the other hand, the fact that $F \subset \mathcal{R}(I - \lambda\mathcal{A})$ for all $\lambda > 0$ implies that the restriction of A_0 on $D(A_0) \cap F$ is in $Q(F)$ (see Def. 5.2) since $\overline{D(A_0) \cap F} \subset F$.   □

Remark 5.1. In view of (29) and since the state feedback operator K_0 is linear, it suffices to check that the function f is Lipschitz continuous, in order to prove that the operator N_0 is Lipschitz continuous too. In addition, when the Hilbert space $X \times U$ is equipped with the inner product $\langle (x, u); (y, v) \rangle := \langle x, y \rangle_H + \langle u, v \rangle_U$, if l_f is a Lipschitz constant for f, then a Lipschitz constant for N_0 is given by $l_0 = l_f \cdot \sqrt{1 + \|K_0\|^2}$.

5.2. **Optimality analysis**

Obviously, the LQ-feedback designed in the previous section is not going to be optimal for the overall nonlinear system since its computation is based on the linearized model with the objective to minimize a specific quadratic criterion. This subsection in concerned with an inverse optimal control problem. This study is inspired by [17], which deals with the inverse optimal control problem for finite-dimensional systems. The question under study in this subsection is the following: what type of modification of the cost criterion can restore optimality? In this subsection it will be assumed that the function f is linear with respect to the input variable *i.e.*

$$f(x) := f_0(x) + Bu.$$

In this case the nonlinear open-loop system can be written as follows:

$$\begin{cases} \dot{x}(t) &= A_0 x(t) + Bu(t) + f_0(x(t)) \\ x(0) &= x_0 \in D(A_0). \end{cases} \tag{5.5}$$

The idea is to write the generator of the latter as the sum of the linearized generator and some nonlinear operator:

$$\begin{cases} \dot{x}(t) &= A x(t) + Bu(t) + \tilde{f}(x(t)) \\ x(0) &= x_0 \in D(A_0) \end{cases} \tag{5.6}$$

where the function $\tilde{f}(x) = f_0(x) - Jf_0(0)x$, where $Jf_0(0)$ denotes the Jacobian of the function f_0 evaluated at 0. The LQ-control law u_{opt} given by (4.6) is not optimal for the nonlinear system (5.6) (except if the function f_0 is linear), but it is optimal with respect to another type of cost criterion that includes the function $\tilde{f}$:

$$\Lambda_0(x_0, u) = \int_0^\infty \pi(x, u)\mathrm{d}t \tag{5.7}$$

where the function $\pi : H \times U \to \mathbb{R}$ is defined as

$$\pi(x, u) := \langle Cx, Cx \rangle - 2\langle \tilde{f}, Q_o x \rangle + \langle u, u \rangle \tag{5.8}$$

where $Q_0 := [\Phi_{i,j} \ I]_{i,j \in \underline{n}}$ is the solution of the operator Riccati algebraic equation (4.2) and Φ is the solution of the matrix Riccati differential equation (4.3). It turns out that the asymptotic stability of the nonlinear closed-loop system (5.6) leads to the optimality of u_{opt} with respect to the cost criterion (5.7).

Theorem 5.4. *If zero is an asymptotically stable state for the nonlinear closed-loop system* (5.6)*, then the LQ-control law u_{opt} given by* (4.6) *is optimal for the nonlinear system* (5.6) *with respect to the cost criterion* (5.7)*.*

Proof. In view of (5.6) and by a straightforward calculation, one gets

$$\begin{aligned} \Lambda_0(x_0, u) &= \int_0^\infty \left\{ \langle Cx, Cx \rangle - 2\langle \tilde{f}, Q_o x \rangle + \langle u, u \rangle \right\} \mathrm{d}t \\ &= \lim_{t_1 \to \infty} \int_0^{t_1} \left\{ \langle Cx, Cx \rangle - 2\langle \dot{x} - Ax - Bu, Q_o x \rangle + \langle u, u \rangle \right\} \mathrm{d}t. \end{aligned}$$

Now by using the fact that Q_o is the unique solution of the ORAE (4.2), it follows that

$$\Lambda_0(x_0, u) = \lim_{t_1 \to \infty} \int_0^{t_1} \left\{ -2\langle \dot{x}, Q_o x \rangle + \|B^* Q_o x\|^2 + 2\langle u, B^* Q_o x \rangle + \|u\|^2 \right\} \mathrm{d}t$$

$$= \langle x_0, Q_o x_0 \rangle + \lim_{t_1 \to \infty} \left[-\langle x(t_1), Q_o x(t_1) \rangle + \int_0^{t_1} \left\{ \langle u - K_o x, u - K_o x \rangle \right\} \mathrm{d}t \right].$$

If we replace u by u_{opt} of (4.6), we obtain

$$\Lambda_0(x_0, u_{opt}) = \langle x_0, Q_o x_0 \rangle - \lim_{t_1 \to \infty} \langle x(t_1), Q_o x(t_1) \rangle.$$

Then

$$\inf \Lambda_0(x_0, u) = \langle x_0, Q_o x_0 \rangle, \tag{5.9}$$

since $x(t) \to 0$ as $t \to \infty$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Comment 5.1. Since the operator Q_o is nonnegative, it follows from equation (5.9) that the lower bound of the cost functional Λ_0, given by (5.7)–(5.8), is nonnegative. Therefore the functional Λ_0 is nonnegative.

In view of Theorems 5.1 and 5.2, the following corollary is an immediate consequence of Theorem 5.4:

Corollary 5.1. *If the operator $\mathcal{N}_0$ is defined everywhere and if one of the following conditions holds:*

  (i) *the operator N_0 is strictly dissipative;*
  (ii) *the operator N_0 is Lipschitz continuous with Lipschitz constant l_0 and $l_0 < \mathrm{e}^{-1}$.*

Then the LQ-control law u_{opt} given by (4.6) is optimal for the nonlinear system (5.6) with respect to the cost criterion (5.7).

6. Concluding remarks

In this paper, we have solved the LQ-optimal control problem for a class of first-order hyperbolic PDE's that includes convection reaction processes arising in chemical engineering. First the dynamical properties, *viz.* exponential stability, reachability and observability of the linearized model have been analyzed. Each of these properties can be characterized by a specific related operator Lyapunov algebraic equation whose solution can be constructed from a matrix Lyapunov differential equation. Next, an optimal control has been designed on the basis of a linearized model by using the corresponding operator Riccati algebraic equation, that can be solved *via* certain matrix Riccati differential equation. The computed LQ-control was applied to the nonlinear model and the closed-loop system performances were analyzed. Some simple conditions have been shown to guarantee the asymptotic stability and the optimality of the closed-loop nonlinear system with respect to a modified cost criterion.

From a physical point of view and in view of all the numerical experiments that we have made so far, we state as an educated guess the conjecture that the asymptotic stability property of the closed-loop system still holds without the condition $l_0 < \mathrm{e}^{-1}$. However, it is not easy to prove this result by following the approach used in this paper. We leave this interesting open question as a topic for further research.

Possible topics for further research include the extension of the approach developed here to reaction-diffusion processes described by models taking axial dispersion phenomena into account, or the design of compensators based on optimal LQ-control of the estimated state.

REFERENCES

[1] H. Abou-Kandil, G. Freiling, V. Ionescu and G. Jank, *Matrix Riccati Equations in Control and Systems Theory, Series: Systems & Control: Foundations & Applications.* Birkhauser (2003).

[2] I. Aksikas, *Analysis and LQ-Optimal Control of Infinite-Dimensional Semilinear Systems: Application to a Plug Flow Reactor.* Ph.D. thesis, Université Catholique de Louvain, Belgium (2005).

[3] I. Aksikas, J. Winkin and D. Dochain, Stability analysis of an infinite-dimensional linearized plug flow reactor model, in *Proceedings of the 43rd IEEE Conference on Decision and Control, CDC* (2004) 2417–2422.

[4] I. Aksikas, J. Winkin and D. Dochain, LQ-optimal feedback regulation of a nonisothermal plug flow reactor infinite-dimensional model. *Int. J. Tomography & Statistics* **5** (2007) 73–78.

[5] I. Aksikas, J. Winkin and D. Dochain, Optimal LQ-feedback regulation of a nonisothermal plug flow reactor model by spectral factorization. *IEEE Trans. Automat. Control* **52** (2007) 1179–1193.

[6] I. Aksikas, J. Winkin and D. Dochain, Asymptotic stability of infinite-dimensional semilinear systems: application to a nonisothermal reactor. *Systems Control Lett.* **56** (2007) 122–132.

[7] V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems.* Boston: Academic Press (1993).

[8] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem.* Oxford University Press (2000).

[9] H. Brezis, *Opéateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert, Mathematics Studies.* North-Holland (1973).

[10] F.M. Callier and C.A. Desoer, *Linear System Theory.* Springer-Verlag, New York (1991).

[11] F.M. Callier and J. Winkin, LQ-optimal control of infinite-dimensional systems by spectral factorization. *Automatica* **28** (1992) 757–770.

[12] P.D. Christofides, *Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Application to Transport-Reaction Processes.* Birkhauser, Boston (2001).

[13] R.F. Curtain and H.J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, New York (1995).

[14] C.M. Dafermos and M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups. *J. Funct. Anal.* **13** (1973) 97–106.

[15] D. Dochain, *Contribution to the Analysis and Control of Distributed Parameter Systems with Application to (Bio)chemical Processes and Robotics.* Thèse d'Agrégation de l'Enseignement Supérieur, Université Catholique de Louvain, Louvain-la-Neuve, Belgium (1994).

[16] G.F. Froment and K.B. Bischoff, *Chemical Reactor Analysis and Design.* 2nd edition, John Wiley, New York (1990).

[17] M. Ikeda and D.D. Siljak, Optimality and robustness of linear quadratic control for nonlinear systems. *Automatica* **26** (1990) 499–511.

[18] M. Laabissi, M.E. Achhab, J. Winkin and D. Dochain, Trajectory analysis of nonisothermal tubular reactor nonlinear models. *Systems Control Lett.* **42** (2001) 169–184.

[19] V. Lakshmikantham and S. Leela, *Nonlinear Differential Equations in Abstract Spaces.* Pergamon, Oxford (1981).

[20] I. Lasiecka and R. Triggiani, *Control Theory for Partial Differential Equations: Continuous and Approximation Theories,* Volume II: *Abstract Hyperbolic-like Systems over a Finite Time Horizon.* Cambridge University Press (2000).

[21] Z. Luo, B. Guo and O. Morgül, *Stability and Stabilization of Infinite Dimensional Systems with Applications.* Springer-Verlag, London (1999).

[22] R.H. Martin, *Nonlinear Operators and Differential Equations in Banach spaces.* John Wiley & Sons, New York (1976).

[23] A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations, Appl. Math. Sci.* **44**. Springer-Verlag, New York (1983).

[24] W.H. Ray, *Advanced Process Control, Series in Chemical Engineering.* Butterworth, Boston (1981).

[25] L.M. Silverman and H.E. Meadows, Controllability and observability in time-variable linear systems. *J. SIAM Control* **5** (1967) 64–73.