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FEEDBACK STABILIZATION OF THE 2-D AND 3-D NAVIER-STOKES EQUATIONS BASED ON AN EXTENDED SYSTEM

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Abstract. We study the local exponential stabilization of the 2D and 3D Navier-Stokes equations in a bounded domain, around a given steady-state flow, by means of a boundary control. We look for a control so that the solution to the Navier-Stokes equations be a strong solution. In the 3D case, such solutions may exist if the Dirichlet control satisfies a compatibility condition with the initial condition. In order to determine a feedback law satisfying such a compatibility condition, we consider an extended system coupling the Navier-Stokes equations with an equation satisfied by the control on the boundary of the domain. We determine a linear feedback law by solving a linear quadratic control problem for the linearized extended system. We show that this feedback law also stabilizes the nonlinear extended system.

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1. Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4, and composed of N connected components $\Gamma^{(1)}, \dots, \Gamma^{(N)}$. Let us consider a stationary motion of an incompressible fluid in Ω which is described by the pair (z_s, p_s), the velocity and the pressure, solution to the stationary Navier-Stokes equations:

$$-\nu\Delta z_s + (z_s \cdot \nabla)z_s + \nabla p_s = f, \quad \nabla \cdot z_s = 0 \text{ in } \Omega \quad \text{and} \quad z_s = v_b \text{ on } \Gamma. \tag{1.1}$$

In the above setting, $\nu > 0$ is the viscosity, f is a function in $\mathbf{L}^2(\Omega)$, v_b belongs to $\mathbf{H}^{3/2}(\Gamma)$ and obeys $\int_{\Gamma^{(j)}} v_b \cdot n = 0$, for all $j = 1 \dots N$, where n denotes the unit normal vector to Γ, exterior to Ω. Notice that here and in the following, we write in bold the spaces of vector fields: $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^{3/2}(\Gamma) = (H^{3/2}(\Gamma))^d$, etc. We recall that a solution to (1.1) is known to exist in $\mathbf{H}^2(\Omega) \times H^1(\Omega)/\mathbb{R}$ [14], Chapter VIII, Theorems 4.1 and 5.2.

If z_s is an unstable equilibrium state, and if we assume that at time $t = 0$ the velocity is equal to $z_0 \neq z_s$, then even if z_0 is close to z_s, the resulting unsteady velocity $\bar{z}(t)$ when $t > 0$ will not necessary stay close to z_s. Hence, the question we address here is: how to obtain a controller localized on the boundary Γ, which makes $\bar{z}(t)$ go back to z_s as $t \to \infty$?

Keywords and phrases. Navier-Stokes equation, feedback stabilization, Dirichlet control, Riccati equation.

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More precisely, we consider a pair $(\bar{z}, \bar{p})$ solution to the instationary Navier-Stokes equations:

$$\partial_t \bar{z} - \nu\Delta\bar{z} + (\bar{z}\cdot\nabla)\bar{z} + \nabla\bar{p} = f \quad \text{and} \quad \nabla\cdot\bar{z} = 0 \ \text{ in } \ Q, \tag{1.2}$$

$$\bar{z} = v_b + u \ \text{ on } \ \Sigma, \quad \bar{z}(0) = z_s + z_0, \tag{1.3}$$

and we assume that z_s is an unstable solution of (1.2)-(1.3) corresponding to $z_0 = 0$ and $u = 0$. In the above setting $Q = \Omega\times(0,\infty)$ and $\Sigma = \Gamma\times(0,\infty)$. Thus, if we make the change of variable $(\bar{z},\bar{p}) = (z_s + z, p_s + p)$, we have:

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z_s + (z_s\cdot\nabla)z + (z\cdot\nabla)z + \nabla p = 0 \ \text{ in } \ Q, \tag{1.4}$$

$$\nabla\cdot z = 0 \ \text{ in } \ Q, \quad z = u \ \text{ on } \ \Sigma, \quad z(0) = z_0, \tag{1.5}$$

and the question of making $\bar{z}(t)$ go back to z_s as $t\to\infty$ is equivalent to the one of making $z(t)$ go back to 0 as $t\to\infty$. The following questions may be addressed:

- Can we find a set of initial conditions $\mathcal{W}_\delta = \{y\in X(\Omega) \mid \|y\|_{X(\Omega)} < \delta\}$, where $\delta > 0$ and $X(\Omega)\subset\{y\in \mathbf{L}^2(\Omega) \mid \nabla\cdot y = 0\}$, and can we find a space of controls $U(\Gamma)$ such that, for $z_0\in\mathcal{W}_\delta$, there exists a boundary control $u\in L^2(0,\infty;U(\Gamma))$ for which the solution to (1.4)-(1.5) satisfies the exponential decay stated below?
$$\|z(t)\|_{X(\Omega)} \le C\mathrm{e}^{-\eta t}\|z_0\|_{X(\Omega)} \qquad \eta > 0. \tag{1.6}$$
- Can we express u in a feedback formulation? More precisely, we are interested in the existence of an operator $F\in\mathcal{L}(X(\Omega),U(\Gamma))$, independent of the time variable $t\ge 0$, and such that
$$u(t) = Fz(t), \quad t\ge 0. \tag{1.7}$$
- Is there a practical way to compute F?

First, let us mention some results partially answering those questions. In the two and three dimensional case, the existence of a pair (z,u), which satisfies (1.4)-(1.5) and (1.6), is stated in [11,12] with $X(\Omega) = \{y\in\mathbf{H}^1(\Omega) \mid \nabla\cdot y = 0,\ y|_{\Gamma_0} = 0,\ t_\Gamma y\cdot n = 0\}$ and $U(\Gamma) = \{y\in\mathbf{H}^{3/2}(\Gamma) \mid y|_{\Gamma_0} = 0,\ \int_\Gamma y\cdot n = 0\}$, $\Gamma = \Gamma\cup\Gamma_0$ and $\Gamma\cap\Gamma_0 = \emptyset$. The key idea in [11,12] relies in an adequate extension operator which maps an initial condition, defined in Ω, to an extended and stable initial condition, defined in an open set G which contains Ω. By this way, the author obtains an operator $F_0\in\mathcal{L}(X(\Omega),L^2(0,\infty;U(\Gamma)))$ such that $u = F_0 z_0$, but he does not obtain a control in the pointwise (in time) formulation (1.7).

In the two dimensional case, the existence of a pair (z,u) satisfying (1.4)-(1.5) and (1.6)-(1.7), and such that u is localized in a part of Γ and has a non vanishing normal component, is proved in [21]. In this paper, $X(\Omega) = \{y\in\mathbf{H}^{1/2-\epsilon}(\Omega) \mid \nabla\cdot y = 0,\ y\cdot n = 0 \text{ on } \Gamma\}$ and $U(\Gamma) = \{my \mid y\in\mathbf{L}^2(\Gamma),\ \int_\Gamma my\cdot n = 0\}$, where $\epsilon\in\,]0,1/4[$ and $m\in C^2(\Gamma)$ is an adequate localization function. The feedback controller is determined by an algebraic Riccati equation which is obtained by solving an optimal control problem. The key point of this approach relies in a reformulation of system (1.4)-(1.5), which only involves Pz, where P is the orthogonal Leray projection operator (see Sect. 2.1). We point out the fact that, since the three dimensional case is more demanding in terms of velocity regularity, and in particular we will see that it requires the compatibility condition $u(0) = z_0|_\Gamma$, it cannot be treated in the same way. In [22] the author overcomes this difficulty, by introducing a time dependent feedback law.

The three dimensional case is treated in [4]. The existence of a pair (z,u), which satisfies (1.4)-(1.5) and (1.6)-(1.7), is stated. In this paper, $X(\Omega) = \{y\in\mathbf{H}^{1/2+\epsilon}(\Omega) \mid \nabla\cdot y = 0,\ y\cdot n = 0 \text{ on } \Gamma\}$ and $U(\Gamma) = \{y\in\mathbf{L}^2(\Gamma) \mid y\cdot n = 0 \text{ on } \Gamma\}$. The authors follow the ideas developed in [3,5], where the case of pointwise feedback stabilization of the 3D Navier-Stokes equations, by means of a distributed control, is investigated. However, the boundary feedback law which is proposed in [4] cannot be numerically calculated. This difficulty is closely linked to the high degree of regularity for the velocity which is necessary to obtain the exponential decrease of the solution of the Navier-Stokes system in the three dimensional case. To obtain the required smoothness degree for the state, the authors solve an optimal control problem involving the velocity norm $L^2(0,\infty;\mathbf{H}^{3/2+\epsilon}(\Omega))$ in the cost functional, and it does not allow to define a feedback law from a well posed Riccati equation. Indeed, the Riccati equation is only defined in $\mathcal{D}(A_R^2)$, where A_R is the infinitesimal generator of the associated closed-loop system, which itself depends on the unknown R of the Riccati equation.

In fact, when $d=3$, obtaining a well-posed closed-loop system (1.4)-(1.5)-(1.7) is not an easy task. Let us give some explanations about the difficulties linked to the three dimensional analysis, which requires a particular compatibility condition between the state and the control. After we compare the maximal order derivatives in time and in space appearing in (1.4), we introduce the function space

$$\mathbf{H}^{\alpha,\beta}(Q)=L^2(0,\infty;\mathbf{H}^{\alpha}(\Omega))\cap H^{\beta}(0,\infty;\mathbf{L}^2(\Omega)),\quad \alpha\ge 0,\beta\ge 0,$$

and we postulate that a strong solution to (1.4)-(1.5) should be searched in $\mathbf{H}^{1+s,1/2+s/2}(Q)$ for $s\ge 0$. This framework is used in [21] to define solutions to the two dimensional closed-loop Navier-Stokes system and in [23] to obtain optimal regularity results for the Oseen system with a nonhomogeneous boundary condition. Hence, z obeys:

$$\partial_t z\in H^{-1/2+s/2}(0,\infty;\mathbf{L}^2(\Omega))\quad \text{and}\quad -\nu\Delta z+(z\cdot\nabla)z_s+(z_s\cdot\nabla)z\in L^2(0,\infty;\mathbf{H}^{s-1}(\Omega)). \tag{1.8}$$

In order to get rid of the pressure term in (1.4) and to simplify our analysis, it is more convenient to evaluate the state equation (1.4) in a space of distributions which is orthogonal to the space of gradient pressures. By projecting equation (1.4) onto $V_0^{s-1}(\Omega)$, the dual space of $V_0^{1-s}(\Omega)=\{y\in\mathbf{H}_0^{1-s}(\Omega)\mid \nabla\cdot y=0,\ y\cdot n=0\ \text{ on }\Gamma\}$ (see Sect. 2.1), we obtain the following necessary condition from (1.8):

$$-P(z\cdot\nabla)z=P(\partial_t z-\nu\Delta z+(z\cdot\nabla)z_s+(z_s\cdot\nabla)z)\in\mathbf{Y}^s(Q),$$

where P is the Leray orthogonal projection operator (see Sect. 2.1), and where

$$\mathbf{Y}^s(Q)=H^{-1/2+s/2}(0,\infty;\mathbf{L}^2(\Omega))+L^2(0,\infty;V_0^{s-1}(\Omega)). \tag{1.9}$$

Thus, by remarking that the free divergence condition $\nabla\cdot z=0$ yields $(z\cdot\nabla)z=\nabla\cdot(z\otimes z)$, we deduce that we shall look for a velocity z solution to (1.4)-(1.5)-(1.7) which obeys:

$$z\in\mathbf{H}^{1+s,1/2+s/2}(Q)\quad \text{and}\quad P\nabla\cdot(z\otimes z)\in\mathbf{Y}^s(Q). \tag{1.10}$$

A brief check of the regularity of $\nabla\cdot(z\otimes z)$ which can be obtained from $z\in\mathbf{H}^{1+s,1/2+s/2}(Q)$, shows that when $d=3$, the value s should be chosen greater than $1/2$. Indeed, from the continuous embedding $\mathbf{H}^{1+s,1/2+s/2}(Q)\hookrightarrow H^{1/4}(0,\infty;\mathbf{H}^{s+1/2}(\Omega))$ and with

$$\left\{uv\mid (u,v)\in H^{1/4}(0,\infty;H^{s+1/2}(\Omega))\times H^{1/4}(0,\infty;H^{s+1/2}(\Omega))\right\}\subset L^2(0,\infty;H^{2s-1/2}(\Omega)),$$

it yields $z\otimes z\in L^2(0,\infty;\mathbf{H}^{2s-1/2}(\Omega))$. Then we obtain $\nabla\cdot(z\otimes z)\in L^2(0,\infty;\mathbf{H}^{2s-3/2}(\Omega))$, and for $s\ge 1/2$ the second statement in (1.10) follows from $L^2(0,\infty;\mathbf{H}^{2s-3/2}(\Omega))\hookrightarrow L^2(0,\infty;\mathbf{H}^{-1+s}(\Omega))$. As a consequence, when $d=3$ the nonlinearity of the Navier-Stokes system imposes to define a solution z which belongs to $\mathbf{H}^{1+s,1/2+s/2}(Q)$ for $s\ge 1/2$. Hence, the trace theorem yields $z|_\Sigma\in\mathbf{H}^{1/2+s,1/4+s/2}(\Sigma)$, and the feedback controller has to obey:

$$z|_\Sigma=Fz\in H^{1/4+s/2}(0,\infty;\mathbf{L}^2(\Gamma))\quad s\ge 1/2. \tag{1.11}$$

Since $1/4+s/2\ge 1/2$, some kind of continuity is required for the control. In the particular case where $s>1/2$, the space $H^{1/4+s/2}(0,\infty;\mathbf{L}^2(\Gamma))$ is a subspace of $C([0,\infty[\,;\mathbf{L}^2(\Gamma))$ (the space of time continuous functions with value in $\mathbf{L}^2(\Gamma)$) and we deduce from (1.11) that the velocity z must satisfy the initial compatibility condition

$z_0|_\Gamma = Fz_0$. This explains why the feedback law which is given in [21] cannot be used in the three dimensional case, and why the author overcomes this difficulty in [22] by introducing a feedback law which is time dependent in an initial transitory time interval. Notice that spaces of initial conditions, for which a stabilization result can be obtain with the Riccati approach, are precisely given in [2].

In fact, finding a feedback controller independent of the time variable and which satisfy $Fz_0 = z_0|_\Gamma$ for a sufficiently large class of initial conditions z_0 is not obvious. That is the reason why in the present paper, we propose to search another type of pointwise (in time) feedback law. We search u as a solution to the following evolution system:

$$\partial_t u - \Delta_b u - \sigma\, n = K(z, u), \quad u(0) = z_0|_\Gamma, \tag{1.12}$$

where the feedback controller K now acts on the pair (z, u). Here Δ_b is the vector-valued Laplace Beltrami operator (see Sect. 5). Formulation (1.12) involves the time derivative of u, so we can fix the initial condition: the initial boundary value $u(0)$ now fits the initial trace $z_0|_\Gamma$. We underline that we had a large degree of freedom in the choice of the boundary system. We have chosen (1.12) for its simplicity and for numerical computational conveniences in view of future implementations (u must be numerically calculated). But one may imagine another boundary system which would have a physical interpretation and which could be concretely constructed. The present paper is a complete and detailed version of [1]. Its main objectives are:

(i) to show the existence of a pair (z, u) satisfying (1.4)-(1.5)-(1.6) and (1.12) in the two and three dimensional case;
(ii) to find an operator K which is provided by a well-posed Riccati equation;
(iii) to find a way to obtain a control u localized on an arbitrary small part of Γ.

• *The Navier-Stokes equations coupled with a boundary system.* We define the space of initial conditions:

$$X(\Omega) = V^s(\Omega) := \left\{ y \in \mathbf{H}^s(\Omega) \mid \nabla \cdot y = 0, \int_\Gamma y \cdot n = 0 \right\}, \quad s \in]1/2, 1], \tag{1.13}$$

and we assume that $z_0 \in X(\Omega)$. In order to impose the compatibility condition $u(0) = z_0|_\Gamma$ and to obtain a sufficient time regularity level for z, we choose to search u as the first component of (u, σ) which satisfies:

$$\partial_t u - \Delta_b u - \sigma\, n = K(z, u) \quad \text{in } \Sigma, \quad u(0) = z_0|_\Gamma \quad \text{and} \quad \int_\Gamma u(t) \cdot n = 0. \tag{1.14}$$

Recall that Δ_b is the vector-valued Laplace Beltrami operator and $\sigma \in L^2(0, T)$ plays the role of the Lagrange multiplier associated with the constraint $\int_\Gamma u \cdot n = 0$. The feedback law K is a linear operator, independent of $t \geq 0$, and it couples (1.14) with (1.4)-(1.5). The state (z, u) now satisfies the following coupled system:

$$\partial_t z - \nu\Delta z + (z \cdot \nabla)z_s + (z_s \cdot \nabla)z + (z \cdot \nabla)z + \nabla p = 0, \;\; \nabla \cdot z = 0 \;\text{ in }\; Q, \tag{1.15}$$

$$\partial_t u - \Delta_b u - \sigma\, n = K(z, u), \quad z = u \;\text{ on }\; \Sigma, \quad \int_\Gamma u(t) \cdot n = 0, \tag{1.16}$$

$$z(0) = z_0 \in V^s(\Omega), \quad u(0) = z_0|_\Gamma. \tag{1.17}$$

We are going to show that we can choose K so that (1.15)-(1.16)-(1.17) is well defined when z_0 is small in $V^s(\Omega)$, and so that z obeys (1.6). The operator K can be considered as a pointwise feedback controller which is acting on (z, u) solution to the extended system (1.15)-(1.16)-(1.17). That is the reason why we shall say that our approach is a compromise between the formulation of a control in the form (1.7), and the treatment of the 3 dimensional case which requires a high regularity level for the control.

• *Calculation of the feedback controller K.* In a first step, we shall simplify our problem and consider the question of stabilizing the linear system obtained from (1.15)-(1.16)-(1.17) by linearizing this one around $(0, 0)$. In other words, we want to find a control $g \in L^2(0, \infty; \mathbf{L}^2(\Gamma))$, which can be expressed in a feedback form, and such that the solution (z, u) to the following linear system is stable:

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z_s + (z_s\cdot\nabla)z + \nabla p = 0, \quad \nabla\cdot z = 0 \ \text{ in } \ Q, \quad z(0) = z_0, \tag{1.18}$$

$$\partial_t u - \Delta_b u - \sigma\, n = g, \quad z|_\Sigma = u \ \text{ on } \ \Sigma, \quad \int_\Gamma u(t)\cdot n = 0, \quad u(0) = u_0. \tag{1.19}$$

In the above setting, (z_0, u_0) is an arbitrary initial pair satisfying $z_0\cdot n = u_0\cdot n$ on Γ. The question of constructing a linear feedback controller stabilizing a linear dynamical system can be answered with the so-called "Riccati approach". It consists in solving an auxiliary optimal control problem, defined over an infinite time horizon, and which involves a linear quadratic cost functional. It provides an optimal control in a feedback form, with a feedback law depending on the solution to an algebraic Riccati equation. Such optimal control theory is developed in [18], Chapter 2. We shall underline the fact that to use the Riccati theory, we shall work with an abstract dynamical model representing equations (1.18)-(1.19) [18], Chapter 2, Section 2.1. Hence, we need to rewrite the system (1.18)-(1.19) as an evolution equation of the type:

$$Y' = \mathcal{A}Y + \Lambda G \ \text{ on } \ \mathcal{D}(\mathcal{A}^*)', \quad Y(0) = Y_0, \tag{1.20}$$

where $\mathcal{A}$ is a linear free dynamic operator, Λ is a linear control operator, and Y and G are the new state and control variables. This can be achieve with the new variable $Y = (y, u)^T = (Pz, u)^T$, where P is the orthogonal Leray projection operator (see Sect. 2.1), with Λ as the canonical projection $(y, u)^T \longmapsto (0, u)^T$ and with an operator $\mathcal{A}$ defined from the free dynamic operators related to each equations (1.18) and (1.19) separately. More precisely, the dynamical system (1.20) can be obtain from the dynamical system related to $y = Pz$ and u:

$$\begin{array}{rcl} y' &=& Ay + Bu \ \text{ on } \ \mathcal{D}(A^*)', \quad y(0) = Pz_0, \\ u' &=& A_b u + g \ \text{ on } \ \mathcal{D}(A_b^*)', \quad u(0) = u_0, \end{array}$$

where A is the free dynamic Oseen operator (see Def. 6.1), A_b is the free dynamic operator defined from Δ_b (see Sect. 5), and B is an input operator which allows to represent the trace condition linking y and u. We shall insist on the fact that the main difficulty relies in the definition of the operator B, which can be done by using the theory developed in [23]. Hence, we look for the control G and the associated state Y solution to (1.20) which minimize the cost functional:

$$\mathcal{J}(Y, G) = \int_0^\infty \|\mathcal{C}Y\|^2_{\mathcal{Z}} + \int_0^\infty \|G\|^2_{\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma)},$$

where $\mathcal{C}$ is an observation operator with value in the space $\mathcal{Z}$. We define the optimal control problem:

$$\inf\left\{\mathcal{J}(Y, G), (Y, G) \text{ satisfies } (1.20)\right\} \tag{1.21}$$

and the resolution of (1.21) provides a feedback control $G = -\Lambda\Pi Y$ where Π is the solution of an algebraic Riccati equation which can be formally written as follows:

$$\Pi\mathcal{A} + \mathcal{A}^*\Pi - \Pi\Lambda\Pi + \mathcal{C}^*\mathcal{C} = 0. \tag{1.22}$$

The precise definition of such a Riccati equation is given in Theorem 7.3. Thus, we apply this feedback control to the nonlinear system, and it yields the following expression of K in (1.15)-(1.16)-(1.17):

$$K(z, u) = -\Pi_2 Pz - \Pi_3 u \quad \text{where} \quad \Pi = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix} \quad \text{is the solution to (1.22).}$$

Notice that (1.22) gives a practical way to calculate the operator K. Finally, we show that K stabilizes (1.15)-(1.16)-(1.17) in a neighborhood of the origin: there is $\delta > 0$ such that, if $\|z_0\|_{V^s(\Omega)} < \delta$, then there exists a unique pair (z, u) satisfying (1.15)-(1.16)-(1.17), which obeys (1.6). We recall that $V^s(\Omega)$ is defined in (1.13).

• *Localization of the control in a part of the boundary.* To treat the case of a boundary control which is localized in an open part of Γ, we replace the boundary condition $z|_\Sigma = u$ by $z|_\Sigma = m(u - \sigma_m(u)\, n)$, where $\sigma_m(u) = (\int_\Gamma m)^{-1} \int_\Gamma mu \cdot n$ and $m \in C^2(\Gamma)$ is an adequate cut-off function with values in $[0, 1]$. By this way, the action of u is localized on $\Gamma_m = \mathrm{Supp}(m)$. Thus, we define the corresponding operator $(\mathcal{A}_m, \mathcal{D}(\mathcal{A}_m))$ and $\mathcal{C}_m$, and the resolution of

$$\inf \left\{ \int_0^\infty \|\mathcal{C}_m Y\|_\Xi^2 + \int_0^\infty \|G\|^2_{\mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Gamma)} \mid \quad Y' = \mathcal{A}_m Y + \Lambda G, \quad Y(0) = Y_0 \right\}, \tag{1.23}$$

provides an operator Π_m satisfying

$$\Pi_m \mathcal{A}_m + \mathcal{A}_m^* \Pi_m - \Pi_m \Lambda \Pi_m + \mathcal{C}_m^* \mathcal{C}_m = 0, \quad \Pi_m = \begin{pmatrix} \Pi_{1,m} & \Pi_{2,m}^* \\ \Pi_{2,m} & \Pi_{3,m} \end{pmatrix}.$$

Hence, we obtain a local stabilization result with the feedback control $K(z, u) = -\Pi_{2,m} P z - \Pi_{3,m} u$. Such a treatment of localized control only adds technical difficulties in the statement of the finite cost condition which guarantees the well-posedness of (1.23). For readability convenience, the main parts of this paper deals with the non localized case, and we postpone the treatment of a localized control to Sections 9 and 10.

The paper is organized as follows. In Section 2 we recall some background material needed throughout the paper and we define spaces of initial conditions. Section 3 is dedicated to the statement of the local stabilization result. We write the Oseen system in the form of an evolution equation in Section 4, and we write the differential boundary system in the form of an evolution equation in Section 5. Next, we define the operator $\mathcal{A}$ and we study the linear system (1.20) in Section 6. Section 7 is dedicated to the study of the optimal control problem (1.21) which provides a feedback controller K. In Section 8, we apply this feedback law to the nonlinear system and we give a proof of the local stabilization result. Finally, we deal with the localization of the control on a part of the boundary in Section 9, and we postpone in a appendix the proof of a finite cost condition ensuring the well-posedness of (1.21) and (1.23).

2. Functional framework

2.1. **Notations**

Let X and Y be two Hilbert spaces. If A is a closed linear mapping in X, we denote its domain by $\mathcal{D}(A)$. We denote by $\mathcal{L}(X, Y)$ the space of all bounded operators from X to Y, and we use the shorter expression $\mathcal{L}(X) = \mathcal{L}(X, X)$. For $0 < T \le \infty$, the space $L^2(0, T; X)$ is the usual vector-valued Lebesgue space and $H^\alpha(0, T; X)$ for $\alpha \ge 0$ is the usual vector-valued Sobolev space. If $C_0^\infty(]0, T[\,; X)$ is the space of infinitely differentiable and compactly supported functions of $t \in\,]0, T[$ with values in X, we denote by $H_0^\alpha(0, T; X)$ the closure of $C_0^\infty(]0, T[\,; X)$ in $H^\alpha(0, T; X)$, and by $H^{-\alpha}(0, T; X')$ the dual space of $H_0^\alpha(0, T; X)$, where X' denotes the dual space of X. We also define:

$$W(0, T; X, Y) = \left\{ y \in L^2(0, T; X) \mid \frac{\mathrm{d}y}{\mathrm{d}t} \in L^2(0, T; Y) \right\}.$$

It is well known that if X is continuously and densely embedded in Y, then the space $W(0, T; X, Y)$ is continuously embedded in $C([0, T]; [X, Y]_{1/2})$ if $T < \infty$ or in $C_b([0, \infty[\,; [X, Y]_{1/2})$ if $T = \infty$, the space of bounded and time continuous functions with values in the interpolation space $[X, Y]_{1/2}$ [19].

Next, let us recall that Ω is a bounded and connected domain in $\mathbb{R}^d$, for $d = 2$ or $d = 3$, with a boundary $\Gamma = \partial\Omega$ of class C^4, and composed of N connected components $\Gamma^{(1)}, \ldots, \Gamma^{(N)}$. We will use the usual function spaces $L^2(\Omega)$, $H^s(\Omega)$, $H_0^s(\Omega)$ and $H^{-s}(\Omega) = (H_0^s(\Omega))'$, and we write in bold the spaces of vector fields $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$, $\mathbf{H}^s(\Omega) = (H^s(\Omega))^d$, $\mathbf{H}_0^s(\Omega) = (H_0^s(\Omega))^d$ and $\mathbf{H}^{-s}(\Omega) = (H^{-s}(\Omega))^d$. The norms are denoted by $\|\cdot\|_{Z(\Omega)}$, where the subscript $Z(\Omega)$ refers to the space which is considered, and we denote the scalar product in $\mathbf{L}^2(\Omega)$ by $(\cdot|\cdot)$. We denote by Δ the vector-valued Laplace operator, with domain $\mathcal{D}(\Delta) = \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$, which is known to be selfadjoint. For all $s \in [0,2]$, its fractional power $(-\Delta)^{s/2}$ is well defined and obeys $\mathcal{D}((-\Delta)^{s/2}) = [\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega), \mathbf{L}^2(\Omega)]_{1-s/2}$, where $[\cdot,\cdot]$ denotes the complex interpolation method. Let $\mathbf{L}^2_{-1/2}(\Omega)$ be the space of functions $y \in \mathbf{L}^2(\Omega)$ such that $\int_\Omega \rho(x)^{-1}|y|^2 < +\infty$, where $\rho(x)$ is the distance from x to Γ. From [15], Theorem 8.1, it can be deduced that $\mathcal{D}((-\Delta)^{s/2}) = \mathbf{H}^s(\Omega)$ if $0 \le s < 1/2$, that $\mathcal{D}((-\Delta)^{1/2}) = \{y \in \mathbf{H}^{1/2}(\Omega) \mid y \in \mathbf{L}^2_{-1/2}(\Omega)\}$ and that $\mathcal{D}((-\Delta)^{s/2}) = \{y \in \mathbf{H}^s(\Omega) \mid y|_\Gamma = 0\}$ if $1/2 < s \le 2$.

Next, if $y \in \mathbf{L}^2(\Omega)$ is such that $\nabla\cdot y \in L^2(\Omega)$, we can define its normal trace $y\cdot n$ in $H^{-1/2}(\Gamma)$ [13], Section III.3, and we introduce the spaces of free divergence functions:

$$
\begin{aligned}
V^s(\Omega) &= \left\{y \in \mathbf{H}^s(\Omega) \mid \nabla\cdot y = 0 \text{ in } \Omega,\ \int_\Gamma y\cdot n = 0\right\}, \quad s \in [0,2],\\
V_n^s(\Omega) &= \left\{y \in \mathbf{H}^s(\Omega) \mid \nabla\cdot y = 0 \text{ in } \Omega,\ y\cdot n = 0 \ \text{ on } \ \Gamma\right\}, \quad s \in [0,2].
\end{aligned}
$$

Moreover, we denote by P the so-called Leray projector which is the orthogonal projector from $\mathbf{L}^2(\Omega)$ onto $V_n^0(\Omega)$ [13], Chapter III, Theorem 1.1, we define the self-adjoint operator $A_0 = P\Delta$ with domain $\mathcal{D}(A_0) = \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\cap V_n^0(\Omega)$, and for $s \in [0,2]$ we introduce the space $V_0^s(\Omega) = \mathcal{D}((-A_0)^{s/2})$. According to [10], we have

$$
V_0^s(\Omega) = \mathcal{D}((-\Delta)^{s/2})\cap V_n^0(\Omega) = \left[\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\,,\,\mathbf{L}^2(\Omega)\right]_{1-s/2}\cap V_n^0(\Omega) \quad \text{for all } s \in [0,2],
$$

which yields the following equalities:

$$
\begin{aligned}
V_0^s(\Omega) &= V_n^s(\Omega), \quad s \in [0,1/2[,\\
V_0^{1/2}(\Omega) &= \left\{y \in V_n^{1/2}(\Omega) \ \mid \ y \in \mathbf{L}^2_{-1/2}(\Omega)\right\},\\
V_0^s(\Omega) &= \left\{y \in V_n^s(\Omega) \ \mid \ y = 0 \ \text{ on } \ \Gamma\right\}, \quad s \in]1/2,2].
\end{aligned}
$$

Notice that the subscript 0 in $V_0^s(\Omega)$ only means that one may have a vanishing Dirichlet boundary condition. The above characterization can also be obtain from the equality $\mathcal{D}((-A_0)^{s/2}) = [\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)\cap V_n^0(\Omega), V_n^0(\Omega)]_{1-s/2}$ for $s \in [0,2]$ (which can be obtain from [6], Chap. 1, Cor. 6.1), by remarking that $K_0 = A_0^{-1}P\Delta$ defines a bounded projector from $\mathbf{L}^2(\Omega)$ into $V_n^0(\Omega)$ which allows to invoke [26], Section 1.17.1, Theorem 1. Finally, for $s > 2$ we define $V_0^s(\Omega) = V_0^2(\Omega)\cap\mathbf{H}^s(\Omega)$ and for $s < 0$ we define $V_0^s(\Omega) = (V_0^{-s}(\Omega))'$, the dual space of $V_0^{-s}(\Omega)$ with respect to the pivot space $V_n^0(\Omega)$. It is equipped with the duality pairing $\langle\cdot|\cdot\rangle_{V_0^{-s}(\Omega),V_0^s(\Omega)}$. We also recall that P can be extended to a bounded linear operator from $\mathbf{H}^{-1}(\Omega)$ onto $V_0^{-1}(\Omega)$ by

$$
Py : w \in V_0^1(\Omega) \longmapsto \langle y \,|\, w\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)}, \quad \text{[4], Appendix A.}
$$

Next, we define the spaces of pressures with free mean

$$
L_0^2(\Omega) = \left\{p \in L^2(\Omega) \mid \int_\Omega p = 0\right\} \quad \text{and} \quad \mathbb{H}^s(\Omega) = H^s(\Omega)\cap L_0^2(\Omega), \qquad s \ge 0,
$$

and we recall that the following *Helmholtz decomposition* holds:

$$
\mathbf{L}^2(\Omega) = V_n^0(\Omega)\oplus\nabla\mathbb{H}^1(\Omega).
$$

Next, we define the following trace spaces with a free mean normal component

$$\begin{aligned} V^s(\Gamma) &= \Big\{ y \in \mathbf{H}^s(\Gamma) \mid \int_\Gamma y\cdot n = 0 \Big\}, \quad s\in[0,3],\\ V^{-s}(\Gamma) &= \Big\{ y \in \mathbf{H}^{-s}(\Gamma) \mid \langle y|n\rangle_{\mathbf{H}^{-s}(\Gamma),\mathbf{H}^s(\Gamma)} = 0 \Big\}, \quad s\in[0,3]. \end{aligned}$$

We make the identification $(V^s(\Gamma))' = \mathbf{H}^{-s}(\Gamma)/(V^s(\Gamma))^\perp$, where $(V^s(\Gamma))'$ denotes the dual space of $V^s(\Gamma)$ with respect to the pivot space $V^0(\Gamma)$ and $(V^s(\Gamma))^\perp = \{y\in \mathbf{H}^{-s}(\Gamma) \mid \langle y|w\rangle_{\mathbf{H}^{-s}(\Gamma),\mathbf{H}^s(\Gamma)} = 0,\ \forall w\in V^s(\Gamma)\}$. We verify that $(V^s(\Gamma))^\perp = \mathbb{R}n$, and that:

$$V^{-s}(\Gamma) = \mathbf{H}^{-s}(\Gamma)/\mathbb{R}n = (V^s(\Gamma))'.$$

Moreover, we introduce the orthogonal projector P_b from $\mathbf{L}^2(\Gamma)$ onto $V^0(\Gamma)$, which is explicitly given by

$$P_b v = v - \frac{1}{|\Gamma|}\Big(\int_\Gamma v\cdot n\Big)n. \tag{2.1}$$

Next, we recall that the normal trace operator $\gamma_n \in \mathcal{L}(\mathbf{L}^2(\Gamma))$ is defined by $\gamma_n(u) = (u\cdot n)n \in \mathbf{L}^2(\Gamma)$, and we extend its definition to $V^{-s}(\Gamma)$ for $s>0$ with the formula

$$\gamma_n(u) = \langle u|\gamma_n(\cdot)\rangle_{V^{-s}(\Gamma),V^s(\Gamma)} \quad \text{for all } u\in V^{-s}(\Gamma).$$

The boundary normal derivative on Γ of a vector field $v\in\mathbf{H}^2(\Omega)$ is defined by $\partial_n v = (\nabla v)n$.
Finally, we shall underline that we will also need the spaces

$$\mathcal{H}^0 = V^0_n(\Omega)\times V^{-1/2}(\Gamma) \quad \text{and} \quad \mathcal{H}^0_* = V^0_n(\Omega)\times V^{1/2}(\Gamma),$$

and the spaces $\mathcal{H}^{2\theta}$ and $\mathcal{H}^{2\theta}_*$, for all $\theta\in[-1,1]$, which will be precisely defined later on in Definition 6.4.

2.2. Space of initial conditions

Definition (1.13) is unnecessary restrictive. We have fixed $s>1/2$ in the introduction for readability convenience but we can also assume that $s\in[0,1/2[$ if $d=2$. The limit case $s=1/2$ may involve some technical difficulties so we choose to avoid it (see Rem. 3.4). In the whole following, we choose $X(\Omega)=V^s(\Omega)$ for $s\in[\frac{d-2}{2},1]\backslash\{1/2\}$ as the space of initial condition, and we introduce the operator $\gamma^s : V^s(\Omega)\to V^{s-1/2}(\Gamma)$ as follows.

Definition 2.1. For all $s\in[0,1]\backslash\{1/2\}$, we define the linear operator $\gamma^s : V^s(\Omega)\to V^{s-1/2}(\Gamma)$ by

$$\gamma^s(y) = \begin{cases} (y\cdot n)n, & \text{if } s\in[0,1/2[,\\ y|_\Gamma & \text{if } s\in]1/2,1]. \end{cases}$$

Proposition 2.2. *The linear operator γ^s satisfies the following regularity properties:*

$$\gamma^s \in \mathcal{L}(V^s(\Omega), V^{s-1/2}(\Gamma)), \quad s\in[0,1]\backslash\{1/2\}. \tag{2.2}$$

Proof. This proposition is a straightforward consequence of the well known trace and normal trace properties. □

3. Main result

In this article, we prove the following local stabilization result.

Theorem 3.1. *Let $s \in [\frac{d-2}{2}, 1]\backslash\{1/2\}$. There exists two linear operators $\Pi_2 \in \mathcal{L}(V_n^0(\Omega), V^{1/2}(\Gamma))$ and $\Pi_3 \in \mathcal{L}(V^{-1/2}(\Gamma), V^{1/2}(\Gamma))$ such that, if we consider the following coupled system:*

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z_s + (z_s\cdot\nabla)z + (z\cdot\nabla)z + \nabla p = 0 \ \ and \ \ \nabla\cdot z = 0 \ \ in \ \ Q, \tag{3.1}$$

$$\partial_t u - \Delta_b u + \Pi_3 u - \sigma\, n = -\Pi_2 Pz \ \ in \ \ \Sigma, \quad z = u \ \ on \ \ \Sigma, \quad \int_\Gamma u(t)\cdot n = 0, \ \ t \geq 0, \tag{3.2}$$

$$z(0) = z_0 \in V^s(\Omega), \quad u(0) = \gamma^s(z_0), \tag{3.3}$$

then the following results hold. There exist $c > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and

$$z_0 \in \mathcal{W}_\delta^s = \Big\{ z_0 \in V^s(\Omega) \ \mid \ \|z_0\|_{V^s(\Omega)} \leq c\delta \Big\}, \tag{3.4}$$

then, (3.1)-(3.2)-(3.3) admits a unique solution in the set

$$\begin{aligned}\mathcal{D}_\delta^s \ \ = \ \ \Big\{ & (z,p,u,\sigma) \in W(0,\infty; V^{s+1}(\Omega), V_0^{s-1}(\Omega)) \times H^{s/2-1/2}(0,\infty;\mathbb{H}^s(\Omega)) \\ & \quad \times W(0,\infty; V^{s+1/2}(\Gamma), V^{s-3/2}(\Gamma)) \times L^2(0,\infty) \ \mid \\ & \|z\|_{W(0,\infty;V^{s+1}(\Omega),V_0^{s-1}(\Omega))} + \|u\|_{W(0,\infty;V^{s+1/2}(\Gamma),V^{s-3/2}(\Gamma))} + \|\sigma\|_{L^2(0,\infty)} \leq \delta, \\ & \|p\|_{H^{s/2-1/2}(0,\infty;H^s(\Omega))} \leq \delta(1+\delta) \Big\}.\end{aligned} \tag{3.5}$$

Moreover, there exist $C > 0$ and $\eta > 0$ such that (z,u) obeys

$$\|z(t)\|_{V^s(\Omega)} + \|u(t)\|_{V^{s-1/2}(\Gamma)} \leq C\|z_0\|_{V^s(\Omega)}\, \mathrm{e}^{-\eta t} \quad \forall t \geq 0. \tag{3.6}$$

Remark 3.2. The linear operators Π_2 and Π_3 are components of Π which is the solution to the Riccati equation (7.7) given later on in Section 7, see Remark 7.4.

Remark 3.3. In fact, assuming Ω of class C^3 [16], Chapter 1, Definition 1.2.1.2, is sufficient to obtain Theorem 3.1. Indeed, the assumption Ω of class C^4 is only needed in the second step of the proof of Theorem 10.2 of the appendix, to treat the case of a control localized on a part of the boundary (see Rem. 10.3). Notice that with Ω of class C^3 the trace space $V^s(\Gamma)$ is well defined for $s \in [0,3]$ [16], Chapter 1, Definition 1.3.3.2. See also Remark 5.3.

Remark 3.4. We decide to avoid the limit case $s = 1/2$ only because it involves technical difficulties. In fact, Theorem 3.1 remains valid for initial condition $(z(0), u(0)) = (z_0, u_0)$ belonging to

$$\Big\{ (z_0, u_0) \in V^{1/2}(\Omega) \times V^0(\Gamma) \mid z_0 - Du_0 \in V_0^{1/2}(\Omega), \ \|z_0\|_{V^{1/2}(\Omega)} + \|u_0\|_{V^0(\Gamma)} \leq c\delta \Big\},$$

where D is the lifting operator given in Section 4. Hence, we have to replace (3.6) by

$$\|z(t)\|_{V^{1/2}(\Omega)} + \|u(t)\|_{V^0(\Gamma)} \leq C(\|z_0\|_{V^{1/2}(\Omega)} + \|u_0\|_{V^0(\Gamma)})\, \mathrm{e}^{-\eta t}, \quad t \geq 0.$$

Remark 3.5. For $s \in [\frac{d-2}{2}, 1]$ and $z_0 \in V_0^s(\Omega)$, Theorem 3.1 holds with $u(0) = 0$.

As explained in the introduction, the case of a boundary control localized in an open part of Γ can be treated by introducing an adequate cut-off function $m \in C^2(\Gamma)$, with values in $[0,1]$. We assume that m is supported in $\Gamma_m \subset \Gamma$, and is equal to 1 in Γ_1, where Γ_1 is an open subset of Γ_m. We introduce the space of initial conditions

$$V_m^s(\Omega) = \{y \in V^s(\Omega) \mid (1-m)\gamma^s(y) = 0\}, \tag{3.7}$$

and we prove the following localized version of Theorem 3.1.

Theorem 3.6. *Let $s \in [\frac{d-2}{2}, 1]\backslash\{1/2\}$. There is two linear operators $\Pi_{m,2} \in \mathcal{L}(V_n^0(\Omega), V^{1/2}(\Gamma))$ and $\Pi_{m,3} \in \mathcal{L}(V^{-1/2}(\Gamma), V^{1/2}(\Gamma))$ such that, if we consider the following coupled system:*

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z_s + (z_s\cdot\nabla)z + (z\cdot\nabla)z + \nabla p = 0 \ \ \textit{and} \ \ \nabla\cdot z = 0 \ \ \textit{in} \ \ Q, \tag{3.8}$$

$$z = m(u - \sigma_m(u)n) \ \ \textit{on} \ \ \Sigma, \quad \sigma_m(u) = \left(\int_\Gamma m\right)^{-1}\int_\Gamma mu\cdot n, \tag{3.9}$$

$$\partial_t u - \Delta_b u + \Pi_{m,3}u - \sigma\, n = -\Pi_{m,2}Pz \ \ \textit{in} \ \ \Sigma, \tag{3.10}$$

$$z(0) = z_0 \in V_m^s(\Omega), \quad u(0) = \gamma^s(z_0), \tag{3.11}$$

then the following result holds. There exist $c > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0,\mu_0)$ and $z_0 \in \mathcal{W}_{m,\delta}^s = \mathcal{W}_\delta^s \cap V_m^s(\Omega)$, then, (3.8)-(3.9)-(3.10)-(3.11) admits a unique solution in the set $\mathcal{D}_\delta^s$, which is defined by (3.5). Moreover, there exist $C > 0$ and $\eta > 0$ such that (z,u) obeys (3.6).

Remark 3.7. The linear operators $\Pi_{m,2}$ and $\Pi_{m,3}$ are components of Π_m which is the solution to the Riccati equation (9.24) given later on in Section 9, see (9.25).

4. The Oseen system

The main objective of this section is to give a precise definition for the solution of the system:

$$\partial_t z - \nu\Delta z + (z\cdot\nabla)z_s + (z_s\cdot\nabla)z + \nabla p = f \ \text{ in } \ Q_T, \tag{4.1}$$

$$\nabla\cdot z = 0 \ \text{ in } \ Q_T, \quad z = u \ \text{ on } \ \Sigma_T, \quad z(0) = z_0. \tag{4.2}$$

In the above setting, $T \in (0,\infty)$ is a fixed time horizon, $Q_T = \Omega\times(0,T)$, $\Sigma_T = \Gamma\times(0,T)$ and $f \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$. By following the ideas introduced in [23], we will rewrite (4.1)-(4.2) as an evolution equation. First, we introduce the following unbounded operators $(\mathcal{D}(A), A)$ and $(\mathcal{D}(A^*), A^*)$ in $V_n^0(\Omega)$:

$$\begin{array}{rclcl} \mathcal{D}(A) & = & V_0^2(\Omega) & \text{and} & Ay = \nu P\Delta y - P(y\cdot\nabla)z_s - P(z_s\cdot\nabla)y, \\ \mathcal{D}(A^*) & = & V_0^2(\Omega) & \text{and} & A^*y = \nu P\Delta y - P(\nabla z_s)^T y + P(z_s\cdot\nabla)y. \end{array}$$

Here, $(b\cdot\nabla)a = (\sum_{i=1}^d b_i\partial_{x_i}a_j)_{1\le j\le d}$ and $(\nabla a)^T b = (\sum_{i=1}^d b_i\partial_{x_j}a_i)_{1\le j\le d}$, and one can verify that $(\mathcal{D}(A^*), A^*)$ is the adjoint of $(\mathcal{D}(A), A)$ with respect to the pivot space $V_n^0(\Omega)$. Throughout the following we denote by $\lambda_0 > 0$ an element in the resolvent set of A satisfying:

$$\langle(\lambda_0 - A)y|y\rangle_{V_0^{-1}(\Omega),V_0^1(\Omega)} \ge \frac{\nu}{2}\|y\|^2_{V_0^1(\Omega)} \quad \text{for all } y \in V_0^1(\Omega). \tag{4.3}$$

Theorem 4.1. *The unbounded operator $(\mathcal{D}(A), A)$ (resp. $(\mathcal{D}(A^*), A^*)$) is the infinitesimal generator of an analytic semigroup on $V_n^0(\Omega)$, and the characterization below holds:*

$$\mathcal{D}((\lambda_0 - A)^\theta) = \mathcal{D}((\lambda_0 - A^*)^\theta) = V_0^{2\theta}(\Omega) \quad \textit{for all } \theta \in [0,1]. \tag{4.4}$$

Proof. See [23], Lemma 4.1. $\square$

We now introduce the Dirichlet operator $D : V^0(\Gamma) \to \mathbf{L}^2(\Omega)$ defined as follows. For $u \in V^0(\Gamma)$, set $Du = w$ where (w, q) satisfies the following system:

$$\lambda_0 w - \nu\Delta w + (w \cdot \nabla)z_s + (z_s \cdot \nabla)w + \nabla q = 0, \quad \nabla \cdot w = 0, \quad w = u \text{ on } \Gamma. \tag{4.5}$$

For rough data $u \in V^0(\Gamma)$, defining a solution to (4.5) can be done with the transposition method. It consists in looking for a velocity $w \in \mathbf{L}^2(\Omega)$ obeying:

$$\int_\Gamma u \cdot (rn - \nu\partial_n\varphi) = \int_\Omega w \cdot f \quad \text{for all } f \in \mathbf{L}^2(\Omega), \tag{4.6}$$

where $(\varphi, r) \in V_0^2(\Omega) \times H^1(\Omega)$ is the unique pair satisfying

$$\lambda_0\varphi - \nu\Delta\varphi + (\nabla z_s)^T\varphi - (z_s \cdot \nabla)\varphi + \nabla r = f \text{ and } \nabla \cdot \varphi = 0 \text{ in } \Omega, \quad \varphi = 0 \text{ on } \Gamma, \quad \int_\Gamma r = 0. \tag{4.7}$$

The existence and uniqueness of $w \in \mathbf{L}^2(\Omega)$ solution to (4.6) is a consequence of the Riesz representation theorem, and an integration by parts allows to prove that a smooth velocity (say $w \in \mathbf{H}^2(\Omega)$ and $u \in V^{3/2}(\Gamma)$) solution to (4.5) in a classical sense is also the solution to (4.6). Moreover, since a smooth solution satisfies $w|_\Gamma = u$, a density argument ensures that this trace condition remains true when $w \in \mathbf{H}^s(\Omega)$ for $s > 1/2$:

$$(Du)|_\Gamma = u \quad \text{if} \quad Du \in \mathbf{H}^s(\Omega), \quad s > 1/2. \tag{4.8}$$

However, if we are only interested in rough solution $w \in \mathbf{L}^2(\Omega)$, it is sufficient to consider a boundary value $u \in V^{-1/2}(\Gamma)$ in (4.6) (where the sign $\int_\Gamma$ must be understood as a duality product), and we can verify that

$$\nabla \cdot Du = 0 \quad \text{and} \quad (Du)|_\Gamma \cdot n = u \cdot n. \tag{4.9}$$

Here is the argument. By choosing $f = \nabla\pi$ in (4.6), successively for $\pi \in H_0^1(\Omega)$ and for $\pi \in H^1(\Omega)$ obeying $\int_\Gamma \pi = 0$, we can deduce that $\varphi = 0$ and $\pi = r$ from (4.7), and integrations by parts allow to recover the free divergence condition $\nabla \cdot w = 0$ and the normal trace condition $w|_\Gamma \cdot n = u \cdot n$. About such a Dirichlet operator D one may refer to [23], Appendix 2, from which the following proposition is taken.

Proposition 4.2. (i) *The operator D is bounded from $V^0(\Gamma)$ into $V^0(\Omega)$ and it satisfies*

$$D \in \mathcal{L}(V^{s-1/2}(\Gamma), V^s(\Omega)) \quad \textit{for all } s \in [0, 2]. \tag{4.10}$$

(ii) *The operator $D^* \in \mathcal{L}(V^0(\Omega), V^0(\Gamma))$, the adjoint of D, is defined by*

$$D^*f = rn - \nu\partial_n\varphi, \quad (\varphi, r) \in V_0^2(\Omega) \times H^1(\Omega) \ \textit{satisfies } (4.7). \tag{4.11}$$

Proof. See [23], Appendix 2. □

Remark 4.3. According to [23], Lemma 7.4, the operator D^* belongs to $\mathcal{L}(V_0^s(\Omega), V^{s+1/2}(\Gamma))$ for all $s \in [0, 2]$. Hence, it allows to extend D by duality to an element of $\mathcal{L}(V^{s-1/2}(\Gamma), V^s(\Omega))$ for $s \in [-2, 0]$.

Remark 4.4. As in the proof of [4], Lemma 3.3.1, one can prove that every $\varphi \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ satisfies $\partial_n\varphi \cdot n = \nabla \cdot \varphi|_\Gamma$. Hence, every φ in $V_0^2(\Omega)$ has a boundary normal derivative $\partial_n\varphi \in V^{1/2}(\Gamma)$ which is tangential. As a consequence, in (4.11) rn and $-\nu\partial_n\varphi$ are respectively the normal and the tangential component of D^*f. In fact, the set of tangential boundary values in $V^{1/2}(\Gamma)$ is totally described by the normal derivatives of vector fields in $V_0^2(\Omega)$: for all $u \in V^{1/2}(\Gamma)$ such that $u \cdot n = 0$, there exists $\varphi_u \in V_0^2(\Omega)$ which obeys

$$\nabla \cdot \varphi_u = 0 \text{ in } \Omega, \quad \partial_n\varphi_u = u, \quad \varphi_u = 0 \text{ on } \Gamma \quad \text{and} \quad \|\varphi_u\|_{\mathbf{H}^2(\Omega)} \le c\|u\|_{V^{1/2}(\Gamma)}, \tag{4.12}$$

where $c > 0$ only depends on the geometry. Such a vector field φ_u can be obtained as follows. In a first step, using a continuous right inverse of the trace and the boundary normal derivative operators [16], Theorem 1.5.1.5, we construct $\phi_u \in \mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$ such that $\partial_n\phi_u = u$. Hence, by recalling that u is tangential we have $\partial_n\phi_u\cdot n = 0$, and since $\phi_u|_\Gamma = 0$ yields the equality $\nabla\cdot\phi_u|_\Gamma = \partial_n\phi_u\cdot n$, we deduce that $\nabla\cdot\phi_u \in H_0^1(\Omega)$. Thus, it allows to construct $\zeta_u \in \mathbf{H}_0^2(\Omega)$ such that $\nabla\cdot\zeta_u = -\nabla\cdot\phi_u$ [13], Chapter III, Theorem 3.2, and the vector field $\varphi_u = \phi_u+\zeta_u$ satisfies (4.12).

Remark 4.5. In fact, the trace equality in (4.8) is still valid for $s \in [0, 1/2]$. Indeed, as in [19], Theorem 6.5, Chapter 2, it can be proved that the trace operator can be extended to a continuous operator from $\{y \in V^0(\Omega) \mid \nu\Delta y - (y\cdot\nabla)z_s - (z_s\cdot\nabla)y \in V_0^{-2}(\Omega)\}$ into $V^{-1/2}(\Gamma)$. Here is the argument. For all $u \in V^{1/2}(\Gamma)$ we construct a pair $(\varphi_u, r_u) \in V_0^2(\Omega)\times H^1(\Omega)$, depending continuously on u, and which obeys $r_u|_\Gamma = u\cdot n$ and $-\nu\partial_n\varphi = u - \gamma_n(u)$ (the tangential component of u, see Rem. 4.4). Thus, for all $y \in V^2(\Omega)$ an integration by parts yields

$$\int_\Gamma (r_u n - \nu\partial_n\varphi_u)\cdot y = \int_\Omega (\nu\Delta y - (y\cdot\nabla)z_s - (z_s\cdot\nabla)y)\cdot\varphi_u + \int_\Omega y\cdot(\nabla r_u - \nu\Delta\varphi_u + (\nabla z_s)^T\varphi_u - (z_s\cdot\nabla)\varphi_u),$$

and by taking the supremum over all $u = r_u n - \nu\partial_n\varphi_u \in V^{1/2}(\Gamma)$, the following estimate can be obtained:

$$\|y|_\Gamma\|_{V^{-1/2}(\Gamma)} \le C(\|y\|_{V^0(\Omega)} + \|\nu\Delta y - (y\cdot\nabla)z_s - (z_s\cdot\nabla)y\|_{V_0^{-2}(\Omega)}).$$

Finally, it remains to extend the trace operator with a density argument.

We are now in position to state the following corollary.

Corollary 4.6. *Let $s \in [0,1]\backslash\{1/2\}$. The linear operator $\gamma^s \in \mathcal{L}(V^s(\Omega), V^{s-1/2}(\Gamma))$, which is given by Definition 2.1, satisfies the following compatibility condition:*

$$y - D\gamma^s(y) \in V_0^s(\Omega) \quad \text{for all } y \in V^s(\Omega). \tag{4.13}$$

Next, let us define solutions to (4.1)-(4.2).

Definition 4.7. Let $z_0 \in V^0(\Omega)$, $u \in L^2(0,T;V^{-1/2}(\Gamma))$ and $f \in L^2(0,T;V_0^{-2}(\Omega))$. We shall say that $z \in L^2(0,T;V^0(\Omega))$ is a weak solution to (4.1)-(4.2), if and only if,

(i) Pz is a weak solution of the evolution equation:

$$(Pz)' = APz + (\lambda_0 - A)PDu + f \ \in L^2(0,T;V_0^{-2}(\Omega)), \tag{4.14}$$
$$Pz(0) = Pz_0 \ \in V_n^0(\Omega). \tag{4.15}$$

(ii) $(I-P)z$ is defined by:

$$(I-P)z = (I-P)D\gamma_n(u) \ \in L^2(0,T;V^0(\Omega)). \tag{4.16}$$

Remark 4.8. Let us underline that (4.16) can be reduced to

$$(I-P)z = (I-P)Du \ \in L^2(0,T;V^0(\Omega)). \tag{4.17}$$

Indeed, by remarking that $\gamma_n(u) - u$ is the tangential component of u, we have $D(\gamma_n(u) - u) \in V_n^0(\Omega)$ which gives $(I-P)D(\gamma_n(u)-u) = 0$, or equivalently $(I-P)D\gamma_n(u) = (I-P)Du$.

Theorem 4.9. *Let $f \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$, $z_0 \in V^0(\Omega)$ and $u \in C([0,T];V^{-1/2}(\Gamma))$ obeying $z_0\cdot n = u(0)\cdot n$.*
(i) If $z \in W(0,T;V^1(\Omega),V_0^{-1}(\Omega))$ is a weak solution in the sense of Definition 4.7, associated with z_0 and u, then there is a unique $p \in H^{-1/2}(0,T;L_0^2(\Omega))$ such that (z,p) satisfies (4.1)-(4.2). Moreover, if $z \in W(0,T;V^2(\Omega),V^0(\Omega))$, then we have $p \in L^2(0,T;\mathbb{H}^1(\Omega))$.

(ii) *Conversely, if* $(z,p) \in W(0,T;V^1(\Omega),V_0^{-1}(\Omega)) \times H^{-1/2}(0,T;L_0^2(\Omega))$ *satisfies* (4.1)-(4.2)*, then* z *is a weak solution in the sense of Definition* 4.7*.*

Remark 4.10. Equation (4.1) is understood as an equality in the distribution space $\mathcal{D}'(0,\infty;\mathbf{H}^{-1}(\Omega))$ and the divergence condition and the trace condition in (4.2) are understood as equalities in $L^2(\Omega\times(0,T))$ and in $L^2(0,T;\mathbf{L}^2(\Gamma))$ respectively.

Proof. (i) Let $z \in W(0,T;V^1(\Omega),V_0^{-1}(\Omega))$ be a weak solution in the sense of Definition 4.7. In a first step, let us prove that for (φ,r) obeying:

$$(\varphi,r)\in V_0^2(\Omega)\times H^1(\Omega) \quad \text{and} \quad \int_\Gamma r = 0, \tag{4.18}$$

we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(t)\cdot\varphi = \int_\Omega z(t)\cdot(\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi - \nabla r) + \int_\Gamma u(t)\cdot(rn - \nu\partial_n\varphi) + \int_\Omega f\cdot\varphi. \tag{4.19}$$

First, by evaluating the equality (4.14) on the test function $\varphi \in V_0^2(\Omega)$ we obtain for all $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(t)\cdot\varphi = \int_\Omega (Pz(t) - PDu(t))\cdot A^*\varphi + \lambda_0\int_\Omega Du(t)\cdot\varphi + \int_\Omega f\cdot\varphi. \tag{4.20}$$

Thus, by successively using $\displaystyle\int_\Omega (Pz(t) - PDu(t))\cdot\nabla r = 0$, the expression of $A^*\varphi$, and the fact that (4.17) guarantees $Pz - PDu = z - Du$, we make the following first calculations:

$$\begin{aligned}\int_\Omega (Pz(t)-PDu(t))\cdot A^*\varphi &= \int_\Omega (Pz(t)-PDu(t))\cdot(A^*\varphi - \nabla r)\\ &= \int_\Omega (z(t)-Du(t))\cdot(\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi - \nabla r).\end{aligned} \tag{4.21}$$

Moreover, from (4.11) we have:

$$\int_\Omega Du(t)\cdot(\lambda_0\varphi - \nu\Delta\varphi + (\nabla z_s)^T\varphi - (z_s\cdot\nabla)\varphi + \nabla r) = \int_\Gamma u(t)\cdot(rn - \nu\partial_n\varphi),$$

and the above equality combined with (4.21) yields:

$$\int_\Omega (Pz(t)-PDu(t))\cdot A^*\varphi + \lambda_0\int_\Omega Du(t)\cdot\varphi = \int_\Omega z(t)\cdot(\nu\Delta\varphi - (\nabla z_s)^T\varphi + (z_s\cdot\nabla)\varphi - \nabla r) + \int_\Gamma u(t)\cdot(rn - \nu\partial_n\varphi).$$

Hence, with (4.20) it gives (4.19). Next, let us prove that the trace condition in (4.2) is true. From (4.19), an integration by parts in space yields:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(t)\cdot\varphi &= -\int_\Omega (\nu\nabla z(t):\nabla\varphi + (z(t)\cdot\nabla)z_s\cdot\varphi + (z_s\cdot\nabla)z(t)\cdot\varphi) + \int_\Omega f\cdot\varphi\\ &\quad + \int_\Gamma (u(t)-z(t))\cdot(rn - \nu\partial_n\varphi),\end{aligned} \tag{4.22}$$

for every (φ,r) obeying (4.18). Hence, in the particular case where $r=0$ and φ is infinitely differentiable, divergence free and compactly supported in Ω, the boundary integral vanishes and we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega z(t)\cdot\varphi = -\int_\Omega (\nu\nabla z(t):\nabla\varphi + (z(t)\cdot\nabla)z_s\cdot\varphi + (z_s\cdot\nabla)z(t)\cdot\varphi) + \int_\Omega f\cdot\varphi. \tag{4.23}$$

Thus, a density argument guarantees that the above equation remains valid for all $\varphi \in V_0^1(\Omega)$, and by comparing it with (4.22), it follows that for all (φ, r) obeying (4.18) we have:

$$\int_\Gamma (z(t) - u(t)) \cdot (rn - \nu \partial_n \varphi) = 0.$$

Finally, since the set $\{rn - \nu\partial_n\varphi,\ (\varphi, r) \text{ obeys (4.18)}\}$ describes the trace space $V^{1/2}(\Gamma)$, it allows to recover the trace condition in (4.2), see Remark 4.4. Next, it remains to prove that z obeys (4.1). First, we define

$$Z(\cdot) = \int_0^{(\cdot)} z(t)\mathrm{d}t \in H^1(0,T; V^1(\Omega)) \quad \text{and} \quad F(\cdot) = \int_0^{(\cdot)} f(t)\mathrm{d}t \in H^1(0,T; \mathbf{H}^{-1}(\Omega)),$$

and because $z \in C([0,T]; \mathbf{H}^{-1}(\Omega))$, we deduce that $z(t) - z_0 - \nu\Delta Z(t) + (\nabla z_s)Z(t) + (\nabla Z(t))z_s - F(t)$ belongs to $C([0,T]; \mathbf{H}^{-1}(\Omega))$. Hence, by recalling that (4.23) is valid for all $\varphi \in V_0^1(\Omega)$, by integrating in time over $(0, t)$ we obtain the pointwise (in time) equality:

$$\langle z(t) - z_0 - \nu\Delta Z(t) + (\nabla z_s)Z(t) + (\nabla Z(t))z_s - F(t)|\varphi\rangle_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = 0 \quad \text{for all} \quad \varphi \in V_0^1(\Omega). \tag{4.24}$$

As a consequence, there exists $P(t) \in L_0^2(\Omega)$ [25], Remark 1.4 (i), Chapter 1, page 15, obeying:

$$\nabla P(t) = z(t) - z_0 - \nu\Delta Z(t) + (\nabla z_s)Z(t) + (\nabla Z(t))z_s - F(t) \in \mathbf{H}^{-1}(\Omega). \tag{4.25}$$

Thus, since $z \in W(0,T; V^1(\Omega), V_0^{-1}(\Omega)) \subset H^{1/2}(0,T; V^0(\Omega))$, (4.25) guarantees $\nabla P \in H^{1/2}(0,T; \mathbf{H}^{-1}(\Omega))$, and from [25], Remark 1.4 (ii), Chapter 1, page 15, we obtain:

$$P \in H^{1/2}(0,T; L_0^2(\Omega)).$$

Hence, we can define the pressure function $p = \frac{\mathrm{d}}{\mathrm{d}t}P$ belonging to $H^{-1/2}(0,T; L_0^2(\Omega))$, and by differentiating (4.25) we obtain that (z, p) satisfies (4.1). Notice that it the case where $z \in W(0,T; V^2(\Omega), V^0(\Omega))$, we deduce that $P \in H^1(0,T; \mathbb{H}^1(\Omega))$ and $p = \frac{\mathrm{d}}{\mathrm{d}t}P \in L^2(0,T; \mathbb{H}^1(\Omega))$ in a similar way. Finally, it remains to recover the initial condition $z(0) = z_0 \in V^0(\Omega)$. On the first hand, from $u \in C([0,T]; V^{-1/2}(\Gamma))$, (4.16) and (4.10) for $s = 0$, we deduce that $(I - P)z(0) = (I - P)D\gamma_n(u)(0) \in V^0(\Omega)$. On the other hand, since we have assumed $u(0)\cdot n = z_0\cdot n$, from (4.9) we obtain $(D\gamma_n(u)(0))\cdot n = z_0\cdot n$. As a consequence, we have $D\gamma_n(u)(0) - z_0 \in V_n^0(\Omega)$ which yields $(I-P)D\gamma_n u(0) = (I-P)z_0$. Then we have proved that $(I-P)z(0) = (I-P)D\gamma_n(u)(0) = (I-P)z_0$, and with $Pz(0) = Pz_0 \in V_n^0(\Omega)$ we can conclude.

(ii) Conversely, if we assume that $z \in W(0,T; V^1(\Omega), V_0^{-1}(\Omega))$ satisfies (4.1)-(4.2), by evaluating (4.1) on $\varphi \in V_0^2(\Omega)$ we get rid of the pressure and obtain (4.23). Thus, by taking into account the trace condition in (4.1) an integration by parts in space yields (4.19), which, in view of the first step in (i), implies (4.20) or equivalently (4.14). Finally, (4.16) is a direct consequence of the trace condition in (4.1). □

5. The system defined on the boundary

The main objective of this section is to give a precise definition of the solution to the system:

$$\partial_t u - \Delta_b u - \sigma\, n = g \ \text{ in } \ \Sigma_T, \quad u(0) = u_0, \quad \int_\Gamma u(t)\cdot n = 0, \quad t \geq 0. \tag{5.1}$$

We recall that $T \in (0, \infty)$ is a fixed time horizon and that $\Sigma_T = \Gamma \times (0,T)$. Notice that in (5.1), σ plays the role of the Lagrange multiplier associated with the constraint $\int_\Gamma u \cdot n = 0$. First, let us consider the gradient operator $\nabla_\Gamma : H^1(\Gamma) \longrightarrow \mathbf{L}^2(\Gamma)$ and the Laplace operator $\Delta_\Gamma : H^1(\Gamma) \longrightarrow H^{-1}(\Gamma)$ (usually called

Laplace Beltrami operator) defined on the Riemannian manifold Γ without boundary which is equipped with the Euclidean metric [24], Chapter 2, page 137. The following equality holds:

$$\langle \Delta_\Gamma \phi | \psi \rangle_{H^{-1}(\Gamma),H^1(\Gamma)} = -\int_\Gamma \nabla_\Gamma \phi \cdot \nabla_\Gamma \psi \quad \text{for all } (\phi,\psi) \in H^1(\Gamma) \times H^1(\Gamma).$$

Hence, we define $\Delta_b : \mathbf{H}^1(\Gamma) \longrightarrow \mathbf{H}^{-1}(\Gamma)$ as the following vector-valued operator

$$\Delta_b u = \big(\Delta_\Gamma u_i\big)^T_{1\le i\le d} \quad \text{for all } u = \big(u_i\big)^T_{1\le i \le d} \in \mathbf{H}^1(\Gamma),$$

and we have:

$$\langle \Delta_b u | v \rangle_{\mathbf{H}^{-1}(\Gamma),\mathbf{H}^1(\Gamma)} = -\sum_{i=1}^d \int_\Gamma \nabla_\Gamma u_i \cdot \nabla_\Gamma v_i \quad \text{for all } u = \big(u_i\big)^T_{1\le i\le d} \in \mathbf{H}^1(\Gamma) \quad \text{and} \quad v = \big(v_i\big)^T_{1\le i\le d} \in \mathbf{H}^1(\Gamma).$$

Remark 5.1. We underline that we consider Γ as a manifold placed in the space $\mathbb{R}^d$ equipped with a fixed orthogonal basis. Hence, each component of Δ_b is the Laplace Beltrami operator Δ_Γ which applies to the corresponding component in $\mathbb{R}^d$ of the vector field $u = (u_1, \ldots, u_d)^T$:

$$\forall x \in \Gamma \quad \Delta_b \begin{pmatrix} u_1(x) \\ \vdots \\ u_d(x) \end{pmatrix} = \begin{pmatrix} \Delta_\Gamma u_1(x) \\ \vdots \\ \Delta_\Gamma u_d(x) \end{pmatrix} \in \mathbb{R}^d.$$

By this way, $\Delta_b u(x)$ does not necessarily belong to the tangent space of Γ, as it is the case for general definition of the Laplace operator for vector fields on manifolds which is based on the notion of Levi-Civita connection, see for instance [8].

Thus, by recalling that P_b is the orthogonal projector from $\mathbf{L}^2(\Gamma)$ onto $V^0(\Gamma)$ whose explicit definition is given by (2.1), we are now in position to introduce the unbounded operator $A_b = P_b\Delta_b$ in $V^0(\Gamma)$ with domain $\mathcal{D}(A_b) = V^2(\Gamma)$.

Theorem 5.2. *The unbounded operator $(\mathcal{D}(A_b), A_b) = (V^2(\Gamma), P_b\Delta_b)$ is the infinitesimal generator of an analytic semigroup on $V^0(\Gamma)$, and it obeys:*

$$\mathcal{D}((\lambda_0 - A_b)^\theta) = V^{2\theta}(\Gamma) \quad \textit{for all } \theta \in [0,1]. \tag{5.2}$$

Proof. For all $\mu > 0$, we introduce the following coercive bilinear form in $V^1(\Gamma)$:

$$a_\mu(v,w) = -\langle \Delta_b v | w \rangle_{\mathbf{H}^{-1}(\Gamma),\mathbf{H}^1(\Gamma)} + \mu \int_\Gamma v \cdot w \quad \text{for all } (v,w) \in V^1(\Gamma) \times V^1(\Gamma).$$

According to the Lax-Milgram Lemma, for any $g \in V^0(\Gamma)$, there exists a unique $u \in V^1(\Gamma)$ satisfying

$$a_\mu(u,v) = \int_\Gamma g \cdot v \quad \text{for all } v \in V^1(\Gamma). \tag{5.3}$$

Thus, replacing $v \in V^1(\Gamma)$ by $v - |\Gamma|^{-1}\big(\int_\Gamma v \cdot n\big)n$ for $v \in \mathbf{H}^1(\Gamma)$ in (5.3), we deduce that $(\mu - \Delta_b)u = g + \sigma\, n$ where $\sigma = |\Gamma|^{-1}\langle(\mu - \Delta_b)u - g | n\rangle_{\mathbf{H}^{-1}(\Gamma),\mathbf{H}^1(\Gamma)}$. Hence, since $g + \sigma\, n \in \mathbf{L}^2(\Gamma)$, classical elliptic regularity results on compact manifold without boundary [24], Chapter 5, Proposition 1.6, yields $u \in V^2(\Gamma)$. Since (5.3) is equivalent to $(\mu - A_b)u = g$ where $\mu > 0$, we have shown that $]0,\infty[$ is included in the resolvent set of A_b. Finally, since $(\mathcal{D}(A_b), A_b) = (V^2(\Gamma), P_b\Delta_b)$ is self-adjoint, it generates an analytic semigroup on $V^0(\Gamma)$ and (5.2) holds. $\square$

Remark 5.3. We shall underline the fact that [24], Chapter 5, Proposition 1.6, which is invoked in the proof of Theorem 5.2, requires the manifold Γ to be of class C^∞. However, since Δ_b only involves second order derivatives, it is sufficient to assume Γ only of class C^2.

We are now in position to give a definition of weak solution to (5.1).

Definition 5.4. Let $u_0 \in V^{-1/2}(\Gamma)$ and $g \in L^2(0,T;V^{-2}(\Gamma))$. We shall say that $u \in L^2(0,T;V^0(\Gamma))$ is a weak solution to (5.1), if and only if, u is a weak solution to the evolution equation

$$u' = A_b u + g \in L^2(0,T;V^{-2}(\Gamma)), \quad u(0) = u_0 \in V^{-1/2}(\Gamma). \tag{5.4}$$

Theorem 5.5. *Let $u_0 \in V^{-1/2}(\Gamma)$ and $g \in L^2(0,T;V^0(\Gamma))$.*
(i) *If $u \in W(0,T;V^{1/2}(\Gamma),V^{-3/2}(\Gamma))$ is a weak solution in the sense of Definition 5.4, associated with u_0 and g, then there is a unique $\sigma \in L^2(0,T)$ such that (u,σ) satisfies the first equation in (5.1) in the distribution sense.*
(ii) *Conversely, if $(u,\sigma) \in W(0,T;V^{1/2}(\Gamma),V^{-3/2}(\Gamma)) \times L^2(0,T)$ satisfies (5.1), then u is a weak solution.*

Proof. Since (ii) is obvious, we focus on (i). First, we define $U(\cdot) = \int_0^{(\cdot)} u(t)\mathrm{d}t$ and $G(\cdot) = \int_0^{(\cdot)} g(t)\mathrm{d}t$, and by integrating in time over $(0,t)$ the first equality in (5.4), we obtain:

$$\langle u(t) - u_0 - \Delta_b U(t) - G(t)|v\rangle_{\mathbf{H}^{-2}(\Gamma),\mathbf{H}^2(\Gamma)} = 0 \quad \text{for all} \quad v \in V^2(\Gamma).$$

Thus, replacing $v \in V^2(\Gamma)$ by $v - \big(|\Gamma|^{-1}\int_\Gamma v\cdot n\big)n$ for $v \in \mathbf{H}^2(\Gamma)$ in the above equality, we deduce that:

$$u - u_0 - \Delta_b U - G = \Sigma\, n \quad \text{where} \quad \Sigma(\cdot) = \frac{1}{|\Gamma|}\langle u(\cdot) - u_0 - \Delta_b U(\cdot) - G(\cdot)|n\rangle_{\mathbf{H}^{-2}(\Gamma),\mathbf{H}^2(\Gamma)} \in L^2(0,T).$$

Moreover, by recalling that $\int_\Gamma u(\cdot)\cdot n = 0$ and that $\langle u_0|n\rangle_{\mathbf{H}^{-2}(\Gamma),\mathbf{H}^2(\Gamma)} = 0$ we deduce that

$$\Sigma(\cdot) = \frac{1}{|\Gamma|}\langle -\Delta_b U(\cdot) - G(\cdot)|n\rangle \in H^1(0,T),$$

and we have $u - u_0 = \Delta_b U + G + \Sigma n$. Finally, we set $\sigma = \frac{\mathrm{d}}{\mathrm{d}t}\Sigma$ and we verify that (u,σ) obeys (5.1). □

6. The extended system

The main objective of this section is to rewrite the system (4.1)-(4.2)-(5.1) in the following form:

$$Y' = \mathcal{A}Y + F, \quad Y(0) = Y_0, \tag{6.1}$$

where Y is the new state variable and F is the new nonhomogeneous source term. First, let us define the linear operator $\mathcal{A}$ of system (6.1) as the following unbounded operator in $\mathcal{H}^0$.

Definition 6.1. Let $(\mathcal{D}(\mathcal{A}),\mathcal{A})$ be the unbounded operator defined in $\mathcal{H}^0 = V^0_n(\Omega) \times V^{-1/2}(\Gamma)$ by

$$\mathcal{D}(\mathcal{A}) = \left\{ (y,u)^T \in V^2_n(\Omega) \times V^{3/2}(\Gamma) \mid y - PDu \in V^2_0(\Omega) \right\}, \tag{6.2}$$

$$\mathcal{A} = \begin{pmatrix} \nu P\Delta - P(\nabla z_s) - P(z_s\cdot\nabla) & (\lambda_0 - \nu P\Delta + P(\nabla z_s) + P(z_s\cdot\nabla))PD \\ 0 & P_b\Delta_b \end{pmatrix}. \tag{6.3}$$

Theorem 6.2. (i) *The domain $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}^0$.*
(ii) *The unbounded operator $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*)$ defined in $\mathcal{H}^0_* = V^0_n(\Omega) \times V^{1/2}(\Gamma)$ by*

$$\mathcal{D}(\mathcal{A}^*) = V^2_0(\Omega) \times V^{5/2}(\Gamma), \tag{6.4}$$

$$\mathcal{A}^* = \begin{pmatrix} \nu P\Delta - P(\nabla z_s)^T + P(z_s \cdot \nabla) & 0 \\ D^*(\lambda_0 - \nu P\Delta + P(\nabla z_s)^T - P(z_s \cdot \nabla)) & P_b\Delta_b \end{pmatrix}, \tag{6.5}$$

is the adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ with respect to the pivot space $V^0_n(\Omega) \times V^0(\Gamma)$.
(iii) *$(\mathcal{D}(\mathcal{A}), \mathcal{A})$ (resp. $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*)$) is the infinitesimal generator of an analytic semigroup on $\mathcal{H}^0$ (resp. $\mathcal{H}^0_*$). Let us set $\widehat{\mathcal{A}} = \lambda - \mathcal{A}$ and $\widehat{\mathcal{A}}^* = \lambda - \mathcal{A}^*$, where $\lambda > \lambda_0 > 0$ is large enough so that $(\mathcal{D}(\mathcal{A}), -\widehat{\mathcal{A}})$ (resp. $(\mathcal{D}(\mathcal{A}^*), -\widehat{\mathcal{A}}^*)$) is the infinitesimal generator of analytic and exponentially stable semigroup on $\mathcal{H}^0$ (resp. $\mathcal{H}^0_*$).*
(iv) *For all $\theta \in [0, 1]$ the following equalities hold:*

$$\mathcal{D}(\widehat{\mathcal{A}}^\theta) = \Big[\mathcal{D}(\mathcal{A}), \mathcal{H}^0\Big]_{1-\theta} = \Big\{ (y, u)^T \in V^{2\theta}_n(\Omega) \times V^{2\theta-1/2}(\Gamma) \mid y - PDu \in V^{2\theta}_0(\Omega) \Big\}, \tag{6.6}$$

$$\mathcal{D}(\widehat{\mathcal{A}}^{*\theta}) = \Big[\mathcal{D}(\mathcal{A}^*), \mathcal{H}^0_*\Big]_{1-\theta} = V^{2\theta}_0(\Omega) \times V^{1/2+2\theta}(\Gamma). \tag{6.7}$$

Remark 6.3. In order to keep a natural gap equal of 1/2 in term of Sobolev index, between the regularity of vector fields defined in Ω and their traces on Γ, we choose $\mathcal{H}^0 = V^0_n(\Omega) \times V^{-1/2}(\Gamma)$ as the state space. However, "duality" and "adjointness" are understood with respect to the $V^0_n(\Omega) \times V^0(\Gamma)$-topology. Indeed, $\mathcal{H}^0_* = V^0_n(\Omega) \times V^{1/2}(\Gamma)$ is the dual space of $\mathcal{H}^0$ with respect to the pivot space $V^0_n(\Omega) \times V^0(\Gamma)$, and $\mathcal{A}^*$, which is defined in $\mathcal{H}^0_*$, is the $V^0_n(\Omega) \times V^0(\Gamma)$-adjoint of $\mathcal{A}$. Hence, the duality pairing $\langle\cdot|\cdot\rangle_{\mathcal{H}^0,\mathcal{H}^0_*}$ between $\mathcal{H}^0$ and $\mathcal{H}^0_*$, is defined from the following scalar product of $V^0_n(\Omega) \times V^0(\Gamma)$:

$$\left(\begin{pmatrix} y \\ u \end{pmatrix}, \begin{pmatrix} w \\ v \end{pmatrix}\right)_{V^0_n(\Omega)\times V^0(\Gamma)} = \int_\Omega y \cdot w + \int_\Gamma u \cdot v.$$

Let us give relations which allow to pass from the $V^0_n(\Omega) \times V^0(\Gamma)$-topology to the $\mathcal{H}^0$-topology. For $s \in \mathbb{R}$, we introduce the following isomorphism:

$$I^{(s)} : V^0_n(\Omega) \times V^s(\Gamma) \longrightarrow V^0_n(\Omega) \times V^0(\Gamma) \quad \text{and} \quad I^{(s)}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ (\lambda_0 - \Delta_b)^{s/2}u \end{pmatrix},$$

and we suppose that $\mathcal{H}^0$ is equipped with the following scalar product:

$$\left(\begin{pmatrix} y \\ u \end{pmatrix}, \begin{pmatrix} w \\ v \end{pmatrix}\right)_{\mathcal{H}^0} = \left(I^{(-1/4)}\begin{pmatrix} y \\ u \end{pmatrix}, I^{(-1/4)}\begin{pmatrix} w \\ v \end{pmatrix}\right)_{V^0_n(\Omega)\times V^0(\Gamma)}. \tag{6.8}$$

Hence, if $(\mathcal{D}(\mathcal{A}^\sharp), \mathcal{A}^\sharp)$ is the $\mathcal{H}^0$-adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$, an easy calculation yield:

$$\mathcal{D}(\mathcal{A}^\sharp) = V^2_0(\Omega) \times V^{3/2}(\Gamma) \quad \text{and} \quad \mathcal{A}^\sharp = I^{(1/2)}\mathcal{A}^* I^{(-1/2)}, \tag{6.9}$$

as well as the following relationship:

$$\mathrm{e}^{\mathcal{A}^\sharp t} = I^{(1/2)}\mathrm{e}^{\mathcal{A}^* t} I^{(-1/2)}, \quad t \geq 0. \tag{6.10}$$

Proof of Theorem 6.2. (a) *Proof of* (i). Let $(y, u)^T \in \mathcal{H}^0$. Since $u \in V^{-1/2}(\Gamma)$, from (4.10) with $s = 0$ we deduce that $Du \in V^0(\Omega)$ and $y - PDu \in V^0_n(\Omega)$. Moreover, the density of $V^2_0(\Omega)$ in $V^0_n(\Omega)$ gives us a sequence

$(z_n)_{n\in\mathbb{N}} \in (V_0^2(\Omega))^{\mathbb{N}}$ converging to $y-PDu$ in $V_n^0(\Omega)$, and the density of $V^{3/2}(\Gamma)$ in $V^{-1/2}(\Gamma)$ gives us a sequence $(u_n)_{n\in\mathbb{N}} \in (V^{3/2}(\Gamma))^{\mathbb{N}}$ converging to u in $V^{-1/2}(\Gamma)$. Therefore, $(z_n + PDu_n, u_n) \in \mathcal{D}(\mathcal{A})$, for all $n \in \mathbb{N}$, and $(z_n + PDu_n, u_n)^T$ converges to $(y,u)^T$ in $\mathcal{H}^0$.

(b) *Proof of* (ii). Let us denote by $(\mathcal{D}(\mathcal{A}^\bigstar), \mathcal{A}^\bigstar)$ the adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ with respect to the pivot space $V_n^0(\Omega)\times V^0(\Gamma)$. We have $\mathcal{D}(\mathcal{A}^\bigstar) \subset \mathcal{H}_*^0 = V_n^0(\Omega)\times V^{1/2}(\Gamma)$, and we must prove that $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*)$, defined by (6.4)-(6.5), is such that $(\mathcal{D}(\mathcal{A}^*), \mathcal{A}^*) = (\mathcal{D}(\mathcal{A}^\bigstar), \mathcal{A}^\bigstar)$. First, for $Y \in \mathcal{D}(\mathcal{A})$, the equality $\langle \mathcal{A}Y|W\rangle_{\mathcal{H}^0,\mathcal{H}^0_*} = \langle Y|\mathcal{A}^*W\rangle_{\mathcal{H}^0,\mathcal{H}^0_*}$ obviously holds for every $W \in V_0^2(\Omega)\times V^{5/2}(\Gamma)$. Thus, the inclusion $V_0^2(\Omega)\times V^{5/2}(\Gamma) \subset \mathcal{D}(\mathcal{A}^\bigstar)$ holds and we have $\mathcal{A}^*W = \mathcal{A}^\bigstar W$ for every $W \in V_0^2(\Omega)\times V^{5/2}(\Gamma)$. Thus, it remains to show that $\mathcal{D}(\mathcal{A}^\bigstar) \subset V_0^2(\Omega)\times V^{5/2}(\Gamma)$. Let $W = (w,v)^T \in \mathcal{D}(\mathcal{A}^\bigstar)$. According to the definition of $\mathcal{D}(\mathcal{A}^\bigstar)$, there is a constant $C_W > 0$, depending on W and obeying

$$|\langle \mathcal{A}Y|W\rangle_{\mathcal{H}^0,\mathcal{H}^0_*}| \le C_W\|Y\|_{\mathcal{H}^0} \quad \text{for all } Y \in \mathcal{D}(\mathcal{A}),$$

or equivalently:

$$\left|\int_\Omega (A-\lambda_0)(y-PDu)\cdot w + \lambda_0\int_\Omega y\cdot w + \langle A_b u\cdot v\rangle_{V^{-1/2}(\Gamma),V^{1/2}(\Gamma)}\right| \le C_W(\|y\|_{V_n^0(\Omega)} + \|u\|_{V^{-1/2}(\Gamma)}), \tag{6.11}$$

for every $(y,u)^T \in \mathcal{D}(\mathcal{A})$. By choosing $y \in V_0^2(\Omega)$ and $u = 0$ in (6.11), we obtain

$$\left|\int_\Omega w\cdot(\lambda_0 - A)y\right| \le (C_W + \lambda_0\|w\|_{V_n^0(\Omega)})\|y\|_{V_n^0(\Omega)},$$

and the Riesz representation theorem yields $(\lambda_0 - A^*)w \in V_n^0(\Omega)$. Then we have $w \in V_0^2(\Omega)$ and it remains to prove that $v \in V^{5/2}(\Gamma)$. By using the following integration by parts in (6.11)

$$\int_\Omega (A-\lambda_0)(y-PDu)\cdot w = \int_\Omega (y-PDu)\cdot(A^*-\lambda_0)w, \quad (y,u)^T \in \mathcal{D}(\mathcal{A}),$$

we obtain the estimate

$$|\langle A_b u\cdot v\rangle_{V^{-1/2}(\Gamma),V^{1/2}(\Gamma)}| \le K_W(\|u\|_{V^{-1/2}(\Gamma)} + \|y\|_{V_n^0(\Omega)}), \tag{6.12}$$

where $K_W = C_W + \lambda_0\|w\|_{V_n^0(\Omega)} + (1+\|D\|_{\mathcal{L}(V^{-1/2}(\Gamma),V_n^0(\Omega))})\|(\lambda_0 - A^*)w\|_{V_n^0(\Omega)}$. Thus, by choosing $(y,u)^T = (PDu,u)^T \in \mathcal{D}(\mathcal{A})$ in (6.12), we deduce that

$$|\langle A_b u\cdot v\rangle_{V^{-1/2}(\Gamma),V^{1/2}(\Gamma)}| \le K_W(1+\|D\|_{\mathcal{L}(V^{-1/2}(\Gamma),V_n^0(\Omega))})\|u\|_{V^{-1/2}(\Gamma)},$$

which guarantees that $A_b v \in V^{1/2}(\Gamma)$. Finally, $v \in V^{5/2}(\Gamma)$, and the inclusion $\mathcal{D}(\mathcal{A}^\bigstar) \subset V_0^2(\Omega)\times V^{5/2}(\Gamma)$ is proved.

(c) *Proof of* (iii). First, analyticity of $(e^{At})_{t\ge 0}$ stated in Theorem 4.1 yields the following resolvent estimate:

$$\|(\nu - A)^{-1}\|_{\mathcal{L}(V_n^0(\Omega))} \le \frac{C_0}{|\nu-\omega|} \quad \text{for all } \nu \in \mathcal{S}_{\theta_0,\omega} = \left\{\nu\in\mathbb{C} \mid \ \nu\neq\omega,\ |\arg(\nu-\omega)| < \theta_0\right\}, \tag{6.13}$$

where $C_0 > 0$, $\omega > 0$ and $\theta_0 \in]\frac{\pi}{2},\pi[$ do not depend on ν. Moreover, invoking the analyticity of $(e^{A_bt})_{t\ge 0}$ on $V^0(\Gamma)$ which is given by Theorem 5.2, and since $\|(\lambda - A_b)^{-1/4}\cdot\|_{V^0(\Gamma)}$ defines a norm on $V^{-1/2}(\Gamma)$, from the equality $(\lambda_0 - A_b)^{-1/4}(\nu - A_b)^{-1} = (\nu - A_b)^{-1}(\lambda_0 - A_b)^{-1/4}$ we obtain:

$$\|(\nu - A_b)^{-1}\|_{\mathcal{L}(V^{-1/2}(\Gamma))} \le \frac{C_1}{|\nu-\omega|} \quad \text{for all } \nu \in \mathcal{S}_{\theta_0,\omega}, \tag{6.14}$$

where $C_1 > 0$ does not depend on ν. Next, we fix $\epsilon > 0$, and for $F = (f,g)^T \in \mathcal{H}^0$ and

$$\nu \in \mathcal{S}_{\theta_0,\omega+\epsilon} = \Big\{\nu \in \mathbb{C} \mid \ \nu \neq \omega+\epsilon, \ |\arg(\nu-\omega-\epsilon)| < \theta_0\Big\},$$

searching a solution $Y \in \mathcal{D}(\mathcal{A})$ to $(\nu - \mathcal{A})Y = F$ is equivalent to search the solution $(y,u)^T \in \mathcal{D}(\mathcal{A})$ to

$$\nu(y - PDu) - A(y - PDu) = f + (\lambda_0 - \nu)PDu \in V^0_n(\Omega), \tag{6.15}$$
$$\nu u - A_b u = g \in V^{-1/2}(\Gamma). \tag{6.16}$$

Hence, with $(\nu - A)^{-1} \in \mathcal{L}(V^0_n(\Omega), V^2_0(\Omega))$, with $(\nu - A_b)^{-1} \in \mathcal{L}(V^{-1/2}(\Gamma), V^{3/2}(\Gamma))$ and with (4.10) for $s = 3/2$, system (6.15)-(6.16) yields:

$$\begin{aligned} y - PDu &= (\nu - A)^{-1}f - (\nu - \lambda_0)(\nu - A)^{-1}PD(\nu - A_b)^{-1}g \in V^2_0(\Omega),\\ y &= (\nu - A)^{-1}f + PD(\nu - A_b)^{-1}g - (\nu - \lambda_0)(\nu - A)^{-1}PD(\nu - A_b)^{-1}g \in V^2_n(\Omega),\\ u &= (\nu - A_b)^{-1}g \in V^{3/2}(\Gamma). \end{aligned}$$

Then, from (6.13), (6.14) and (4.10) with $s = 0$, we deduce the existence of $C > 0$, independent of ν, such that

$$\begin{aligned} \|(\nu - \mathcal{A})^{-1}F\|_{\mathcal{H}^0} &= \|y\|_{V^0_n(\Omega)} + \|u\|_{V^{-1/2}(\Gamma)} \le \frac{C}{|\nu-\omega|}\|f\|_{V^0_n(\Omega)} + \left(1 + \frac{|\nu-\lambda_0|}{|\nu-\omega|}\right)\frac{C}{|\nu-\omega|}\|g\|_{V^{-1/2}(\Gamma)},\\ &\le C \sup_{\nu\in\mathcal{S}_{\theta_0,\omega+\epsilon}} \left\{\left(2 + \frac{|\nu-\lambda_0|}{|\nu-\omega|}\right)\frac{|\nu-\omega-\epsilon|}{|\nu-\omega|}\right\}\frac{\|F\|_{\mathcal{H}^0}}{|\nu-\omega-\epsilon|}. \end{aligned}$$

This last estimate proves that $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ is the infinitesimal generator of an analytic semigroup on $\mathcal{H}^0$. Finally, by invoking [20], Chapter 1, Lemma 10.1-2, we have $\|R(\nu, \mathcal{A}^*)\|_{\mathcal{L}(\mathcal{H}^0_*)} = \|R(\nu, \mathcal{A})^*\|_{\mathcal{L}(\mathcal{H}^0_*)} = \|R(\nu, \mathcal{A})\|_{\mathcal{L}(\mathcal{H}^0)}$, for every $\nu \in \mathcal{S}_{\theta_0,\omega+\epsilon}$, and the analyticity of $(\mathrm{e}^{\mathcal{A}^*t})_{t\ge0}$ on $\mathcal{H}^0_*$ is a direct consequence of the analyticity of $(\mathrm{e}^{\mathcal{A}t})_{t\ge0}$ on $\mathcal{H}^0$.

(d) *Proof of* (iv). Let us equip $\mathcal{H}^0$ with the scalar product (6.8) and let us consider $(\mathcal{D}(\mathcal{A}^\sharp), \mathcal{A}^\sharp)$, the adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ with respect to the pivot space $\mathcal{H}^0$ (see Rem. 6.3). According to (6.9) we have $\mathcal{D}(\mathcal{A}^\sharp) = V^2_0(\Omega) \times V^{3/2}(\Gamma)$ which yields $[\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta} = [\mathcal{D}(\mathcal{A}^\sharp), \mathcal{H}^0]_{1-\theta} = V^{2\theta}_n(\Omega) \times V^{2\theta-1/2}(\Gamma)$ for $0 < \theta < 1/4$. Then it allows to invoke [27], Theorem $\mathcal{B}$, (i), and to obtain the identity $\mathcal{D}(\widehat{\mathcal{A}}^\theta) = [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_{1-\theta}$, for every $\theta \in [0,1]$. According to [27], Theorem $\mathcal{B}$, (iv), this last identity is equivalent to the fact that the function $z \in \{z \in \mathbb{C} \mid Re\ z > 0\} \mapsto \|\widehat{\mathcal{A}}^{-z}\|_{\mathcal{L}(\mathcal{H}^0)}$ can be extended to a strongly continuous function on $\{z \in \mathbb{C} \mid Re\ z \ge 0\}$. By invoking [20], Chapter 1, Lemma 10.1-2, we obtain $\|\widehat{\mathcal{A}}^{*-z}\|_{\mathcal{L}(\mathcal{H}^0_*)} = \|\widehat{\mathcal{A}}^{-z*}\|_{\mathcal{L}(\mathcal{H}^0_*)} = \|\widehat{\mathcal{A}}^{-z}\|_{\mathcal{L}(\mathcal{H}^0)}$. So we deduce that $z \in \{z \in \mathbb{C} \mid Re\ z > 0\} \mapsto \|\widehat{\mathcal{A}}^{*-z}\|_{\mathcal{L}(\mathcal{H}^0_*)}$ can be extended to a strongly continuous function on $\{z \in \mathbb{C} \mid Re\ z \ge 0\}$, and we conclude that $\mathcal{D}(\widehat{\mathcal{A}}^{*\theta}) = [\mathcal{D}(\mathcal{A}^*), \mathcal{H}^0]_{1-\theta}$, for all $\theta \in [0,1]$, from [27], Theorem $\mathcal{B}$, (iv). Now, it remains to prove the second equality in (6.6) and in (6.7). According to [15], Definition 2.2, we have $(y,u)^T \in [\mathcal{D}(\mathcal{A}), \mathcal{H}^0]_\theta$, if and only if, there exists

$$(y^*, u^*)^T \in L^2(\mathbb{R}^+; \mathcal{D}(\mathcal{A})) \cap H^{1/2(1-\theta)}(\mathbb{R}^+; \mathcal{H}^0), \quad (y^*(0), u^*(0)) = (y,u),$$

which is equivalent to

$$\begin{aligned} y^* &\in L^2(\mathbb{R}^+; V^2_n(\Omega)) \cap H^{1/2(1-\theta)}(\mathbb{R}^+; V^0_n(\Omega)), && y^*(0) = y,\\ u^* &\in L^2(\mathbb{R}^+; V^{3/2}(\Gamma)) \cap H^{1/2(1-\theta)}(\mathbb{R}^+; V^{-1/2}(\Gamma)), && u^*(0) = u,\\ y^* - PDu^* &\in L^2(\mathbb{R}^+; V^2_0(\Omega)) \cap H^{1/2(1-\theta)}(\mathbb{R}^+; V^0_n(\Omega)), && z^*(0) = y - PDu. \end{aligned}$$

Hence, still by invoking [15], Definition 2.2, the above setting is equivalent to

$$y \in [V_n^2(\Omega), V_n^0(\Omega)]_{1-\theta}, \quad u \in [V^{3/2}(\Gamma), V^{-1/2}(\Gamma)]_{1-\theta}, \quad y - PDu \in [V_n^2(\Omega), V_n^0(\Omega)]_{1-\theta},$$

and with $[V_n^2(\Omega), V_n^0(\Omega)]_{1-\theta} = V_n^{2\theta}(\Omega)$, with $[V^{3/2}(\Gamma), V^{-1/2}(\Gamma)]_{1-\theta} = V^{2\theta-1/2}(\Gamma)$ and with $[V_n^2(\Omega), V_n^0(\Omega)]_{1-\theta} = V_n^{2\theta}(\Omega)$, we obtain the second equality in (6.6). Finally, the second equality in (6.7) follows from

$$[\mathcal{D}(\mathcal{A}^*), \mathcal{H}_*^0]_{1-\theta} = [V_0^2(\Omega) \times V^{5/2}(\Gamma), V_n^0(\Omega) \times V^{1/2}(\Gamma)]_{1-\theta} = [V_0^2(\Omega), V_n^0(\Omega)]_{1-\theta} \times [V^{5/2}(\Gamma), V^{1/2}(\Gamma)]_{1-\theta},$$

with $[V_0^2(\Omega), V_n^0(\Omega)]_{1-\theta} = V_0^{2\theta}(\Omega)$ and with $[V^{5/2}(\Gamma), V^{1/2}(\Gamma)]_{1-\theta} = V^{1/2+2\theta}(\Gamma)$. □

Next, we introduce a shorter notation for the function spaces defined in (6.6) and (6.7).

Definition 6.4. For $\theta \in [0, 1]$, we define the function spaces:

$$\begin{aligned} \mathcal{H}^{2\theta} &= \Big[\mathcal{D}(\mathcal{A})\,,\, \mathcal{H}^0\Big]_{1-\theta} = \Big\{ (y,u)^T \in V_n^{2\theta}(\Omega) \times V^{2\theta-1/2}(\Gamma) \mid y - PDu \in V_0^{2\theta}(\Omega) \Big\}, \\ \mathcal{H}_*^{2\theta} &= \Big[\mathcal{D}(\mathcal{A}^*)\,,\, \mathcal{H}_*^0\Big]_{1-\theta} = V_0^{2\theta}(\Omega) \times V^{1/2+2\theta}(\Gamma), \end{aligned}$$

and

$$\mathcal{H}^{-2\theta} = (\mathcal{H}_*^{2\theta})' = V_0^{-2\theta}(\Omega) \times V^{-1/2-2\theta}(\Gamma) \quad \text{and} \quad \mathcal{H}_*^{-2\theta} = (\mathcal{H}^{2\theta})'.$$

The following theorem is a consequence of the analyticity of $(e^{\mathcal{A}t})_{t\geq 0}$ (resp. $(e^{\mathcal{A}^* t})_{t\geq 0}$) on $\mathcal{H}^0$ (resp. $\mathcal{H}_*^0$).

Theorem 6.5. *For every $0 \leq \theta \leq 1$, the following mapping is an isomorphism:*

$$\begin{array}{rcl} W(0,\infty;\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}) & \longrightarrow & L^2(0,\infty;\mathcal{H}^{2(\theta-1)}) \times [\mathcal{H}^{2\theta},\mathcal{H}^{2(\theta-1)}]_{1/2}, \\ Y & \longmapsto & (Y' + \widehat{A}Y, Y(0)). \end{array}$$

Proof. It is a consequence of maximal regularity results for analytic semigroups which can be found in [6], Chapter 3, Theorem 2.2, page 166, where we can take $T = \infty$ because $(e^{-\widehat{\mathcal{A}}t})_{t\geq 0}$ is exponentially stable on $\mathcal{H}^0$. □

Next, we determine the spaces of initial conditions $[\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}]_{1/2}$.

Lemma 6.6. *The following equality holds:*

$$\big[\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}\big]_{1/2} = \mathcal{H}^{2\theta-1} \quad \textit{for all } \theta \in [0,1]. \tag{6.17}$$

Proof. According to [15], Definition 2.2, we have $(y,u)^T \in [\mathcal{H}^{2\theta}, \mathcal{H}^{2(\theta-1)}]_{1/2}$, if and only if, there is (y^*, u^*) obeying:

$$y^* \in L^2(\mathbb{R}^+; V_n^{2\theta}(\Omega)) \cap H^1(\mathbb{R}^+; V_0^{2\theta-2}(\Omega)), \tag{6.18}$$
$$u^* \in L^2(\mathbb{R}^+; V^{2\theta-1/2}(\Gamma)) \cap H^1(\mathbb{R}^+; V^{2\theta-5/2}(\Gamma)), \tag{6.19}$$
$$y^* - PDu^* \in L^2(\mathbb{R}^+; V_0^{2\theta}(\Omega)), \tag{6.20}$$
$$(y^*(0), u^*(0)) = (y, u). \tag{6.21}$$

Moreover, from (6.19) and Remark 4.3, we also have $y^* - PDu^* \in H^1(\mathbb{R}^+; V_0^{2\theta-2}(\Omega))$, and the use of [15], Definition 2.2, with $[V^{2\theta-1/2}(\Gamma), V^{2\theta-5/2}(\Gamma)]_{1/2} = V^{2\theta-3/2}(\Gamma)$ and with $[V_0^{2\theta}(\Omega), V_0^{2\theta-2}(\Omega)]_{1/2} = V_0^{2\theta-1}(\Omega)$, ensures that (6.19)-(6.20)-(6.21) is equivalent to

$$y \in [V_n^{2\theta}(\Omega), V_0^{2\theta-2}(\Omega)]_{1/2}, \quad u \in V^{2\theta-3/2}(\Gamma), \quad y - PDu \in V_0^{2\theta-1}(\Omega). \tag{6.22}$$

Thus, for $\theta \in [1/2, 1]$, observing that

$$y = y - PDu + PDu \in V_0^{2\theta-1}(\Omega) + V_n^{2\theta-1}(\Omega) \subset V_n^{2\theta-1}(\Omega),$$

and

$$V_n^{2\theta-1}(\Omega) = [V_n^{2\theta}(\Omega), (V_n^{2(1-\theta)}(\Omega))']_{1/2} \subset [V_n^{2\theta}(\Omega), V_0^{2\theta-2}(\Omega)]_{1/2},$$

we conclude that (6.22) is true, if and only if

$$(y, u) \in \left\{ (y, u)^T \in V_n^{2\theta-1}(\Omega) \times V^{2\theta-3/2}(\Gamma) \mid y - PDu \in V_0^{2\theta-1}(\Omega) \right\} = \mathcal{H}^{2\theta-1}.$$

The case $\theta \in [0, 1/2]$ may be treated similarly, by remarking that

$$y = y - PDu + PDu \in V_0^{2\theta-1}(\Omega) + V_0^{2\theta-1}(\Omega) = V_0^{2\theta-1}(\Omega),$$

and

$$V_0^{2\theta-1}(\Omega) = [V_0^{2\theta}(\Omega), V_0^{2\theta-2}(\Omega)]_{1/2} \subset [V_n^{2\theta}(\Omega), V_0^{2\theta-2}(\Omega)]_{1/2}.$$

Then we conclude that (6.22) is true, if and only if

$$(y, u) \in \left\{ (y, u)^T \in V_0^{2\theta-1}(\Omega) \times V^{2\theta-3/2}(\Gamma) \mid y - PDu \in V_0^{2\theta-1}(\Omega) \right\},$$

and by observing that $(y, u) \in V_0^{2\theta-1}(\Omega) \times V^{2\theta-3/2}(\Gamma)$ imply $y - PDu \in V_0^{2\theta-1}(\Omega)$, it follows that (6.22) is equivalent to

$$(y, u) \in V_0^{2\theta-1}(\Omega) \times V^{2\theta-3/2}(\Gamma) = (\mathcal{H}_*^{1-2\theta})' = \mathcal{H}^{2\theta-1}.$$

□

Let us collect some useful results in the following corollary.

Corollary 6.7. (i) *The linear mapping $Y_0 \longmapsto \mathrm{e}^{-\widehat{\mathcal{A}}t} Y_0$ is bounded from $\mathcal{H}^1$ into $L^2(0, \infty; \mathcal{H}^2)$.*
(ii) *The operators $\mathcal{K} : L^2(0, \infty; \mathcal{H}^0) \longrightarrow L^2(0, \infty; \mathcal{H}^0)$ and $\mathcal{K}^* : L^2(0, \infty; \mathcal{H}_*^0) \longrightarrow L^2(0, \infty; \mathcal{H}_*^0)$ defined by*

$$\mathcal{K} : F \longmapsto \int_0^t \mathrm{e}^{-\widehat{\mathcal{A}}(t-\tau)} F(\tau) \mathrm{d}\tau \quad \textit{and} \quad \mathcal{K}^* : F \longmapsto \int_t^\infty \mathrm{e}^{-\widehat{\mathcal{A}}^*(\tau-t)} F(\tau) \mathrm{d}\tau \tag{6.23}$$

obey $\mathcal{K} \in \mathcal{L}(L^2(0, \infty; \mathcal{H}^0), W(0, \infty; \mathcal{H}^2, \mathcal{H}^0))$ and $\mathcal{K} \in \mathcal{L}(L^2(0, \infty; \mathcal{H}_*^0), W(0, \infty; \mathcal{H}_*^2, \mathcal{H}_*^0))$.
(iii) *For $T \in (0, \infty)$, $F \in L^2(0, T; \mathcal{H}^0)$ and $Y_0 \in \mathcal{H}^0$, there exists a unique $Y \in W(0, T; \mathcal{H}^1, \mathcal{H}^{-1})$ solution to*

$$Y' = \mathcal{A}Y + F \quad \textit{on} \quad \mathcal{D}(\mathcal{A}^*)', \quad Y(0) = Y_0. \tag{6.24}$$

Proof. Part (i) follows from Theorem 6.5 and Lemma 6.6 when $\theta = 1$. Part (ii) follows from Theorem 6.5 when $\theta = 1$: one can verify that $\mathcal{K}(F)$ is the solution to $Y + \widehat{\mathcal{A}}Y = F$ and $Y(0) = 0$, and that $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$ with respect to the pivot space $L^2(0, \infty; V_n^0(\Omega) \times V^0(\Gamma))$. Finally, part (iii) follows from Theorem 6.5 and Lemma 6.6 when $\theta = 1/2$, by remarking that $Y = \mathrm{e}^{\lambda_0 t} \widehat{Y}$ where $\widehat{Y} + \widehat{\mathcal{A}}\widehat{Y} = \mathrm{e}^{-\lambda_0 t} F$ and $\widehat{Y}(0) = Y_0$. □

Remark 6.8. In (6.24), $\mathcal{A}$ abusively denotes the extension of the operator $\mathcal{A}$ to $\mathcal{D}(A^*)'$ (the dual space of $\mathcal{D}(A^*)$ with respect to the pivot space $V_n^0(\Omega) \times V^0(\Gamma)$), obtained with the extrapolation method [18], Section 0.3.

We are now in position to rewrite the two coupled systems (4.14)-(4.15)-(4.16) and (5.4) as an evolution system.

Theorem 6.9. *Let $(z_0, u_0) \in V^0(\Omega) \times V^{-1/2}(\Gamma)$ and $F = (f, g)^T \in L^2(0, T; \mathcal{D}(\mathcal{A}^*)')$. Then u is a weak solution to (5.1) associated with (u_0, g) in the sense of Definition 5.4, and z is a weak solution to (4.1)-(4.2) associated with (z_0, u, f) in the sense of Definition 4.7, if and only if:*

(i) *the state $Y = (Pz, u)^T$ is a weak solution of the evolution equation:*

$$Y' = \mathcal{A}Y + F \in L^2(0,T;\mathcal{D}(\mathcal{A}^*)'), \quad Y(0) = \begin{pmatrix} Pz_0 \\ u_0 \end{pmatrix} \in \mathcal{H}^0; \tag{6.25}$$

(ii) *the pair (z, u) obeys:*

$$(I-P)z(\cdot) = (I-P)D\gamma_n(u)(\cdot) \ \in L^2(0,T;V^0(\Omega)).$$

Proof. The first statement in (6.25) can be rewritten as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y(t)|V\rangle_{\mathcal{H}^0,\mathcal{H}^0_*} = \langle Y(t)|\mathcal{A}^*V\rangle_{\mathcal{H}^0,\mathcal{H}^0_*} + \langle F(t)|V\rangle_{\mathcal{D}(\mathcal{A}^*)',\mathcal{D}(\mathcal{A}^*)} \quad \text{for all } V \in \mathcal{D}(\mathcal{A}^*),$$

or equivalently, with $Y = (Pz, u)^T$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega Pz(t)\cdot\varphi + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma u(t)\cdot v \quad &= \quad \int_\Omega Pz(t)\cdot A^*\varphi + \int_\Omega PDu(t)\cdot(\lambda_0 - A^*)\varphi + \int_\Gamma u(t)\cdot P_b\Delta_b v \\ &\quad + \int_\Omega f\cdot\varphi + \int_\Gamma g(t)\cdot v \quad \text{for all} \quad (\varphi, v) \in V_0^2(\Omega)\times V^{5/2}(\Gamma).\end{aligned}$$

Finally, we extend the above equality by density to $(\varphi, v) \in V_0^2(\Omega)\times V^2(\Gamma)$, and we conclude by setting $\varphi = 0$ and $v = 0$ alternatively. □

7. The control problem

The goal of this section is to find a control $G \in L^2(0,\infty;V_n^0(\Omega)\times V^0(\Gamma))$, which can be expressed in a feedback form, and which stabilizes the system:

$$Y' = \mathcal{A}Y + \Lambda G \ \text{ on } \ \mathcal{D}(\mathcal{A}^*)', \quad Y(0) = Y_0 \in \mathcal{H}^0, \tag{7.1}$$

where Λ is the control operator defined as the following canonical projection:

$$\Lambda : \mathcal{H}^0 \longrightarrow \mathcal{H}^0 \quad \text{ and } \quad \Lambda\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix}. \tag{7.2}$$

Hence, we introduce the optimal control problem:

$$(\mathcal{P}_{Y_0}) \quad \inf\left\{\mathcal{J}(Y,G) \ \mid \ G \in L^2(0,\infty;V_n^0(\Omega)\times V^0(\Gamma)), \ (Y,G) \text{ satisfies (7.1)}\right\},$$

where

$$\mathcal{J}(Y,G) = \int_0^\infty \|\mathcal{C}Y\|^2_{\mathcal{Z}} + \int_0^\infty \|G\|^2_{V_n^0(\Omega)\times V^0(\Gamma)}. \tag{7.3}$$

In (7.3), the observation space and the observation operator are given by $\mathcal{Z} = \mathbf{L}^2(\Omega)\times L^2(\Omega,\mathbb{R}^{d^2})$ and

$$\mathcal{C} : \mathcal{H}^1 \longrightarrow \mathcal{Z} \qquad \text{and} \qquad \mathcal{C}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y + (I-P)D\gamma_n(u) \\ \nabla(y + (I-P)D\gamma_n(u)) \end{pmatrix}. \tag{7.4}$$

Proposition 7.1. *The operator $\mathcal{C}$ belongs to $\mathcal{L}(\mathcal{H}^1,\mathcal{Z})$, and the following properties hold:*

$$\|\mathcal{C}\cdot\|_{\mathcal{Z}} \sim \|.\|_{\mathcal{H}^1} \quad \textit{and} \quad \mathcal{C}^*\mathcal{C} \in \mathcal{L}(\mathcal{H}^1,\mathcal{H}_*^{-1}) \cap \mathcal{L}(\mathcal{H}^2,\mathcal{H}^0_*). \tag{7.5}$$

Proof. The first inequality $\|\mathcal{C}\cdot\|_{\mathcal{Z}} \leq C_1\|\cdot\|_{\mathcal{H}^1}$ in (7.5) is a straightforward consequence of (4.10), and conversely, the trace inequality $\|u\|_{V^{1/2}(\Gamma)} \leq C\|y+(I-P)D\gamma_n(u)\|_{V^1(\Omega)}$ provides the second inequality $\|\cdot\|_{\mathcal{H}^1} \leq C_2\|\mathcal{C}\cdot\|_{\mathcal{Z}}$. Next, since the first statement in (7.5) holds, we obviously have $\mathcal{C}^*\mathcal{C} \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^{-1}_*)$. Finally, for $(y,u)^T \in \mathcal{H}^2$ and $(w,v)^T \in \mathcal{H}^2$, we consider the scalar product $((y,u)^T|(w,v)^T)_{\mathcal{Z}}$ whose expression is given by:

$$\int_\Omega (y+(I-P)D\gamma_n(u))\cdot(w+(I-P)D\gamma_n(v)) + \int_\Omega \nabla(y+(I-P)D\gamma_n(u)) : \nabla(w+(I-P)D\gamma_n(v)).$$

By integrating by parts we deduce that

$$\mathcal{C}^*\mathcal{C}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} P(-\Delta y + y) + P(-\Delta + I)(I-P)D\gamma_n(u) \\ (\gamma_n D^*(I-P)(-\Delta y + y) + \partial_n y) + (\gamma_n D^*(I-P)(-\Delta + I) + \partial_n)(I-P)D\gamma_n(u) \end{pmatrix},$$

and $\mathcal{C}^*\mathcal{C} \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^0_*)$ follows from (4.10), $D^* \in \mathcal{L}(V^0(\Omega), V^{1/2}(\Gamma))$ and $\partial_n \in \mathcal{L}(V^2(\Omega), V^{1/2}(\Gamma))$. □

Remark 7.2. Assume that $Y_0 = (Pz_0, u_0)^T$, where $z_0 \in V^0(\Omega)$ and $u_0 \in V^{-1/2}(\Gamma)$ obeys $z_0|_\Gamma \cdot n = u_0 \cdot n$. From Theorems 6.9, 5.5 and 4.9 one can verify that problem $(\mathcal{P}_{Y_0})$ is equivalent to the following control problem:

$$(\mathcal{Q}_{z_0,u_0}) \quad \inf\left\{ \mathcal{I}(z,g) \mid g \in L^2(0,\infty;V^0(\Gamma)) \right\} \quad \text{where} \quad \mathcal{I}(z,g) = \int_0^\infty \|z\|^2_{\mathbf{H}^1(\Omega)} + \int_0^\infty \|g\|^2_{\mathbf{L}^2(\Gamma)},$$

and $z \in W(0,\infty;V^1(\Omega),V^{-1}_0(\Omega))$ satisfies (1.18)-(1.19). Indeed, for $Y=(y,u)^T$ and $z = y+(I-P)D\gamma_n(u)$ we have the following equalities

$$\|z\|^2_{\mathbf{H}^1(\Omega)} \;=\; \|z\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla z\|^2_{\mathbf{L}^2(\Omega)} = \|y+(I-P)D\gamma_n(u)\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla(y+(I-P)D\gamma_n u)\|^2_{\mathbf{L}^2(\Omega)} = \|\mathcal{C}Y\|^2_{\mathcal{Z}},$$

which prove that functionals $\mathcal{I}(z,g)$ and $\mathcal{J}(Y,G)$ are equal.

In order to characterize the solution of $(\mathcal{P}_{Y_0})$, we are going to use the optimal control theory over an infinite time horizon which is developed in [18], Chapter 2. However, we shall underline that we are not exactly in the framework given there. Indeed, since we have $\mathcal{D}(\mathcal{C}) = \mathcal{D}(\widehat{\mathcal{A}}^{1/2})$ and $\mathcal{A}$ is not self-adjoint, $\mathcal{C}$ does not fit the assumption [18], Chapter 2, equation (2.5.5) (where we are in the case $\delta = 1/2$ and $\gamma = 0$), and we cannot directly apply [18], Chapter 2, Theorem 2.5.1. However, with Corollary 6.7(i) and $\mathcal{C}\widehat{\mathcal{A}}^{-1/2} \in \mathcal{L}(\mathcal{H}^0, \mathcal{Z})$, it can be shown that

$$\mathcal{C}\mathrm{e}^{-\widehat{\mathcal{A}}t} : \text{ continuous } \mathcal{H}^0 \longrightarrow L^2(0,\infty;\mathcal{H}^0),$$

and assumption [18], Chapter 2, equation (2.5.1), is recovered. The second reason why we are not in the framework of [18], Chapter 2, is that we use the pivot space $V^0_n(\Omega)\times V^0(\Gamma)$ to define adjointness (see Rem. 6.3). Indeed, it is explained in [18], Chapter 2, page 122, that adjointness must be understood with respect to the $\mathcal{H}^0$-topology.

Theorem 7.3. *Let $Y_0 \in \mathcal{H}^0$. The following results hold.*
(i) *There exists a unique operator Π in the space*

$$\mathcal{X} = \left\{ L \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^0_*) \mid \langle L\xi|\zeta\rangle_{\mathcal{H}^0_*,\mathcal{H}^0} = \langle \xi|L\zeta\rangle_{\mathcal{H}^0,\mathcal{H}^0_*} \text{ and } \langle L\xi|\xi\rangle_{\mathcal{H}^0_*,\mathcal{H}^0} \geq 0 \text{ for all } (\xi,\zeta) \in \mathcal{H}^1\times\mathcal{H}^1 \right\}, \quad (7.6)$$

solution to the following Riccati equation:

$$\langle \Pi\xi|\mathcal{A}\zeta\rangle_{\mathcal{H}^0_*,\mathcal{H}^0} + \langle \mathcal{A}\xi|\Pi\zeta\rangle_{\mathcal{H}^0,\mathcal{H}^0_*} - (\Lambda\Pi\xi|\Lambda\Pi\zeta)_{V^0_n(\Omega)\times V^0(\Gamma)} + (\mathcal{C}\xi|\mathcal{C}\zeta)_{\mathcal{Z}} = 0 \quad \forall(\xi,\zeta) \in \mathcal{D}(\mathcal{A})\times\mathcal{D}(\mathcal{A}). \quad (7.7)$$

(ii) *The problem $(\mathcal{P}_{Y_0})$ admits a unique optimal pair (Y_{Y_0}, G_{Y_0}) which obeys $G_{Y_0} = -\Lambda\Pi Y_{Y_0}$ and*

$$\mathcal{J}(Y_{Y_0}, G_{Y_0}) = \langle \Pi Y_0|Y_0\rangle_{\mathcal{H}^0_*,\mathcal{H}^0} = \inf\left\{ \mathcal{J}(Y,G) \mid (Y,G) \text{ satisfies (7.1)} \right\}. \quad (7.8)$$

(iii) *The unbounded operator* $(\mathcal{D}(\mathcal{A}_\Pi), -\mathcal{A}_\Pi)$ *defined by*

$$\mathcal{D}(\mathcal{A}_\Pi) = \mathcal{H}^2 \qquad \textit{and} \qquad \mathcal{A}_\Pi = \Lambda\Pi - \mathcal{A},$$

is the infinitesimal generator of an analytic and exponentially stable semigroup on $\mathcal{H}^0$*, and the optimal state* Y_{Y_0} *is the unique solution to the closed-loop system:*

$$Y' + \mathcal{A}_\Pi Y = 0, \quad Y(0) = Y_0. \tag{7.9}$$

Moreover, the following equalities hold:

$$\mathcal{D}(\mathcal{A}_\Pi^\theta) = \mathcal{H}^{2\theta}, \quad \mathcal{D}(\mathcal{A}_\Pi^{*\theta}) = \mathcal{H}_*^{2\theta} \qquad \textit{for all } \theta \in [0,1]. \tag{7.10}$$

(iv) *The operator* Π *obeys:*

$$\Pi \in \mathcal{L}(\mathcal{H}^{2\theta}, \mathcal{H}_*^{2\theta}) \qquad \textit{for all } \theta \in [0, 1/2]. \tag{7.11}$$

Remark 7.4. Since Π is the solution to an extended Riccati equation, involving extended operator $\mathcal{A}$, $\mathcal{C}$ and Λ, it can be viewed as an extended operator. One easily verify that there is a triplet (Π_1, Π_2, Π_3) which obeys:

$$\Pi = \begin{pmatrix} \Pi_1 & \Pi_2^* \\ \Pi_2 & \Pi_3 \end{pmatrix}, \quad (\Pi_1, \Pi_2, \Pi_3) \in \mathcal{L}(V_n^0(\Omega)) \times \mathcal{L}(V_n^0(\Omega), V^{1/2}(\Gamma)) \times \mathcal{L}(V^{-1/2}(\Gamma), V^{1/2}(\Gamma)). \tag{7.12}$$

Proof. (a) *Auxiliary control problem* $(\bar{\mathcal{P}}_{\bar{Y}_0})$. For all $Y_0 \in \mathcal{H}^0$, the existence and uniqueness of the optimal pair (Y_{Y_0}, G_{Y_0}) solution to $(\mathcal{P}_{Y_0})$ is a direct consequence of the finite cost condition given in the appendix in Corollary 10.4. In order to characterize such an optimal pair, let us use the change of variable method of [18], Chapter 2, Section 2.5. We recall that to fit the framework given there, the $\mathcal{H}^0$-topology should be used to define the adjoint of $\mathcal{A}$. In the following, $(\mathcal{D}(\mathcal{A}^\sharp), \mathcal{A}^\sharp)$ denotes the $\mathcal{H}^0$-adjoint of $(\mathcal{D}(\mathcal{A}), \mathcal{A})$ (see Rem. 6.3), we set $\widehat{\mathcal{A}}^\sharp = \lambda - \mathcal{A}^\sharp$ where $\lambda > 0$ is given in (iii) in Theorem 6.2, and for a given Hilbert space X we denote by $X^\bullet$ the dual X with respect to the $\mathcal{H}^0$-topology. Let us consider the system:

$$\bar{Y}' = \mathcal{A}\bar{Y} + \bar{B}G \quad \text{on} \quad \mathcal{D}(\mathcal{A}^\sharp)^\bullet \quad \text{and} \quad Y(0) = \bar{Y}_0 \in \mathcal{H}^0, \tag{7.13}$$

where

$$\bar{B} \in \mathcal{L}(V_n^0(\Omega) \times V^0(\Gamma), \mathcal{D}(\widehat{\mathcal{A}}^{\sharp 1/2})^\bullet) \quad \text{and} \quad \bar{B}G = \widehat{\mathcal{A}}^{1/2}\Lambda G,$$

and let us define the auxiliary control problem:

$$(\bar{\mathcal{P}}_{\bar{Y}_0}) \quad \inf\left\{\bar{\mathcal{J}}(\bar{Y}, G) \mid G \in L^2(0,\infty; V_n^0(\Omega) \times V^0(\Gamma)) \text{ and } (\bar{Y}, G) \text{ satisfies (7.13)}\right\},$$

where

$$\bar{\mathcal{J}}(\bar{Y}, G) = \int_0^\infty \|\bar{\mathcal{C}}\,\bar{Y}\|_{\mathcal{Z}}^2 + \int_0^\infty \|G\|_{V_n^0(\Omega)\times V^0(\Gamma)}^2 \quad \text{with} \quad \bar{\mathcal{C}} = \mathcal{C}\widehat{\mathcal{A}}^{-1/2} \in \mathcal{L}(\mathcal{H}^0, \mathcal{Z}). \tag{7.14}$$

Problem $(\bar{\mathcal{P}}_{\bar{Y}_0})$ now fits the framework of [18], Chapter 2 (where with the notations there we are in the case $\gamma := 1/2$, $Y := \mathcal{H}^0$, $U := V_n^0(\Omega) \times V^0(\Gamma)$, $Z := \mathcal{Z}$, $R := \bar{\mathcal{C}}$). If $Y_0 \in \mathcal{H}^1$, then we have $\bar{Y}_0 = \widehat{\mathcal{A}}^{1/2} Y_0 \in \mathcal{H}^0$, and for all admissible pairs $(\bar{Y}, G)$ and (Y, G) of $(\bar{\mathcal{P}}_{\bar{Y}_0})$ and $(\mathcal{P}_{Y_0})$ respectively we have $\bar{\mathcal{J}}(\bar{Y}, G) = \mathcal{J}(Y, G)$. As a consequence, existence and uniqueness of $(\bar{Y}_{\bar{Y}_0}, \bar{G}_{\bar{Y}_0})$ solution to $(\bar{\mathcal{P}}_{\bar{Y}_0})$ can be deduced from Corollary 10.4, and

$$\mathcal{J}(Y_{Y_0}, G_{Y_0}) = \bar{\mathcal{J}}(\bar{Y}_{\bar{Y}_0}, \bar{G}_{\bar{Y}_0}), \quad G_{Y_0} = \bar{G}_{\bar{Y}_0} \quad \text{and} \quad Y_{Y_0} = \widehat{\mathcal{A}}^{-1/2}\bar{Y}_{\bar{Y}_0} \quad \text{where} \quad \bar{Y}_0 = \widehat{\mathcal{A}}^{1/2} Y_0. \tag{7.15}$$

From [18], Chapter 2, Theorems 2.2.1(a_2) and 2.2.1(b_2), we have $\bar{G}_{\bar{Y}_0} = -\bar{B}^\sharp \bar{\Pi} Y_{\bar{Y}_0}$ where $\bar{\Pi}$ is the unique operator of $\mathcal{L}(\mathcal{H}^0)$, within the class of $\mathcal{H}^0$-self-adjoint operator L such that $\widehat{\mathcal{A}}^{\sharp 1/2} L \in \mathcal{L}(\mathcal{H}^0)$, solution to:

$$(\bar{\Pi}\xi|\mathcal{A}\zeta)_{\mathcal{H}^0} + (\mathcal{A}\xi|\bar{\Pi}\zeta)_{\mathcal{H}^0} - (\bar{B}^\sharp\,\bar{\Pi}\xi|\bar{B}^\sharp\,\bar{\Pi}\zeta)_{V^0(\Gamma)} + (\bar{\mathcal{C}}\xi|\bar{\mathcal{C}}\zeta)_{\mathcal{Z}} = 0 \quad \text{for all } (\xi,\zeta) \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A}). \tag{7.16}$$

Moreover, [18], Chapter 2, Theorem 2.2.1(a_4), (a_8), with the first statement in (7.15) yields

$$(\bar{\Pi}\bar{Y}_0|\bar{Y}_0)_{\mathcal{H}^0} = \mathcal{J}(Y_{Y_0}, G_{Y_0}), \tag{7.17}$$

and from [18], Chapter 2, Theorem 2.2.1(a_8), we obtain that $\bar{\Pi}$ obeys:

$$\bar{\Pi} Y_0 = \bar{Q}(0) \quad \text{where} \quad \bar{Q}(t) = \int_t^\infty \mathrm{e}^{-\widehat{\mathcal{A}}^\sharp(\tau-t)}(\bar{\mathcal{C}}^\sharp\bar{\mathcal{C}} + 2\lambda_0\bar{\Pi})(\mathrm{e}^{-\lambda_0\tau}\bar{Y}_{\bar{Y}_0}(\tau))\mathrm{d}\tau. \tag{7.18}$$

In the above setting, the operators $\bar{B}^\sharp \in \mathcal{L}(\mathcal{D}(\widehat{\mathcal{A}}^{\sharp 1/2}), V_n^0(\Omega) \times V^0(\Gamma))$ and $\bar{\mathcal{C}}^\sharp \in \mathcal{L}(\mathcal{Z}, \mathcal{H}^0)$ are the $\mathcal{H}^0$-adjoints of $\bar{B}$ and $\bar{C}$ respectively, which are given by:

$$\bar{B}^\sharp = \Lambda\widehat{\mathcal{A}}^{*1/2} I^{(-1/2)} \quad \text{and} \quad \bar{\mathcal{C}}^\sharp = I^{(1/2)}\widehat{\mathcal{A}}^{*-1/2}\mathcal{C}^*, \tag{7.19}$$

where $I^{(-1/2)}$ and $I^{(1/2)}$ are the isomorphisms which have been introduced in Remark 6.3. Indeed, because we have $\widehat{\mathcal{A}}^{\sharp 1/2} = I^{(1/2)}\widehat{\mathcal{A}}^{*1/2}I^{(-1/2)}$ (see (6.9)), equalities in (7.19) are obtained from the following calculations:

$$\begin{aligned} \langle \bar{B}G|\xi\rangle_{\mathcal{D}(\widehat{\mathcal{A}}^{\sharp 1/2})^\bullet, \mathcal{D}(\widehat{\mathcal{A}}^{\sharp 1/2})} &= (\Lambda G|\widehat{\mathcal{A}}^{\sharp 1/2}\xi)_{\mathcal{H}^0} = (G|\Lambda\widehat{\mathcal{A}}^{*1/2}I^{(-1/2)}\xi)_{V_n^0(\Omega)\times V^0(\Gamma)}, \\ (\bar{\mathcal{C}}\zeta|Z)_{\mathcal{Z}} &= (\mathcal{C}\widehat{\mathcal{A}}^{-1/2}\zeta|Z)_{\mathcal{Z}} = (\zeta|\widehat{\mathcal{A}}^{*-1/2}\mathcal{C}^*Z)_{V_n^0(\Omega)\times V^0(\Gamma)} = (\zeta|I^{(1/2)}\widehat{\mathcal{A}}^{*-1/2}\mathcal{C}^*Z)_{\mathcal{H}^0}, \end{aligned}$$

where $(G,\xi) \in (V_n^0(\Omega) \times V^0(\Gamma)) \times \mathcal{D}(\widehat{\mathcal{A}}^{\sharp 1/2})$ and $(\zeta, Z) \in \mathcal{H}^0 \times \mathcal{Z}$. Finally, the optimal state is given by the expression $\bar{Y}_{\bar{Y}_0} = \bar{\Phi}(t)\bar{Y}_0$, where $(\bar{\Phi}(t))_{t\geq t}$ is an analytic and exponentially stable semigroup on $\mathcal{H}^0$ [18], Chapter 2, Theorems 2.2.1(a_6) and 2.2.1(b_1). Hence, we have

$$Y_{Y_0}(t) = \Phi(t)Y_0, \quad t \geq 0, \quad \text{and} \quad \Phi = \widehat{\mathcal{A}}^{-1/2}\bar{\Phi}\widehat{\mathcal{A}}^{1/2} \text{ is analytic and exponentially stable on } \mathcal{H}^1. \tag{7.20}$$

(b) *Definition of* Π *(proof of* (i) *and* (iv)*).* Let us define the operator Π as follows:

$$\Pi = \widehat{\mathcal{A}}^{*1/2}I^{(-1/2)}\bar{\Pi}\widehat{\mathcal{A}}^{1/2} \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^0_*). \tag{7.21}$$

Since $\bar{\Pi}$ is self-adjoint with respect to the $\mathcal{H}^0$-topology, it is easy to see that Π belongs to $\mathcal{X}$. Moreover, from equation (7.16) with (6.8) and (7.19), we deduce that Π is solution to the Riccati equation (7.7), and its uniqueness follows from the uniqueness of $\bar{\Pi}$. Next, by setting $\bar{Y}_0 = \widehat{\mathcal{A}}^{1/2}Y_0$ for $Y_0 \in \mathcal{H}^1$ in (7.17), we obtain:

$$\mathcal{J}(Y_{Y_0}, G_{Y_0}) = \langle \Pi Y_0|Y_0\rangle_{\mathcal{H}^0_*, \mathcal{H}^0} \quad \text{for all } Y_0 \in \mathcal{H}^1, \tag{7.22}$$

and from (7.18) with (6.10), (6.23), (7.15) and (7.19) we deduce that $Q = \widehat{\mathcal{A}}^{*1/2}I^{(-1/2)}\bar{Q}$ satisfies

$$\Pi Y_0 = Q(0) \quad \text{where} \quad Q = \mathcal{K}^*(\mathcal{C}^*\mathcal{C} + 2\lambda_0\Pi)(\mathrm{e}^{-\widehat{\mathcal{A}}(\cdot)}Y_0 + \mathcal{K}(\Lambda G_{Y_0})).$$

Hence, with Corollary 6.7, $\mathcal{C}^*\mathcal{C} \in \mathcal{L}(\mathcal{H}^2, \mathcal{H}^0_*)$, $\Pi \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^0_*)$ and (7.22), we can make the following calculation

$$\|Q(0)\|_{\mathcal{H}^1_*} \leq C_1(\|Y_0\|_{\mathcal{H}^1} + \|G_{Y_0}\|_{L^2(0,\infty;V_n^0(\Omega)\times V^0(\Gamma))}) \leq C_2(\|Y_0\|_{\mathcal{H}^1} + (\langle \Pi Y_0|Y_0\rangle_{\mathcal{H}^0_*,\mathcal{H}^0})^{1/2}) \leq C_3\|Y_0\|_{\mathcal{H}^1},$$

and we deduce that $\Pi \in \mathcal{L}(\mathcal{H}^1, \mathcal{H}^1_*)$. Finally, since $\Pi \in \mathcal{X}$, an easy duality argument yields $\Pi \in \mathcal{L}(\mathcal{H}^{-1}, \mathcal{H}^{-1}_*)$, and (7.11) follows by interpolation.

(c) *Closed-loop system (proof of* (iii)*).* If $Y_0 \in \mathcal{H}^1$ and $\bar{Y}_0 = \widehat{\mathcal{A}}^{1/2} Y_0$, then (7.15), (7.19) and (7.21) yields:

$$G_{Y_0} = \bar{G}_{\bar{Y}_0} = -\bar{B}^\sharp \bar{\Pi} Y_{\bar{Y}_0} = -\Lambda \widehat{\mathcal{A}}^{*1/2} I^{(-1/2)} \bar{\Pi} \widehat{\mathcal{A}}^{1/2} Y_{Y_0} = -\Lambda \Pi Y_{Y_0}.$$

As a consequence, for all $Y_0 \in \mathcal{H}^1$ the optimal state $Y_{Y_0} = \Phi(\cdot) Y_0$ is solution to (7.9). Moreover, since $\Lambda\Pi \in \mathcal{L}(\mathcal{H}^0)$, we obviously have $\mathcal{D}(\mathcal{A}_\Pi) = \mathcal{D}(\mathcal{A} - \Lambda\Pi) = \mathcal{D}(\mathcal{A}) = \mathcal{H}^2$ and the analyticity of $(\mathrm{e}^{-\mathcal{A}_\Pi t})_{t \geq 0}$ on $\mathcal{H}^0$ follows from [20], Chapter 3, page 81, Corollary 2.2. Notice that since $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0$ is the unique solution to (7.9) for all $Y_0 \in \mathcal{H}^0$ [20], Corollary 4.1.5, the semigroups $\Phi(\cdot)$ and $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)}$ coincide on $\mathcal{H}^1$, and $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)}$ is the unique extension of $\Phi(\cdot)$ to $\mathcal{H}^0$. Finally, since $\mathcal{D}(\mathcal{A}_\Pi) = \mathcal{H}^2$ and $\mathcal{D}(\mathcal{A}^*_\Pi) = \mathcal{H}^2_*$ (which can be deduced from $\Pi\Lambda \in \mathcal{L}(\mathcal{H}^0_*)$), then proving (7.10) can be reduced to proving equalities $\mathcal{D}({\mathcal{A}_\Pi}^\theta) = [\mathcal{D}(\mathcal{A}_\Pi), \mathcal{H}^0]_{1-\theta}$ and $\mathcal{D}({\mathcal{A}_\Pi}^{*\theta}) = [\mathcal{D}({\mathcal{A}_\Pi}^*), \mathcal{H}^0_*]_{1-\theta}$ for all $\theta \in [0,1]$. According to [27], it is equivalent to show that the holomorphic function $z \in \{z \in \mathbb{C} \mid Re(z) > 0\} \mapsto {\mathcal{A}_\Pi}^{-z} \in \mathcal{L}(\mathcal{H}^0)$ can be extended to a strongly continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ into $\mathcal{L}(\mathcal{H}^0)$. We verify that:

$$(t + \mathcal{A}_\Pi)^{-1} = (t + \lambda_0 - \mathcal{A})^{-1} + (t + \lambda_0 - \mathcal{A})^{-1}(\lambda_0 - \Lambda\Pi)(t + \mathcal{A}_\Pi)^{-1} \qquad t \geq 0,$$

which yields the following equality:

$${\mathcal{A}_\Pi}^{-z} = (\lambda_0 - \mathcal{A})^{-z} + I(z), \qquad I(z) = \frac{\sin \pi z}{\pi} \int_0^{+\infty} t^{-z}(t + \lambda_0 - \mathcal{A})^{-1}(\lambda_0 - \Lambda\Pi)(t + \mathcal{A}_\Pi)^{-1}\mathrm{d}t.$$

Hence, it remains to show that $z \mapsto I(z)$ can be extended to a strongly continuous function from $\{z \in \mathbb{C} \mid Re(z) \geq 0\}$ into $\mathcal{L}(\mathcal{H}^0)$. The values ρ and σ being respectively the real and imaginary part of z, we invoke the resolvent property of the generators $\lambda_0 - \mathcal{A}$ and $\mathcal{A}_\Pi$ of analytic semigroups [20], Chapter 2, equation (6.2), to obtain $\|I(z)\|_{\mathcal{L}(\mathcal{H}^0)} \leq C\mathrm{e}^{\pi\sigma} \int_0^\infty \frac{\mathrm{d}t}{t^{-\rho}(1+t)^2} < +\infty$. We conclude by virtue of [17], Theorem 17.9.1, Chapter 17.

(d) *Extension to* $Y_0 \in \mathcal{H}^0$ *(proof of* (ii)*).* Let $Y_0 \in \mathcal{H}^0$ and $(Y, G) \in L^2(0,\infty;\mathcal{H}^1) \times L^2(0,\infty;V^0_n(\Omega) \times V^0(\Gamma))$ be an admissible pair for $(\mathcal{P}_{Y_0})$. Since the analyticity of $(\mathrm{e}^{\mathcal{A}t})_{t\geq 0}$ on $\mathcal{H}^0$ ensures that $Y \in C([1/n, T]; \mathcal{H}^1) \cap C([0,T];\mathcal{H}^0)$ for all $n \in \mathbb{N}^*$ [6], Chapter 1, Proposition 3.8, then $Y(1/n)_{n \in \mathbb{N}^*}$ is a sequence of $\mathcal{H}^1$ converging to Y_0 in $\mathcal{H}^0$. Thus, for all $n \in \mathbb{N}^*$ we verify that $t \mapsto (Y_n(t), G_n(t)) = (Y(t + 1/n), G(t + 1/n))$ is admissible for $(\mathcal{P}_{Y(1/n)})$, and (7.22) (where we set $Y_0 := Y(1/n) \in \mathcal{H}^1$) with the optimality of $(Y_{Y(1/n)}, G_{Y(1/n)})$ yields:

$$\langle \Pi Y(1/n) | Y(1/n)\rangle_{\mathcal{H}^0_*, \mathcal{H}^0} = \mathcal{J}(Y_{Y(1/n)}, G_{Y(1/n)}) \leq \mathcal{J}(Y_n, G_n) = \int_{1/n}^\infty \|\mathcal{C}Y\|^2_{\mathcal{Z}} + \int_{1/n}^\infty \|G\|^2_{V^0_n(\Omega)\times V^0(\Gamma)}. \tag{7.23}$$

Thus, by passing to the limit sup in (7.23), we obtain the strong convergence of $Y_{Y(1/n)} = \mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y(1/n)$ to $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0$ in $L^2(0,\infty;\mathcal{H}^1)$, and of $G_{Y(1/n)} = -\Lambda\Pi\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y(1/n)$ to $-\Lambda\Pi\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0$ in $L^2(0,\infty; V^0_n(\Omega) \times V^0(\Gamma))$. Hence, by taking the inf over all admissible pair (Y, G) in the resulting inequality, we obtain (7.8) and $(Y_{Y_0}, G_{Y_0}) = (\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0, -\Lambda\Pi\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0)$. It follows that $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)} Y_0 \in L^2(0,\infty;\mathcal{H}^1)$ for all $Y_0 \in \mathcal{H}^0$, and the exponential stability of $\mathrm{e}^{-\mathcal{A}_\Pi(\cdot)}$ on $\mathcal{H}^0$ can be deduced from [20], Theorem 4.4.1. □

8. Stabilization of the Navier-Stokes equations

The goal of this section is to prove that for initial conditions belonging to an adequate neighborhood of the origin, the nonlinear system (3.1)-(3.2)-(3.3) admits a unique solution which is exponentially stable. For $s \in [\frac{d-2}{2}, 1]$ and $z_0 \in V^s(\Omega)$ the initial condition $Y_0 = (Pz_0, \gamma^s(z_0))^T$ belongs to $\mathcal{H}^s$, and according to Theorems 6.9, 4.9, 5.5 and (7.12) system (3.1)-(3.2)-(3.3) can be rewritten in the following abstract form:

$$Y' + \mathcal{A}_\Pi Y = B(Y,Y), \quad Y(0) = Y_0 \in \mathcal{H}^s, \tag{8.1}$$

where $B(\cdot,\cdot)$ is defined by

$$B\left(\left(\begin{array}{c} y \\ u \end{array}\right),\left(\begin{array}{c} w \\ v \end{array}\right)\right) = -\left(\begin{array}{c} \big((y+(I-P)D\gamma_n(u))\cdot\nabla\big)(w+(I-P)D\gamma_n(v)) \\ 0 \end{array}\right). \tag{8.2}$$

The proof of the stability of the solution to (8.1) relies in an adequate choice of the norm of $\mathcal{H}^s$ and $\mathcal{H}^{1+s}$. We first need to define the following operator.

Definition 8.1. For $s \in [0,1]$, we define the following linear operator:

$$\Pi^{(s)} : \mathcal{H}^s \longrightarrow \mathcal{H}_*^{-s} \quad \text{ and } \quad \Pi^{(s)} = {\mathcal{A}_\Pi}^{*s/2}\,\Pi\,{\mathcal{A}_\Pi}^{s/2}.$$

Proposition 8.2. *For $s \in [0,1]$, the linear operator $\Pi^{(s)}$ obeys:*

$$\Pi^{(s)} \in \mathcal{L}(\mathcal{H}^{2\theta+s},\mathcal{H}_*^{2\theta-s}) \quad \textit{for all } \theta \in [0,1/2]. \tag{8.3}$$

Proof. (8.3) is a direct consequence of (7.10) and (7.11). □

We are now in position to define the following new scalar product $(\cdot|\cdot)_{\Pi,s}$ on $\mathcal{H}^s$:

$$(\xi|\zeta)_{\Pi,s} = \langle \Pi^{(s)}\xi|\zeta\rangle_{\mathcal{H}^{-s},\mathcal{H}_*^s} \quad \text{ for all } (\xi,\zeta)\in\mathcal{H}^s\times\mathcal{H}^s. \tag{8.4}$$

Proposition 8.3. *For all $s \in [0,1]$, the bilinear form $(\cdot|\cdot)_{\Pi,s}$ defined by (8.4) is a scalar product on $\mathcal{H}^s$. If we define $\|\xi\|_{\Pi,s} = \big((\xi|\xi)_{\Pi,s}\big)^{1/2}$, then the following norm equivalence holds:*

$$\|\cdot\|_{\Pi,s} \sim \|\cdot\|_{\mathcal{H}^s}. \tag{8.5}$$

Moreover, we also have:

$$(\mathcal{A}_\Pi\cdot|\cdot)_{\Pi,s} \sim \|\cdot\|^2_{\mathcal{H}^{1+s}}. \tag{8.6}$$

Proof. Let us show (8.5) for $s=0$. The inequality $\|\cdot\|_{\Pi,0} \le C_1\|\cdot\|_{\mathcal{H}^0}$ is a straightforward consequence of (7.11) with $\theta=0$. The converse one follows from the next calculation where we invoke successively a trace theorem, the first equation in (7.9), the first statement in (7.5), and (7.8):

$$\begin{aligned}
\|\xi\|^2_{\mathcal{H}^0} &\le C_2(\|Y_\xi\|^2_{L^2(0,\infty;\mathcal{H}^1)} + \|Y'_\xi\|^2_{L^2(0,\infty;\mathcal{H}^{-1})}) \\
&\le C_3(\|Y_\xi\|^2_{L^2(0,\infty;\mathcal{H}^1)} + \|\Lambda\Pi Y_\xi\|^2_{L^2(0,\infty;\mathcal{H}^{-1})}) \\
&\le C_4\langle \Pi\xi|\xi\rangle_{\mathcal{H}_*^0,\mathcal{H}^0}.
\end{aligned}$$

Next, from $\|\cdot\|_{\Pi,0}\sim\|\cdot\|_{\mathcal{H}^0}$, we obtain $\|\cdot\|_{\Pi,s} = \|\mathcal{A}_\Pi^{s/2}\cdot\|_{\Pi,0} \sim \|\mathcal{A}_\Pi^{s/2}\cdot\|_{V_n^0(\Omega)}$, and (8.5) follows from (7.10). Finally, we invoke the density of $\mathcal{D}(\mathcal{A})$ in $\mathcal{H}^1$ with $\Pi\in\mathcal{L}(\mathcal{H}^1,\mathcal{H}_*^1)$ to extend the validity of (7.7) to $(\xi,\zeta)\in\mathcal{H}^1\times\mathcal{H}^1$, and we replace ξ and ζ by $\mathcal{A}_\Pi^{s/2}\xi$ to obtain the following explicit expression of $(\mathcal{A}_\Pi\xi|\xi)_{\Pi,s}$:

$$(\mathcal{A}_\Pi\xi|\xi)_{\Pi,s} = \langle\mathcal{A}_\Pi\xi|\Pi^{(s)}\xi\rangle_{\mathcal{H}^{-1+s},\mathcal{H}_*^{1-s}} = \frac{1}{2}\|\mathcal{C}{\mathcal{A}_\Pi}^{s/2}\xi\|^2_{\mathcal{Z}} + \frac{1}{2}\|\Lambda\Pi{\mathcal{A}_\Pi}^{s/2}\xi\|^2_{V^0(\Gamma)} \;\text{ for all }\; \xi\in\mathcal{H}^{1+s}.$$

Hence, (8.6) is a consequence of the first statement in (7.5) with (7.10). □

Next, in order to prove the well posedness of (8.1), as well as a local exponential stabilization result for (8.1), we also need some estimates of the nonlinearity $B(\cdot,\cdot)$.

Lemma 8.4. *Let* $(s_1, s_2, s_3) \in [0,1]^3$ *such that* $s_1 + s_2 + s_3 \geq \frac{d}{2}$ *if* $s_i \neq \frac{d}{2}$, $i = 1, 2, 3$ *or* $s_1 + s_2 + s_3 > \frac{d}{2}$ *if* $s_i = \frac{d}{2}$, *for at least one* i. *There is* $C > 0$ *such that:*

$$|\langle B(Y_1, Y_2)|Y_3\rangle_{\mathcal{H}^{-s_3},\mathcal{H}_*^{s_3}}| \leq C\|Y_1\|_{\mathcal{H}^{s_1}}\|Y_2\|_{\mathcal{H}^{1+s_2}}\|Y_3\|_{\mathcal{H}_*^{s_3}} \quad \forall (Y_1, Y_2, Y_3) \in \mathcal{H}^{s_1} \times \mathcal{H}^{1+s_2} \times \mathcal{H}_*^{s_3}. \tag{8.7}$$

Proof. According to [7], Chapter 6, Section 6.9, for such s_1, s_2 and s_3 there exists a constant $c > 0$ depending on $(s_1, s_2, s_3, \Omega, d)$, such that:

$$\left|\int_\Omega (w_1 \cdot \nabla) w_2 \cdot w_3\right| \leq c\|w_1\|_{V^{s_1}(\Omega)}\|w_2\|_{V^{1+s_2}(\Omega)}\|w_3\|_{V^{s_3}(\Omega)}, \tag{8.8}$$

for all $(w_1, w_2, w_3) \in V^{s_1}(\Omega) \times V^{1+s_2}(\Omega) \times V^{s_3}(\Omega)$. Hence, since for $(Y_1, Y_2, Y_3) \in \mathcal{H}^{s_1} \times \mathcal{H}^{1+s_2} \times \mathcal{H}_*^{s_3}$ where $Y_i = (y_i, u_i)^T$, $i = 1, 2, 3$, we have

$$\langle B(Y_1, Y_2)|Y_3\rangle_{\mathcal{H}^{-s_3},\mathcal{H}_*^{s_3}} = -\int_\Omega \big((y_1 + (I-P)D\gamma_n(u_1)) \cdot \nabla\big)(y_2 + (I-P)D\gamma_n(u_2)) \cdot y_3,$$

from estimate (8.8) we deduce that:

$$|\langle B(Y_1, Y_2)|Y_3\rangle_{\mathcal{H}^{-s_3},\mathcal{H}_*^{s_3}}| \leq c\|y_1 + (I-P)D\gamma_n(u_1)\|_{V^{s_1}(\Omega)}\|y_2 + (I-P)D\gamma_n(u_2)\|_{V^{1+s_2}(\Omega)}\|y_3\|_{V_0^{s_3}(\Omega)}.$$

Finally, (8.7) follows from $\|y_1 + (I-P)D\gamma_n(u_1)\|_{V^{s_1}(\Omega)} \leq C_1(\|y_1\|_{V_n^{s_1}(\Omega)} + \|u_1\|_{V^{s_1-1/2}(\Gamma)}) = C_1\|Y_1\|_{\mathcal{H}^{s_1}}$, from $\|y_2 + (I-P)D\gamma_n(u_2)\|_{V^{1+s_2}(\Omega)} \leq C_2(\|y_2\|_{V_n^{1+s_2}(\Omega)} + \|u_2\|_{V^{s_2+1/2}(\Gamma)}) = C_2\|Y_2\|_{\mathcal{H}^{1+s_2}}$, and from $\|y_3\|_{V_0^{s_3}(\Omega)} \leq \|Y_3\|_{\mathcal{H}_*^{s_3}}$. ☐

Finally, Theorem 3.1 is a consequence of the following theorem.

Theorem 8.5. *Let* $s \in [\frac{d-2}{2}, 1]$. *There exist* $c_0 > 0$ *and* $\mu_0 > 0$ *such that, if* $\delta \in (0, \mu_0)$ *and*

$$Y_0 \in \mathcal{V}_\delta^s = \Big\{Y \in \mathcal{H}^s \quad | \quad \|Y\|_{\mathcal{H}^s} < c_0\delta\Big\}, \tag{8.9}$$

system (8.1) *admits a unique solution in the set*

$$\mathcal{S}_\delta^s = \Big\{Y \in W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s}) \quad | \quad \|Y\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \leq \delta\Big\}. \tag{8.10}$$

Moreover, there exist $C > 0$ *and* $\eta > 0$ *such that*

$$\|Y(t)\|_{\mathcal{H}^s} \leq C\|Y_0\|_{\mathcal{H}^s}\mathrm{e}^{-\eta t}. \tag{8.11}$$

Proof. Let us treat the cases $s > 0$ and $s = 0$ separately.
(i) *The case* $s > 0$.

Since $-\mathcal{A}_\Pi$ is the infinitesimal generator of an analytic semigroup of negative type, the following application

$$\begin{array}{ccc} W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s}) & \longrightarrow & L^2(0,\infty;\mathcal{H}^{-1+s}) \times \mathcal{H}^s, \\ Y & \longmapsto & (Y' + \mathcal{A}_\Pi Y, Y(0)), \end{array} \tag{8.12}$$

is an isomorphism, see [6], Chapter 3, Theorem 2.2, where we can set $T = \infty$ because $(\mathrm{e}^{-\mathcal{A}_\Pi t})_{t\geq 0}$ is exponentially stable. Thus, we consider the mapping

$$\Psi : Z \in W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s}) \longmapsto Y_Z,$$

where, for all $T \in (0,\infty)$, $Y_Z \in W(0,T;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})$ is the solution to

$$Y' + \mathcal{A}_\Pi Y = B(Z,Z), \quad Y(0) = Y_0 \in \mathcal{H}^s.$$

We look for values $c_0 > 0$ and $\mu_0 > 0$ such that, for every $Y_0 \in \mathcal{V}^s_\delta$ with $\delta \in (0,\mu_0)$, Ψ is a contraction in $\mathcal{S}^s_\delta$. Since (8.12) is an isomorphism, and according to (8.7) for $(s_1,s_2,s_3) = (s,s,1-s)$, there is $C_0 > 0$ such that

$$\|\Psi(Z)\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \le C_0(\|Z\|_{L^\infty(0,\infty;\mathcal{H}^s)}\|Z\|_{L^2(0,\infty;\mathcal{H}^{1+s})} + \|Y_0\|_{\mathcal{H}^s}). \tag{8.13}$$

Hence, the continuous embedding $W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s}) \hookrightarrow L^\infty(0,\infty;\mathcal{H}^s)$ gives $C_1 > 0$ such that

$$\|\Psi(Z)\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \le C_0(C_1\|Z\|^2_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} + \|Y_0\|_{\mathcal{H}^s}),$$

and since $Z \in \mathcal{S}^s_\delta$ and $Y_0 \in \mathcal{V}^s_\delta$, we have

$$\|\Psi(Z)\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \le C_0(C_1\mu_0 + c_0)\delta. \tag{8.14}$$

Next, for Z_1 and Z_2 in $\mathcal{S}^s_\delta$ we verify that $Y = \Psi(Z_1) - \Psi(Z_2)$ satisfies

$$Y' + \mathcal{A}_\Pi Y = B(Z_1 - Z_2, Z_1) + B(Z_2, Z_1 - Z_2), \quad Y(0) = 0,$$

and since (8.12) is an isomorphism, according to (8.7) when $(s_1,s_2,s_3) = (s,s,1-s)$, there is $C_2 > 0$ such that

$$\begin{aligned}\|\Psi(Z_1) - \Psi(Z_2)\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \quad \le \quad & C_2(\|Z_1 - Z_2\|_{L^\infty(0,\infty;\mathcal{H}^s)}\|Z_1\|_{L^2(0,\infty;\mathcal{H}^{1+s})} \\ & + \|Z_2\|_{L^\infty(0,\infty;\mathcal{H}^s)}\|Z_1 - Z_2\|_{L^2(0,\infty;\mathcal{H}^{1+s})}).\end{aligned} \tag{8.15}$$

Hence, we invoke the continuous embedding $W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s}) \hookrightarrow L^\infty(0,\infty;\mathcal{H}^s)$, and since Z_1 and Z_2 belong to S^s_δ, we obtain the existence of $C_3 > 0$, such that:

$$\|\Psi(Z_1) - \Psi(Z_2)\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} \le C_2C_3\mu_0\|Z_1 - Z_2\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})}. \tag{8.16}$$

Then for $\mu_0 = \min(\frac{1}{2C_0C_1}, \frac{1}{2C_2C_3})$ and $c_0 < \frac{1}{2C_0}$ in (8.14) and (8.16), Ψ is a contraction in S^s_δ and system (8.1) admits a unique solution in S^s_δ. Next, we multiply the first equation in (8.1) by $\Pi^{(s)}Y(t)$ and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2_{\Pi,s} + (\mathcal{A}_\Pi Y(t)|Y(t))_{\Pi,s} = \langle B(Y(t),Y(t))|\Pi^{(s)}Y(t)\rangle_{\mathcal{H}^{s-1},\mathcal{H}^{1-s}_*}. \tag{8.17}$$

Thus, from (8.7) with $(s_1,s_2,s_3) = (s,s,1-s)$, from (8.3) with $\theta = 1/2$ and from (8.5) and (8.6), we obtain

$$|\langle B(Y(t),Y(t))|\Pi^{(s)}Y(t)\rangle| \le K_s\|Y(t)\|_{\Pi,s}(\mathcal{A}_\Pi Y(t)|Y(t))_{\Pi,s}, \tag{8.18}$$

and (8.17) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2_{\Pi,s} + 2(1 - K_s\|Y(t)\|_{\Pi,s})(\mathcal{A}_\Pi Y(t)|Y(t))_{\Pi,s} \le 0. \tag{8.19}$$

If we choose Y_0 so that $\|Y_0\|_{\Pi,s} \le \frac{1}{2K_s}$, then the mapping $t \mapsto \|Y(t)\|_{\Pi,s}$ is a nonincreasing function with values less than $\frac{1}{2K_s}$. Finally, let $C_4 > 0$ and $\eta > 0$ such that $\|\cdot\|_{\Pi,s} \le C_4\|\cdot\|_s$ and $2\eta\|\cdot\|^2_{\Pi,s} \le (\mathcal{A}_\Pi \cdot|\cdot)_{\Pi,s}$. If we choose $\mu_0 = \min(\frac{1}{2C_0C_1}, \frac{1}{2C_2C_3}, \frac{1}{2c_0C_4K_s})$, then $\|Y_0\|_s \le c_0\mu_0$ implies $\|Y_0\|_{\Pi,s} \le C_4c_0\mu_0 \le \frac{1}{2K_s}$ and (8.19) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|Y(t)\|^2_{\Pi,s} + 2\eta\|Y(t)\|^2_{\Pi,s} \le 0. \tag{8.20}$$

Finally, (8.11) follows from (8.5).

(ii) *The case $s = 0$ and $d = 2$.*

For all $Z = (z, \chi)^T \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1})$ and $V = (w, v)^T \in \mathcal{H}^1_*$, an integration by parts yields

$$\langle B(Z(t), Z(t))|V\rangle_{\mathcal{H}^{-1},\mathcal{H}^1_*} = \begin{pmatrix} \displaystyle\int_\Omega ((z + (I-P)D\gamma_n(\chi)) \cdot \nabla)w \cdot (z + (I-P)D\gamma_n(\chi)) \\ 0 \end{pmatrix}$$

and with (8.8) for $(s_1, s_2, s_3) = (1/2, 0, 1/2)$ we obtain

$$\langle B(Z(t), Z(t))|V\rangle_{\mathcal{H}^{-1},\mathcal{H}^1_*} \leq C\|Z(t)\|_{\mathcal{H}^{1/2}}\|w\|_{V^1_0(\Omega)}\|Z(t)\|_{\mathcal{H}^{1/2}}.$$

Thus, from the interpolation inequality

$$\|\cdot\|_{\mathcal{H}^{1/2}} \leq C\|\cdot\|^{1/2}_{\mathcal{H}^0}\|\cdot\|^{1/2}_{\mathcal{H}^1}, \tag{8.21}$$

we deduce that $\|B(Z(t), Z(t))\|_{\mathcal{H}^{-1}} \leq C\|Z(t)\|_{\mathcal{H}^0}\|Z(t)\|_{\mathcal{H}^1}$ and we obtain (8.13) when $s = 0$. A similar argument also yields (8.15) when $s = 0$ and existence and uniqueness of a solution to (8.1) can be deduce as in the case $s > 0$. Finally, (8.7) with $(s_1, s_2, s_3) = (1/2, 0, 1/2)$ and (8.3) with $s = 0$ and $\theta = 1/2$ yields

$$|\langle B(Y(t), Y(t))|\Pi^{(0)}Y(t)\rangle| \leq C\|Y(t)\|^2_{\mathcal{H}^{1/2}}\|Y(t)\|_{\mathcal{H}^1},$$

and (8.18) when $s = 0$ follows from (8.21) and from (8.5) and (8.6) with $s = 0$. □

Remark 8.6. The mapping $\xi \in \mathcal{H}^s \mapsto \|\xi\|^2_{\Pi,s}$ is a Lyapunov function of system (8.1). Indeed, according to (8.20), for all $Y_0 \in \mathcal{V}^s_\delta$ the solution $Y \in \mathcal{S}^s_\delta$ of (8.1) is such that $t \mapsto \|Y(t)\|^2_{\Pi,s}$ decreases to 0 with values in $\mathbb{R}^+$.

Proof of Theorem 3.1. Let $s \in [\frac{d-2}{2}, 1]\backslash\{1/2\}$ and set $Y_0 = (Pz_0, \gamma^s(z_0))^T$. According to Theorems 6.9, 4.9 and 5.5 and to (7.12), the formulation (8.1) is equivalent to (3.1)-(3.2)-(3.3) where $z = y + (I-P)D\gamma_n(u)$. Moreover, (2.2) and (4.13) guarantee that $Y_0 \in \mathcal{H}^s$, and that there exists $c_1 > 0$ such that

$$\|Y_0\|_{\mathcal{H}^s} \leq c_1\|z_0\|_{V^s(\Omega)}.$$

Next, with $Y = (y, u)^T$ and $z = y + (I-P)D\gamma_n(u)$, we obtain $c_2 > 0$ such that

$$\|z\|_{W(0,\infty;V^{s+1}(\Omega),V^{s-1}_0(\Omega))} + \|u\|_{W(0,\infty;V^{s+1/2}(\Gamma),V^{s-3/2}(\Gamma))} \leq c_2\|Y\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})}.$$

Moreover, from the continuous embedding $W(0, \infty; V^{s+1}(\Omega), V^{s-1}_0(\Omega)) \hookrightarrow H^{s/2+1/2}(0, \infty; V^0(\Omega))$ we deduce that $\partial_t u \in H^{s/2-1/2}(0, \infty; V^0(\Omega))$, and by recalling that p obeys (3.1) we obtain $\nabla p \in H^{s/2+1/2}(0, \infty; \mathbf{H}^{s-1}(\Omega))$ and $p \in H^{s/2-1/2}(0, \infty; \mathbb{H}^s(\Omega))$. Hence, (3.1) and (3.2) with $Y = (y, u)^T$ and $z = y + (I-P)D\gamma_n(u)$, provide $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned} \|p\|_{H^{s/2-1/2}(0,\infty;\mathbb{H}^s(\Omega))} &\leq c_3(\|Y\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})} + \|Y\|^2_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})}), \\ \|\sigma\|_{L^2(0,\infty)} &\leq c_4\|Y\|_{W(0,\infty;\mathcal{H}^{1+s},\mathcal{H}^{-1+s})}. \end{aligned}$$

As a consequence, for $c_5 = \max(1, c_2, c_3, c_4)$ the above inequalities with Theorem 8.5 guarantee that if z_0 obeys $\|z_0\|_{V^s(\Omega)} \leq \frac{c_0}{c_1c_5}\delta$ then we successively obtain $\|Y_0\|_{\mathcal{H}^s} \leq \frac{c_0}{c_5}\delta$, $Y \in \mathcal{S}^s_{\frac{\delta}{c_5}}$ and $(z, u, p, \sigma) \in \mathcal{D}^s_\delta$. Finally, Theorem 3.1 holds with $c = \frac{c_0}{c_1c_5}$ in (3.4). □

9. Localization of the control on a part of the boundary

In the previous sections, we deal with a boundary control u acting on the whole boundary Γ. Nevertheless, it is possible to treat the case of a boundary control which is localized in an open subset of Γ. We introduce a weight function $m \in C^2(\Gamma)$ with values in $[0,1]$, with support in $\Gamma_m \subset \Gamma$ and equal to 1 in Γ_1, where Γ_1 is an open subset of Γ_m. Thus, we define $D_m : V^0(\Gamma) \longrightarrow V^0(\Omega)$ by $D_m u = w$ where (w,q) is the solution to

$$\begin{aligned}&\lambda_0 w - \nu\Delta w + (w\cdot\nabla)z_s + (z_s\cdot\nabla)w + \nabla q = 0, \quad \nabla\cdot w = 0,\\ &w|_\Gamma = m(u - \sigma_m(u)\,n), \quad \sigma_m(v) = \left(\int_\Gamma m\right)^{-1}\int_\Gamma mv\cdot n.\end{aligned}$$

Notice that for all $u \in V^0(\Gamma)$ the boundary value $m(u-\sigma_m(u)\,n)$ is supported in Γ_m and belongs to $V^0(\Gamma)$. Hence, we define the operator $(\mathcal{D}(\mathcal{A}_m),\mathcal{A}_m)$ in $\mathcal{H}^0$ by

$$\begin{aligned}\mathcal{D}(\mathcal{A}_m) &= \left\{ (y,u) \in V^2_n(\Omega)\times V^{3/2}(\Gamma) \mid y - PD_m u \in V^2_0(\Omega) \right\},\\ \mathcal{A}_m &= \begin{pmatrix} \nu P\Delta - P(\nabla z_s) - P(z_s\cdot\nabla) & (\lambda_0 - \nu P\Delta + P(\nabla z_s) + P(z_s\cdot\nabla))PD_m \\ 0 & P_b\Delta_b \end{pmatrix}.\end{aligned}$$

By following the path of Section 6, we can prove that the $V^0_n(\Omega)\times V^0(\Gamma)$-adjoint of $(\mathcal{D}(\mathcal{A}_m),\mathcal{A}_m)$ is defined by

$$\mathcal{D}(\mathcal{A}^*_m) = V^2_0(\Omega)\times V^{5/2}(\Gamma), \quad \mathcal{A}^*_m = \begin{pmatrix} \nu P\Delta - P(\nabla z_s)^T + P(z_s\cdot\nabla) & 0 \\ D^*_m(\lambda_0 - \nu P\Delta + P(\nabla z_s)^T - P(z_s\cdot\nabla)) & P_b\Delta_b \end{pmatrix}, \tag{9.22}$$

and that $(\mathcal{D}(\mathcal{A}_m),\mathcal{A}_m)$ (resp. $(\mathcal{D}(\mathcal{A}^*_m),\mathcal{A}^*_m)$) is the infinitesimal generator of an analytic semigroup on $\mathcal{H}^0$ (resp. $\mathcal{H}^0_*$). Thus, for $\theta\in[0,1]$, we introduce the spaces

$$\mathcal{H}^{2\theta}_m = \left\{ (y,u)\in V^{2\theta}_n(\Omega)\times V^{2\theta-1/2}(\Gamma) \mid y - PD_m u \in V^{2\theta}_0(\Omega) \right\}, \quad \mathcal{H}^{2\theta}_{*,m} = V^{2\theta}_0(\Omega)\times V^{2\theta+1/2}(\Gamma) = \mathcal{H}^{2\theta}_*,$$

$\mathcal{H}^{-2\theta}_m = (\mathcal{H}^{2\theta}_{*,m})'$ and $\mathcal{H}^{-2\theta}_{*,m} = (\mathcal{H}^{2\theta}_m)'$, and we remark that $\mathcal{H}^{2\theta}_m = \mathcal{H}^{2\theta}$ when $\theta\in[0,1/4[$. As in Corollary 6.7, we show that for $Y_0\in\mathcal{H}^0$ and $G\in L^2(0,T;V^0_n(\Omega)\times V^0(\Gamma))$, there is a unique $Y\in W(0,T;\mathcal{H}^1_m,\mathcal{H}^{-1}_m)$ solution to

$$Y' = \mathcal{A}_m Y + \Lambda G \ \text{ on } \ \mathcal{D}(\mathcal{A}^*_m)', \quad Y(0) = Y_0\in\mathcal{H}^0. \tag{9.23}$$

Next, we define the observation space $\Xi = \mathbf{L}^2(\Omega)\times L^2(\Omega,\mathbb{R}^{d^2})\times L^2(\Omega,\mathbb{R}^{d^2})$ and the observation operator:

$$\mathcal{C}_m : \mathcal{H}^1_m \longrightarrow \Xi \quad \text{and} \quad \mathcal{C}_m Y = \begin{pmatrix} y + (I-P)D_m\gamma_n(u) \\ \nabla(y + (I-P)D_m\gamma_n(u)) \\ \nabla Du \end{pmatrix},$$

and we easily verify that $\|\mathcal{C}_m\cdot\|_\Xi \sim \|\cdot\|_{\mathcal{H}^1_m}$ and $\mathcal{C}^*_m\mathcal{C}_m \in \mathcal{L}(\mathcal{H}^1_m,\mathcal{H}^{-1}_{*,m})\cap\mathcal{L}(\mathcal{H}^2_m,\mathcal{H}^0_m)$. We have added the third component ∇Du in order to control $\|u\|_{V^{1/2}(\Gamma)}$, and so that the inequality $\|\cdot\|_{\mathcal{H}^1_m} \le C\|\mathcal{C}_m\cdot\|_\Xi$ be true. Finally, we define the optimal control problem:

$$(\mathcal{P}_{m,Y_0}) \quad \inf\left\{ \mathcal{J}_m(Y,G) \ \mid \ G\in L^2(0,\infty;V^0_n(\Omega)\times V^0(\Gamma)), \ (Y,G) \text{ satisfies (9.23)} \right\},$$

where

$$\mathcal{J}_m(Y,G) = \int_0^\infty \|\mathcal{C}_m Y\|^2_\Xi + \int_0^\infty \|G\|^2_{V^0_n(\Omega)\times V^0(\Gamma)}.$$

Solving $(\mathcal{P}_{m,Y_0})$ provides $\Pi_m \in \mathcal{L}(\mathcal{H}^0, \mathcal{H}^0_*)$ which is the unique solution in $\mathcal{X}$ (defined by (7.6)) to the following Riccati equation:

$$\langle \Pi_m\xi|\mathcal{A}_m\zeta\rangle_{\mathcal{H}^0_*,\mathcal{H}^0}+\langle \mathcal{A}_m\xi|\Pi_m\zeta\rangle_{\mathcal{H}^0,\mathcal{H}^0_*}-(\Lambda\Pi_m\xi|\Lambda\Pi_m\zeta)_{V^0_n(\Omega)\times V^0(\Gamma)}+(\mathcal{C}_m\xi|\mathcal{C}_m\zeta)_{\Xi}=0,\ \forall(\xi,\zeta)\in\mathcal{D}(\mathcal{A}_m)\times\mathcal{D}(\mathcal{A}_m). \tag{9.24}$$

As in Remark 7.4 one shall also underline that there is a triplet $(\Pi_{m,1}, \Pi_{m,2}, \Pi_{m,3})$ which obeys:

$$\Pi_m = \begin{pmatrix} \Pi_{m,1} & \Pi_{m,2} \\ \Pi^*_{m,2} & \Pi_{m,3} \end{pmatrix}, \quad (\Pi_{m,1}, \Pi_{m,2}, \Pi_{m,3}) \in \mathcal{L}(V^0_n(\Omega)) \times \mathcal{L}(V^0_n(\Omega), V^{1/2}(\Gamma)) \times \mathcal{L}(V^{-1/2}(\Gamma), V^{1/2}(\Gamma)). \tag{9.25}$$

Thus, as what has been done in Sections 7 and 8, we introduce the unbounded operator $\mathcal{A}_{m,\Pi_m} = \Lambda\Pi_m - \mathcal{A}_m$ in $\mathcal{H}^0$ with domain $\mathcal{D}(\mathcal{A}_{m,\Pi_m}) = \mathcal{H}^2_m$, and for $s \in [\frac{d-2}{2}, 1]$ we introduce the operator $\Pi^{(s)}_m = \mathcal{A}^{*\,s/2}_{m,\Pi_m}\Pi_m\mathcal{A}^{s/2}_{m,\Pi_m}$, the scalar product $(\cdot|\cdot)_{\Pi_m,s} = \langle \Pi^{(s)}_m \cdot |\cdot\rangle_{\mathcal{H}^{-s}_m,\mathcal{H}^s_{*,m}}$, and the two norms $\|\cdot\|_{\Pi_m,s} = (\cdot|\cdot)^{1/2}_{\Pi_m,s}$ and $(\mathcal{A}_{m,\Pi_m}\cdot|\cdot)^{1/2}_{\Pi_m,s}$, which are respectively equivalent to $\|\cdot\|_{\mathcal{H}^s_m}$ and $\|\cdot\|_{\mathcal{H}^{1+s}_m}$. An obvious adaptation of the proof of Theorem 8.5 shows that there exist $C > 0$, $\eta > 0$, $c_0 > 0$ and $\mu_0 > 0$ such that, if $\delta \in (0, \mu_0)$ and

$$Y_0 \in \mathcal{V}^s_{m,\delta} = \Big\{ Y \in \mathcal{H}^s_m \mid \|Y\|_{\mathcal{H}^s_m} < c_0\delta \Big\},$$

then the system

$$Y' + \mathcal{A}_{m,\Pi_m}Y = B(Y,Y), \quad Y(0) = Y_0, \tag{9.26}$$

admits a unique solution in

$$\mathcal{S}^s_{m,\delta} = \Big\{ Y \in W(0,\infty; \mathcal{H}^{1+s}_m, \mathcal{H}^{-1+s}_m) \ \mid\ \|Y\|_{W(0,\infty;\mathcal{H}^{1+s}_m,\mathcal{H}^{-1+s}_m)} \le \delta \Big\},$$

which obeys $\|Y(t)\|_{\mathcal{H}^s_m} \le C\|Y_0\|_{\mathcal{H}^s_m}\mathrm{e}^{-\eta t}$. Finally, if $s \neq 1/2$, we easily verify that for $z_0 \in V^s_m(\Omega)$ (defined in (3.7)) we have $z_0 - D_m\gamma^s(z_0) \in V^s_0(\Omega)$ and $(Pz_0, \gamma^s(z_0))^T \in \mathcal{H}^s_m$. As a consequence, for $Y_0 = (Pz_0, \gamma^s(z_0))^T$ we rewrite (9.26) in the equivalent formulation (3.8)-(3.9)-(3.10)-(3.11), and Theorem 3.6 follows.

10. Appendix

The goal of the present appendix is to prove a finite cost condition ensuring that $(\mathcal{P}_{m,Y_0})$ admits solutions. The main argument of the proof relies in a geometrical extension procedure which consists in working with a system defined in a larger domain $\widetilde{\Omega} = \mathrm{Int}(\overline{\Omega} \cup \overline{\omega})$, where ω is an open bounded domain of $\mathbb{R}^d$ such that $\widetilde{\Omega}$ is of class C^4, $\omega \cap \Omega = \emptyset$ and $\sigma = \partial\omega \cap \Gamma$ is an open subset of Γ_1. See Figure 1, where σ is the part of Γ going from A to B in the clockwise direction, where Γ_1 is the part of Γ going from A_1 to B_1 in the clockwise direction and where Γ_m is the part of Γ going from A_m to B_m in the clockwise direction. We recall that Γ_1 is an open subset of Γ on which the cut-off function m is equal to 1. With such a choice of ω, it will be possible to construct a boundary control u supported in Γ_1, and so that $u = m(u - \sigma_m(u)n)$. We set $\widetilde{\Gamma} = \cup_{j=1}^k \widetilde{\Gamma}^{(j)} = \partial\widetilde{\Omega}$ where $\widetilde{\Gamma}^{(1)}, \dots, \widetilde{\Gamma}^{(N)}$ denote the connected components of $\partial\widetilde{\Omega}$, and for $s \ge 0$ we define the spaces $V^s_n(\widetilde{\Omega})$, $V^s_0(\widetilde{\Omega})$ and $V^s(\widetilde{\Gamma})$ in the same way as $V^s_n(\Omega)$, $V^s_0(\Omega)$ and $V^s(\widetilde{\Gamma})$, and we introduce the spaces:

$$\widehat{V}^2(\Omega) = \big\{ y \in V^s(\Omega) \mid \langle y\cdot n|1\rangle_{H^{-1/2}(\Gamma^{(j)}),H^{1/2}(\Gamma^{(j)})} = 0,\ j = 1,\dots,N \big\}, \tag{10.27}$$

$$\widehat{V}^2(\widetilde{\Omega}) = \big\{ y \in V^s(\widetilde{\Omega}) \mid \langle y\cdot n|1\rangle_{H^{-1/2}(\widetilde{\Gamma}^{(j)}),H^{1/2}(\widetilde{\Gamma}^{(j)})} = 0,\ j = 1,\dots,N \big\}. \tag{10.28}$$

Thus, we establish an extension lemma that is lifting a function $z \in \widehat{V}^2(\Omega)$ to a function $\widetilde{z} \in \widehat{V}^2(\widetilde{\Omega})$.

Lemma 10.1. *There exists an operator $E \in \mathcal{L}(\widehat{V}^2(\Omega), \widehat{V}^2(\widetilde{\Omega}))$ such that $E(z)|_\Omega = z$ for all $z \in \widehat{V}^2(\Omega)$.*

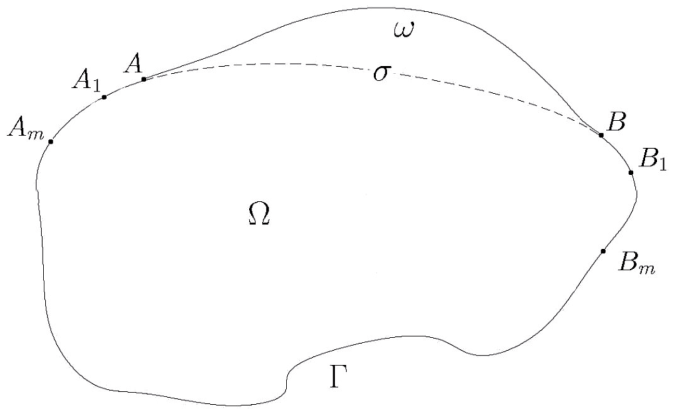

FIGURE 1. Geometrical extended domain $\widetilde{\Omega}$.

Proof. Let $z \in \widehat{V}^2(\Omega)$. According to [25], Appendix I, Proposition 1.3, Remark 1.5 and Proposition 1.4, page 467, there exists $F \in (H^3(\Omega))^k$ such that $\nabla \times F = z$, where $k = 1$ if $d = 2$ and $k = 3$ if $d = 3$. Thus, we extend F to a function $\widetilde{F} \in (H^3(\widetilde{\Omega}))^k$ by classical ways, and we define $E(z) = \nabla \times \widetilde{F}$. □

We now define the Oseen operator in $\widetilde{\Omega}$. From $z_s \in \widehat{V}^2(\Omega)$ we define $\widetilde{z}_s = E(z_s) \in \widehat{V}^2(\widetilde{\Omega})$ and $(\mathcal{D}(\widetilde{A}), \widetilde{A}) = (V_0^2(\widetilde{\Omega}), \nu\widetilde{P}\Delta - \widetilde{P}(\nabla\widetilde{z}_s) - \widetilde{P}(\widetilde{z}_s \cdot \nabla))$ in $V_n^0(\widetilde{\Omega})$, where $\widetilde{P}$ is the orthogonal projector from $\mathbf{L}^2(\widetilde{\Omega})$ into $V_n^0(\widetilde{\Omega})$. The following theorem states the existence of a pair (Y, G) obeying (9.23) and $\mathcal{J}(Y, G) < +\infty$.

Theorem 10.2. *For all $Y_0 \in \mathcal{H}^0$, there is $G \in L^2(0, \infty; V_n^0(\Omega) \times V^0(\Gamma))$ such that $Y \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1})$ and*

$$Y' = \mathcal{A}_m Y + \Lambda G \quad on \quad \mathcal{D}(\mathcal{A}_m^*)', \quad Y(0) = Y_0 \in \mathcal{H}^0. \tag{10.29}$$

Proof. In three steps, we are going to exhibit a pair $(Y, G) \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1}) \times L^2(0, \infty; V_n^0(\Omega) \times V^0(\Gamma))$ satisfying (10.29). Let us fix $0 < t_1 < t_2 < +\infty$.

Step 1. Here, we exhibit a control $G \in L^2(0, t_1; V_n^0(\Omega) \times V^0(\Gamma))$ which brings $Y(0) = Y_0 \in \mathcal{H}^0$ to $Y(t_1) = (z_1, 0)^T$ where $z_1 \in V_0^2(\Omega)$. Let us assume that $0 < \tau' < \tau'' < t_1$. First, by setting $G = (0, 0)^T$ on $[0, \tau']$, the analyticity of $(\mathrm{e}^{\mathcal{A}_m t})_{t \geq 0}$ on $\mathcal{H}^0$ ensures that $Y(\tau') = (y(\tau'), u(\tau'))^T \in \mathcal{D}(\widehat{\mathcal{A}}_m^{3/4}) = \mathcal{H}_m^{3/2}$. Hence, since $u(\tau') \in V^1(\Gamma)$ we can choose $v \in W(\tau', \tau''; V^2(\Gamma), V^0(\Gamma))$ obeying $v(\tau') = u(\tau')$ and $v(\tau'') = 0$. Thus, by choosing the control $G = (0, v' - A_b v)^T \in L^2(\tau', \tau''; V_n^0(\Omega) \times V^0(\Gamma))$ on $[\tau', \tau'']$ we obtain $Y(\tau'') = (y(\tau''), 0)^T$ where $y(\tau'') \in V_0^1(\Omega)$. As a consequence, by applying $G = (0, 0)^T$ on $[\tau'', t_1]$, we have $Y(t) = (\mathrm{e}^{A(t-\tau'')} y(\tau''), 0)^T$ on $[\tau'', t_1]$, and with the analyticity of $(\mathrm{e}^{At})_{t \geq 0}$ on $V_n^0(\Omega)$, we deduce that $Y(t_1) = (z_1, 0)^T$ where $z_1 \in V_0^2(\Omega)$.

Step 2. Here, we exhibit a control $G \in L^2(t_1, t_2; V_n^0(\Omega) \times V^0(\Gamma))$ which brings $Y(t_1) = (z_1, 0)^T$ where $z_1 \in V_0^2(\Omega)$ to $Y(t_2) = (P z_2, z_2|_\Gamma)^T$ where z_2 is the restriction to Ω of a function $\widetilde{z}_2 \in V_0^2(\widetilde{\Omega})$. First, $z_1 \in V_0^2(\Omega) \subset \widehat{V}^2(\Omega)$ with Lemma 10.1 ensures that we can define $\widetilde{z}_1 = E(z_1) \in \widehat{V}^2(\widetilde{\Omega})$. Hence, we have $\widetilde{z}_1|_{\widetilde{\Gamma}} \in V^{3/2}(\widetilde{\Gamma}) = [V^{5/2}(\widetilde{\Gamma}), V^0(\widetilde{\Gamma})]_{2/5}$ and there exists $\widetilde{u} \in L^2(t_1, t_2; V^{5/2}(\widetilde{\Gamma})) \cap H^{5/4}(t_1, t_2; V^0(\widetilde{\Gamma}))$ such that $\widetilde{u}(t_1) = \widetilde{z}_1|_{\widetilde{\Gamma}}$ [15]. Notice that since $\widetilde{z}_1|_{\widetilde{\Gamma}}$ is equal to zero outside $\widetilde{\sigma} = \partial\omega \cap \partial\widetilde{\Omega}$, even by replacing $\widetilde{u}$ by $\rho\widetilde{u}$ where $\rho \in C^\infty(\widetilde{\Gamma} \times (t_1, t_2))$ is an adequate cut-off function, on can suppose that $\widetilde{u}$ is equal to zero on $\Gamma \backslash \Gamma_1$ and obeys $\widetilde{u}(t_2) = 0$. By this way, the velocity $\widetilde{z}$ solution to the following Oseen system defined in $\widetilde{\Omega}$:

$$\begin{aligned} &\partial_t \widetilde{z} - \nu\Delta\widetilde{z} + (\widetilde{z} \cdot \nabla)\widetilde{z}_s + (\widetilde{z}_s \cdot \nabla)\widetilde{z} + \nabla\widetilde{p} = 0, \quad \nabla \cdot \widetilde{z} = 0 \ \text{ in } \ \widetilde{\Omega} \times (t_1, t_2), \\ &\widetilde{z} = \widetilde{u} \ \text{ on } \ \widetilde{\Gamma} \times (t_1, t_2), \quad \widetilde{z}(t_1) = \widetilde{z}_1, \end{aligned}$$

admits on Γ a trace $u = \widetilde{z}|_\Gamma$ supported in Γ_1 and such that $u = m(u - \sigma_m(u)n)$. Moreover, with $\widetilde{u} \in L^2(t_1, t_2; V^{5/2}(\widetilde{\Gamma})) \cap H^{5/4}(t_1, t_2; V^0(\widetilde{\Gamma}))$, with $\widetilde{z}_1 \in V^2(\widetilde{\Omega})$ and with $\widetilde{\Omega}$ of class C^4, one can use regularity results for the Oseen system [23], Theorem 4.1(v), which yields $\widetilde{z} \in L^2(t_1, t_2; V^3(\widetilde{\Omega})) \cap H^{3/2}(t_1, t_2; V^0(\widetilde{\Omega}))$. Hence, we have $\widetilde{z} \in C([t_1, t_2]; V^2(\widetilde{\Omega}))$, and since $\widetilde{u}$ obeys $\widetilde{u}(t_2) = 0$, we deduce that $\widetilde{z}(t_2) = \widetilde{z}_2 \in V_0^2(\widetilde{\Omega})$. Then by setting $z_2 = \widetilde{z}_2|_\Omega$, $z = \widetilde{z}|_\Omega$ and recalling $u = \widetilde{z}|_\Gamma$ and $u = m(u - \sigma_m(u)n)$, one easily verify that $Y = (Pz, u)^T$ satisfies the first equation in (10.29) for $G = (0, u' - A_b u)^T \in L^2(t_1, t_2; V_n^0(\Omega) \times V^0(\Gamma))$, and we have $Y(t_2) = (Pz_2, z_2|_\Gamma)^T$ where $z_2 = \widetilde{z}_2|_\Omega$ and $\widetilde{z}_2 \in V_0^2(\widetilde{\Omega})$.

Step 3. Here, we exhibit a control $G \in L^2(t_2, \infty; V_n^0(\Omega) \times V^0(\Gamma))$ which brings $Y(t_2) = (Pz_2, z_2|_\Gamma)^T$ to zero at infinity. To overcome this goal, it is sufficient to construct a control $\widehat{f} \in L^2(0, \infty; \mathbf{L}^2(\widetilde{\Omega}))$ supported in ω, for which the solution $\widetilde{z}$ to the Oseen system in $\widetilde{\Omega}$, which obeys $\widetilde{z}(t_2) = \widetilde{z}_2 \in V_0^2(\widetilde{\Omega})$, belongs to $W(t_2, \infty; V_0^3(\widetilde{\Omega}), V_0^1(\widetilde{\Omega}))$. By this way, $z = \widetilde{z}|_\Omega \in W(t_2, \infty; V^3(\Omega), V^1(\Omega))$ obeys $z(t_2) = z_2$ and is solution to the Oseen system in Ω for the boundary control $u = \widetilde{z}|_\Gamma$ supported in Γ_1, and so that $u = m(u - \sigma_m(u)n)$. Moreover, one can verify that $G = (0, u' - A_b u)^T \in L^2(0, \infty; V_n^0(\Omega) \times V^0(\Gamma))$ and that $Y = (Pz, u)^T \in W(t_2, \infty; \mathcal{H}^1, \mathcal{H}^{-1})$ satisfies (10.29). Let $M \in C^\infty(\widetilde{\Omega})$ be supported in ω and obeying $M|_{\omega_1} = 1$, where ω_1 is an open subset of ω, and let us choose $\widehat{f}$ which minimizes the cost

$$\mathcal{F}(y, f) = \int_0^{+\infty} \int_{\widetilde{\Omega}} |y|^2 + \int_0^{+\infty} \int_{\widetilde{\Omega}} |f|^2,$$

where

$$y' = \widetilde{A}y + \widetilde{P}Mf \quad \text{on} \quad \mathcal{D}(\widetilde{A}^*)', \quad y(0) = y_0 \in V_n^0(\widetilde{\Omega}). \tag{10.30}$$

An exact controllability result [9] ensures the existence of a finite time $T_0 \in]0, \infty[$, and of a control $f_{T_0} \in L^2(0, \infty; \mathbf{L}^2(\widetilde{\Omega}))$ supported in $\omega_1 \times (0, T_0)$ for which the solution to (10.30) is zero past T_0. Notice that since $M|_{\omega_1} = 1$ we have $f_{T_0} = Mf_{T_0}$. Hence, we have a finite cost $\mathcal{F}(y_{T_0}, f_{T_0}) < +\infty$, and we can apply the theory of [18], Chapter 2. There exists an optimal control given by the feedback expression $\widehat{f} = -\widetilde{P}M^2Ry$, where R is a linear operator in $\mathcal{L}(V_n^0(\widetilde{\Omega}), V_0^1(\widetilde{\Omega}))$, and if we set $\widetilde{A}_R = \widetilde{A} - \widetilde{P}M^2R$, the semigroup $(\mathrm{e}^{\widetilde{A}_R t})_{t \geq 0}$ is analytic and exponentially stable on $V_n^0(\widetilde{\Omega})$. Hence, classical regularity results [6] ensures that $\mathrm{e}^{\widetilde{A}_R(\cdot)}y_0 \in W(0, \infty; \mathcal{D}(\widetilde{A}_R^{3/2}), \mathcal{D}(\widetilde{A}_R^{1/2}))$ for all $y_0 \in \mathcal{D}(\widetilde{A}_R)$. Moreover, since $\widetilde{A}_R$ is a bounded perturbation of $\widetilde{A}$, we deduce that $\mathcal{D}(\widetilde{A}_R) = \mathcal{D}(\widetilde{A}) = V_0^2(\widetilde{\Omega})$ and that $\mathcal{D}(\widetilde{A}_R^{1/2}) = V_0^1(\widetilde{\Omega})$, and since equation $\widetilde{A}_R^{3/2}y = \chi$ is equivalent to

$$-\nu\Delta y + \nabla p = -(\widetilde{z}_s \cdot \nabla)y - (y \cdot \nabla)\widetilde{z}_s - M^2Ry - \widetilde{A}_R^{-1/2}\chi,$$

from $\widetilde{z}_s \in \mathbf{H}^2(\widetilde{\Omega})$, $M^2R \in \mathcal{L}(V_n^0(\widetilde{\Omega}), \mathbf{H}^1(\widetilde{\Omega}))$, $\widetilde{A}_R^{-1/2}\chi \in \mathcal{D}(\widetilde{A}_R^{1/2}) = V_0^1(\widetilde{\Omega})$ for all $\chi \in V_n^0(\widetilde{\Omega})$, and $\widetilde{\Omega}$ of class C^3, regularity results for the Stokes system [13] yields the continuous embedding $\mathcal{D}(\widetilde{A}_R^{3/2}) \hookrightarrow V_0^3(\widetilde{\Omega})$. As a consequence, for all $y_0 \in V_0^2(\widetilde{\Omega})$ we have $\mathrm{e}^{\widetilde{A}_R(\cdot)}y_0 \in W(0, \infty; V_0^3(\widetilde{\Omega}), V_0^1(\widetilde{\Omega}))$, and $\widetilde{z} = \mathrm{e}^{\widetilde{A}_R(\cdot - t_2)}\widetilde{z}_2$ belongs to $W(t_2, \infty; V_0^3(\widetilde{\Omega}), V_0^1(\widetilde{\Omega}))$. □

Remark 10.3. Since E does not map $V_0^2(\Omega)$ onto $V_0^2(\widetilde{\Omega})$, we cannot claim that $\widetilde{z}_1 = E(z_1) \in V_0^2(\widetilde{\Omega})$ and skip the second step of the proof of Theorem 10.2. However, in the particular case of non localized control (where $\Gamma_m = \Gamma_1 = \Gamma$), we have $\partial\Omega \cap \partial\omega = \Gamma$ and we can obtain an extension operator $E_0 \in \mathcal{L}(V_0^2(\Omega), V_0^2(\widetilde{\Omega}))$. Indeed, in the proof of Lemma 10.1, it suffices to set $E_0(z) = \nabla \times (\rho\widetilde{F})$ where $\rho \in C^\infty(\widetilde{\Omega})$ is an adequate cut-off function.

Corollary 10.4. *For all $Y_0 \in \mathcal{H}^0$, there is $G \in L^2(0, \infty; V_n^0(\Omega) \times V^0(\Gamma))$ such that $Y \in W(0, \infty; \mathcal{H}^1, \mathcal{H}^{-1})$ and*

$$Y' = \mathcal{A}Y + \Lambda G \quad \textit{on} \quad \mathcal{D}(\mathcal{A}^*)', \quad Y(0) = Y_0 \in \mathcal{H}^0.$$

Proof. It is a consequence of Theorem 10.2 in the particular case $m = 1$ and $\Gamma_1 = \Gamma_m = \Gamma$. □

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