

MORPHISMS FIXING WORDS ASSOCIATED WITH EXCHANGE OF THREE INTERVALS

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Abstract. We consider words coding exchange of three intervals with permutation $(3,2,1)$, here called 3iet words. Recently, a characterization of substitution invariant 3iet words was provided. We study the opposite question: what are the morphisms fixing a 3iet word? We reveal a narrow connection of such morphisms and morphisms fixing Sturmian words using the new notion of amicability.

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1. INTRODUCTION

Words coding exchange of three intervals represent one of possible generalizations of Sturmian words to a ternary alphabet. An exchange of three intervals is given by a permutation π on the set $\{1, 2, 3\}$, and a triplet of positive numbers α, β, γ , corresponding to lengths of intervals I_A, I_B, I_C , respectively, which define a division of the interval I . In this paper we study infinite words coding exchange of three intervals with the permutation $(3, 2, 1)$. Such words are called here 3iet words. Properties of 3iet words have been studied from various points of view in papers [1, 8, 10–12].

Recently, articles [5] and [3] gave a characterization of 3iet words invariant under a substitution. Recall that a similar question for Sturmian words (*i.e.*, words coding exchange of two intervals) has been partially solved in [9, 14, 16]. Complete solution to the task was provided by Yasutomi [19]. An alternative proof valid for bidirectional Sturmian words is given in [4], yet another proof in [7].

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One has also asked the question from another angle: what are the substitutions fixing a Sturmian word, this problem has been studied in a wider context. One considers the so-called Sturmian morphisms, *i.e.*, morphisms that preserve the set of Sturmian words. The monoid of Sturmian morphisms has been described in [15,18]. It turns out that it is generated by three simple morphisms, namely

$$\varphi : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}, \quad \psi : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}, \quad \text{and} \quad E : \begin{array}{l} 0 \mapsto 1 \\ 1 \mapsto 0 \end{array}. \quad (1.1)$$

It is known [6] that a morphism ξ such that $\xi(u)$ is Sturmian for at least one Sturmian word u belongs also to the monoid. In particular, all morphisms fixing Sturmian word are Sturmian morphisms.

The aim of this paper is to describe morphisms over the alphabet $\{A, B, C\}$ fixing a 3iet word. The main tool which we use is a narrow connection between 3iet words and Sturmian words over the alphabet $\{0, 1\}$ by means of morphisms $\sigma_{01}, \sigma_{10} : \{A, B, C\}^* \rightarrow \{0, 1\}^*$ given by

$$\begin{array}{l} A \mapsto 0 \\ \sigma_{01} : B \mapsto 01, \\ C \mapsto 1 \end{array}, \quad \text{and} \quad \begin{array}{l} A \mapsto 1 \\ \sigma_{10} : B \mapsto 10. \\ C \mapsto 0 \end{array}. \quad (1.2)$$

In [3] the following statement is proved.

Theorem 1.1 [3]. *A ternary word u is a 3iet word if and only if both $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are Sturmian words.*

Another important statement connecting 3iet words and Sturmian words is taken from [5].

Theorem 1.2 [5]. *A non-degenerate 3iet word u is invariant under a substitution if and only if both $\sigma_{01}(u)$ and $\sigma_{10}(u)$ are invariant under substitution.*

The paper is organized as follows. In Section 2 we recall the definitions of 3iet words and morphisms and the geometric representation of a fixed point of a morphism. In Section 3 we define a relation on the set of Sturmian morphisms with a given incidence matrix, called amicability, and we show how to construct from a pair of amicable morphisms a morphism over the alphabet $\{A, B, C\}$ with a 3iet fixed point (Thm. 3.5). In Section 4 we show that any morphism η fixing a non-degenerate 3iet word (or its square η^2) is constructed in this way (Thm. 4.1).

2. PRELIMINARIES

2.1. THREE INTERVAL EXCHANGE

A transformation $T : I \rightarrow I$ of an exchange of three intervals is usually defined as a mapping with the domain $I = [0, \alpha + \beta + \gamma)$, where α, β, γ are arbitrary positive numbers determining the splitting of I into three disjoint subintervals

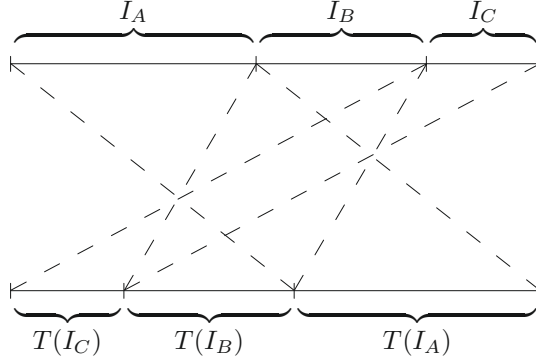


FIGURE 1. Graph of a 3-interval exchange transformation.

$I = I_A \cup I_B \cup I_C$. An infinite word associated with such a transformation is given as a coding of an initial point $x_0 \in I$ in a ternary alphabet $\{A, B, C\}$. Properties of the transformation T and the corresponding infinite word do not depend on absolute values of α, β, γ , but rather on their relative sizes. As well, translation of the interval I on the real line does not influence the corresponding dynamical system. For the study of substitution properties of 3iet words it proved useful to consider the definition of a 3iet mapping with parameters normalized by $\alpha + 2\beta + \gamma = 1$ and a translation of the interval I such that the initial point x_0 is the origin.

Definition 2.1. Let ε, l, c be real numbers satisfying

$$\varepsilon \in (0, 1), \quad \max\{\varepsilon, 1 - \varepsilon\} < l < 1, \quad 0 \in [c, c + l) =: I.$$

The mapping

$$T(x) = \begin{cases} x + 1 - \varepsilon & \text{for } x \in [c, c + l - 1 + \varepsilon) & =: I_A, \\ x + 1 - 2\varepsilon & \text{for } x \in [c + l - 1 + \varepsilon, c + \varepsilon) & =: I_B, \\ x - \varepsilon & \text{for } x \in [c + \varepsilon, c + l) & =: I_C, \end{cases} \quad (2.1)$$

is called exchange of three intervals with permutation $(3, 2, 1)$, see Figure 1.

Note that the parameter ε represents the length of the interval $I_A \cup I_B$, and $1 - \varepsilon$ corresponds to the length of $I_B \cup I_C$. The number l is the length of the interval $I = I_A \cup I_B \cup I_C$.

The orbit of the point $x_0 = 0$ under the transformation T of (2.1) can be coded by an infinite word $(u_n)_{n \in \mathbb{Z}}$ in the alphabet $\{A, B, C\}$, where

$$u_n = \begin{cases} A & \text{if } T^n(0) \in I_A, \\ B & \text{if } T^n(0) \in I_B, \\ C & \text{if } T^n(0) \in I_C, \end{cases} \quad \text{for } n \in \mathbb{Z}. \quad (2.2)$$

The infinite word $(u_n)_{n \in \mathbb{Z}}$ is non-periodic exactly in the case that the parameter ε is irrational. Words coding the orbit of 0 under an exchange of intervals with the permutation $(3, 2, 1)$ and an irrational parameter ε are called 3iet words.

2.2. WORDS AND MORPHISMS

An alphabet \mathcal{A} is a finite set of symbols. In this paper we shall systematically use the alphabet $\{A, B, C\}$ for 3iet words, and the alphabet $\{0, 1\}$ for Sturmian words. A finite word in the alphabet \mathcal{A} is a concatenation $v = v_1 v_2 \cdots v_n$, where $v_i \in \mathcal{A}$ for all $i = 1, 2, \dots, n$. The length of the word v is denoted by $|v| = n$. The set of all finite words over \mathcal{A} , including the empty word ϵ will be denoted by \mathcal{A}^* . Equipped with the operation of concatenation, \mathcal{A}^* is a monoid. Sequences $u_0 u_1 u_2 \dots \in \mathcal{A}^{\mathbb{N}}$, $\dots u_{-3} u_{-2} u_{-1} \in \mathcal{A}^{\mathbb{Z}_{<0}}$, $\dots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \dots \in \mathcal{A}^{\mathbb{Z}}$ are called right-sided, left-sided and bidirectional infinite word, respectively.

If for a finite word w there exist (finite or infinite) words $v^{(1)}$ and $v^{(2)}$ such that $w = v^{(1)} w v^{(2)}$, then w is said to be a factor of the (finite or infinite) word v . If $v^{(1)}$ is the empty word, then w is a prefix of v , if $v^{(2)} = \epsilon$, then w is a suffix of v . The set of all factors of an infinite word u is called the language of u and denoted $\mathcal{L}(u)$. Factors of u of length n form the set $\mathcal{L}_n(u)$; obviously $\mathcal{L}_n(u) = \mathcal{L}(u) \cap \mathcal{A}^n$. The mapping $\mathcal{C} : \mathbb{N} \rightarrow \mathbb{N}$ given by the prescription $n \mapsto \#\mathcal{L}_n(u)$ is called the factor complexity of the infinite word u .

Infinite words u such that the set $\{wv \in \mathcal{L}(u) \mid w \text{ is not a factor of } v\}$ is finite for every $w \in \mathcal{L}(u)$ are called uniformly recurrent. Right-sided Sturmian words are defined as right-sided infinite words with factor complexity $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$. Bidirectional Sturmian words are uniformly recurrent bidirectional infinite words satisfying $\mathcal{C}(n) = n + 1$ for all $n \in \mathbb{N}$.

For the factor complexity \mathcal{C} of a 3iet word it holds that

- (i) either $\mathcal{C}(n) = n + K$ for all sufficiently large n ;
- (ii) or $\mathcal{C}(n) = 2n + 1$ for all $n \in \mathbb{N}$.

3iet words with complexity $\mathcal{C}(n) = n + K$ belong to the set of the so-called quasisturmian words, which are images of Sturmian words under suitable morphisms. 3iet words with complexity $\mathcal{C}(n) = 2n + 1$ are called non-degenerate 3iet words or regular 3iet words. The factor complexity of a 3iet word is given by (i) or (ii) according to the parameters ε, l : a 3iet word is non-degenerate if and only if $l \notin \mathbb{Z}[\varepsilon] := \mathbb{Z} + \varepsilon\mathbb{Z}$, see [1].

A mapping $\xi : \mathcal{A}^* \rightarrow \mathcal{B}^*$ satisfying $\xi(wv) = \xi(w)\xi(v)$ for all $w, v \in \mathcal{A}^*$ is called a morphism. A morphism is uniquely determined by the images $\xi(a)$ of all letters $a \in \mathcal{A}$. The action of a morphism can be naturally extended to infinite words by

$$\begin{aligned} \xi(u_0 u_1 u_2 \dots) &= \xi(u_0) \xi(u_1) \xi(u_2) \dots, \\ \xi(\dots u_{-3} u_{-2} u_{-1}) &= \dots \xi(u_{-3}) \xi(u_{-2}) \xi(u_{-1}), \\ \xi(\dots u_{-3} u_{-2} u_{-1} | u_0 u_1 u_2 \dots) &= \dots \xi(u_{-3}) \xi(u_{-2}) \xi(u_{-1}) | \xi(u_0) \xi(u_1) \xi(u_2) \dots \end{aligned}$$

With every morphism ξ one can associate a matrix M_ξ . The matrix has $\#\mathcal{A}$ rows and $\#\mathcal{B}$ columns, and

$$(M_\xi)_{ab} = \text{number of letters } b \text{ in } \xi(a).$$

A morphism $\xi : \mathcal{A}^* \rightarrow \mathcal{A}^*$ is called primitive if some power of the square matrix M_ξ has all elements positive. In other words, there exists a positive integer k such that for all $a, b \in \mathcal{A}$, the letter a is a factor of the k -th iteration $\xi^k(b)$. For properties of matrices of morphisms consult [17].

An infinite word u in $\mathcal{A}^{\mathbb{N}}$, $\mathcal{A}^{\mathbb{Z}_{<0}}$, $\mathcal{A}^{\mathbb{Z}}$ is said to be a fixed point of a morphism $\xi : \mathcal{A}^* \rightarrow \mathcal{A}^*$, if $\xi(u) = u$. It is obvious that if $u = u_0u_1u_2\dots$ is a fixed point of a primitive morphism ξ , then $\xi(u_0) = u_0w$ for a non-empty word w , and u is the limit of finite words $\xi^n(u_0)$, which is usually denoted by $\xi^\infty(u_0) = \lim_{n \rightarrow \infty} \xi^n(u_0)$. Analogous properties must be satisfied by primitive morphisms fixing left-sided or bidirectional infinite words.

A morphism ξ such that $\xi(a) \neq \epsilon$ for all $a \in \mathcal{A}$ and for which there exists $b \in \mathcal{A}$ such that $\xi(b) = bw$ for a non-empty word $w \in \mathcal{A}^*$ is sometimes called a substitution. It is quite obvious that the only non-primitive morphism which can fix a 3iet word or a Sturmian word is the identity. Therefore it is not misleading not to distinguish between notions of primitive morphism and substitution when speaking about substitution invariant Sturmian or 3iet words.

Substitution invariance of non-degenerate bidirectional 3iet words has been studied in [5]. Similarly as in the case of Sturmian words, one needs the notion of Sturm numbers. The original definition of a Sturm number uses continued fractions. We cite the equivalent definition given in [2]: a real number $\varepsilon \in (0, 1)$ is called a Sturm number, if it is a quadratic irrational with algebraic conjugate $\varepsilon' \notin (0, 1)$.

Let us cite here the characterization of substitution invariant 3iet words from [5].

Theorem 2.2 [5]. *Let u be a non-degenerate 3iet word with parameters ε, l, c . Then u is invariant under a primitive morphism if and only if*

- ε is a Sturm number;
- $c, l \in \mathbb{Q}(\varepsilon)$;
- $\min\{\varepsilon', 1 - \varepsilon'\} \leq -c' \leq \max\{\varepsilon', 1 - \varepsilon'\}$ and $\min\{\varepsilon', 1 - \varepsilon'\} \leq c' + l' \leq \max\{\varepsilon', 1 - \varepsilon'\}$, where x' is the field conjugate of x in $\mathbb{Q}(\varepsilon)$.

2.3. GEOMETRIC REPRESENTATION OF A FIXED POINT OF A MORPHISMS

It is useful to reformulate the task of searching for a substitution fixing a given infinite word $u = (u_n)_{n \in \mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$ in geometric terms. Let us associate with letters of the alphabet mutually distinct lengths by an injective mapping $\ell : \mathcal{A} \rightarrow (0, +\infty)$. Then, with the infinite word u we associate a strictly increasing sequence $(t_n)_{n \in \mathbb{Z}}$ such that

$$t_0 = 0 \quad \text{and} \quad t_{n+1} - t_n = \ell(u_n) \quad \text{for all } n \in \mathbb{Z}.$$

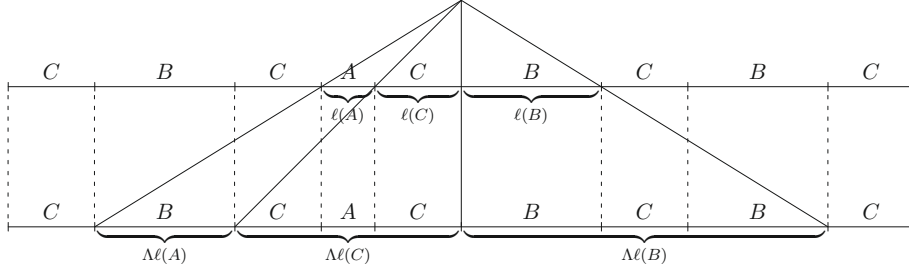


FIGURE 2. Geometric representation of a ternary word u fixed by a substitution η with the self-similarity factor Λ . In our example, $\eta(A) = B$, $\eta(B) = BCB$, $\eta(C) = CAC$.

A number $\Lambda > 1$ satisfying

$$\Sigma := \{t_n \mid n \in \mathbb{Z}\} \supset \{\Lambda t_n \mid n \in \mathbb{Z}\} =: \Lambda\Sigma,$$

is called a self-similarity factor of the sequence $(t_n)_{n \in \mathbb{Z}}$. Let us suppose that the assignment of lengths ℓ and the self-similarity factor Λ satisfy that to every $a \in \mathcal{A}$ there exists a finite set $P_a \subset (0, +\infty)$ such that

$$[\Lambda t_n, \Lambda t_{n+1}] \cap \Sigma = \Lambda t_n + P_a \quad \text{for all } n \in \mathbb{Z} \text{ with } u_n = a. \quad (2.3)$$

It means that the gap between t_n and t_{n+1} is after stretching by Λ filled by members of the original sequence $(t_n)_{n \in \mathbb{Z}}$ in the same way for all gaps corresponding to the letter a . An infinite word u for which one can find a mapping ℓ and a factor Λ with the above described properties is obviously invariant under a substitution ξ , where the image $\xi(a)$ is determined by the distances between consecutive elements of the set P_a . We call the set $\{t_n \mid n \in \mathbb{Z}\}$ with the property (2.3) the geometric representation of the word u with the factor Λ .

On the other hand, if an infinite word u is invariant under a primitive substitution ξ with the matrix M_ξ , then the eigenvector of M_ξ corresponding to the dominant eigenvalue Λ is a column of length $\#\mathcal{A}$ with all components x_a , $a \in \mathcal{A}$, positive, cf. [13]. The correspondence $\ell : a \rightarrow x_a$ results in a sequence $(t_n)_{n \in \mathbb{Z}}$ having Λ as its self-similarity factor and satisfying (2.3). Therefore the set $\{t_n \mid n \in \mathbb{Z}\}$ is the geometric representation of the infinite word u with the factor Λ . We illustrate the concept of the geometric representation in Figure 2.

In [3], the authors derive (in their Cors. 7.1 and 7.2) several properties of matrices of substitutions fixing a 3iet word.

Theorem 2.3 [3]. *Let u be a non-degenerate 3iet word with parameters ε, l, c which is invariant under a primitive substitution η . Then for the dominant eigenvalue Λ of the matrix M_η one has*

- (1) Λ is a quadratic unit;
- (2) $(1 - \varepsilon, 1 - 2\varepsilon, -\varepsilon)^T$ is the right eigenvector of M_η , corresponding to Λ' the algebraic conjugate of Λ .

$$\begin{array}{rcccccc}
u = & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{01} & \boxed{0} & \boxed{1} \\
v = & \boxed{0} & \boxed{1} & \boxed{0} & \boxed{10} & \boxed{0} & \boxed{1} \\
w = & A & C & A & B & A & C
\end{array}$$

FIGURE 3. Finite words $u = 0100101$ and $v = 0101001$ satisfy $u \propto v$ and their ternarization is equal to $w = \text{ter}(u, v) = ACABAC$.

Item (2) of the above theorem implies for the matrix M_η that $\mathbf{v} := (1 - \varepsilon', 1 - 2\varepsilon', -\varepsilon')^T$ is its right eigenvector corresponding to the dominant eigenvalue Λ . Using Theorem 2.2, we see that the parameter ε is a Sturm number, and so $\varepsilon' \notin (0, 1)$. The vector \mathbf{v} has thus all components positive or all components negative. In any case, in the geometric representation of the fixed point of the substitution η of Theorem 2.3, the length $\ell(B)$ corresponding to the letter B is the sum $\ell(A) + \ell(C)$.

3. AMICABLE MORPHISMS

The narrow connection of 3iet words and Sturmian words and their invariance under morphisms is described in Theorems 1.1 and 1.2 by means of morphisms σ_{01}, σ_{10} , see (2.1). These morphisms also allow us the description of morphisms fixing a 3iet word using Sturmian morphisms. For that, several notions need to be defined.

Definition 3.1. Let u, v be finite or infinite words over the alphabet $\{0, 1\}$. We say that u is amicable to v , and denote it by $u \propto v$, if there exist a ternary word w over $\{A, B, C\}$ such that $u = \sigma_{01}(w)$ and $v = \sigma_{10}(w)$. In such a case we denote $w := \text{ter}(u, v)$ and say that w is the ternarization of u and v .

Note that the relation \propto is not symmetric. For example, $u = 01$ is amicable to $v = 10$, but not *vice versa*. It is also interesting to notice that if two finite words u, v satisfy $u \propto v$, then they are of the same length and the number of letters a in u and v are equal for both $a = 0, 1$.

Figure 3 illustrates an easy way how to recognize amicability of two words and how to construct their ternarization. According to the definition, $u \propto v$ if u can be written as a concatenation $u = u^{(1)}u^{(2)}u^{(3)} \dots$ and v as a concatenation $v = v^{(1)}v^{(2)}v^{(3)} \dots$ such that for all $i = 1, 2, 3, \dots$ we have either $u^{(i)} = v^{(i)} = 0$ or $u^{(i)} = v^{(i)} = 1$ or $u^{(i)} = 01$ and $v^{(i)} = 10$. The ternarization w is then constructed by associating letters in the alphabet $\{A, B, C\}$ to the blocks, namely it associates A , if $u^{(i)} = v^{(i)} = 0$; it gives C if $u^{(i)} = v^{(i)} = 1$, and it gives B , if $u^{(i)} = 01$ and $v^{(i)} = 10$.

We introduce the notion of amicability and ternarization also for morphisms.

Definition 3.2. Let $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be two morphisms. We say that φ is amicable to ψ , and denote it by $\varphi \propto \psi$, if the three following relations hold

$$\begin{aligned}\varphi(0) &\propto \psi(0), \\ \varphi(1) &\propto \psi(1), \\ \varphi(01) &\propto \psi(10).\end{aligned}\tag{3.1}$$

The morphism $\eta : \{A, B, C\}^* \rightarrow \{A, B, C\}^*$ given by

$$\begin{aligned}\eta(A) &:= \text{ter}(\varphi(0), \psi(0)), \\ \eta(B) &:= \text{ter}(\varphi(01), \psi(10)), \\ \eta(C) &:= \text{ter}(\varphi(1), \psi(1)),\end{aligned}$$

is called the ternarization of φ and ψ and denoted by $\text{ter}(\varphi, \psi)$.

As an example, consider two basic Sturmian morphisms φ, ψ from (1.1),

$$\varphi : \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array}, \quad \psi : \begin{array}{l} 0 \mapsto 10 \\ 1 \mapsto 0 \end{array}.$$

It can be easily checked that $\varphi \propto \psi$ and that their ternarization $\eta = \text{ter}(\varphi, \psi)$ is of the form

$$\begin{aligned}A &\mapsto \text{ter}(01, 10) = B, \\ \eta : B &\mapsto \text{ter}(010, 010) = ACA, \\ C &\mapsto \text{ter}(0, 0) = A.\end{aligned}\tag{3.2}$$

From the definition of amicability of words it follows that if $u \propto v$ and $u' \propto v'$ then for their concatenation we have $uu' \propto vv'$. As a simple consequence of this idea, we have the following lemma.

Lemma 3.3. *Let u, v be two (finite or infinite) words over $\{0, 1\}$ such that $u \propto v$, and let $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be two morphisms such that $\varphi \propto \psi$. Then $\varphi(u) \propto \psi(v)$. Moreover, if $w = \text{ter}(u, v)$, then $\text{ter}(\varphi(u), \psi(v)) = \eta(w)$, where $\eta = \text{ter}(\varphi, \psi)$.*

Remark 3.4. Note that if $\varphi \propto \psi$ and $\eta = \text{ter}(\varphi, \psi)$, then

$$\varphi : \begin{array}{l} 0 \mapsto \sigma_{01}\eta(A) \\ 1 \mapsto \sigma_{01}\eta(C) \end{array} \quad \text{and} \quad \psi : \begin{array}{l} 0 \mapsto \sigma_{10}\eta(A) \\ 1 \mapsto \sigma_{10}\eta(C) \end{array}.$$

Theorem 3.5. *Let $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ be two primitive Sturmian morphisms having fixed points such that $\varphi \propto \psi$. Then the morphism $\eta : \{A, B, C\}^* \rightarrow \{A, B, C\}^*$ given by $\eta = \text{ter}(\varphi, \psi)$ has a 3iet fixed point.*

Proof. The first step is to prove that a fixed point of φ , say u , is amicable to a fixed point of ψ , say v . We prove the statement for right-sided words only, the

proof for left-sided and bidirectional fixed points follows the same lines. We will discuss two separate cases.

Case A. Let there exists a letter $X \in \{0, 1\}$ such that $\varphi(X)$ starts with X and $\psi(X)$ starts with X . Primitivity of φ and ψ implies that both $\varphi(X)$ and $\psi(X)$ have at least two letters. Therefore

$$u = \lim_{k \rightarrow \infty} \varphi^k(X) \quad \text{and} \quad v = \lim_{k \rightarrow \infty} \psi^k(X).$$

Since $X \propto X$ we have $u \propto v$ by Lemma 3.3.

Case B. Let the negation of Case A hold.

- a) Let $\varphi(1)$ start with 1. Then necessarily $\psi(1)$ starts with 0 which is in contradiction with $\varphi(1) \propto \psi(1)$.
- b) Let $\varphi(1)$ start with 0. Since φ has a fixed point, $\varphi(0)$ must start with 0. Thus $\psi(0)$ does not start with 0, which implies that $\psi(1)$ starts with 1 since ψ also has a fixed point.

Consider $\varphi(01)$ and $\psi(10)$. Clearly, $\varphi(01) = \varphi(0)\varphi(1)$ starts with 0 and $\psi(10) = \psi(1)\psi(0)$ starts with 1. Moreover, since $\varphi(01) \propto \psi(10)$, the word $\varphi(01)$ must have the prefix 01 and the word $\psi(10)$ must have the prefix 10. Therefore $u = \lim_{k \rightarrow \infty} \varphi^k(01)$ and $v = \lim_{k \rightarrow \infty} \psi^k(10)$. Now $01 \propto 10$ and therefore by Lemma 3.3, it follows that $u \propto v$.

We have shown in all cases that fixed points u, v of the Sturmian morphisms $\varphi \propto \psi$ satisfy $u \propto v$. Moreover, Lemma 3.3 implies that if $w = \text{ter}(u, v)$, then $w = \eta(w)$, i.e., w is the fixed point of the ternarization of φ and ψ . But since $\sigma_{01}(w) = u$, $\sigma_{10}(w) = v$ are fixed points of primitive Sturmian morphisms, they are Sturmian words, and therefore the infinite word w must be a 3iet word, as follows from Theorem 1.1. \square

4. MORPHISMS WITH 3IET FIXED POINT

The aim of this section is to prove the following theorem.

Theorem 4.1. *Let η be a primitive substitution fixing a non-degenerate 3iet word u . Then there exist Sturmian morphisms φ and ψ having fixed points, such that $\varphi \propto \psi$ and η or η^2 is equal to $\text{ter}(\varphi, \psi)$.*

Before proving this theorem we illustrate on the following example that considering both η and η^2 in the statement of the theorem is inevitable.

Example 4.2. Consider substitution $\eta : \{A, B, C\}^* \rightarrow \{A, B, C\}^*$ given by

$$\begin{aligned} A &\mapsto AB \\ \eta : B &\mapsto ACAAC \\ C &\mapsto AC. \end{aligned}$$

We show that η (with 3iet word as its fixed point) is not ternarization of any two Sturmian morphisms, whereas η^2 is. Let us suppose that $\eta = \text{ter}(\varphi, \psi)$, for some

Sturmian morphisms φ and ψ . Then the form of $\eta(A)$ and $\eta(C)$ implies

$$\begin{aligned}\varphi &: 0 \mapsto 001, & 1 \mapsto 01, \\ \psi &: 0 \mapsto 010, & 1 \mapsto 01.\end{aligned}$$

Nevertheless $\eta(B) \neq \text{ter}(\varphi(01), \psi(10)) = ABB$, and therefore η did not arise by ternarization of Sturmian morphisms.

On the other hand, it is not difficult to verify that

$$\begin{aligned}A &\mapsto ABACAAC \\ \eta^2 : B &\mapsto ABACABABAC \\ C &\mapsto ABAC\end{aligned}$$

is the ternarization of Sturmian morphisms

$$\begin{aligned}\varphi &: 0 \mapsto 00101001, & 1 \mapsto 00101, \\ \psi &: 0 \mapsto 01001001, & 1 \mapsto 01001.\end{aligned}$$

Finally, to show that the fixed point of η is a 3iet word it suffices to realize that the fixed point of η^2 is a 3iet word by Theorem 3.5 and that the fixed points of η and η^2 coincide.

The proof of Theorem 4.1 will combine results of papers [3,5] concerning substitution invariance of non-degenerate 3iet words and of the paper [4] which solves the same question for Sturmian words. We shall study infinite words defined by (2.2) under a transformation T from (2.1) where parameters ε, l satisfy additional conditions

$$\varepsilon \in (0, 1) \setminus \mathbb{Q} \quad \text{and} \quad l \notin \mathbb{Z}[\varepsilon] = \mathbb{Z} + \varepsilon\mathbb{Z}. \quad (4.1)$$

These conditions mean precisely that the corresponding infinite 3iet word is non-degenerate.

According to Theorem 1.1, the images of a 3iet word u under morphisms σ_{01}, σ_{10} are Sturmian words. Let us determine parameters of the Sturmian words $\sigma_{01}(u), \sigma_{10}(u)$ (*i.e.*, the corresponding exchanges of two intervals), provided that the parameters of u are ε, l, c . The procedure is illustrated in Figure 4.

Define the mapping $T_{01} : [c, c+1) \rightarrow [c, c+1)$ by

$$T_{01}(x) = \begin{cases} x + 1 - \varepsilon & \text{for } x \in [c, c + \varepsilon) & =: I_0 \\ x - \varepsilon & \text{for } x \in [c + \varepsilon, c + 1) & =: I_1. \end{cases}$$

Comparing T_{01} and T we obtain (see Fig. 4)

$$x \in I_B \quad \iff \quad T_{01}(x) \in [c + l, c + 1).$$

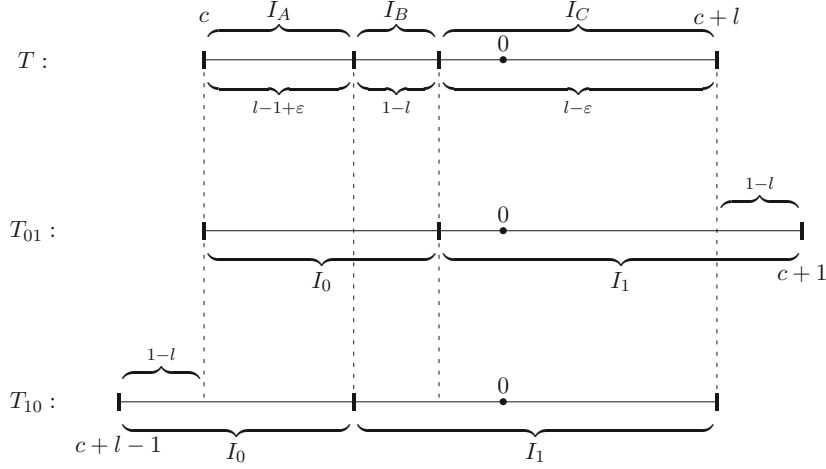


FIGURE 4. Exchanges of intervals corresponding to a 3iet word u and Sturmian words $\sigma_{01}(u)$, $\sigma_{10}(u)$.

For $x \in [c, c+l)$ we have

$$\begin{aligned} x \in I_A &\implies x \in I_0 \text{ and } T_{01}(x) = T(x), \\ x \in I_B &\implies x \in I_0, T_{01}(x) \in I_1 \text{ and } T(x) = T_{01}^2(x), \\ x \in I_C &\implies x \in I_1 \text{ and } T_{01}(x) = T(x). \end{aligned}$$

Therefore $\sigma_{01}(u)$ is the infinite word coding the orbit of 0 under the exchange T_{01} of intervals with lengths ε and $1-\varepsilon$. Such a word is a Sturmian word of the slope ε and intercept $-c$ (i.e., the distance of the initial point of the orbit and the left end-point of the interval $[c, c+1)$ which is the domain of T_{01}).

In a similar way, we derive that the infinite word $\sigma_{10}(u)$ is the coding of the orbit of 0 under the exchange of two intervals $T_{10} : [c+l-1, c+l) \rightarrow [c+l-1, c+l)$. In particular, it is a Sturmian word of the slope ε and intercept $-c-l+1$.

Let us cite the result characterizing substitution invariant Sturmian words. Comparing [19] and [4] we obtain that a right-sided Sturmian word with the slope α and intercept β is substitution invariant if and only if the bidirectional Sturmian word with the same slope and intercept is substitution invariant.

Theorem 4.3 [19]. *Let $\alpha \in (0, 1)$ be irrational and $\beta \in [0, 1)$. A Sturmian word with the slope α and intercept β is invariant under a primitive morphism if and only if*

- (1) α is a Sturm number;
- (2) $\beta \in \mathbb{Q}(\alpha)$;
- (3) $\min\{\alpha', 1-\alpha'\} \leq \beta' \leq \max\{\alpha', 1-\alpha'\}$, where α', β' denote the field conjugates of α, β in $\mathbb{Q}(\alpha)$.

Note that the inequalities in item (3) are satisfied for β' if and only if they are satisfied replacing β' by $1 - \beta'$. Knowing the slope and intercept of Sturmian words $\sigma_{01}(u)$, $\sigma_{10}(u)$ we can deduce from Theorem 2.2 the statement of Theorem 1.1, namely that a non-degenerate 3iet word is invariant under a primitive substitution if and only if both Sturmian words $\sigma_{01}(u)$, $\sigma_{10}(u)$ are substitution invariant.

We will now put into relation the substitutions fixing infinite words u , $\sigma_{01}(u)$, and $\sigma_{10}(u)$. First we consider the self-similarity factors and geometric representations of these substitutions.

Lemma 4.4. *Let η be a primitive substitution over the alphabet $\{A, B, C\}$ having as its fixed point a non-degenerate 3iet word u . Let us denote its parameters ε, l, c . Denote by Λ the dominant eigenvalue of the matrix \mathbf{M}_η and by $(\ell(A), \ell(B), \ell(C))^T$ its positive right eigenvector corresponding to Λ . If $\Lambda' > 0$, then there exist substitutions $\varphi, \psi : \{0, 1\}^* \rightarrow \{0, 1\}^*$ fixing $\sigma_{01}(u)$, $\sigma_{10}(u)$, respectively, and such that Λ is the dominant eigenvalue of \mathbf{M}_φ and \mathbf{M}_ψ , and $(\ell(A), \ell(C))^T$ is their common right eigenvector corresponding to Λ . Moreover, $\ell(B) = \ell(A) + \ell(C)$.*

Proof. Theorems 2.2 and 2.3 imply that $\mathbf{v} = (\ell(A), \ell(B), \ell(C))^T = (1 - \varepsilon', 1 - 2\varepsilon', -\varepsilon')^T$ is a right eigenvector of \mathbf{M}_η corresponding to Λ . Recall that $\sigma_{01}(u)$ is the Sturmian word of the slope ε and intercept $-c$, and $\sigma_{10}(u)$ the Sturmian word of the slope ε and intercept $-c - l + 1$. By Theorem 1.2 they are invariant under substitutions, say φ, ψ . Since ε and $1 - \varepsilon$ are densities of letters 0 and 1 respectively, the substitution matrices \mathbf{M}_φ and \mathbf{M}_ψ must have the eigenvector $(1 - \varepsilon', -\varepsilon')^T = (\ell(A), \ell(C))^T$. Obviously $\ell(B) = \ell(A) + \ell(C)$.

It remains to show that φ, ψ can be chosen so that the dominant eigenvalue of η , i.e., Λ , is also the dominant eigenvalue of $\mathbf{M}_\varphi, \mathbf{M}_\psi$. As a consequence of the equality $\mathbf{M}_\eta \mathbf{v} = \Lambda \mathbf{v}$ and the fact that Λ is a quadratic unit, we have $\mathbb{Z} + \varepsilon' \mathbb{Z} =: \mathbb{Z}[\varepsilon'] = \Lambda \mathbb{Z}[\varepsilon'] = \Lambda' \mathbb{Z}[\varepsilon']$, which, after conjugation, gives

$$\mathbb{Z}[\varepsilon] = \Lambda' \mathbb{Z}[\varepsilon] = \Lambda \mathbb{Z}[\varepsilon]. \quad (4.2)$$

Proposition 5.6 of [5] (see also Rems. 5.7 and 6.5 ibidem) implies that $\Lambda' c \in c + \mathbb{Z}[\varepsilon]$ and $\Lambda'(c + l - 1 + \varepsilon) \in c + l - 1 + \varepsilon + \mathbb{Z}[\varepsilon]$. This, together with (4.2) gives

$$\begin{aligned} \Lambda'(c + \mathbb{Z}[\varepsilon]) &= c + \mathbb{Z}[\varepsilon], \\ \Lambda'(c + l - 1 + \varepsilon + \mathbb{Z}[\varepsilon]) &= c + l - 1 + \varepsilon + \mathbb{Z}[\varepsilon]. \end{aligned} \quad (4.3)$$

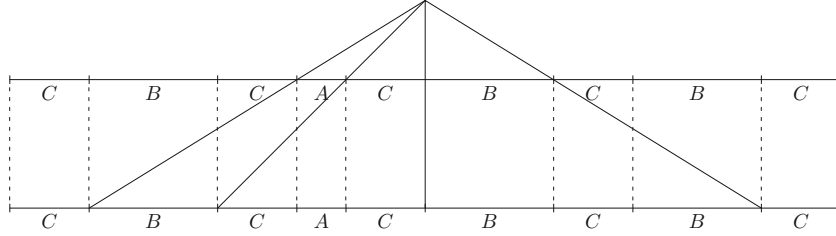
Note that the assumption $\Lambda' > 0$ is required in order that we can use results from [5].

Note that substitution invariance of $\sigma_{01}(u)$ and $\sigma_{10}(u)$ implies by Theorem 4.3 that their parameters satisfy

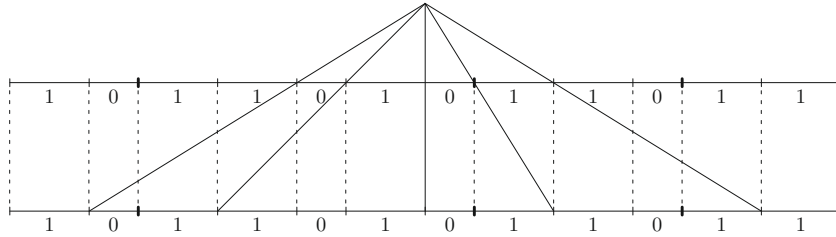
$$\min\{\varepsilon', 1 - \varepsilon'\} \leq -c' \leq \max\{\varepsilon', 1 - \varepsilon'\}, \quad \min\{\varepsilon', 1 - \varepsilon'\} \leq c' + l' \leq \max\{\varepsilon', 1 - \varepsilon'\}.$$

These inequalities, together with (4.2), already imply that there exist substitutions φ and ψ with factor Λ (see proof of Prop. 5.3 in [4]). \square

Ternary substitution $\eta : A \mapsto B, B \mapsto BCB, C \mapsto CAC$ and its fixed point u



Sturmian substitution $\varphi = \sigma_{01} \circ \eta : 0 \mapsto 01, 1 \mapsto 101$ and its fixed point $\sigma_{01}(u)$



Sturmian substitution $\psi = \sigma_{10} \circ \eta : 0 \mapsto 10, 1 \mapsto 101$ and its fixed point $\sigma_{10}(u)$

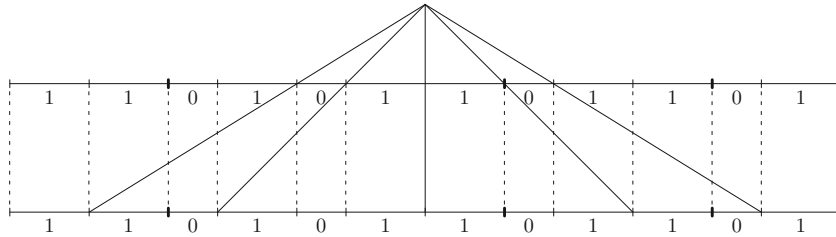


FIGURE 5. Geometric representation of infinite words u , $\sigma_{01}(u)$ and $\sigma_{10}(u)$, and the substitutions η , φ , ψ (all with the same self-similarity factor Λ) fixing them. We have $u = \text{ter}(\sigma_{01}(u), \sigma_{10}(u))$ and $\eta = \text{ter}(\varphi, \psi)$.

Proof of Theorem 4.1. The dominant eigenvalue of the matrix M_η is a quadratic unit Λ . If $\Lambda' > 0$, we shall prove the statement for η . If $\Lambda' < 0$, we will consider the second iteration η^2 . Therefore we consider without loss of generality $\Lambda' > 0$.

With the help of geometric representation of infinite words we will show that morphisms φ, ψ found by Lemma 4.4 are amicable, *i.e.*, $\varphi \propto \psi$, and that η is their ternarization. We use the fact that all of the considered substitutions, η, φ and ψ have the same factor Λ . The idea of the proof is illustrated in Figure 5.

Let u be a fixed point of η and let $\{t_n\}_{n=0}^\infty$ be the geometric representation of the substitution η with the dominant eigenvalue Λ and the right eigenvector

$(\ell(A), \ell(B), \ell(C))^T$ for which

$$t_{n+1} - t_n = \ell(X) \iff u_n = X \in \{A, B, C\}.$$

Morphisms φ and ψ are Sturmian substitutions with fixed points $\sigma_{01}(u)$, $\sigma_{10}(u)$, respectively. The geometric representation of the infinite word $\sigma_{01}(u)$ is

$$\{t_n^{01}\}_{n=0}^\infty := \{t_n\}_{n=0}^\infty \cup \{t_n + \ell(A) \mid u_n = B\},$$

and the geometric representation of the infinite word $\sigma_{10}(u)$ is

$$\{t_n^{10}\}_{n=0}^\infty := \{t_n\}_{n=0}^\infty \cup \{t_n + \ell(C) \mid u_n = B\}.$$

If $t_{n+1} - t_n = \ell(A)$, *i.e.*, $u_n = A$, then the segment in $\{t_n\}_{n=0}^\infty$ between Λt_n and Λt_{n+1} (both in $\{t_n\}_{n=0}^\infty$) contains points ordered according to $\eta(A)$. And the segment in $\{t_n^{01}\}_{n=0}^\infty$ between Λt_n and Λt_{n+1} (both in $\{t_n^{01}\}_{n=0}^\infty$) contains points ordered according to $\sigma_{01}(\eta(A))$. Analogically, for n such that $u_n = C$, points in $\{t_n^{01}\}_{n=0}^\infty$ between Λt_n and Λt_{n+1} are ordered according to $\sigma_{01}(\eta(C))$.

From what was said above it is obvious, that the substitution φ with factor Λ fixing the Sturmian word $\sigma_{01}(u)$ must be of the form

$$\varphi: \begin{array}{l} 0 \mapsto \sigma_{01}\eta(A), \\ 1 \mapsto \sigma_{01}\eta(C). \end{array}$$

In a similar way, we can deduce that the substitution ψ under which the infinite word $\sigma_{10}(u)$ is invariant is of the form

$$\psi: \begin{array}{l} 0 \mapsto \sigma_{10}\eta(A), \\ 1 \mapsto \sigma_{10}\eta(C). \end{array}$$

By Definition 3.1, we have that $\varphi(0) \propto \psi(0)$ and $\varphi(1) \propto \psi(1)$, and that $\eta(A) = \text{ter}(\varphi(0), \psi(0))$, $\eta(C) = \text{ter}(\varphi(1), \psi(1))$.

In order to complete the proof of the theorem, we have to show that $\varphi(01) \propto \psi(10)$ and $\eta(B) = \text{ter}(\varphi(01), \psi(10))$. For that, consider $n \in \mathbb{Z}$ such that $t_{n+1} - t_n = \ell(B) = \ell(A) + \ell(C)$, *i.e.*, $u_n = B$. The segment between Λt_n and $\Lambda(t_n + \ell(A))$ in the geometric representation $\{t_n^{01}\}_{n=0}^\infty$ of $\sigma_{01}(u)$ contains the points arranged according to $\sigma_{01}\eta(A)$. Similarly, the segment between $\Lambda(t_n + \ell(A))$ and Λt_{n+1} contains the points arranged according to $\sigma_{01}\eta(C)$. Of course, the segment between Λt_n and Λt_{n+1} in the geometric representation $\{t_n\}_{n=0}^\infty$ of the original infinite word u is arranged according to $\eta(B)$. Altogether, we have

$$\sigma_{01}\eta(B) = \sigma_{01}\eta(A)\sigma_{01}\eta(C) = \varphi(0)\varphi(1).$$

Analogously,

$$\sigma_{10}\eta(B) = \sigma_{10}\eta(C)\sigma_{10}\eta(A) = \psi(1)\psi(0).$$

This means that $\varphi(01) \propto \psi(10)$, and the word $\eta(B)$ is the ternarization of words $\varphi(01)$ and $\psi(10)$. Consequently, φ is amicable to ψ , and the substitution η is the ternarization of φ and ψ . \square

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REFERENCES

- [1] B. Adamczewski, Codages de rotations et phénomènes d'autosimilarité. *J. Théor. Nombres Bordeaux* **14** (2002) 351–386.
- [2] C. Allauzen, Une caractérisation simple des nombres de Sturm. *J. Théor. Nombres Bordeaux* **10** (1998) 237–241.
- [3] P. Arnoux, V. Berthé, Z. Masáková and E. Pelantová, Sturm numbers and substitution invariance of 3iet words. *Integers* **8** (2008) 17 (electronic).
- [4] P. Baláži, Z. Masáková and E. Pelantová, Complete characterization of substitution invariant Sturmian sequences. *Integers* **5** (2005) 23 (electronic).
- [5] P. Baláži, Z. Masáková and E. Pelantová, Characterization of substitution invariant 3iet words. *Integers* **8** (2008) 21 (electronic).
- [6] J. Berstel and P. Séébold, Morphismes de Sturm. *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994) 175–189. Journées Montoises (Mons, 1992).
- [7] V. Berthé, H. Ei, S. Ito and H. Rao, On substitution invariant Sturmian words: an application of Rauzy fractals. *RAIRO-Theor. Inf. Appl.* **41** (2007) 329–349.
- [8] M.D. Boshernitzan and C.R. Carroll, An extension of Lagrange's theorem to interval exchange transformations over quadratic fields. *J. Anal. Math.* **72** (1997) 21–44.
- [9] D. Crisp, W. Moran, A. Pollington and P. Shiue, Substitution invariant cutting sequences. *J. Théor. Nombres Bordeaux* **5** (1993) 123–137.
- [10] S. Ferenczi, C. Holton and L.Q. Zamboni, Structure of three interval exchange transformations. I. An arithmetic study. *Ann. Inst. Fourier* **51** (2001) 861–901.
- [11] S. Ferenczi, C. Holton and L.Q. Zamboni, Structure of three-interval exchange transformations. II. A combinatorial description of the trajectories. *J. Anal. Math.* **89** (2003) 239–276.
- [12] S. Ferenczi, C. Holton and L.Q. Zamboni, Structure of three-interval exchange transformations III: ergodic and spectral properties. *J. Anal. Math.* **93** (2004) 103–138.
- [13] M. Fiedler, Special matrices and their applications in numerical mathematics. Martinus Nijhoff Publishers, Dordrecht (1986). Translated from the Czech by Petr Příkryl and Karel Segeth.
- [14] T. Komatsu and A.J. van der Poorten, Substitution invariant Beatty sequences. *Jpn J. Math. (N.S.)* **22** (1996) 349–354.
- [15] F. Mignosi and P. Séébold, Morphismes sturmiens et règles de Rauzy. *J. Théor. Nombres Bordeaux* **5** (1993) 221–233.
- [16] B. Parvaix, Substitution invariant Sturmian bisequences. *J. Théor. Nombres Bordeaux* **11** (1999) 201–210. Les XXèmes Journées Arithmétiques (Limoges, 1997).
- [17] M. Queffélec, *Substitution dynamical systems—spectral analysis*. Lect. Notes Math. **1294** (1987).
- [18] P. Séébold, Fibonacci morphisms and Sturmian words. *Theoret. Comput. Sci.* **88** (1991) 365–384.
- [19] S.-I. Yasutomi, On Sturmian sequences which are invariant under some substitutions. In *Number theory and its applications* (Kyoto, 1997), *Dev. Math.* **2**, Kluwer Acad. Publ. (1999) 347–373.