# ON THE HARDNESS OF GAME EQUIVALENCE UNDER LOCAL ISOMORPHISM\*

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**Abstract.** We introduce a type of isomorphism among strategic games that we call *local isomorphism*. Local isomorphisms is a weaker version of the notions of strong and weak game isomorphism introduced in J. Gabarro, A. Garcia and M. Serna, *Theor. Comput. Sci.* **412** (2011) 6675–6695]. In a local isomorphism it is required to preserve, for any player, the player's preferences on the sets of strategy profiles that differ only in the action selected by this player. We show that the game isomorphism problem for local isomorphism is equivalent to the same problem for strong or weak isomorphism for strategic games given in: general, extensive and formula general form. As a consequence of the results in [J. Gabarro, A. Garcia and M. Serna, Theor. Comput. Sci. **412** (2011) 6675–6695] this implies that local isomorphism problem for strategic games is equivalent to (a) the circuit isomorphism problem for games given in general form, (b) the boolean formula isomorphism problem for formula games in general form, and (c) the graph isomorphism problem for games given in explicit form.

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#### 1. INTRODUCTION

The isomorphism problem is a classical complexity question for several combinatorial structures. One of the fundamental problems in complexity is the graph isomorphism problem for graphs. Two graphs are *isomorphic* if there is a oneto-one correspondence between their vertices and there is an edge between two vertices of one graph if and only if there is an edge between the two corresponding vertices in the other graph. The GRAPHISO problem asks whether there is an isomorphism among two given graphs. It is well known that GRAPHISO belongs to NP but it is not expected to be NP-hard [12]. For circuits and formulas it is well known that the problem is harder. Recall that two circuits  $C_1(x_1,\ldots,x_n)$  and  $C_2(x_1,\ldots,x_n)$  are isomorphic if there is a permutation  $\pi$  of  $\{1,\ldots,n\}$  such that, for any truth assignment  $x \in \{0,1\}^n$ ,  $C_1(x) = C_2(\pi(x))$ . The CIRCUITISO problem has been studied by B. Borchert, D. Ranjan and F. Stephan in [5], among many other results they show that CIRCUITISO  $\in \Sigma_2^p$ . M. Agrawal and T. Thierauf prove that the CIRCUITISO problem cannot be  $\Sigma_2^p$ -hard unless the polynomial hierarchy collapses (see Cor. 3.5 in [1]). For boolean formulas, B. Borchert, D. Ranjan and F. Stephan in [5] show that FORMULAISO  $\in \Sigma_2^p$ . Again the FORMULAISO problem cannot be  $\Sigma_2^p$ -hard unless the polynomial hierarchy collapses (see Cor. 3.4 in [1]).

We are interested in analyzing the complexity of deciding whether two strategic games are equivalent. In defining a concrete equivalence between strategic games we have to pay attention to the structural properties that are preserved in equivalent games. Two notions of isomorphisms that preserve at different levels the structure of the Nash equilibria strong and weak isomorphism were introduced in [8]. Isomorphisms are defined on the basis of game mappings formed by a bijection among players, and for each player a bijection among its action set. A strong isomorphism is a mapping that preserves the player's utilities in correspondence to the notion of isomorphism introduced in [14]. In consequence strong isomorphisms preserve both pure and mixed Nash equilibria, however the notion is not well adapted to compare games with ordinal preferences (look at Def. 13.1 in [15]). In [7, 8] we consider weak isomorphisms as one notion better suited to games with ordinal preferences. In a weak isomorphism it is required to preserve player's preferences and in consequence only the structure of the pure strategy profiles, including pure Nash equilibria, is preserved. This approach was implicitly undertaken to classify all strategic games with two players and two strategies per player  $(2 \times 2 \text{ games})$  [11].

In this paper we consider another notion of isomorphism that we call *local isomorphism*. A local isomorphism is a mapping that preserves, for each player and each strategy profile, the preferences of the player on the "close" neighborhood of the strategy profile. This condition, which is weaker than the requirements for weak isomorphism, still guarantees that the local structure of the pure strategy profiles, including the pure Nash equilibria, is preserved. However, as for the case of weak isomorphisms, mixed Nash equilibria are not preserved. Our definition of local isomorphism constitutes a generalization of the concept of *better response equivalence* [3] restricted to pure strategies, see definitions in Section 2. This last

restriction on pure strategies has been used in comparing potential games [21]. In this paper, our objective is to analyze the computational complexity of the local isomorphism problem.

In the context of computational complexity it is very important to fix how games are represented as problem inputs. We consider here two of the game representations considered in [2]. For giving a game  $\Gamma$  we have to provide a listing of the set of actions allowed to each player and the corresponding utilities. The two representations differ on the form in which utilities are given. In the *general form* utilities are given implicitly by a Turing machine. In the *explicit form* utilities are provided explicitly by giving the value corresponding to each profile. We also consider another succinct representation of games introduced in [8], formula game in general form in which the utility of a player is defined by a collection of boolean formulas, each one of them providing a bit of the player's utility. This is one of the many ways in which games have been described in terms of formulas. In [4], player i has a goal  $\varphi_i$  to fulfill. Goals are usually described by boolean formulas. The utility of the player is binary. It is 1 if the goal is satisfied and 0 otherwise. Another model for strategic games that use boolean formula was introduced in [13], the weighted boolean formula games. Along the lines suggested by circuit games [17] the formula strategic form, whose representation is close to a game given in general form but with utilities defined by formulas was introduced in [8].

In [8] we obtained a classification of the complexity of the game isomorphism problem according to the level of succinctness of the representation of the game. Our results show that the isomorphism problem for strategic games and strong or weak isomorphism is equivalent to (a) the circuit isomorphism problem for games given in general form, (b) the boolean formula isomorphism problem for formula games in general form, and (c) the graph isomorphism problem for games given in explicit form. Thus showing that the classification differ depending on the game representation.

In this paper we show the computational equivalence between the isomorphism problem for strong isomorphism and the isomorphism problem for local isomorphism. This equivalence is proved for any of the three game representation considered above. Thus showing that the local isomorphism problem for strategic games has the same classification as the other stronger notions of isomorphism. Our proof shows the equivalence through a series of steps. First we reduce the **Stronglso** problem for *binary games*, games whose actions and utilities are binary, to the **Locallso** problem. Then we show how to reduce the **Locallso** problem to the **Locallso** problem for *binary actions games*, games whose actions are binary. Finally we reduce the **Locallso** problem for binary actions games to the **Stronglso** problem for binary games and the output games are written in explicit, general, and formula general form (see definitions later).

The paper is organized as follows, in Section 2 we give the definitions and results needed in the paper, we also introduce the isomorphism problem. In Section 3 we show how to transform a binary game into another strategic game in such

a way that strongly isomorphic binary games are mapped to locally isomorphic games. Sections 4 and 5 are devoted to show transformation that preserve local isomorphism. In Section 4 we transform a strategic game into a binary actions game and in Section 5 a binary actions game into binary games. Section 6 is devoted to show that descriptions (in the adequate form) of the transformed games can be computed in polynomial time and conclude with our main result. Finally, in Section 7, we conclude with some open questions and research directions.

## 2. Preliminaires

In this section we provide the basic definitions and results needed in the paper. We start stating the mathematical definition of strategic game as given in [16] and introduce further notation following [8].

A strategic game  $\Gamma$  is a tuple  $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ . The set of players is  $N = \{1, \ldots, n\}$ . Player  $i \in N$  has a finite set of actions  $A_i$ , we note  $a_i$  any action belonging to  $A_i$ . The elements  $a = (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$  are the strategy profiles. The *utility* (or *payoff*) function  $u_i$ , for each player  $i \in N$ , is a mapping from  $A_1 \times \ldots \times A_n$  to the rationals.

Let  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$  and  $\Gamma' = (N, (A'_i)_{i \in N}, (u'_i)_{i \in N})$  be two strategic games. A game mapping  $\psi$  from  $\Gamma$  to  $\Gamma'$  is a tuple  $\psi = (\pi, (\varphi_i)_{i \in N})$  where  $\pi$  is a bijection from N to N, the player's bijection, and, for any  $i \in N$ ,  $\varphi_i$  is a bijection from  $A_i$  to  $A'_{\pi(i)}$ , the *i*-th player actions bijection. In the case that, for any  $i \in N$ ,  $A_i = A'_i$ , we can consider the *identity mapping* which is defined as  $I = (\pi, (\varphi)_{i \in N})$ such that  $\pi(i) = i$  and  $\varphi_i(a_i) = a_i$  for all  $i \in N$  and for all  $a_i \in A_i$ .

Observe that the player's bijection identifies a player  $i \in N$  with a player  $\pi(i)$ and the corresponding actions bijection  $\varphi_i$  maps the set of actions of player i to the set of actions of player  $\pi(i)$ . A game mapping  $\psi$  from  $\Gamma$  to  $\Gamma'$  induces, in a natural way, a bijection on strategy profiles, where strategy profile  $(a_1, \ldots, a_n)$ is mapped into the strategy profile  $(a'_1, \ldots, a'_n)$  defined as  $a'_{\pi(i)} = \varphi_i(a_i)$ , for all  $1 \leq i \leq n$ . We note this bijection as  $\psi(a_1, \ldots, a_n) = (a'_1, \ldots, a'_n)$ , overloading the use of  $\psi$ .

**Example 2.1.** Consider the set of two player games in which each player has actions  $\{0, 1\}$ , and the set of all possible game mappings between them. For a mapping  $\psi = (\pi, \varphi_1, \varphi_2)$  the only possibilities for  $\pi$ ,  $\varphi_1$  and  $\varphi_2$  are the identity or the swap (denoted shortly as *i* and *s*). Given a mapping  $\psi = (i, s, i)$  we write shorty  $\psi = isi$ . The 8 bijections defined on the set of strategy profiles by the 8 possible mappings are summarized in the following table where  $a = (a_1, a_2)$ .

$iii(a) = (a_1, a_2)$	$sii(a) = (a_2, a_1)$
$isi(a) = (\overline{a}_1, a_2)$	$ssi(a) = (a_2, \overline{a}_1)$
$iis(a) = (a_1, \overline{a}_2)$	$sis(a) = (\overline{a}_2, a_1)$
$iss(a) = (\overline{a}_1, \overline{a}_2)$	$sss(a) = (\overline{a}_1, \overline{a}_2)$

Consider the games  $\Gamma$  and  $\Gamma'$  defined as follows:

Player 1 
$$\begin{array}{c} \begin{array}{c} Player 2 \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{c} 0 \\ e \\ c,g \end{array} \begin{array}{c} d,h \end{array} \end{array}$$
 Player 1  $\begin{array}{c} 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{c} 0 \\ g,c \end{array} \begin{array}{c} e,a \\ h,d \end{array} \begin{array}{c} f,b \end{array} \end{array}$  Player 1  $\begin{array}{c} 0 \\ h,d \end{array} \begin{array}{c} f,b \end{array}$ 

Observe that ssi is a game mapping from  $\Gamma$  to  $\Gamma'$ .

Isomorphisms are game mappings fulfilling some additional restrictions on utilities or preferences.

A game mapping  $\psi : \Gamma \to \Gamma'$  with  $\psi = (\pi, (\varphi_i)_{i \in N})$  is called a

- strong isomorphism when, for any player  $1 \le i \le n$  and any strategy profile a, it holds  $u_i(a) = u'_{\pi(i)}(\psi(a))$ ;
- local isomorphism when, for any player  $1 \leq i \leq n$ , any strategy profile a and any action  $a_i \in A_i$ , it holds that  $u_i(a) \leq u_i(a_{-i}, a_i)$  iff  $u'_{\pi(i)}(\psi(a)) \leq u'_{\pi(i)}(\psi(a_{-i}, a_i))$ .

We use the notation  $\Gamma \sim_s \Gamma'$  to say that there is a strong isomorphism between  $\Gamma$  and  $\Gamma'$ . Similarly we use  $\sim_{\ell}$  to denote local isomorphism.

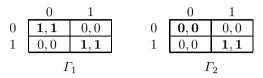
**Example 2.2.** Observe that the mapping ssi, given in Example 2.1, is a strong isomorphism between the games  $\Gamma$  to  $\Gamma'$ . We show now that the following games are locally isomorphic but not strongly isomorphic.

To prove that  $\Gamma_1$  and  $\Gamma_2$  are not strongly isomorphic we check all the mappings given in Example 2.1. As for each  $(\pi, \varphi_1, \varphi_2)$  it holds  $u_1(0,0) \neq u'_{\pi(1)}(\psi(0,0))$  or  $u_2(0,0) \neq u'_{\pi(2)}(\psi(0,0))$  there is no strong isomorphism.

On the other hand the mapping *iii* is a local isomorphism, as the local preference relations are preserved:

$$u_1(0,0) = 0 < 1 = u_1(1,0) \text{ and } u'_1(0,0) = 0 < 1 = u'_1(1,0)$$
$$u_1(0,1) = 0 < 1 = u_1(1,1) \text{ and } u'_1(0,1) = 0 < 1 = u'_1(1,1)$$
$$u_2(0,0) = 1 = 1 = u_2(0,1) \text{ and } u'_2(0,0) = 0 = 0 = u'_2(0,1)$$
$$u_2(1,0) = 0 = 0 = u_2(1,1) \text{ and } u'_2(1,0) = 1 = 1 = u'_2(1,1)$$

From the definitions it is easy to see that, as a local isomorphism  $\psi$  between games  $\Gamma_1$  and  $\Gamma_2$  preserves the local preferences, we have that, for any strategy profile a of  $\Gamma_1$ , a is a PNE iff  $\psi(a)$  is a PNE. However the reverse statement, if the sets of pure PNE coincide there exist a local isomorphism between them, is false. Consider the games  $\Gamma_1$  and  $\Gamma_2$  defined as follows:



The utilities corresponding to the pure Nash equilibria are boldfaced. In both cases  $PNE = \{(0,0), (1,1)\}$  It is easy to see that there is no local isomorphism between  $\Gamma_1$  and  $\Gamma_2$ .

Our definition of local isomorphism can be seen as a generalization of the notion of better response equivalence for pure strategy profiles. We recall here the definition adapted from [3]. Given a game  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ , for any  $a_i, a'_i \in A_i$ , define

$$\Lambda_i^P(a_i, a_i' \mid u_i) = \{a_{-i} \in A_{-i} \mid u_i(a_i, a_{-i}) \ge u_i(a_i', a_{-i})\}.$$

The games  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$  and  $\Gamma' = (N, (A_i)_{i \in N}, (u'_i)_{i \in N})$  are better response equivalent under pure strategies if, for any  $i \in N$  and  $a_i, a'_i \in A_i$ ,

$$\Lambda_i^P(a_i, a_i' \mid u_i) = \Lambda_i^P(a_i, a_i' \mid u_i').$$

Introducing a mapping  $\psi = (\pi, (\varphi_i)_{i \in N})$  in replacement of the identity function and allowing the games to have different sets of actions, now  $\Gamma' = (N, (A'_i)_{i \in N}, (u'_i)_{i \in N})$ , the condition can be rewritten as, for any  $i \in N$  and  $a_i, a'_i \in A_i$ ,

$$\psi(\Lambda_i^P(a_i, a_i' \mid u_i)) = \Lambda_{\pi(i)}^P\left(\varphi_i(a_i), \varphi_i(a_i') \mid u_{\pi(i)}'\right),$$

which is equivalent to require that  $\psi$  is a local isomorphism.

We consider the following computational problems related to games and morphisms.

Strong Isomorphism problem (STRONGISO). Given two strategic games  $\Gamma$ ,  $\Gamma'$ , decide whether  $\Gamma \sim_s \Gamma'$ .

Local Isomorphism problem (LOCALISO). Given two strategic games  $\Gamma$ ,  $\Gamma'$ , decide whether  $\Gamma \sim_{\ell} \Gamma'$ .

Finally we have to decide the representation of the input games. We consider the game representations considered in [8]. A strategic game  $\Gamma$  is given in

• explicit form by a tuple  $\langle 1^n, A_1, \ldots, A_n, (T_{i,a})_{1 \le i \le n, a \in A} \rangle$ .  $\Gamma$  has n players, and for each player  $i, 1 \le i \le n$ , their set of actions  $A_i$  is given by listing all its elements. The utility  $u_i(a)$ , for player i of strategy profile a, is the value  $T_{i,a}$ ;

- general form by a tuple  $\langle 1^n, A_1, \ldots, A_n, M, 1^t \rangle$ .  $\Gamma$  has n players, and for each player  $i, 1 \leq i \leq n$ , their set of actions  $A_i$  is given by listing all its elements. The utility  $u_i(a)$ , for player i of strategy profile a, is the output of M on input  $\langle a, i \rangle$  after t steps;
- formula general form by a tuple  $\langle 1^n, A_1, \ldots, A_n, 1^\ell, (\varphi_{i,j})_{1 \le i \le n, 0 \le j < \ell} \rangle$ . The set of actions for player i,  $1 \le i \le n$ , is  $A_i = \{0, 1\}^{m_i}$ . The utility of player *i* is given by the boolean formulas  $\varphi_{i,j}(a_1, \ldots, a_n) \in \{0, 1\}, 0 \le j < \ell$ , by the equation  $u_i(a_1, \ldots, a_n) = \sum_{0 \le i \le \ell} \varphi_{i,j}(a_1, \ldots, a_n) 2^j$ .

We assume for the rest of the paper that all the games have 2 or more players. Observe that, in the case that the number of players is constant, with respect to the number of actions, we can obtain an explicit representation in polynomial time from a given general form representation, otherwise the transformation requires exponential time.

A binary actions game is a game in which the set of actions for each player is  $\{0, 1\}$ . A binary game is a binary actions game in which the utility functions range is  $\{0, 1\}$ .

#### **Theorem 2.3** ([8]). The STRONGISO problem is polynomially equivalent to

- the CIRCUITISO problem, for strategic games given in general form,
- the FORMULAISO problem, for strategic games given in formula general form, and
- to the GRAPHISO problem, for strategic games given in explicit form.

The equivalence is also valid for binary games and binary actions games.

In the following sections we prove the computational equivalence among the two game isomorphism problems. For doing so we provide a series of game transformations that preserve local isomorphism or strong isomorphism. Finally, we show that, for any of those transformations, given an strategic game in form F, a description in form F of the transformed game can be obtained in polynomial time, when F is any of the three game representations considered in the paper.

#### 3. From strong isomorphism to local isomorphism

Assume that  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$  is a binary game where  $N = \{1, \ldots, n\}$ . For  $a_i \in \{0, 1\}$  we define  $\overline{a_i} = 1 - a_i$ .

CHECKL( $\Gamma$ ) =  $(N', (A'_i)_{i \in N'}, (u'_i)_{i \in N})$  has n+1 players, that is  $N' = \{0, 1, \ldots, n\}$ . Player 0 has  $A'_0 = \{0, 1\}$  and for i > 0 the set of actions is  $A'_i = \{0, 1\}^2$ . A profile factors  $c = (a_0, a')$  with  $a_0 \in A'_0$ ,  $a' = (a_1b_1, \ldots, a_nb_n)$  with  $a_ib_i \in A'_i$ . Note that the part  $a = (a_1, \ldots, a_n)$  extracted from c is a profile in  $\Gamma$ .

Let us define the utilities for any player  $i, 0 \le i \le n$ , and any profile  $c = (a_0, b)$ . The utility of player 0 is defined as  $u'_0(c) = a_0$ , observe that it depends only on  $a_0$ and player 0 prefers 1 to 0. To define  $u'_i$ , for i > 0 we consider separately the cases  $a_0 = 1$  and  $a_0 = 0$ . When  $a_0 = 1$ , we set  $u'_i(c) = 7$  when  $b_i = 0$  and  $u'_i(c) = 8$  when  $b_i = 1$ . When  $a_0 = 0$ , we have three cases:

• When  $u_i(a) \neq u_i(a_{-i}, \overline{a}_i)$ , define

$$u_i'(c) = \begin{cases} 0 & \text{when } b_i = 0 \text{ and } u_i(a) = 1, \\ 1 & \text{when } b_i = 0 \text{ and } u_i(a) = 0, \\ 2 & \text{when } b_i = 1 \text{ and } u_i(a) = 0, \\ 3 & \text{when } b_i = 1 \text{ and } u_i(a) = 1. \end{cases}$$

- When  $u_i(a) = u_i(a_{-i}, \overline{a}_i) = 0$  we define u'(c) = 4.
- When  $u_i(a) = u_i(a_{-i}, \overline{a}_i) = 1$ , set

$$u_i'(c) = \begin{cases} 5 & \text{when } b_i = 0, \\ 6 & \text{when } b_i = 1. \end{cases}$$

**Example 3.1.** Let us construct  $CHECKL(\Gamma)$  for the following game  $\Gamma$ 

Player 2  
0 1  
Player 1 0 0,1 0,1  
1,0 1,0  

$$\Gamma_1$$

As  $\Gamma$  has two players, the game CHECKL( $\Gamma_1$ ) has three players  $N' = \{0, 1, 2\}$ . Any profile in CHECKL factors as  $c = (a_0, a_1b_1, a_2b_2)$  and the utilities are defined by two tables: one corresponds to  $a_0 = 0$  and another to  $a_0 = 1$ .

When  $a_0 = 0$ , as  $u_1(0,0) = 0 \neq u_1(1,0)$  the first player utilities are defined, for any  $a \in \{0,1\}$ , as

$$u_1'(0,0b_1,0a) = \begin{cases} 1 \text{ if } b_1 = 0\\ 2 \text{ if } b_1 = 1 \end{cases} \quad u_1'(0,1b_1,0a) = \begin{cases} 0 \text{ if } b_1 = 0\\ 3 \text{ if } b_1 = 1. \end{cases}$$

When  $a_2 = 0$  the analysis is similar and we get, for any  $a \in \{0, 1\}$ ,

$$u_1'(0,0b_1,1a) = \begin{cases} 1 \text{ if } b_1 = 0\\ 2 \text{ if } b_1 = 1 \end{cases} \quad u_1'(0,1b_1,1a) = \begin{cases} 0 \text{ if } b_1 = 0\\ 3 \text{ if } b_1 = 1. \end{cases}$$

Let us write the utilities for the second player. When  $a_1 = 0$ , as  $u_2(0,0) = u_2(0,1) = 1$ the utility of  $u'_2$  depends on  $b_2$ , but when  $a_1 = 1$ , as  $u_2(1,0) = u_2(1,1) = 0$  the utility is independent of  $b_2$ . Thus, for any  $a, b, c \in \{0, 1\}$ ,

$$u_2'(0,0a,bb_2) = \begin{cases} 5 \text{ if } b_2 = 0\\ 6 \text{ if } b_2 = 1 \end{cases} \quad u_2'(0,1a,bc) = 4.$$

Collecting all this information we get the first table.

		Player 2							
		00 01 10 11							
Player 1	00	0, 1, 5	0, 1, 6	0, 1, 5	0, 1, 6				
	01	0, 2, 5	0, 2, 6	0, 2, 5	0, 2, 6				
	10	0, 0, 4	0, 0, 4	0, 0, 4	0, 0, 4				
	11	0, 3, 4	0, 3, 4	0, 3, 4	0, 3, 4				

Player 0 chooses 0

When  $a_0 = 1$  the utilities  $u'_1$  and  $u'_2$  are defined as follows, for any  $a, b, c \in \{0, 1\}$ ,

$$u_1'(1, ab_1, bc) = \begin{cases} 7 \text{ if } b_1 = 0\\ 8 \text{ if } b_1 = 1 \end{cases} \quad u_2'(1, ab, cb_2) = \begin{cases} 7 \text{ if } b_2 = 0\\ 8 \text{ if } b_2 = 1. \end{cases}$$

This gives the table

		Player 2					
		00 01 10 11					
Player 1	00	1, 7, 7	1, 7, 8	1, 7, 7	1, 7, 8		
	01	1, 8, 7	1, 8, 8	1, 8, 7	1, 8, 8		
	10	1, 7, 7	1, 7, 8	1, 7, 7	1, 7, 8		
	11	1, 8, 7	1, 8, 8	1, 8, 7	1, 8, 8		

Player 0 chooses 1

Before showing the correctness of the transformation let's introduce some *ad-hoc* notation to deal with profiles in CHECKL( $\Gamma$ ). Given  $a' = (a_1b_1, \ldots, a_nb_n)$  we note  $a' = a \uparrow b$  with  $a = (a_1, \ldots, a_n)$  and  $b = (b_1, \ldots, b_n)$ . Given a player *i*, we factor a profile *c* in CHECKL( $\Gamma$ ) as  $c = (a_0, a_{-i} \uparrow b_{-i}, a_ib_i)$  adopting here the criterion  $-i = N' \setminus \{0, i\}$  (-i is the adversary team of *i* in game  $\Gamma$ ). Given  $\psi$  and defining  $\mu = (\pi, id_1, \ldots, id_n)$ , using that for i > 0 player *i* maps into player  $\pi(i)$  it holds

$$\psi'(a_0, a_{-i} \uparrow b_{-i}, a_i b_i) = (a_0, \psi(a_{-i}) \uparrow \mu(b_{-i}), \varphi(a_i) b_i).$$

**Lemma 3.2.** Let  $\Gamma_1$  and  $\Gamma_2$  be two binary games such that  $\Gamma_1 \sim_s \Gamma_2$ , then we have that  $\text{CHECKL}(\Gamma_1) \sim_{\ell} \text{CHECKL}(\Gamma_2)$ .

*Proof.* Let  $\Gamma'_1 = \text{CHECKL}(\Gamma_1)$  and  $\Gamma'_2 = \text{CHECKL}(\Gamma_2)$  and let  $\psi : \Gamma_1 \to \Gamma_2$  be a strong isomorphism, assume that  $\psi = (\pi, \varphi_1, \dots, \varphi_n)$ . Consider the mapping  $\psi' : \text{CHECKL}(\Gamma_1) \to \text{CHECKL}(\Gamma_2), \ \psi' = (p, f_0, \dots, f_n)$  were.

- p(0) = 0 and  $p(i) = \pi(i)$ , for i > 0, and
- $f_0(a_0) = a_0$  and  $f_i(a_i b_i) = \varphi_i(a_i) b_i$ , for i > 0.

Let us show that the mapping  $\psi'$  verifies  $u_i(c) = u_{p(i)}(\psi'(c))$ .

Let us consider the player i = 0. As p(0) = 0 and  $f_0(a_0) = a_0$ , given  $c = (a_0, ...)$  it holds  $u_0(c) = u_{p(0)}(\psi'(c)) = a_0$ .

Consider the case i > 0 with  $c = (0, a_{-i} \uparrow b_{-i}, a_i b_i)$ . It holds  $u_i(c) = u_{p(i)}(\psi'(c))$ . The prove is done by case analysis.

- When  $u_i(a) \neq u_i(a_{-i}, \overline{a}_i)$  it holds that  $u_{\pi(i)}(\psi(a)) \neq u_{\pi(i)}(\psi(a_{-i}), \varphi(a_i))$ . As  $\psi$  is a strong morphism we have,  $u_i(a) = u_{\pi(i)}(\psi(a))$  and  $u_i(a_{-i}, \overline{a}_i) = u_{\pi(i)}(\psi(a_{-i}), \varphi(\overline{a}_i))$ . As  $a_i \in \{0, 1\}$  and  $\varphi_i$  is a bijection  $\varphi_i(\overline{a}_i) = \overline{\varphi_i(a_i)}$ . Then we conclude the inequality. Let us consider the case  $u'_i(c) = 0$ , other cases are similar. When  $u'_i(c) = 0$  the profile c verifies  $b_i = 0$  and  $u_i(a) = 1$ . As  $\mu_i(b_i) = b_i$  and  $u_{\pi(i)}(\psi(a)) = 1$  it holds  $u'_{p(i)}(\psi'(c)) = u'_i(c) = 0$ .
- When  $u_i(a) = u_i(a_{-i}, a_i) = 0$  it holds  $u_{\pi(i)}(\psi(a)) = u_{\pi(i)}(\psi(a_{-i}), \varphi(a_i)) = 0$ . Therefore  $u'_{p(i)}(\psi'(c)) = u'_i(c) = 4$ . The proof is similar to the preceding case.

• When  $u_i(a) = u_i(a_{-i}, a_i) = 1$  it holds  $u_{\pi(i)}(\psi(a)) = u_{\pi(i)}(\psi(a_{-i}), \overline{\varphi(a_i)}) = 1$ . As  $\mu_i(b_i) = b_i$  it also holds  $u'_{\pi(i)}(\psi'(c)) = u'_i(c) \in \{5, 6\}$ .

Consider the case i > 0 with  $c = (1, a_{-i} \uparrow b_{-i}, a_i b_i)$ . As  $\mu_i(b_i) = b_i$  it holds  $u'_{p(i)}(\psi'(c)) = u'_i(c) \in \{7, 8\}.$ 

Therefore we conclude that  $\psi'$  is a strong isomorphism, and indeed a local isomorphism.

Now we prove the reverse implication.

**Lemma 3.3.** Let  $\Gamma_1$  and  $\Gamma_2$  be two binary games. If  $CHECKL(\Gamma_1) \sim_{\ell} CHECKL(\Gamma_2)$ , then we have that  $\Gamma_1 \sim_s \Gamma_2$ .

*Proof.* Assume that  $\psi' : \Gamma'_1 \to \Gamma'_2$  is a local isomorphism with  $\psi' = (p, f_0, \ldots, f_n)$ . As all the  $f_i$ , for  $0 \le i \le n$  are bijections, the players bijection p has to map player 0 into player 0 because player 0 is the only one with 2 actions. Therefore p(0) = 0 and players  $\{1, \ldots, n\}$  are bijectively mapped into  $\{1, \ldots, n\}$  by p, so we can define  $\pi(i) = p(i)$ , for  $1 \le i \le n$ , and get a player's bijection.

Consider two profiles  $c_1 = (0, a'_1)$  and  $c_2 = (1, a'_2 \dots)$ , it holds  $c_1 \prec_0 c_1$ . Suppose that  $f_0(a_0) = \overline{a}_0$ . As p(0) = 0, it holds  $\psi'(c) = (1, \dots)$ ,  $\psi'(\hat{c}) = (0, \dots)$  and  $\psi'(c) \prec_0 \psi'(\hat{c})$  getting a contradiction. So it holds that  $f_0$  is the identity.

Let us prove now that, for  $1 \leq i \leq n$ ,  $f_i(a_i 0) = \varphi_i^0(a_i) 0$  and  $f_i(a_i 1) = \varphi_i^1(a_i) 1$ . Given  $a'_{-i} = a_{-i} \uparrow b_{-i}$  it holds  $(1, a'_{-i}, a_i 0) \prec_i (1, a'_{-i}, a'_i 1)$  because  $u'_i(1, a'_{-i}, a_i 0) = 7$  and  $u'_i(1, a'_{-i}, a'_i 1) = 8$ . As  $\psi'$  is a local isomorphism and  $f_0(1) = 1$  we necessarily have  $(1, \psi'(a'_{-i}), f_i(a_i 0)) \prec_{p(i)} (1, \psi'(a'_{-i}), f_i(a'_i 1))$ . This forces the factorizations  $f_i(a_i 0) = \varphi_i^0(a_i) 0$  and  $f_i(a_i 1) = \varphi_i^1(a_i) 1$ .

Finally, let us prove that the mapping  $\psi^0 = (\pi, \varphi^0, \dots, \varphi^0)$  between  $\Gamma_1$  and  $\Gamma_2$  is a strong isomorphism. We prove by cases that  $u_i(a) = u_{\pi(i)}(\psi^0(a))$ .

Case 1.  $a = (a_{-i}, a_i)$  with  $u_i(a_{-i}, a_i) \neq u_i(a_{-i}, \overline{a_i})$ .

In the following we use b = (0, ..., 0), the array containing n - 1 zeros.

We give the proof for  $u_i(a_{-i}, a_i) = 1$  and  $u_i(a_{-i}, \overline{a}_i) = 0$ . For  $u_i(a_{-i}, a_i) = 0$ and  $u_i(a_{-i}, \overline{a}_i) = 1$  the proof is similar. As

$$u'_i(0, a_{-i} \uparrow b, a_i 0) = 0, \ u'_i(0, a_{-i} \uparrow b, \overline{a}_i 0) = 1$$
  
 $u'_i(0, a_{-i} \uparrow b, \overline{a}_i 1) = 2, \ u'_i(0, a_{-i} \uparrow b, a_i 1) = 3$ 

we have

$$(0, a_{-i} \uparrow b, a_i 0) \prec_i (0, a_{-i} \uparrow b, \overline{a}_i 0) \prec_i (0, a_{-i} \uparrow b, \overline{a}_i 1) \prec_i (0, a_{-i} \uparrow b, a_i 1).$$

As  $\psi'$  is a local morphism

$$(0,\psi^0(a_{-i})\uparrow b,\varphi^0_i(a_i)0)\prec_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^0_i(\overline{a}_i)0)\prec_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^1_i\overline{a}_i1)\prec_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^1_i(a_i)1)$$

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This forces us to the values

$$\begin{aligned} u'_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^0_i(a_i)0) &= 0, \ u'_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^0_i(\overline{a}_i)0) = 1\\ u'_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^1_i\overline{a}_i1) &= 3, \ u'_{\pi(i)}(0,\psi^0(a_{-i})\uparrow b,\varphi^1_i(a_i)1) = 4. \end{aligned}$$

As  $u_{\pi(i)}(0, \psi^0(a_{-i}) \uparrow b, \varphi_i^0(a_i)0) = 0$  and  $u_{\pi(i)}(0, \psi^0(a_{-i}) \uparrow b, \varphi_i^0(\overline{a}_i)0) = 1$  the difference in values forces that, in  $\Gamma_2$ , we must have  $u_{\pi(i)}(\psi^0(a_{-i}), \varphi_i^0(a_i)) = u_{\pi(i)}(\psi^0(a)) = 0$  and  $u_{\pi(i)}(\psi^0(a_{-i}), \varphi_i^0(\overline{a}_i)) = 1$ .

**Case 2.**  $a = (a_{-i}, a_i)$  with  $u_i(a_{-i}, a_i) = u_i(a_{-i}, \overline{a}_i) = 1$ . As before we use  $b_{-i} = (0, \ldots, 0)$ . In this case the utilities verify:

$$\begin{aligned} u_i'(0, a_{-i} \uparrow b, a_i 0) &= u_i'(0, a_{-i} \uparrow b, \overline{a}_i 0) = 5\\ u_i'(0, a_{-i} \uparrow b, \overline{a}_i 1) &= u_i'(0, a_{-i} \uparrow b, a_i 1) = 6 \end{aligned}$$

we have

$$(0, a_{-i} \uparrow b, a_i 0) \sim_i (0, a_{-i} \uparrow b, \overline{a}_i 0) \prec_i (0, a_{-i} \uparrow b, \overline{a}_i 1) \sim_i (0, a_{-i} \uparrow b, a_i 1)$$

As  $\psi'$  is a local isomorphism

$$(0, \psi^{0}(a_{-i}) \uparrow b, \varphi^{0}_{i}(a_{i})0) \sim_{\pi(i)} (0, \psi^{0}(a_{-i}) \uparrow b, \varphi^{0}_{i}(\overline{a}_{i})0) \\ \prec_{\pi(i)} (0, \psi^{0}(a_{-i}) \uparrow b, \varphi^{1}_{i}\overline{a}_{i}1) \sim_{\pi(i)} (0, \psi^{0}(a_{-i}) \uparrow b, \varphi^{1}_{i}(a_{i})1).$$

This preorder structure forces the value of the utilities, in particular we have that  $u'_{\pi(i)}(0, \psi^0(a_{-i}) \uparrow b, \varphi^0_i(a_i)0) = 5$  but then  $u_{\pi(i)}(\psi^0(a)) = 1$ .

**Case 3.**  $a = (a_{-i}, a_i)$  with  $u_i(a_{-i}, a_i) = u_i(a_{-i}, \overline{a_i}) = 0$ . In this case we have

$$(0, a_{-i} \uparrow b, a_i 0) \sim_i (0, a_{-i} \uparrow b, \overline{a}_i 0) \sim_i (0, a_{-i} \uparrow b, \overline{a}_i 1) \sim_i (0, a_{-i} \uparrow b, a_i 1)$$

This forces  $u'_{\pi(i)}(0, \psi^0(a_{-i}) \uparrow b, \varphi^0_i(a_i)0) = 4$  but then  $u_{\pi(i)}(\psi^0(a)) = 0$ . Which concludes the proof.

## 4. FROM GENERAL GAMES TO BINARY ACTIONS GAMES

Our next step is to transform a strategic game into a binary actions game preserving local isomorphism. The game construction follows the same lines as in the BINARYACT in [8], but now we have to adapt the definition in order to guarantee an adequate local preference relation for each player. Again we follow notation from [8].

Given a strategic game  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ , assume without loss of generality that  $N = \{1, \ldots, n\}$  and that, for any  $i \in N$ ,  $A_i = \{1, \ldots, k_i\}$  for suitable values. Given  $A_i = \{1, \ldots, k_i\}$  we "binify" an action  $j \in A_i$  coding it with  $k_i$  bits, as binify $(j) = 0^{j-1}10^{k_i-j}$ . Thus binify $(A_i) \subseteq A^i = \{0, 1\}^{k_i}$ . The binify process can be used in a strategy profile, given  $a = (a_1, \ldots, a_n)$ , we

write  $\operatorname{binify}(a) = \operatorname{binify}(a_1) \cdots \operatorname{binify}(a_n)$ . Observe that by setting  $k = \sum_{i \in N} k_i$ , we have  $\operatorname{binify}(a) \in A' = \{0,1\}^k = A^{k_1} \times \cdots \times A^{k_2}$ . We define  $\operatorname{good}(A') = \{\operatorname{binify}(a) | a \in A\}$  and  $\operatorname{bad}(A') = A' \setminus \operatorname{good}(A')$ . Note that  $\operatorname{binify} : A \to \operatorname{good}(A')$ is a bijection and therefore its inverse function is also a bijection. Observe that, for  $a' \in \operatorname{good}(A')$ ,  $\operatorname{binify}^{-1}(a') = \operatorname{binify}^{-1}(b_1) \cdots \operatorname{binify}^{-1}(b_n)$ . Assume that  $\Gamma = (N, A_1, \ldots, A_n, (u_i)_{1 \le i \le n})$ , we construct the following game:

BINARYACTL( $\Gamma$ ) =  $(N', (A'_i)_{i \in N'}, (u'_i)_{i \in N'})$  where  $N' = \{1, \ldots, k\}$  and, for any  $i \in N', A'_i = \{0, 1\}$  and thus the set of action profiles is  $A' = \{0, 1\}^k$ .

For a player *i* with  $|A_i| > 2$ , we associate to  $A_i$  a block  $B_i$  and a block  $C_i$  of  $k_i = |A_i|$  players in each block, each player takes care of one bit. For a player *i* with  $|A_i| \le 2$ , we associate to player *i* a block  $B_i$  formed by three players,  $k_i = 3$ . Thus,  $k = 2(k_1 + \cdots + k_n)$ .

We decompose a strategy profile a' into 2n blocks, 2 blocks per player, so that  $a' = (b_1, \ldots, b_n, c_1, \ldots, c_n)$  where  $b_i, c_i \in \{0, 1\}^{k_i}$ , often we will also decompose a' = (b, c). We keep  $a'_j$  to refer to the strategy of player j in the profile a', but sometimes we refer to its strategy by the position inside its corresponding B or C block.

For a player  $\alpha$  that occupies position j in block  $B_i$ , the utility function is defined as

$$u'_{\alpha}(a') = \begin{cases} 0 & \text{if } b_i \in \mathsf{bad}(A^i), \\ 1 & \text{if } b_i \in \mathsf{good}(A^i)b \in \mathsf{bad}(A'), \\ 2 & \text{if } b \in \mathsf{good}(A'). \end{cases}$$

For a player  $\beta$  that occupies position j in block  $C_i$ , the utility function is defined as

$$u_{\beta}'(a') = \begin{cases} 1 - a_{\beta} & \text{if } b_i \neq c_i b \in \mathsf{bad}(A'), \\ 3 - a_{\beta}' & \text{if } b_i = c_i b \in \mathsf{bad}(A'), \\ 4 + u_i(\mathsf{binify}^{-1}(b)) & \text{if } a_{\beta}' = 1b \in \mathsf{good}(A'), \\ 4 + u_i(\mathsf{binify}^{-1}(b)_{-i}, j) & \text{if } a_{\beta}' = 0b \in \mathsf{good}(A'). \end{cases}$$

Let us comment the case with alphabets having a small number of actions. When  $A_i = \{1\}$ , by definition  $B_i$  contains 3 players,  $A^i = \{0, 1\}^3$  and  $good(A^i) = \{binify(1)\} = \{100\}$ . When  $A_i = \{1, 2\}$  we also have  $A^i = \{0, 1\}^3$  and  $good(A^i) = \{binify(1), binify(2)\} = \{100, 010\}$ . Finally when  $A_i = \{1, 2, 3\}$  we also have  $A^i = \{0, 1\}^3$  but  $good(A^i) = \{100, 010, 001\}$ .

**Example 4.1.** Consider a version  $\Gamma$  of rock-paper-scissors where we add 1 to all the utilities to get non negative values. The set of actions is  $A_i = \{1, 2, 3\}$  where 1 corresponds to rock, 2 to paper and 3 to scissors.

		Player 2				
		1 2 3				
	1	1, 1	0, 2	2, 0		
Player 1	2	2, 0	1, 1	0, 2		
	3	0, 2	2, 0	1, 1		
			Г			

BINARYACTL( $\Gamma$ ) has 12 players having binary actions. The strategy profiles are  $a' = (a_1, \ldots, a_{12}) = (b_1, b_2, c_1, c_2)$ , with  $b_i, c_i \in \{0, 1\}^3$ . In this case

$$good(A^3) = \{100, 010, 001\},$$
  $bad(A^3) = \{0, 1\}^3 \setminus good(A^3)$ 

As the number of strategy profiles is  $2^{12}$  we give some examples.

Consider the profile  $a' = (b_1, b_2, c_1, c_2) = (000, 010, 100, 010)$ . The utility  $u'_{\alpha}$  for a player  $\alpha$  in block  $B_1$  or  $B_2$  depends only on the *b* part of the profile. As  $b_1 = 000 \in bad(A^3)$ ,  $b_2 = 010 \in good(A^3)$  and  $b = 000010 \in bad(A^6)$  therefore

$$u'_{\alpha}(a') = \begin{cases} 0 \text{ if } \alpha \in \{1, 2, 3\}\\ 1 \text{ if } \alpha \in \{4, 5, 6\}. \end{cases}$$

For a player  $\beta$  that occupies position j in block  $C_i$ .

$$u'_{\beta}(a') = \begin{cases} 1 - a'_{\beta} \text{ if } \beta \in \{10, 11, 12\} \\ 3 - a'_{\beta} \text{ if } \alpha \in \{13, 14, 15\} \end{cases}$$

and these results can be summarized in the table

ſ	player	$\beta$	7	8	9	10	11	12
l	action	$a'_{\beta}$	1	0	0	0	1	0
ſ	utility	$u'_{\beta}$	0	1	1	3	2	3

When a' = (010, 100, 011, 010),  $b = 010100 \in \text{good}(A^6)$  and  $u'_{\alpha}(a') = 2$  for  $\alpha \in \{1, \ldots, 6\}$ . Furthermore, set  $a = \text{binify}^{-1}(010100) = (2, 1) \in A$ , that is the associated action profile in  $\Gamma$ . Then,

$$u_{\beta}'(a') = \begin{cases} 4 + u_1(a_{-1}, 1) = 4 + u_1(1, 1) = 5 \text{ if } \beta = 7\\ 4 + u_1(a) = 4 + u_1(2, 1) = 6 & \text{if } \beta \in \{8, 9\}\\ 4 + u_2(a_{-2}, 1) = 4 + u_1(2, 1) = 5 & \text{if } \beta = 10\\ 4 + u_2(a) = 4 + u_2(2, 1) = 5 & \text{if } \beta = 11\\ 4 + u_1(a_{-2}, 3) = 4 + u_2(2, 3) = 6 & \text{if } \beta = 12 \end{cases}$$

and these results can be summarized in the table

player	$\beta$	7	8	9	10	11	12
block number	i	1	1	1	2	2	2
position into the block	j	1	2	3	1	2	3
action	$a'_{\beta}$	0	1	1	0	1	0
utility	$u'_{\beta}$	5	6	6	5	5	6

Given a player  $\alpha$  in BINARYACTL( $\Gamma$ ) we define the *local indifference set* of  $\alpha$  as  $I(\alpha) = \{a'_{-\alpha} | (a'_{-\alpha}, 0) \sim_{\alpha} (a'_{-\alpha}, 1)\}.$ 

**Lemma 4.2.** Let  $\Gamma_1$  and  $\Gamma_2$  be two strategic games such that  $\Gamma_1 \sim_{\ell} \Gamma_2$ , then we have BINARYACTL $(\Gamma_1) \sim_{\ell}$  BINARYACTL $(\Gamma_2)$ .

Proof. Let  $\Gamma'_1 = \text{BINARYACTL}(\Gamma_1)$  and  $\Gamma'_2 = \text{BINARYACTL}(\Gamma_2)$ . Assume that  $\psi = (\pi, \varphi_1, \ldots, \varphi_n)$  is a local isomorphism between  $\Gamma_1$  and  $\Gamma_2$ . Consider the mapping  $\psi' = (p, f_1, \ldots, f_k)$  where, for  $1 \leq i \leq n$ , p maps the bits in block  $B_i$  of  $\Gamma'_1$  to the bits in block  $B_{\pi(i)}$  of  $\Gamma'_2$  and the bits in block  $C_i$  of  $\Gamma'_1$  to the bits in block  $C_{\pi}(i)$  of  $\Gamma'_2$ , so that bit j goes to bit  $\varphi_i(j)$ , and for  $1 \leq \alpha \leq k$ ,  $f_{\alpha}$  is the identity function. It is straightforward to show that  $\psi'$  is a strong (therefore also a local) isomorphism between  $\Gamma'_1$  and  $\Gamma'_2$ .

Before proving the reverse implication we analyze some properties of the constructed game. Let  $\Gamma$  be a game and let  $\Gamma' = \text{BINARYACTL}(\Gamma)$ . As usual we assume n > 1. Consider a player  $\alpha$  that occupies position j in block  $B_i$ , for u = 0, 1, 2 define  $X_u(\alpha) = \{a' \mid u'_{\alpha}(a') = u\}$ .

**Lemma 4.3.** Given an  $\alpha$  player belonging to a *B*-block, it holds  $|X_0(\alpha)| > |X_1(\alpha)| > |X_2(\alpha)|$ .

*Proof.* Note that any strategy a' factors a' = (b, c). As the utility  $u'_{\alpha}(a')$  only depends on the b part, the c part give a common multiplicative constant. We analyze only the b part of a'. Suppose that  $\alpha$  occupies position j in block  $B_i$ , we factor  $b = (b_{-i}, b_i)$ .

First, let us prove  $|X_0(\alpha)| > |X_1(\alpha)|$ . When  $a' \in X_0(\alpha)$  the profile factors a' = (b, c) such that  $b_i \in \mathsf{bad}(A^i)$  and there is no special restrictions on  $b_{-i}$ . When  $\hat{a}' \in X_1(\alpha)$  the profile factors  $\hat{a}' = (\hat{b}, \hat{c})$  such that  $\hat{b}_i \in \mathsf{good}(A^i)$  and  $\hat{b}_{-i} \in \mathsf{bad}(A')$ . As  $|\mathsf{bad}(A^i)| = 2_i^k - k_i$  and  $|\mathsf{good}(A^i)| = k_i$  and  $k_i > 3$  it holds  $|\mathsf{bad}(A^i)| > |\mathsf{good}(A^i)|$ . As there are  $2^{k/2-k_i}$  profiles  $b_{-i}$  with no special restrictions (note that  $k/2 - k_i = k_1 + \cdots + k_{i-1} + k_{i+1} + \cdots + k_n$ ) and  $|\mathsf{bad}(A')| = 2^{k/2-k_i} - k/2 - k_i$  the inequality also holds for the second part. As there are more elements in both parts (parts are i and -i) in the first case inequality holds.

It remains to prove  $|X_1(\alpha)| > |X_2(\alpha)|$ . We have  $a' \in X_1(\alpha)$  iff a' = (b, c) and b factors as  $(b_{-i}, b_i)$  with  $b_i \in \text{good}(A^i)$  and  $b_{-i} \in \text{bad}(A^{-i})$ . We have  $\hat{a}' \in X_2(\alpha)$  iff  $\hat{a}' = (\hat{b}, \hat{c})$  and  $\hat{b}$  factors as  $(\hat{b}_{-i}, \hat{b}_i)$  with  $\hat{b}_i \in \text{good}(A^i)$  and  $\hat{b}_{-i} \in \text{good}(A^{-i})$ . As  $\text{bad}(A^{-i}) > \text{good}(A^{-i})$  inequality holds.

Given a player  $\alpha$  in  $\Gamma'$ , the local indifference set  $I(\alpha)$  for this player is the set  $I(\alpha) = \{a'_{-\alpha} | (a'_{-\alpha}, 0) \sim_{\alpha} (a'_{-\alpha}, 1)\}.$ 

**Lemma 4.4.** Let  $\alpha$  be a player belonging to a *B*-block and let  $\beta$  be a player belonging to a *C*-block, it holds

- the indifference set  $I(\alpha)$  has  $2^{k-3}$  elements,
- the indifference set  $I(\beta)$  has at most  $k_1 \dots k_n 2^{k/2-1}$  elements and
- it holds  $|I(\beta)| < |I(\alpha)|$ .

*Proof.* Consider a player  $\alpha$  that occupies position j in block  $B_i$ , observe that for any strategy profile a' of  $\Gamma'$ , we have:

- if  $(a'_{-\alpha}, 1) \in X_2(\alpha)$  then  $(a'_{-\alpha}, 0) \in X_0(\alpha)$ , as we are eliminating the unique 1 in  $B_i$ ,
- if  $(a'_{-\alpha}, 0) \in X_2(\alpha)$  then  $(a'_{-\alpha}, 1) \in X_0(\alpha)$ , as after the transformation  $B_1$  will have two 1's
- if  $(a'_{-\alpha}, 1) \sim_{\alpha} (a'_{-\alpha}, 0)$  iff  $(a'_{-\alpha}, 1), (a'_{-\alpha}, 0) \in X_0(\alpha)$ . Note that profile  $b_i$  corresponding to block  $B_i$  in  $a'_{\alpha}$  factors  $b_i = b_{i_1} \dots b_{i_j} \dots b_{i_{k_i}}$ . Profile  $a'_{-\alpha}$  ignores the action  $b_{i_j}$  corresponding to player  $\alpha$ . Ignoring player  $\alpha$  in  $b_i$  we write  $(b_i)_{-\alpha} = b_{i_1} \dots b_{i_j-1} b_{i_j+1} \dots b_{i_{k_i}}$ , the preceding indifference condition is equivalent to  $1 \in \{b_{i_1}, \dots, b_{i_j-1}\}$  and  $1 \in \{b_{i_j+1}, \dots, b_{i_{k_i}}\}$ . As  $|A^i| \geq 3$  we always have at least 3 positions.

Let us analyze  $I(\alpha)$ . As any profile  $(b_i)_{-\alpha}$  contains  $k_i - 1$  players and (at least) two players have to choose 1 it remains  $k_i - 3$  "free" choices. The profile  $a'_{-\alpha}$  contains a profile c and this give us k/2 possibilities. Finally the profile  $a'_{-\alpha}$  contains also a part  $b_{-i}$  and this give us  $k/2 - k_i$  extra choices. Based on that, the cardinal of  $I(\alpha)$  is  $2^{k-3}$ . Thus the number of  $a'_{-\alpha}$  which give raise to an indifferent pair for player  $\alpha$  is  $2^{k-3}$ . Note that this value is independent of the chosen block  $B_i$ .

Consider a player  $\beta$  that occupies position j in block  $C_i$  and consider the set  $I(\beta)$ . The cardinal of  $I(\beta)$  is at most  $k_1 \dots k_n 2^{k/2-1}$  because, to give raise to an indifferent pair for player  $\alpha$  the profile a' must verify  $b \in good(A')$ . As n > 1 it holds  $|I(\beta)| < |I(\alpha)|$ . 

Given  $\Gamma_1' = \text{BinaryActL}(\Gamma_1)$  and  $\Gamma_2' = \text{BinaryActL}(\Gamma_2)$  let  $\psi'$ = $(p, f_1, \ldots, f_k)$  be a local morphism between  $\Gamma'_1$  and  $\Gamma'_2$ .

**Lemma 4.5.** Local morphisms map bijectively indifference into indifference sets, that is

- given a player α₁ in Γ₁' it holds ψ'(I(α₁)) = I(p(α₁)) and
  given a player α₂ in Γ₂' it holds ψ'<sup>-1</sup>(I(α₂)) = I(p<sup>-1</sup>(α₂)).

*Proof.* Take a player  $\alpha_1$  in  $\Gamma_1$ . A profile  $a'_{-\alpha_1}$  belongs to  $I(\alpha_1)$  iff  $(a'_{-\alpha_1}, 0) \sim_{\alpha_1}$  $(a'_{-\alpha_1}, 1)$ . As  $\psi'$  is a local morphism  $\psi'(a'_{-\alpha_1}, 0) \sim_{p(\alpha_1)} \psi'(a'_{-\alpha_1}, 1)$ . As player  $\alpha_1$  is mapped into  $p(\alpha_1)$  we have the factorizations  $\psi'(a'_{-\alpha_1}, 0) = (\psi'(a'_{-\alpha_1}), f_{\alpha}(0))$ and  $\psi'(a'_{-\alpha_1}, 1) = (\psi'(a'_{-\alpha_1}), f_{\alpha}(1))$ . As  $f_{\alpha_1}$  is a bijection in  $\{0, 1\}$  we have that  $\psi'(a'_{-\alpha_1}) \in I(p(\alpha_1))$  and therefore  $\psi'(I(\alpha_1))$  is included in  $I(p(\alpha_1))$ . Suppose that inclusion is strict, then exists  $a'_{-p(\alpha_1)} \in I(p(\alpha_1)) \setminus \psi'(I(\alpha_1))$ . It is straightforward to prove that  $\psi'^{-1}(a'_{-p(\alpha_1)})$  belongs to  $I(\alpha_1)$  getting a contradiction. The case  $\psi'^{-1}$  is similar. 

**Lemma 4.6.** Let  $\psi' = (p, f_1 \dots, k_k)$  a local morphism between  $\Gamma'_1$  and  $\Gamma'_2$ ,

- $\psi'$  preserves B-blocks and  $f_{\alpha}$  is the identity when  $\alpha$  is a B-player,
- $\psi'$  preserves C-blocks and  $f_{\beta}$  is the identity when  $\beta$  is a C-player,
- the permutation  $\pi$  induced in the B-blocks and the C-blocks by p is the same,
- Let  $\varphi_i$  be the mapping p restricted to  $B_i$  and let  $\hat{\varphi}_i$  be the mapping p restricted to  $C_i$ , it holds  $\varphi_i = \hat{\varphi}_i$ .

*Proof.* We proceed proving a sequence of claims.

The local morphism  $\psi'$  preserves B-blocks. Suppose that block  $B_i$  in  $\Gamma'_1$  is only partially mapped into block  $B_{\ell}$  in  $\Gamma'_2$ . There exists player  $\alpha$  in  $B_i$  and another B-player  $\alpha'$  such that  $p(\alpha)$  and  $p(\alpha')$  belong to  $B_{\ell}$ . Suppose that player  $\alpha$  occupies the position j in  $B_i$ . Choose in  $\Gamma'_1$  a profile a' = (b, c) such that  $b_i = 0^{j-1} 10^{k_i-j}$  and choose  $b_{-i}$  in such a way that in  $\psi'(b) = (b_{-\ell}, b_{\ell})$  the profile  $b_{\ell}$  is bad. This is always possible because we are free to set the action belonging to  $\alpha'$  in such a way that  $f_{\alpha'}(a_{\alpha'})$  forces  $b_{\ell}$  to be bad. Notice that  $u'_{\alpha}(a') \in \{1,2\}$  but  $u'_{n(\alpha)}(\psi'(a')) = 0$ . Notice that  $a' = (a'_{-\alpha}, 1)$ . The profile  $\hat{a}' = (a'_{-\alpha}, 0)$  factors as  $\hat{a}' = (\hat{b}, \hat{c})$  with  $b_i \in \mathsf{bad}(A^i)$  because  $b_i = 0^{k_i}$  and therefore  $u'_{\alpha}(\hat{a}) = 0$ . It holds  $a' = (a'_{-\alpha}, 1) \succ_{\alpha} (a'_{-\alpha}, 0) = \hat{a}'$ . As  $\psi'$  is a local we should have  $\psi'(a') \succ_{p(\alpha)} \psi'(a'_{-\alpha}, 0)$ . This cannot be true because  $\psi'(a')$  is a minimal elements among preferences (because  $u'_{p(\alpha)}(\psi'(a')) = 0$ ). Therefore  $\psi'$  preserves *B*-blocks.

For any *B*-player  $\alpha$  the mapping  $f_{\alpha}$  is the identity. Notice that functions  $f_{\alpha}$  can be only identities or negations. We do a case analysis based on the number of negations.

- Case with 1 negation. Suppose that  $f_{\alpha}$  is the only negation among players in  $B_i$ and suppose that  $\alpha$  plays the position j in the block. Consider a' = (b, c) with  $b = (b_{-i}, b_i)$  and  $b_i = 0^{j-1} 10^{k_i - j}$ , then  $a' = (a'_{-\alpha}, 1)$ . Consider  $\hat{a}' = (a'_{-\alpha}, 0) =$  $(\hat{b}, \hat{c})$  with  $\hat{b}_i = 0^{k_i}$ . As  $a' \succ_{\alpha} \hat{a}'$  but  $\psi'(b_i) = 0^{k_i}$  we get a contradiction.
- Case with 2 negations. Suppose that only  $\alpha$  and  $\alpha'$  have negating player functions. Note that we have  $I(\alpha) = I(p(\alpha))$ . In  $B_i$  there are at least three players, therefore exists  $\gamma$  such that  $f_{\gamma}$  is the identity. Consider a  $b_i$  such that actions corresponding to  $\alpha$ ,  $\alpha'$  and  $\gamma$  are 1 and all other actions are 0, clearly such  $b_i$  is bad. Let a' = (b, c) a profile containing the preceding  $b_i$ , clearly  $a'_{-\alpha} \in I(\alpha)$  because the part of the profile corresponding to the  $b_i$  has two ones (the one corresponding to player  $\alpha$  is missing). The profile  $\psi'(a'_{-\alpha})$  contains just one 1 and therefore  $\psi'(a'_{-\alpha}) \notin I(p(\alpha))$  because player  $p(\alpha)$  can force a good (choosing 0) or a bad (choosing 1) block and therefore  $(\psi'(a'_{-\alpha}), 0) \not\sim_{p(\alpha)} (\psi'(a'_{-\alpha}), 1)$ .
- Case with at least 3 negations. Let  $\alpha$  having a negation for  $f_{\alpha}$ . As in the case with one negation, consider a profile  $a' = (a'_{-\alpha}, 1)$  containing  $b_i = 0^{j-1}10^{k_i-j}$ , then  $u'_{\alpha}(a')$  is 1 or 2. The profile  $\psi'(a'_{-\alpha}, 1) = (\hat{b}, \hat{c})$  contains at least two ones in the corresponding *B*-block therefore  $u_{p(\alpha)}(\psi'(a'_{-\alpha}, 1)) = 0$ . We get a contradiction.

All the  $f_{\beta}$  in the C-blocks are identities. Suppose that there is a player  $\beta \in C_i$  such that  $f_{\beta}$  is a negation. Let be  $C_{\ell}$  the block such that  $p(\beta) \in C_{\ell}$ . There are two exclusive possibilities for the other players in  $C_{\ell}$ ,

- there is at least one player  $\gamma$  such that  $f_{\gamma}$  is the identity and  $p(\gamma) \in C_{\ell}$ ,
- all the players  $\gamma$  mapped into  $C_{\ell}$  have functions  $f_{\gamma}$  corresponding to negations.

Let us consider the first possibility. Fix  $a'_{-\beta}$  such that all the *B*-blocks are fixed to 1 and all the *C*-blocks are fixed to 0. Note that independently of the  $a'_{\beta} \in \{0, 1\}$ it holds  $c_i \neq b_i$ . As  $u'_{\beta}(a'_{-\beta}, 0) > u'_{\beta}(a'_{-\beta}, 1)$  it holds  $(a'_{-\beta}, 0) \succ_{\beta} (a'_{-\beta}, 1)$ . As  $\gamma$ enters  $C_{\ell}$  and  $f_{\gamma}$  is the identity, the profile  $\psi'(a'_{-\beta})$  corresponding to the team  $-p(\beta)$  verifies  $b_{\ell} \neq c_{\ell}$  (independently of the value of the action chosen by  $p(\beta)$ ) and  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 0)) = u'_{p(\beta)}(\psi'(a'_{-\beta}), 1) = 0$  and  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 1)) = 1$  getting a contradiction. Let us consider the second possibility. Choose a profile  $a'_{-\beta}$  such that all the players in *B*-blocks choose 1 and all the players in the *C* blocks choose 0 except player  $\gamma$  choosing 1. The analysis follows similarly.

The players bijection p induces a bijection between C-blocks and moreover this bijection coincides with the bijection for the B-blocks. Let us denote by  $\pi$  the

bijection induced by the *B*-blocks. If this not holds, exists *i* such that players in  $C_i$  are only partially mapped into  $C_{\pi(i)}$ . Let  $\beta$  be a player mapped into  $C_{\ell}$  with  $\ell \neq \pi(i)$ . Take a profile  $a'_{-\beta}$  such that  $b_i$  contains only ones and  $(c_i)_{-\beta}$  contains only ones, in  $(a'_{-\beta}, 1)$  we have  $b_i = c_i$  and  $u'_{\beta}(a'_{-\beta}, 1) = 2$ . As in  $(a'_{-\beta}, 0)$  we have  $b_i \neq c_i$  and  $u'_{\beta}(a'_{-\beta}, 1) = 1$ , therefore  $(a'_{-\beta}, 0) \prec_{\beta} (a'_{-\beta}, 1)$ . As  $B_i$  is mapped into  $B_{\pi(i)}$  there is no intersection between  $B_{\pi(i)}$  and  $B_{\ell}$ . Let us denote by  $\hat{c}_{\ell}$  the profile corresponding to the players  $C_{\ell}$  in  $\psi'(a'_{-\beta}, 0)$ , note that in such a profile the action chosen by player  $\beta$  is 0. Fix in  $a'_{-\beta}$  the profile  $b_{\pi^{-1}(\ell)}$  corresponding to  $B_{\pi^{-1}(\ell)}$  such that  $\psi'(b_{\pi^{-1}(\ell)}) = \hat{c}_{\ell}$ . Note that  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 0)) = 3$  because profiles belonging to  $B_{\ell}$  and  $C_{\ell}$  coincide. Note also that  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 1)) = 0$  because profiles in  $B_{\ell}$  and  $C_{\ell}$  are different. Profiles for player  $p(\beta)$  factor as  $\psi'(a'_{-\beta}, 0) = (\psi'(a'_{-\beta}), 0)$  and  $\psi'(a_{-\beta}, 1) = (\psi'(a'_{-\beta}), 1)$ . Therefore we get  $(\psi'(a'_{-\beta}), 0) \succ_{p(\beta)} (\psi'(a'_{-\beta}), 1)$  and we obtain a contradiction.

Let  $\varphi_i$  be the mapping p restricted to  $B_i$  and let  $\hat{\varphi}_i$  be the mapping p restricted to  $C_i$ , it holds  $\varphi_i = \hat{\varphi}_i$ . Note that  $\varphi_i : B_i \to B_{\pi(i)}$  and  $\hat{\varphi}_i : C_i \to C_{\pi(i)}$ . Suppose that  $\varphi_i(r) = s$  and  $\hat{\varphi}_i(r) = t$  with  $s \neq t$ . Consider a profile a' = (b, c). Fix  $b = (b_{-i}, b_i)$  with  $b_i = \text{binify}(r)$  and the  $b_{-i}$  filled with zeros in order to get a bad b. Consider  $c = (c_{-i}, c_i)$  with  $c_i = \text{binify}(r)$ . Let  $\beta$  be the player belonging to the  $C_i$  block controlling the position r then  $a' = (a'_{-\beta}, 1)$ , moreover  $u_\beta(a'_{-\beta}, 1) = 2$  because b is bad,  $b_i = c_i$ , and  $a'_\beta = 1$ . Also  $u_\beta(a'_{-\beta}, 0) = 1$ because b is bad,  $b_i \neq c_i$ , and  $a'_\beta = 0$ . Finally  $(a'_{-\beta}, 1) \succ_\beta (a'_{-\beta}, 0)$ . Consider the profile  $\psi'(a') = (\hat{b}, \hat{c})$ . As  $B_i$  is mapped into  $B_{\pi(i)}$  we factor  $\hat{b} = (\hat{b}_{-\pi(i)}, \hat{b}_{\pi(i)})$ with  $b_{\pi(i)} = \text{binify}(s)$ . Similarly  $\hat{c} = (\hat{c}_{-\pi(i)}, \hat{c}_{\pi(i)})$  with  $\hat{c}_{\pi(i)} = \text{binify}(t)$ . In  $\psi'$  player  $\beta$  is mapped into  $p(\beta)$  controlling the position t in  $C_{\pi(i)}$ . Moreover  $\psi'(a') = (\psi'(a')_{-p(\beta)}, 1)$  and  $u'_{p(\beta)}(\psi'(a')_{-p(\beta)}, 0)$  and we get a contradiction.  $\Box$ 

**Lemma 4.7.** Let  $\Gamma_1$  and  $\Gamma_2$  be two strategic games.  $\Gamma_1 \sim_{\ell} \Gamma_2$  if and only if BINARYACTL $(\Gamma_1) \sim_{\ell} BINARYACTL(\Gamma_2)$ .

Proof. Let  $\Gamma'_1 = \text{BINARYACTL}(\Gamma_1)$  and  $\Gamma'_2 = \text{BINARYACTL}(\Gamma_2)$  and let  $\psi' = (p, f_1, \ldots, f_k)$  be a local isomorphism between  $\Gamma'_1$  and  $\Gamma'_2$ . Let  $\alpha$  be a player belonging to a *B*-block and let  $\beta$  be a player belonging to a *C*-block. As  $\psi'$  preserves local indifference sets and  $|I(\beta)| < |I(\alpha)|$  players  $\alpha$  and  $\beta$  cannot be mixed (look at the Lemmata 4.3, 4.4, and 4.5). It must happen that  $p(\alpha)$  belongs to a *B*-block in  $\Gamma'_2$  and  $p(\beta)$  belongs to a *C*-block in  $\Gamma'_2$ . Moreover the structure of  $\psi'$  verifies the conditions established in Lemma 4.6. Notice that,  $\psi'$  induces a permutation  $\pi$  on  $\{1, \ldots, n\}$ . Moreover, for a player  $\alpha$  in position j inside block  $B_i$ , the mapping  $\varphi(j)$  give us the position of player  $p(\beta)$  in block  $\pi(i)$ . Let us show that the mapping  $\psi = (\pi, \varphi_1, \ldots, \varphi_n)$  is a local isomorphism between  $\Gamma_1$  and  $\Gamma_2$ 

Suppose  $(a_{-i}, r) \prec_i (a_{-i}, s)$  with  $r, s \in \{1, \ldots, k_i\}$ , then  $u_i(a_{-i}, r) < u_i(a_{-i}, s)$ . Consider a' = (b, c) such that  $b = \text{binify}(a_{-i}, r)$  and  $c = (c_{-i}, c_i)$  with  $c_{-i}$  is all filled with zeros and  $c_i = \text{binify}(s)$ . Let  $\beta$  be the player controlling the the 1 in  $c_i$  (player  $\beta$  controls the position s in C-block i) it holds  $u'_{\beta}(a') = 4 + u_i(a_{-i}, r)$ . Factorizing player  $\beta$  we have  $a' = (a'_{-\beta}, 1)$ . Now player  $\beta$  can change the action from 1 to 0 getting the profile  $(a'_{-\beta}, 0)$  such that  $u'_{\beta}(a'_{-\beta}, 0) = 4 + u_i(a_{-i}, s)$ . Therefore we have  $(a'_{-\beta}, 1) \prec_{\beta} (a'_{-\beta}, 0)$ . As  $\psi'$  is a local morphism it holds  $\psi'(a'_{-\beta}, 1) \prec_{p(\beta)} \psi'(a'_{-\beta}, 0)$ .

We have to prove that  $(\psi(a_{-i}), \varphi_i(r)) \prec_{\pi(i)} (\psi(a_{-i}), \varphi_i(s))$  holds. As *i* is mapped to  $\pi(i)$  block  $C_i$  is mapped to block  $C_{\pi(i)}$ . Actions for player *i* are mapped  $\varphi_i(r) = r'$  and  $\varphi_i(s) = s'$ . The *B*-blocks are mapped,

$$\psi'(b) = \psi'(\mathsf{binify}(a_{-i}, r)) = (\mathsf{binify}(\psi(a_{-i})), \mathsf{binify}(\varphi_i(r)) = (\hat{b}_{-\pi(i)}, \hat{b}_{\pi(i)})$$

The C-blocks are mapped,  $\psi'(c) = (\hat{c}_{-\pi(i)}, \hat{c}_{\pi(i)})$  with  $\hat{c}_{-\pi(i)}$  filled with zeros and  $\hat{c}_{\pi(i)} = \psi'(\text{binify}(s)) = \text{binify}(s')$ . Note that  $p(\beta)$  controls the unique one in binify(s'). Then  $\psi'(a'_{-\beta}, 1) = (\hat{b}, \hat{c})$  and  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 1)) = 4 + u_{\pi(i)}(\psi(a_{-i}), \varphi_i(r))$ When player  $p(\beta)$  changes the action corresponding to 1 in binify(s') into a 0 we obtain  $u'_{p(\beta)}(\psi'(a'_{-\beta}, 0)) = 4 + u_{\pi(i)}(\psi(a_{-i}), \varphi_i(s))$ . As  $\psi'(a'_{-\beta}, 1) \prec_{p(\beta)} \psi'(a'_{-\beta}, 0)$ we obtain  $u_{\pi(i)}(\psi(a_{-i}), \varphi_i(r)) < u_{\pi(i)}(\psi(a_{-i}), \varphi_i(s))$  and we conclude the result.

We conclude that the mapping  $\psi = (\pi, \varphi_1, \dots, \varphi_n)$  is a local isomorphism between  $\Gamma_1$  and  $\Gamma_2$ .

## 5. From local isomorphism on binary action games to strong isomorphism

Next step is to transform two binary actions locally isomorphic games into two strongly isomorphic games.

FLIPL( $\Gamma$ ) =  $(N, (A'_i)_{i \in N}, (u'_i)_{i \in N})$  is defined as follows given a binary actions game  $\Gamma = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$  where  $N = \{1, \ldots, n\}$  and  $A_i = \{0, 1\}$ . Actions are  $A'_i = \{0, 1\}$  and for  $a' \in A'$  and  $i \in N$  define flip<sub>i</sub> $(a') = (a'_{-i}, 1 - a_i)$ . Utilities are defined as follows ( $\prec_i$  and  $\sim_i$  are defined using  $u_i$  in  $\Gamma$ ):

$$u_{i}'(a') = \begin{cases} 2 & \text{if } a' \prec_{i} \text{flip}_{i}(a'), \\ 1 & \text{if } a' \sim_{i} \text{flip}_{i}(a'), \\ 0 & \text{if flip}_{i}(a') \prec_{i} a'. \end{cases}$$

**Example 5.1.** The flip<sub>i</sub> function over A' for  $i \in \{1, 2\}$  is given by

A'	(0, 0)	(0, 1)	(1, 0)	(1, 1)
$flip_1$	(1, 0)	(1, 1)	(0, 0)	(0, 1)
$flip_2$	(0, 1)	(0, 0)	(1, 1)	(1, 0)

Following we give an example of  $\text{FLIPL}(\Gamma)$ .

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**Lemma 5.2.** If  $\Gamma_1$  and  $\Gamma_2$  are two binary actions games.  $\Gamma_1 \sim_{\ell} \Gamma_2$  iff  $\operatorname{FLIPL}(\Gamma_1) \sim_s \operatorname{FLIPL}(\Gamma_2)$ .

*Proof.* Let  $\Gamma'_1 = \text{FLIPL}(\Gamma_1)$  and  $\Gamma'_2 = \text{FLIPL}(\Gamma_2)$ .

Given a game mapping  $\psi' = (p, f_1, \ldots, f_n)$  between  $\Gamma'_1$  and  $\Gamma'_2$  it holds the equality  $\psi'(\mathsf{flip}_i(a')) = \mathsf{flip}_{p(i)}(\psi'(a'))$ . To prove it, note first, that as  $f_i$  is the identity or a negation it holds  $f_i(1 - a_i) = 1 - f(a_i)$  and therefore

$$\psi'(\mathsf{flip}_i(a')) = \psi'(a'_{-i}, 1 - a'_i) = (\psi'(a'_{-i}), f_i(1 - a'_i)) = \mathsf{flip}_{p(i)}(\psi'(a')).$$

Assume that  $\psi = (\pi, \varphi_1, \ldots, \varphi_n)$  is a local isomorphism between games  $\Gamma_1$  and  $\Gamma_2$ . Consider the mapping  $\psi' = (p, f_1, \ldots, f_n)$  between  $\Gamma'_1$  and  $\Gamma'_2$  defined by  $p = \pi$  and for all players  $f_i = \varphi_i$ . We have to prove that  $\psi'$  is a strong isomorphism. Let a' be a profile in  $\Gamma'_1$  and suppose that  $u'_i(a') = 2$  (other cases are similar). Condition  $u'_i(a') = 2$  forces  $a' \prec_i \operatorname{flip}_i(a')$ . As  $\psi$  is a local morphism and a' is also a profile in  $\Gamma_1$  it holds  $\psi(a') \prec_{\pi(i)} \psi(\operatorname{flip}_i(a'))$ . As  $\psi'$  behaves identically to  $\psi$  it holds the relation  $\psi'(a') \prec_{p(i)} \operatorname{flip}_{p(i)}(\psi'(a'))$  and then  $u'_{p(i)}(\psi'(a')) = 2$ .

Assume that  $\psi' = (p, f_1, \ldots, f_n)$  is a strong isomorphism between games  $\Gamma'_1$ and  $\Gamma'_2$ . Let  $\psi = (\pi, \varphi_1, \ldots, \varphi_n)$  a copy of  $\psi'$ . Note that a local preference in  $\Gamma_1$  like  $a = (a_{-i}, a_i) \prec_i (a_{-i}, 1 - a_i)$  can be rewritten in  $\Gamma'_2$  (formally a' = a) as  $a' \prec_i$  flip<sub>i</sub>(a') and then  $u'_i(a') = 2$ . As  $\psi'$  is a strong isomorphism it holds  $u'_{p(i)}(\psi(a')) = 2$  but this forces  $(\psi(a_{-i}), \varphi_i(a_i)) \prec_{\pi(i)} (\psi(a_{-i}), \varphi_i(1 - a_i))$  and therefore  $\psi$  is a local isomorphism.  $\Box$ 

## 6. The complexity of local isomorphism

In this section we show that the game transformations defined in the previous sections can be computed in polynomial time, this settles the complexity classification of the Locallso problem.

**Lemma 6.1.** Let  $\Gamma$  be a game given in explicit form. A description in explicit form of the games  $\operatorname{BINARYACT}(\Gamma)$ ,  $\operatorname{FLIPL}(\Gamma)$  when  $\Gamma$  is a binary actions game,  $\operatorname{CHECKL}(\Gamma)$  when  $\Gamma$  is a binary game, can be computed in polynomial time.

*Proof.* Recall that a representation in explicit form of a game is

$$\Gamma = \langle 1^n, A_1, \dots, A_m, (T_{i,a})_{1 \le i \le n, a \in A} \rangle.$$

We have to show how to compute a representation in explicit form of the games  $BINARYACT(\Gamma)$ ,  $FLIPL(\Gamma)$ , for a given binary actions game  $\Gamma$ ,  $CHECKL(\Gamma)$ , for a given binary game  $\Gamma$ .

Observe that in the three constructions, the number of players increases at most polynomially in the size of  $\Gamma$ . Furthermore, the size of the set of actions, for any player, is polynomial in the size of the set of actions of the players in the original game. As a consequence of those facts we have that the number of strategy profiles is polynomial in the size of  $\Gamma$ . Therefore the tabulated utility functions can be computed in polynomial time.

**Lemma 6.2.** Let  $\Gamma$  be a game given in general form. A description in general form of the games  $\operatorname{BINARYACT}(\Gamma)$ ,  $\operatorname{FLIPL}(\Gamma)$  when  $\Gamma$  is a binary actions game,  $\operatorname{CHECKL}(\Gamma)$  when  $\Gamma$  is a binary game, can be computed in polynomial time.

*Proof.* When the game  $\Gamma$  is given in general form,  $\Gamma = \langle 1^n, A_1, \ldots, A_m, M, 1^t \rangle$  in all three constructions we have that the number of players and the size of the sets of actions are polynomial. Now we have to construct, in each case, a Turing machine that computes the utility functions. From the definitions of the games it is straightforward to write the codification of the TMs in polynomial time, given a description of machine M.

**Lemma 6.3.** Let  $\Gamma$  be a game given in formula general form. A description in formula general form of the games  $\operatorname{BINARYACT}(\Gamma)$ ,  $\operatorname{FLIPL}(\Gamma)$  when  $\Gamma$  is a binary actions game,  $\operatorname{CHECKL}(\Gamma)$  when  $\Gamma$  is a binary game, can be computed in polynomial time.

**Proof.** We show how to compute a description in formula general form of the game CHECKL( $\Gamma$ ) when  $\Gamma$  is a binary game. So, we assume that the game is given as  $\Gamma = \langle 1^n, A_1, \ldots, A_m, 1^{\ell}, (\varphi_i)_{1 \leq i \leq n} \rangle$ , where  $A_i = \{0, 1\}$ . In the following table we recall the definition of the utility functions of the game  $\Gamma' = \text{CHECKL}(\Gamma)$ , as defined in Section 3, and reformulate its utility functions so as to describe the formula required to compute a bit of the utility value.

Recall that a profile for  $\Gamma'$  has the form  $c = (a_0, a')$  where  $a' = (a_1b_1, \ldots, a_nb_n)$ . We also set  $a = (a_1, \ldots, a_n)$  as the original strategy profile in  $\Gamma$ . The range of utility values is  $\{0, \ldots, 8\}$  and therefore we need 4 formulas, as we have 9 different utility values,  $\phi_i^j$ , for  $0 \le i \le n$  and  $0 \le j \le 3$ . In the following table we give the utility functions, according to the definitions but rewritten so to make clear the conditions for each value and the utility value written in binary.

		$u_i'(c)$	$\phi_i^3$	$\phi_i^2$	$\phi_i^1$	$\phi_i^0$		
Player 0								
		$a_0$	0	0	0	$a_0$		
	Player $i > 0$							
	$b_i = 0 \land u_i(a) = 1$	0	0	0	0	0		
$a_0 = 0 \land$	$b_i = 0 \land u_i(a) = 0$	1	0	0	0	1		
$u_i(a) \neq u_i(a_{-i}, \overline{a}_i)$	$b_i = 1 \wedge u_i(a) = 0$	2	0	0	1	0		
	$b_i = 1 \land u_i(a) = 1$	3	0	0	1	1		
$a_0 = 0 \wedge u_i(a) = u_i(a_{-i})$	$(\overline{a}_i) = 0$	4	0	1	0	0		
$a_0 = 0 \land$	$b_i = 0$	5	0	1	0	1		
$u_i(a) = u_i(a_{-i}, \overline{a}_i) = 1$	$b_i = 1$	6	0	1	1	0		
$a_0 = 1$	$b_i = 0$	7	0	1	1	1		
	$b_i = 1$	8	1	0	0	0		

Using the above table it is easy to write an expression of the boolean formulas, for each player an bit, as a disjunction of the corresponding minterms. Indeed, those formulas have a constant number of minterms and thus can be written in polynomial time.

For the remaining constructions working in a similar (but a bit more involved) way it can be shown the claimed result.  $\Box$ 

**Theorem 6.4.** The LOCALISO problem and the STRONGISO problem are polynomially equivalents for strategic games given in general form, formula general form and explicit form. The equivalence holds even for binary games.

As a consequence of the results in Theorem 2.3 and the previous theorem, we have the following result.

Theorem 6.5. The LOCALISO problem is polynomially equivalent to

- the CIRCUITISO problem, for strategic games given in general form,
- the FORMULAISO problem, for strategic games given in formula general form, and
- to the GRAPHISO problem, for strategic games given in explicit form.

The equivalence is also valid for binary games.

As a corollary of the above result we have that the LOCALISO problem belongs to  $\Sigma_2^p$  for games given in general form or formula general form and to NP for games in explicit form. In any of the three cases the problem is not expected to be hard.

## 7. Conclusions and future work

We have analyzed the general question of determining when two games are the same. As games with ordinal preferences are fundamental to analyze qualitative strategic behavior our analysis of equivalence have been based on preferences. In [7,8] we consider weak isomorphisms which preserve player's preferences. In this paper we have introduced local isomorphisms that provide a coarser classification of strategic games. Local isomorphism maintain the structure of the pure strategy profiles and therefore the set of pure Nash equilibria as well as weak isomorphism. We have shown that the complexity of the the isomorphism problem is the same as, for weak and strong isomorphism in all the representation forms considered in this paper.

The study of the equivalence between games under other different criteria (risk dominance, rationability) has also been undertaken [10] it will be of interest to analyze the computational complexity of the corresponding notions of isomorphisms.

Another direction of interest is whether it is possible to build (at least until some point) a catalog of games. For the cases in which we are just interested in the set of PNE, the problem of finding all the equivalence classes under some class of isomorphism which preserve the set of PNE. For instance [22] is a very entertaining partial answer to that question. The case of two players and 2 strategies per player has been consider under different approaches [6, 7, 11]. In [9] some preliminary results for 3 players having 2 and 2 or 3 actions each under local isomorphisms

are given. In this framework, this paper provides a notion of isomorphism with a smaller number of classes. We expect that a complete classification under local isomorphism can be obtained for a small number of players and a small number of actions, extending the results given in [9]. However, the complexity results indicate that it will be difficult to scale the number of players or the number of actions. However we feel that ever partial results are of interest in this direction.

There are several fields in computer science developing games, strategic or extensive, that can be used to attain different goals in the Semantic Web. One clear example of this direction is the games with a purpose approach [18–20]. Those games are used, for instance, to label a image, thus facilitating the acquisition of terms for the semantic web. Games, strategic or extensive, are used in this approach to learn from the strategic behavior of the players. The games are defined in such a way that term agreement provides higher utility. Observe that in this setting games designed by different research teams might lead to different definitions on the game corresponding to the labeling of the same image. To asses the validity of the final results we should check the equivalence among the games. This might lead to different notions of equivalence from the ones presented in this paper. We believe that the results on this paper will provide the basis for the analysis of the complexity of equivalence of such games and other web games.

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