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DIGITAL SEMIGROUPS

Horst Brunotte[1]

Abstract. The well-known expansion of rational integers in an arbitrary integer base different from $0, 1, -1$ is exploited to study relations between numerical monoids and certain subsemigroups of the multiplicative semigroup of nonzero integers.

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1. Introduction

Recently, Rosales *et al.* [7] investigated sets of positive integers and their relations to the number of decimal digits. More precisely, they introduced and thoroughly studied digital semigroups which are defined as follows. A digital semigroup D is a subsemigroup of the semigroup $(\mathbb{N} \setminus \{0\}, \cdot)$ such that for all $d \in D$ the set $\{n \in \mathbb{N} : \ell(n) = \ell(d)\}$ is contained in D; here $\mathbb{N}$ is the set of nonnegative rational integers and $\ell(n)$ denotes the number of digits of n in the usual decimal expansion. Among other things, the smallest digital semigroup containing a set of positive integers is determined, and for this purpose a bijective map θ between the set of digital semigroups and a certain subset $\mathcal{L}$ of numerical monoids, namely LD-semigroups, is constructed. Recall that a numerical monoid is a submonoid of $(\mathbb{N}, +)$ whose complement in $\mathbb{N}$ is finite, and an LD-semigroup S is a numerical monoid such that there exists a digital semigroup D with the property $S = \{\ell(d) : d \in D\} \cup \{0\}$. It is shown that $\mathcal{L}$ is a Frobenius variety and that the elements of $\mathcal{L}$ can be arranged in a tree. Moreover, LD-semigroups are characterized by the fact that the minimum element in each interval of nongaps belongs to the minimal set of generators. Finally, it is observed that certain combinatorial configurations introduced by Bras–Amorós and Stokes [1] are in fact LD-semigroups.

It is well-known that every positive integer can be represented in an arbitrary integer base larger than one. Expansions of integers in negative integer bases have apparently been introduced by Grünwald [3] and rediscovered by several authors; the reader is referred to Knuth [4] for more details. In view of these facts we extend the notions of digital semigroups and LD-semigroups coined by Rosales, Branco and Torrão for decimal expansions to expansions of integers in an arbitrary integer base, *i.e.*, instead of the base $b = 10$ we consider an integer base $b \neq 0, 1, -1$. Consequently, we replace the digit set $\{0, 1, \ldots, 9\}$ by the canonically chosen set $\{0, 1, \ldots, |b| - 1\}$ and simply apply the prefix b (subscript b, respectively) at appropriate places; clearly, by omitting b the original notions are recovered.

It turns out that for positive base b essentially all results coincide with the respective results presented by Rosales *et al.* [8]; however, for negative base b some modifications have to be taken. In particular, bijective maps θ_b between the set of certain b-digital semigroups and specified subsets of $\mathcal{L}$ play an important role here.

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[1] Haus-Endt-Straße 88, 40593 Düsseldorf, Germany. brunoth@web.de

2. b-digital semigroups

In this article we always let $b \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and denote by $N_b := \{0, 1, \ldots, |b| - 1\}$ the set of all nonnegative integers less than $|b|$. It is well-known that for positive b every positive integer z can uniquely be represented in the form

$$z = \sum_{i=0}^{n} u_i b^i \qquad (u_0, \ldots, u_n \in N_b,\ u_n \neq 0); \tag{2.1}$$

similarly, if b is negative then every non-zero integer z can uniquely be written in the form (2.1). Putting[2] $Z_b := \mathbb{N} \setminus \{0\}$ for $b > 0$ $(Z_b := \mathbb{Z} \setminus \{0\}$ for $b < 0$, respectively) the positive integer

$$\ell_b(z) := n + 1$$

is called the length of the representation of $z \in Z_b$ in base b, and we consistently set $\ell_b(0) := 1$. Thus, for every $z \in Z_b \cup \{0\}$ the integer $\ell_b(z)$ denotes the number of digits of the representation of z in base b. Some elementary properties of the length function are collected in the last section.

We now generalize the fundamental notion of a digital semigroup in the sense explained in the introduction. Further, for these new objects we present some examples and properties which will be used in the sequel.

Definition 2.1. A b-digital semigroup D is a subsemigroup of $(Z_b, \cdot)$ such that $\Delta_b(\ell_b(d)) \subseteq D$ for all $d \in D$. Here we introduce the notation

$$\Delta_b(n) := \{z \in Z_b \ : \ \ell_b(z) = n\} \qquad (n \in \mathbb{N} \setminus \{0\}).$$

Following [7] we let

$$L_b(A) := \{\ell_b(a) \ : \ a \in A\}$$

for the set $A \subseteq Z_b$, and we apply the commonly used abbreviation

$$\{z_1, \ldots, z_k, \rightarrow\} := \{z_1, \ldots, z_k\} \cup \{z \in \mathbb{Z} \ : \ z > z_k\}$$

for integers $z_1 < \ldots < z_k$.

Before listing some properties of b-digital semigroups we present several examples. In particular, these examples show that the analogue of ([7], Prop. 2) does not hold unrestrictedly.

Example 2.2.

(i) Let $D := \{1\}$ be the trivial subgroup of $(\mathbb{Z} \setminus \{0\}, \cdot)$. If $|b| = 2$ then D is a b-digital semigroup; however, $L_b(D)$ is not additively closed. Trivially, if $|b| > 2$ then D is not a b-digital semigroup.

(ii) The set $Z_b \setminus N_b$ is a b-digital semigroup, and $L_b(Z_b \setminus N_b) = \{2, \rightarrow\}$ is a subsemigroup of $(\mathbb{N}, +)$.

(iii) Let $b < -1$, $\ell_0 \geq 3$ and $D := \{d \in \mathbb{Z} \setminus \{0\} \ : \ \ell_b(d) \text{ odd},\ \ell_b(d) \geq \ell_0\}$. Then $D \subset \mathbb{N} \setminus \{0\}$ by Proposition 6.1 below, D is a b-digital semigroup by Lemma 6.7, but

$$L_b(D) = \{2n + 1 \ : \ n \in \mathbb{N},\ n \geq (\ell_0 - 1)/2\}$$

is not additively closed.

(iv) The set

$$D := \{z \in \mathbb{Z} \ : \ \ell_{-2}(z) \geq 3\} = \mathbb{Z} \setminus (\Delta_{-2}(1) \cup \Delta_{-2}(2) \cup \{0\})$$

is a (-2)-digital semigroup, and $L_{-2}(D) = \{3, \rightarrow\}$ is additively closed.

The essential ideas for the proof of the following statements are taken from ([7], Prop. 2).

[2] Obviously, this and some other notions in the sequel depend only on the sign of b. However, our notion facilitates subsequent formulations.

Lemma 2.3. *Let D be a b-digital semigroup.*

(i) *If $x \in L_b(D)$ and $u \in N_b \setminus \{0\}$ then $ub^{x-1} \in D$.*
(ii) *If $x, y \in L_b(D)$ then $x + y - 1 \in L_b(D)$.*
(iii) *There exist $x, y \in L_b(D)$ such that $\gcd(x, y) = 1$.*
(iv) *Let $x, y \in L_b(D)$. If $b \geq 3$ then $x + y \in L_b(D)$, and if $b \leq -3$ then $x + y + 1 \in L_b(D)$.*

Proof.

(i) By definition we have $\ell_b(ub^{x-1}) = x \in L_b(D)$, hence $ub^{x-1} \in D$.
(ii) By (i) we have b^{x-1}, $b^{y-1} \in D$, hence $b^{x+y-2} \in D$ which yields

$$x + y - 1 = \ell_b(b^{x+y-2}) \in L_b(D).$$

(iii) Pick $x \in L_b(D)$ such that $x > 0$. By (ii) we have $y := 2x - 1 \in L_b(D)$, and clearly $\gcd(x, y) = 1$.
(iv) Pick $u, v \in N_b$ such that $|b| \leq uv < 2\,|b|$. Then there exists $w \in N_b$ such that

$$uv = |b| + w.$$

By (i) we have ub^{x-1}, $vb^{y-1} \in D$, hence

$$d := (|b| + w)b^{x+y-2} = uvb^{x+y-2} = (ub^{x-1})(vb^{y-1}) \in D.$$

If $b > 0$ we deduce

$$x + y = \ell_b(b^{x+y-1}) = \ell_b(b \cdot b^{x+y-2}) = \ell_b(d) \in L_b(D),$$

and if $b < 0$ we have

$$\begin{aligned} x + y + 1 = \ell_b(b^{x+y}) &= \ell_b(b^2 \cdot b^{x+y-2}) = \ell_b(((|b| - 1)b + b^2) \cdot b^{x+y-2}) \\ &= \ell_b(d) \in L_b(D), \end{aligned}$$

since

$$|b| + w = b^2 + (|b| - 1)b + w.$$ □

Our interest concerns the structure of the set of the lengths of the b-adic representations of the elements of a b-digital semigroup.

Proposition 2.4. *Let D be a b-digital semigroup. Then $L_b(D) \cup \{0\}$ is a numerical monoid provided that one of the following conditions holds.*

(i) *$L_b(D)$ is additively closed.*
(ii) *$b \geq 3$.*
(iii) *$b = 2$ and $2 \cdot \min(L_2(D)) \in L_2(D)$.*
(iv) *For all $n, m \in \mathbb{N}$ the relation b^n, $b^m \in D$ implies $b^{n+m+1} \in D$.*

Proof. Set $S := L_b(D) \cup \{0\}$.

(i) Pick $x \in S \setminus \{0\}$. Then Lemma 2.3 yields $2x - 1 \in S$. In view of $\gcd(x, 2x - 1) = 1$ our assertion now follows from ([7], Lem. 1).
(ii) Lemma 2.3 shows that S is additively closed, and then (i) implies our assertion.
(iii) Let $n, m \in S \setminus \{0\}$.

Case 1 $\quad n = 1 \quad$ or $\quad m = 1$

Then we have $\min(S \setminus \{0\}) = 1 \in D$. By assumption this yields $\ell_2(d) = 2$ for some $d \in D$, thus $2 \in D$ and further $2^k \in D$ for all $k \in \mathbb{N}$. But then we have $S = \mathbb{N}$, and we are done.

Case 2 $n, m > 1$

In view of Proposition 6.3 we have $2^n - 1$, $2^m - 1 \in D$, thus

$$d := (2^n - 1)\,(2^m - 1) \in D.$$

We easily check

$$2^{n+m-1} \le d < 2^{n+m},$$

and we conclude

$$n + m = \ell_2(d) \in S$$

by Proposition 6.3, and again we are done by (i).

(iv) Clear by Lemma 2.3 and (i). □

3. b-LD-semigroups

In this section we adapt the notion of an LD-semigroup introduced in [7]. We characterize b-LD-semigroups and construct a correspondence between b-digital semigroups and b-LD-semigroups. Further, several examples and properties of b-LD-semigroups for negative b are listed.

Definition 3.1. Let S be a submonoid of $(\mathbb{N}, +)$. We call S a b-LD-semigroup if there exists a b-digital semigroup D such that $S = L_b(D) \cup \{0\}$.

Now we are in a position to extend ([7], Thm. 4) and provide the crucial characterization of b-LD-semigroups. For ease of notation, we put $E_b := \{-1\}$ for $b > 1$ and $E_b := \{-3, -1, 1\}$ for $b < -1$.

Theorem 3.2. *Let S be a submonoid of $(\mathbb{N}, +)$. Then the following statements are equivalent:*

(i) *S is a b-LD-semigroup.*
(ii) *$S \neq \{0\}$ and $s + t + e \in S$ for all $s, t \in S \setminus \{0, 1\}$ and $e \in E_b$.*

Proof. (i) $\implies$ (ii) Let D be a b-digital semigroup such that $S = L_b(D) \cup \{0\}$. Then we clearly have $S \neq \{0\}$. Let $s, t \in S \setminus \{0, 1\}$ and $e \in E_b$. By Lemma 6.7 there exist $a, c \in Z_b$ such that $s = \ell_b(a)$, $t = \ell_b(c)$ and $\ell_b(ac) = s + t + e$. By the properties of D we know that $a, c \in D$, thus

$$s + t + e = \ell_b(ac) \in L_b(D) \subset S.$$

(ii) $\implies$ (i) Since $S \neq \{0\}$ the set

$$D_b := \{z \in Z_b \;:\; \ell_b(z) \in S\}$$

is nonempty, and we immediately convince ourselves that $S = L_b(D_b) \cup \{0\}$. By construction we have $\Delta_b(\ell_b(d)) \subseteq D_b$ for all $d \in D_b$. Therefore we are left to show that D_b is multiplicatively closed.

Let $a, c \in D_b$, thus $s := \ell_b(a)$, $t := \ell_b(c) \in S$. If $s = 1$ or $t = 1$ then $\mathbb{N} \subseteq S$, and we are done. Therefore we may assume $s, t > 1$. If $b > 1$ then our prerequisites and Lemma 6.7 yield some $e \in \{-1, 0\}$ such that

$$\ell_b(ac) = s + t + e \in S. \tag{3.1}$$

Similarly, if $b < -1$ then there is some $e \in E_b$ such that (3.1) holds. Thus, in both cases we have shown $ac \in D_b$. □

Let us list some direct consequences of this result.

Corollary 3.3. *Let S be a b-LD-semigroup.*

(i) *If $b > 1$ then S is a c-LD-semigroup for all $c > 1$.*
(ii) *If $b < -1$ then S is a c-LD-semigroup for all $c \in \mathbb{Z} \setminus \{-1, 0, 1\}$.*

Corollary 3.4. *Every b-LD-semigroup is a numerical monoid.*

Proof. Using Theorem 3.2 the proof is analogous to ([7], Prop. 2) and left to the reader. □

Remark 3.5.

(i) Let $b > 1$ and S be a b-LD-semigroup. Then S need not be a c-LD-semigroup for $c < -1$, *e.g.*, consider $S = \{0, 4, 7, \rightarrow\}$.

(ii) Let $b < -1$, D be a b-digital semigroup, $n, m \in L_b(D)$. Then there do not exist $d, e \in D$ such that $\ell_b(d) = n,\ \ell_b(e) = m$ and $\ell_b(de) = n + m$. Indeed, if $n + m$ is even then either both n, m are odd or both n, m are even. In any case the product de is positive, hence $\ell_b(de)$ is odd (*cf*. Prop. 6.1). We similarly argue in the case $n + m$ odd.

In view of Theorem 3.2 we let

$$\mathcal{L} := \{S \text{ submonoid of } \mathbb{N} \ :\ S \neq \{0\}\,,\ s + t - 1 \in S \text{ for all } s, t \in S \setminus \{0, 1\}\}$$

be the set of all b-LD-semigroups for $b > 1$, and

$$\mathcal{L}_- := \{S \text{ submonoid of } \mathbb{N} \ :\ S \neq \{0\}\,,\ s + t - 3,\ s + t - 1,\ s + t + 1 \in S$$
$$\text{for all } s, t \in S \setminus \{0, 1\}\}$$

be the set of all b-LD-semigroups for $b < -1$. By what we have seen above, $\mathcal{L}$ coincides which the respective set in ([7], Sect. 2). Moreover, $\mathcal{L}_-$ is a proper subset of $\mathcal{L}$ (see Ex. 3.6 below), and by ([7], Prop. 12) the set $\mathcal{L}$ is a Frobenius variety which has been investigated in detail in [7]. Recall that a Frobenius variety is a nonempty set $\mathcal{V}$ of numerical semigroups with the following properties:

(i) If $S, T \in \mathcal{V}$, then $S \cap T \in \mathcal{V}$.

(ii) If $S \in \mathcal{V}$ and $S \neq \mathbb{N}$, then $S \cup \{F(S)\} \in \mathcal{V}$.

Here, for $A \subseteq \mathbb{N}$ such that $\mathrm{Card}\,(\mathbb{N} \setminus A) < \infty$ we let $F(A)$ denote the Frobenius number of A, *i.e.*, the greatest integer which does not belong to A.

In view of our remark above, we now mainly concentrate on the subset $\mathcal{L}_-$ of the Frobenius variety $\mathcal{L}$.

Some examples which also illustrate subsequent results seem appropriate. As usual, we denote by $\mathrm{msg}(S)$ the (unique) minimal set of generators of the numerical monoid S.

Example 3.6.

(i) Let $n \in \mathbb{N} \setminus \{0\}$. The LD-semigroups $S_n := \{0, n, \rightarrow\}$ appear as the left-most branch in the tree of LD-semigroups presented in ([7], Fig. 1); note that $S_n \in \mathcal{L}_-$ if and only if $n \neq 2$ since $2 + 2 - 3 \notin S_2$. Clearly, $\mathrm{msg}(S_n) = \{n, \ldots, 2n - 1\}$, and for $n \geq 2$ we have $F(S_n) = n - 1$ and $S_n \setminus \{n\} = S_{n+1} \in \mathcal{L}_-$, but $S_n \setminus \{2n - 1\} = \{0, n, \ldots, 2n - 2, 2n, \rightarrow\} \notin \mathcal{L}$.

(ii) $< 3, 5, 7 >, < 4, 5, 7 > \in \mathcal{L}_-$, but $< 4, 6, 7, 9 > \in \mathcal{L} \setminus \mathcal{L}_-$ since $4 + 4 - 3 = 5 \notin < 4, 6, 7, 9 >$.

(iii) Trivially, we have $\mathbb{N} \in \mathcal{L}_-$. By (i) we have $S := S_3 \in \mathcal{L}_-$, and we easily check $\mathrm{msg}(S) = \{3, 4, 5\}$ and $S \setminus \{5\} = \{0, 3, 4, 6, \rightarrow\} \notin \mathcal{L}$, since $3 + 3 - 1 \notin S \setminus \{5\}$. Further, we have $S \cup \{F(S)\} = S_2 \notin \mathcal{L}$. We remark in passing that $\mathcal{L}_-$ is not a Frobenius pseudo-variety (see [6] for details).

Motivated by the last example we establish the following observation.

Proposition 3.7. *$\mathcal{L}_- \setminus \{\{0, 3, \rightarrow\}\}$ is a Frobenius variety.*

Proof. The proof follows the same lines as ([7], Prop. 12). Set $S_3 := \{0, 3, \rightarrow\}$. Clearly, $\mathcal{V} := \mathcal{L}_- \setminus \{S_3\} \neq \emptyset$ since $\mathbb{N} \in \mathcal{V}$.

It is immediate that $S, T \in \mathcal{V}$ implies $S \cap T \in \mathcal{V}$. Indeed, $S \cap T \in \mathcal{L}_-$ by Theorem 3.2, and by ([7], Sect. 3) the assumption $S \cap T = S_3$ implies $S = S_3$ or $T = S_3$ which is impossible.

Now, let $S \in \mathcal{V}$ such that $S \neq \mathbb{N}$. Note that $2 \notin S$, since otherwise $1 = 2 + 2 - 3 \in S$ which we excluded. Let $e \in \{-3, -1, 1\}$ and $s, t \in S \cup \{F(S)\}$ such that $s, t > 1$. If $s, t \in S$ then certainly $s + t + e \in S$. Therefore it remains to consider the case $F(S) \in \{s, t\}$. If $F(S) = s$ then $s > 2$ because otherwise $F(S) = 2$ and $S = S_3$ which is impossible. Thus we may assume $s, t \geq 3$, hence $s + t + e \geq F(S)$, and we are done. □

Applying the ideas of ([7], Prop. 14) we can derive the following result without difficulty.

Proposition 3.8. *Let $S \in \mathcal{L}_-$ such that $3 \notin S$, and let $s \in \operatorname{msg}(S)$. Then $S \setminus \{s\} \in \mathcal{L}_-$ if and only if $s - 1,\ s + 1,\ s + 3 \in (\mathbb{N} \setminus S) \cup \operatorname{msg}(S)$.*

Proof. Note that $S \cap \{-1, 1, 3\} = \emptyset$ by our prerequisites.

Let $S \setminus \{s\} \in \mathcal{L}_-$ and assume $s + e \notin (\mathbb{N} \setminus S) \cup \operatorname{msg}(S)$ for some $e \in \{-1, 1, 3\}$. Then $s + e \in S \setminus \operatorname{msg}(S)$ and there exist $t, r \in S$ such that $s + e = t + r$. In view of

$$t + r - e = s \notin S \setminus \{s\}$$

we infer $S \setminus \{s\} \notin \mathcal{L}_-$ from Theorem 3.2: Contradiction.

Conversely, let $t, r \in S \setminus \{0, s\}$, thus in particular $t, r \neq 1$. Using Theorem 3.2 again we see $t + r - e \in S$ for each $e \in \{-1, 1, 3\}$. The assumption $t + r - e = s$ leads to $s + e = t + r \in S \setminus \{s\}$ which implies the contradiction $s + e \notin (\mathbb{N} \setminus S) \cup \operatorname{msg}(S)$. Thus we have shown $t + r \in S \setminus \{s\}$, and we are done by Theorem 3.2. □

Analogously as ([7], Cor. 15) we can formulate:

Corollary 3.9. *Let $S \in \mathcal{L}_-$ such that $3 \notin S$, and let $s \in \operatorname{msg}(S)$ with $s > F(S)$. Then $S \setminus \{s\} \in \mathcal{L}_-$ if and only if $s - 1 \in (\mathbb{N} \setminus S) \cup \operatorname{msg}(S)$ and $s + 1,\ s + 3 \in \operatorname{msg}(S)$.*

Remark 3.10. Note that we cannot renounce the assumption $3 \notin S$ in our two last results. Indeed, choose $s = 3$ and consider the semigroups $< 3, 5, 7 >$ for Proposition 3.8 and $< 3, 4, 5 >$ for Corollary 3.9.

Let $\mathcal{D}_b$ be the set of all b-digital semigroups which satisfy the condition stated in Proposition 2.4 (iii). An inspection of the proof of Theorem 3.2 immediately yields the following extensions of the respective results of ([7], Sect. 2).

Corollary 3.11. *The correspondence $\theta_b : \mathcal{L} \rightarrow \mathcal{D}_b$ given by*

$$\theta_b(S) := \{z \in Z_b \ :\ \ell_b(z) \in S\}$$

is a bijective map, and its inverse $\varphi_b : \mathcal{D}_b \rightarrow \mathcal{L}$ is defined by

$$\varphi_b(D) := L_b(D) \cup \{0\} \,.$$

Corollary 3.12. *For every $D \in \mathcal{D}_b$ the set $Z_b \setminus D$ is finite.*

Proof. By what we have seen so far we know that $S := \varphi_b(D)$ is a numerical monoid. If $b > 1$ then analogously as in the proof of ([7], Cor. 8) we show that $\left\{b^{F(S)}, \rightarrow\right\} \subseteq D$. Now, let $b < -1$ and $n \in \mathbb{N}$ be even such that $n \geq F(S)$. Then Corollary 6.4 yields $\{b^n, \rightarrow\} \subseteq D$. Moreover, $b^{2n+1} \in D$ by Proposition 2.4, hence $(-\infty,\ b^{2n-1}) \cap \mathbb{Z} \subseteq D$ by Lemma 6.5, and we are done. □

Example 3.13. We have $Z_b \in \mathcal{D}_b$, but $\mathbb{N} \setminus \{0\} \in \mathcal{D}_b$ if and only if $b > 1$.

Recall that a (v,b,r,k)-configuration is an incidence structure with v points, b lines, r lines through each point and k points on each line. Let $S_{(r,k)}$ be the set of all integers d such that there exists a $(d\cdot \frac{k}{\gcd(r,k)}, d\cdot \frac{r}{\gcd(r,k)}, r, k)$-configuration. Bras–Amorós and Stokes ([1], Thm. 2) showed that $S_{(r,k)}$ is a numerical monoid provided $r,k \geq 2$. By ([7], Introduction) $S_{(r,r)}$ is an LD-semigroup if $r \geq 2$, and this statement is slightly sharpened now.

Theorem 3.14. *If $r \geq 2$ then $S_{(r,r)}$ belongs to $\mathcal{L}_-$.*

Proof. Let $S := S_{(r,r)}$ and $s,t \in S \setminus \{0,1\}$. By ([9], Sect. 2) we know that $s+t-1, s+t+1 \in S$. Therefore, in view of Theorem 3.2 it suffices to show that $s+t-3 \in S$.

If $r=2$ then we infer $S =< 3,4,5 >$ from ([1], Cor. 1), and we easily deduce our claim. Now, let $r>2$ and m be the multiplicity of S, *i.e.*, the least positive integer belonging to S. Then we have

$$m \geq r^2 - r + 1 \geq 3$$

by ([9], Lem. 1). Since we may assume $s \geq t \geq m$ we find $s+t-3 \in S$ by ([9], Thm. 9). □

4. Generating b-digital semigroups

This section is devoted to a description of the set $\mathcal{D}_b$ which is very closely related to the respective result in [7]. Let us start with the analogue of ([7], Lem. 16) which can immediately be verified.

Lemma 4.1. *The intersection of b-digital semigroups which belong to $\mathcal{D}_b$ is a b-digital semigroup in $\mathcal{D}_b$.*

In view of this result, given $A \subseteq Z_b$ the set

$$\mathcal{D}_b(A) := \bigcap_{D \in \mathcal{D}_b,\ A \subseteq D} D$$

is the smallest element of $\mathcal{D}_b$ which contains A.

For $A \subseteq \mathbb{N} \setminus \{0\}$ we let $\mathcal{L}_b(A)$ denote the intersection of all b-LD-semigroups which contain A. Analogously as ([7], Prop. 17, Cor. 18) we write down the following result based on Theorem 3.2 and ([7], Lem. 1).

Proposition 4.2. *If $A \subseteq \mathbb{N} \setminus \{0\}$ is nonempty then $\mathcal{L}_b(A)$ is the smallest b-LD-semigroup which contains A.*

Now we straightforwardly extend ([7], Prop. 19).

Proposition 4.3. *Let $S \in \mathcal{L}$ and $A \subseteq \mathbb{N} \setminus \{0\}$ be nonempty. Then S is the smallest b-LD-semigroup containing $\mathcal{L}_b(A)$ if and only if $\theta_b(S)$ is the smallest element of $\mathcal{D}_b$ which contains A.*

Let A be a subset of the b-digital semigroup D. Following ([7], Sect. 4) we call A a $\mathcal{D}_b$-system of generators of D if $\mathcal{D}_b(A) = D$; we say that A is a minimal $\mathcal{D}_b$-system of generators of D if no proper subset of A is a $\mathcal{D}_b$-system of generators of D. Analogously as ([7], Thm. 21) we can prove the following theorem using Lemma 4.1, Corollary 3.11, Corollary 3.4 and Proposition 4.3.

Theorem 4.4. *We have*

$$\mathcal{D}_b = \{\mathcal{D}_b(A)\ :\ A \text{ finite nonempty subset of } Z_b\}.$$

5. b-LD-semigroups containing prescribed integers

In this section we treat b-LD-semigroups which contain a prescribed set of positive integers. In particular, we derive an algorithm calculating the smallest element of $\mathcal{L}_-$ which contains given positive integers. Due to the fact that E_b may contain a positive element we present a restricted b-adic version of ([7], Prop. 28).

Proposition 5.1. *Let $S \neq \mathbb{N}$ be a numerical monoid and* $\mathrm{msg}(S) = \{n_1, \ldots, n_p\}$. *Then the following statements are equivalent:*

(i) *S is a b-LD-semigroup.*
(ii) *If $e \in E_b$ and $i, j \in \{1, \ldots, p\}$ then $n_i + n_j + e \in S$.*
(iii) *If $e \in E_b$ and $s \in S \setminus \{0, n_1, \ldots, n_p\}$ then $s + e \in S$.*

Proof.

(i) $\implies$ (ii): Clear by Theorem 3.2.
(ii) $\implies$ (iii): Let $t \in S$ and $i, j \in \{1, \ldots, p\}$ such $s = n_i + n_j + t$. Then we clearly have

$$s + e = (n_i + n_j + e) + t \in S.$$

(iii) $\implies$ (i): Let $s, t \in S \setminus \{0, 1\}$. Then $s + t \in S \setminus \{0, n_1, \ldots, n_p\}$, hence $s + t + e \in S$, and we are done by Theorem 3.2. □

It does not seem obvious how ([7], Prop. 28 (iv)) can be modified for a characterization of the semigroups in $\mathcal{L}_-$. In fact, both numerical monoids $S := \langle 3, 4, 5\rangle$ and $T := \langle 4, 5, 7\rangle$ belong to $\mathcal{L}_-$ and satisfy the conditions given in Proposition 5.1 and ([7], Prop. 28 (iv)). Furthermore, we have

$$3 \in S, \quad 3 - (P(3) + 1) = 1 \notin S, \quad 3 - (P(3) - 3) = 5 \in S,$$

but

$$4 \in T, \quad 4 - (P(4) + 1) = 2 \notin T, \quad 4 - (P(4) - 3) = 6 \notin T;$$

here we set

$$P(s) := \max \{c_1 + \cdots + c_p \ : \ c_1, \ldots, c_p \in \mathbb{N} \text{ and } s = c_1 n_1 + \cdots + c_p n_p\}$$

where s is an element of the numerical monoid with minimal system of generators $\{n_1, \ldots, n_p\}$.

On the other hand, for $U := < 2, 3 > \in \mathcal{L} \setminus \mathcal{L}_-$ we have

$$2 \in U \quad \text{and} \quad 2 - (P(2) - 3),\, 2 - (P(2) - 1),\, 2 - (P(2) + 1) \ \in U.$$

Clearly, in view of Proposition 5.1, Theorem 3.2 and ([7], Prop. 28 (iv)) we can immediately formulate the following result.

Proposition 5.2. *Let S be a numerical monoid and $b > 1$. Then S is a b-LD-semigroup if and only if $s - \{0, \ldots, P(s) - 1\} \subset S$ for all $s \in S \setminus \{0\}$.*

The algorithm below computes the smallest element of $\mathcal{L}_-$ containing a given finite set of integers larger than 1. After choosing a large heuristic bound the algorithm closely follows ([7], Algorithm 32) for the determination of the smallest LD-semigroup containing a set of positive integers, and in view of our previous results the justification of its behavior is analogous to the one in ([7], Sect. 5).

Let us illustrate this algorithm by an easy example.

Example 5.3. We determine the minimal system of generators of the smallest element S of $\mathcal{L}_-$ containing 8. Our algorithm requires the following three steps:

- $B = \{8\}$, $A = B \cup \{13, 15, 17\}$.
- $B = \{8, 13, 15, 17\}$, $A = B \cup \{18, 20, 22, 27\}$.
- $B = \{8, 13, 15, 17, 18, 20, 22, 27\}$, $A = B$.

Therefore

$$S = \langle 8, 13, 15, 17, 18, 20, 22, 27\rangle = \{0, 8, 13, 15, 16, 17, 18, 20, \rightarrow\}.$$

It seems worthwile to remark that S is not an Arf numerical semigroup (see [8]), because $2 \times 16 - 13 = 19 \notin S$.

Algorithm 1. Computation of the smallest element of $\mathcal{L}_-$ containing given positive integers.

Input: Non-void finite subset $A \subset \mathbb{N} \setminus \{0,1\}$, bound $\in \mathbb{N}$.
Output: The minimal system of generators of the smallest element of $\mathcal{L}_-$ containing A or "overflow"

```
k ← 0
E ← {−3, −1, 1}
repeat
   k ← k + 1
   B ← msg(A)
   A ← B ∪ {x + y + e : x, y ∈ B, e ∈ E, x + y + e ∉ < B >}
until k > bound or B = A
if k > bound then
   return "overflow"
else
   return "Minimal system of generators:" B
end if
```

6. Auxiliary results on the lengths of b-adic representations

The considerations presented in the previous sections are based on the knowledge of some facts on the lengths of b-adic representations of integers. These facts are certainly well-known, but are collected here for the sake of completeness. First we recall a fundamental observation which is tacitly used in this paper.

Proposition 6.1 ([2], Prop. 3.1)**.** *Let $b < -1$ and $z \in \mathbb{Z}$. If $z > 0$ then $\ell_b(z)$ is odd, and if $z < 0$ then $\ell_b(z)$ is even.*

Example 6.2. Let $b < -1$ and $u \in N_b \setminus \{0\}$. Then we have $-u = b + v$ with some $v \in N_b$, thus $\ell_b(-u) = 2$. In particular, we have $-1 = b + (|b| - 1)$, hence the base b representation of $|b|$ is

$$|b| = (-1) \cdot b = b^2 + (|b| - 1)\, b,$$

and we have $\ell_b(|b|) = 3$.

Using ([5], Lem. 7) the following bounds for the length of the b-adic representation of an integer z can immediately be derived:

$$\frac{\log|z| - \log(|b| - 1)}{\log|b|} \le \ell_b(z) \le \frac{\log|z|}{\log|b|} + 4. \qquad (z \in Z_b).$$

However, our purposes require bounds which depend on the signs of the integers b and z. Note that the next result yields an explicit description of the sets $\Delta_b(n)$.

Proposition 6.3. *Let $b \in \mathbb{Z} \setminus \{-1, 0, 1\}$ and $a \in \mathbb{Z}$.*

(i) *If $b > 1$ and $a > 0$ then $\ell_b(a) = \ell$ if and only if*

$$b^{\ell-1} \le a \le b^{\ell} - 1.$$

In this case we have

$$\frac{\log a}{\log b} < \ell \le \frac{\log a}{\log b} + 1.$$

(ii) *If $b < -1$ and $a > 0$ then $\ell_b(a) = \ell$ if and only if*

$$\frac{b(b^{\ell-2} - 1)}{1 - b} \le a \le \frac{b^{\ell+1} - 1}{1 - b}.$$

In this case we have

$$\frac{\log\bigl((|b|+1)a+1\bigr)}{\log|b|}-1\le\ell\le\frac{\log\bigl((1+1/|b|)a-1\bigr)}{\log|b|}+2.$$

(iii) *If $b<-1$ and $a<0$ then $\ell_b(a)=\ell$ if and only if*

$$\frac{b(b^\ell-1)}{1-b}\le a\le\frac{b^{\ell-1}-1}{1-b}.$$

In this case we have

$$\frac{\log\bigl((1+1/|b|)\,|a|+1\bigr)}{\log|b|}\le\ell\le\frac{\log\bigl((1+|b|)\,|a|-1\bigr)}{\log|b|}+1.$$

Proof.

(i) This is well-known and easy to check.

(ii) We observe

$$a\le(|b|-1)\sum_{i=0}^{(\ell-1)/2}b^{2i}=-(b+1)\frac{b^{2((\ell-1)/2+1)}-1}{b^2-1}=-\frac{b^{\ell+1}-1}{b-1}$$

and

$$\begin{aligned}a&\ge b^{2\cdot(\ell-1)/2}+(|b|-1)\sum_{i=1}^{(\ell-1)/2}b^{2i-1}\\&=b^{\ell-1}-\frac{b+1}{b}\Bigl(\frac{b^{2((\ell-1)/2+1)}-1}{b^2-1}-1\Bigr)=\frac{b(1-b^{\ell-2})}{b-1},\end{aligned}$$

from which the estimates for ℓ are derived straightforwardly.

(iii) Noting

$$(|b|-1)\bigl(b^{2\cdot\ell/2-1}+\sum_{i=1}^{\ell/2-1}b^{2i-1}\bigr)\le a\le b^{\ell-1}+(|b|-1)\sum_{i=0}^{\ell/2-1}b^{2i}$$

we complete the proof as above. □

Corollary 6.4. *Let $b<-1$ and $a,n\in\mathbb{N}$. If n is even and $a\ge b^n$ then we have $\ell_b(a)>n$.*

Proof. Assume the contrary. Then Proposition 6.1 yields $\ell_b(a)\le n-1$, hence $n\ge2$ and we infer the impossible inequality $b^n\le(b^n-1)/(1-b)$ from the Proposition. □

Now we compare the sizes of integers to the lengths of their b-adic representation.

Lemma 6.5. *Let $a,c\in\mathbb{Z}$.*

(i) *If $0\le a<c$ then $\ell_b(a)\le\ell_b(c)$.*
(ii) *If $a,c\ge0$ and $\ell_b(a)<\ell_b(c)$ then $a<c$.*
(iii) *If $b<-1$ and $a>0$ then we have $\ell_b(-a)=\ell_b(a)+1$.*
(iv) *Let $b<-1$.*
  (a) *$a<c<0\implies\ell_b(a)\ge\ell_b(c)$.*
  (b) *$a,c\le0$ and $\ell_b(a)>\ell_b(c)\implies a<c$.*

Proof.

(i) This is well-known and easy to check.
(ii) – (iv) This is straightforwardly derived from Proposition 6.3. □

Lemma 6.6. *Let $b < -1$ and n, m be even positive integers such that $n \le m$. If*

$$\frac{b(b^m - 1)}{1 - b} \le z \le \frac{b^{n-1} - 1}{1 - b}$$

then we have

$$n \le \ell_b(z) \le m.$$

Proof. Let $y \in \mathbb{Z}$ such that $\ell_b(y) = n$ and assume $n > \ell_b(z)$. Then Proposition 6.3 and Lemma 6.5 yield

$$\frac{b(b^n - 1)}{1 - b} \le y < z \le \frac{b^{n-1} - 1}{1 - b}$$

and then $n = \ell_b(z)$: Contradiction.

The second inequality is proved analogously. □

Further, we need the length of the b-adic representation of the product of two elements.

Lemma 6.7.

(i) *Let $b > 1$ and $a, c \in \mathbb{N} \setminus \{0\}$. Then we have*

$$\ell_b(ac) = \ell_b(a) + \ell_b(c) + e \tag{6.1}$$

for some $e \in \{-1, 0\}$.

(ii) *Let $b < -1$ and $a, c \in \mathbb{Z} \setminus N_b$. Then there exists some $e \in \{-3, -1, 1\}$ such that (6.1) holds.*

(iii) *If $n, m \ge 2$ and $e \in E_b$ then there exist $a, c \in Z_b$ such that $\ell_b(a) = n,\ \ell_b(c) = m$ and (6.1) holds.*

Proof.

(i) For $b = 2$ this is immediately checked using Proposition 6.3, and for $b > 2$ the proof of ([7], Lem. 3) can easily be extended.

(ii) Set $n := \ell_b(a)$ and $m := \ell_b(c)$. Certainly it suffices to consider the subsequent cases.

Case 1 $a > 0$

Then n is odd and we infer

$$\frac{b(b^{n-2} - 1)}{1 - b} \le a \le \frac{b^{n+1} - 1}{1 - b}$$

from Proposition 6.3.

Case 1.1 $c > 0$

Then m is odd and as above we have

$$\frac{b(b^{m-2} - 1)}{1 - b} \le c \le \frac{b^{m+1} - 1}{1 - b}.$$

Now we easily verify

$$\begin{aligned} \frac{b(b^{n+m-5} - 1)}{1 - b} &\le \frac{b^2(b^{n+m-4} - b^{n-2} - b^{m-2} + 1)}{(1 - b)^2} \le ac \\ &\le \frac{b^{n+m+2} - b^{n+1} - b^{m+1} + 1}{(1 - b)^2} \le \frac{b^{n+m+2} - 1}{1 - b}. \end{aligned}$$

Then Proposition 6.3 yields

$$n+m-3 \leq \ell_b(ac) \leq n+m+1,$$

and our assertion follows from Proposition 6.1.

Case 1.2 $c<0$

As above we verify

$$\frac{b(b^{n+m+1}-1)}{1-b} \leq \frac{b(b^{n+m+1}-b^{n+1}-b^m+1)}{(1-b)^2} \leq ac$$
$$\leq \frac{b^{n+m-3}-b^{n-2}-b^{m-1}+1}{(1-b)^2} \leq \frac{b^{n+m-4}-1}{1-b}$$

keeping in mind that m is even, and then we conclude using Lemma 6.6.

Case 2 $a<0$

We may suppose $c<0$ and proceed as in Case 1.1.

(iii) The case $b>1$ is well-known. Now, let $b<-1$. For the positive integers

$$a = \frac{b^{2n}-1}{1-b} \quad \text{and} \quad c = \frac{b^{2m}-1}{1-b}$$

we have

$$\ell_b(ac) = \ell_b(a) + \ell_b(c) + 1.$$

Similarly, for the negative integers

$$a = \frac{b^{2n-1}-1}{1-b} \quad \text{and} \quad c = \frac{b(b^{2m}-1)}{1-b}$$

we verify

$$\ell_b(ac) = \ell_b(a) + \ell_b(c) - 1,$$

and for

$$a = \frac{b(b^{2n-3}-1)}{1-b} \quad \text{and} \quad c = \frac{b(b^{2m-1}-1)}{1-b}$$

we see

$$\ell_b(ac) = \ell_b(a) + \ell_b(c) - 3. \qquad \square$$

We close this section by an easy application of Proposition 6.3 the details of which we leave to the reader (cf. the special case $b=10$ in [7], proof of Cor. 9).

Proposition 6.8. *For $n \in \mathbb{N} \setminus \{0\}$ we have*

$$\operatorname{Card}(\Delta_b(n)) = \begin{cases} (b-1)\, b^{n-1} & (b>1), \\ -(b+1)\, b^{n-1} & (b<-1,\ n \ \textit{odd}), \\ (b+1)\, b^{n-1} & (b<-1,\ n \ \textit{even}). \end{cases}$$

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