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RELATIONSHIPS BETWEEN VERTEX ATTACK TOLERANCE AND OTHER VULNERABILITY PARAMETERS

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Abstract. Let $G(V, E)$ be a simple undirected graph. Recently, the vertex attack tolerance (VAT) of G has been defined as $\tau(G) = \min\left\{\frac{|S|}{|V-S-C_{\max}(G-S)|+1} : S \subset V\right\}$, where $C_{\max}(G-S)$ is the order of a largest connected component in $G-S$. This parameter has been used to measure the vulnerability of networks. The vertex attack tolerance is the only measure that fully captures both the major bottlenecks of a network and the resulting component size distribution upon targeted node attacks. In this article, the relationships between the vertex attack tolerance and some other vulnerability parameters, namely connectivity, toughness, integrity, scattering number, tenacity, binding number and rupture degree have been determined.

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1. Introduction

In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. It is known that communication systems are often exposed to failures and attacks. Then the robustness of network topology is a key aspect in the design of computer networks [10, 13]. A network is described as an undirected and unweighted graph in which vertices represent the processing and edges represent the communication channel between them. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. In the literature, various measures have been defined to measure the robustness of networks, and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability [10, 13].

Let $G = (V, E)$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V$, the open neighborhood of v is $N_G(v) = \{u \in V \mid \ uv \in E\}$ and closed neighborhood of v is $N_G[v] = N_G(v) \cup \{v\}$. The degree of vertex v in G denoted by $d_G(v)$ is the size of its open neighborhood, and the minimum degree of G is denoted by $\delta(G)$ [6]. The components of a graph G are its maximal connected subgraphs. A vertex cut of a graph G is a set $S \subset V$ such that $G - S$ has more than one

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component [6]. A set $C \subseteq V$ is a covering set that contains at least one endpoint of every edge. The minimum cardinality taken over all covering sets of G is called the covering number of G and is denoted by $\alpha(G)$ [6].

Graph vulnerability relates to the study of a graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how a deletion of one or more network elements changes properties of the network. The best known measure of reliability of a graph is its connectivity. The vertex connectivity is defined to be the minimum number of vertices whose deletion results in a disconnected or trivial graph [8]. Then many graph theoretic parameters have been introduced for the measurement of vulnerability. Some of them have been toughness [3], integrity [1], binding number [7], scattering number [9], tenacity [2] and rupture degree [14]. Although they were developed independently over many years, these measures are similar in that they combine, in various ways, an attack set, the graph's largest connected component, and the number of connected components. These parameters have been defined for any graph G as follows:

(i) *Connectivity*: $k(G) = \min\{|S| : S \subset V \textit{ and } w(G-S) > 1\}$;
(ii) *Toughness*: $t(G) = \min\{\frac{|S|}{w(G-S)} : S \subset V \textit{ and } w(G-S) > 1\}$;
(iii) *Binding number*: $b(G) = \min\{\frac{|N_G(S)|}{|S|} : S \subseteq V\}$;
(iv) *Integrity*: $I(G) = \min\{|S| + C_{\max}(G-S) : S \subset V\}$;
(v) *Tenacity*: $T(G) = \min\{\frac{|S|+C_{\max}(G-S)}{w(G-S)} : S \subset V \textit{ and } w(G-S) > 1\}$;
(vi) *Scattering number*: $s(G) = \max\{w(G-S) - |S| : S \subset V \textit{ and } w(G-S) > 1\}$;
(vii) *Rupture degree*: $r(G) = \max\{w(G-S) - |S| - C_{\max}(G-S) : S \subset V \textit{ and } w(G-S) > 1\}$,

where $C_{\max}(G-S)$ is the order of a largest connected component in $G-S$ and $w(G-S)$ is the number of components of $G-S$, respectively.

A new measure of robustness called vertex attack tolerance has been proposed by Ercal and Matta [5]. VAT is a vertex-based measure of robustness. The VAT of a graph G, denoted by $\tau(G)$, is defined as:

$$\tau(G) = \min\left\{\frac{|S|}{|V - S - C_{\max}(G-S)| + 1} : S \subset V\right\},$$

where $C_{\max}(G-S)$ is the order of a largest connected component in $G-S$.

The definition of VAT represents a worst case scenario of the proportionally smallest number of vertices that must be attacked in order to disconnect the largest number of vertices from the network (in particular, those disconnected from the remaining largest connected component) [12]. Furthermore VAT is meant specially as a measure of a graph's vulnerability to the removal of vertices, and can be seen as minimizing the adversarial relationship between the size of an attack set and the resulting portion of a network that can continue to communicate. In [11], Matta *et al.* have showed VAT is a more sensitive vulnerability parameter than some other parameters, namely toughness, integrity, scattering number and tenacity for the star graph, the barbell graph, the wheel graph, the HOTnet graph, the big barbell graph and the PLOD graph. Furthermore, Ercal has studied the VAT of regular graphs in [4].

Our aim in this paper is to study a new vulnerability parameter that has been defined recently, so-called vertex attack tolerance. In the Section 2, well-known basic results are given for VAT. In the Section 3, experimental comparisons of graph vulnerability measures are given. In the Section 4, we determine upper and lower bounds of the relationships between the vertex attack tolerance and some other vulnerability parameters.

1.1. Basic results

Theorem 1.1. [5] *If $S_{1,n-1}$ is a star graph of order n, then $\tau(S_{1,n-1}) = \frac{1}{n-1}$*

Theorem 1.2. [5] *For any connected, undirected graph G on $n \geqslant 3$, $\tau(G) \geqslant \frac{1}{n-1}$.*

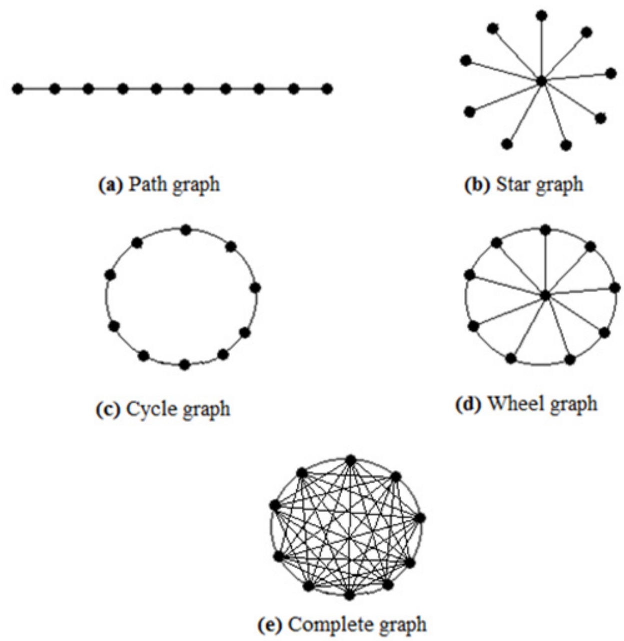

FIGURE 1. The graphs P_{10}, $S_{1,9}$, C_{10}, $W_{1,9}$ and K_{10}used in comparions.

TABLE 1. Comparisons of measures on 5 graphs.

Graph Types	$\tau(G)$	$k(G)$	$t(G)$	$b(G)$	$I(G)$	$T(G)$	$s(G)$	$r(G)$
$G \cong P_{10}$	0.2	1	0.5	1	5	1.2	1	-1
$G \cong C_{10}$	0.4	2	1	1	6	1.2	0	-1
$G \cong S_{1,9}$	0.11	1	0.11	0.11	2	0.22	8	-1
$G \cong W_{1,9}$	0.75	3	1.25	1.2	6	1.75	0	-3
$G \cong K_{10}$	1	9	∞	9	10	∞	∞	-9

1.2. Experimental comparisons of graph vulnerability measures

In this section, experimental computations are handled for graph vulnerability measures vertex attack tolerance, connectivity, toughness, binding number, integrity, tenacity, scattering number and rupture degree, and the results are compared and discussed in detail. Let P_n, C_n, $S_{1,n-1}$, $W_{1,n-1}$, and K_n be the path, cycle, star, wheel, and the complete graph of order n, respectively. These specific types of graphs are shown in Figure 1, where the path, cycle, star, wheel and complete graphs are of order 10. The experimental computation results of the vulnerability notions for these graph types are presented in Table 1.

It is certainly meaningful to compare graphs with each other of similar sizes. In the context of edge attacks, such a disconnection would be most significant because each vertex removal can disconnect one new component at most. In the context of vertex attacks it is not severe as a situation in which many small components result. Nonetheless, amongst the set of attacks that result in exactly two remaning components, the worst case situation is for the components to be of similar sizes with each other, as that situation maximizes the disconnected pairs [5].

Since the scattering number and rupture degree give the maximum cost to disrupt the network, we will leave them out of comparison. They will not be a matter of further discussion in relation to this paper to discuss.

Connectivity parameter is the best known graph vulnerability parameter. This parameter, however, deals only with the fact that the graph is fragmented or not when graph's vertices and edges are attacked. At the end of the attack, if the graph is fragmented, we can talk about the connectivity value of the graph. This process which can be done with minimum number of vertices or edges, is called connectivity. Connectivity, like the other parameters in Table 1, is not interested in the number of components of the remaining structure after fragmentation, or the number of vertices of the largest component. So it is better to interpret this parameter alone. Now let's look at the connectivity values for the graphs discussed in Table 1. According to the results, wheel graph is a very well connected graph for its size than the path, cycle and star graphs, since we have $k(W_{1,n-1}) > k(C_n) > k(P_n) = k(S_{1,n-1})$. If attacking half of the vertices of a graph is necessary to cause a serious disruption, then that graph is very robust. Consequently, a complete graph has the most robust structure depending on the connectivity measure values.

The remaining parameters are toughness, binding number, integrity, tenacity and VAT in Table 1. If the values of these parameters are examined for the graph being considered, it is seen that these values are equal to each other for some graphs. For example, the robustness values of path and cycle for binding number, the robustness values of path and cycle for tenacity and the robustness values of cycle and wheel for integrity are equal to each other. The VAT parameter informs us which of these structures is more robust. We say that cycle is stronger than path, since $\tau(C_n) > \tau(P_n)$, and wheel is stronger than cycle, since $\tau(W_{1,n-1}) > \tau(C_n)$.

According to Table 1, we can say that the values of VAT and toughness are very close to each other. While the toughness parameter does not give us any information for the complete graph, the VAT parameter can give. Furthermore, we get two results from Table 1. The first of these is related to parameters and the other is related to graph structures. These results are given as follows:

- As a result of comparing the parameters, we say that VAT is a more precise measurement. It is shown that all measures except for VAT in Table 1 do not distinguish between considered graphs, while VAT does, and it is concluded that the VAT is more sensitive measure for vulnerability than the others. Since VAT is considered to be a reasonable measure for the vulnerability of graphs, it is of particular interest to evaluate the VAT of different classes of graphs. Note that VAT is a normalized measure in $(0, 1]$ [4]. Structures with VAT value closer to 1 are more robust.
- We say that the complete graph is the most robust from the structures studied.

1.3. Upper and lower bounds

In this section, we consider the relationships between the vertex attack tolerance and the connectivity, toughness, binding number, integrity, rupture degree and scattering number.

Theorem 1.3. *Let G be a connected graph of order n such that $\tau(G) = \tau$ and $k(G) = k$. Then $\tau \geqslant \dfrac{k}{2n}$.*

Proof. Let S be a vertex cut of G. Then, from the definition of the absolute value, we know that

$$|n - (|S| + C_{\max}(G - S))| \leqslant |n| + ||S| + C_{\max}(G - S)|$$

Since the values of the n and $|S| + C_{\max}(G - S)$ are always positive, we have

$$|n - (|S| + C_{\max}(G - S))| \leqslant n + |S| + C_{\max}(G - S)$$

However, it is not difficult to see that

$$|S| + C_{\max}(G - S) \leqslant n + 1 - \omega(G - S).$$

Therefore,

$$\begin{aligned} |n - (|S| + C_{\max}(G - S))| &\leqslant n + n + 1 - \omega(G - S) \\ &\leqslant 2n + 1 - \omega(G - S). \end{aligned}$$

Since $\omega(G-S) \geqslant 2$, we get
$$|n-(|S|+C_{\max}(G-S))| \leqslant 2n+1-2.$$
When we collect both sides of the inequality by positive integer, inequality does not change direction.
$$\begin{aligned}|n-(|S|+C_{\max}(G-S))|+1 &\leqslant 2n-1+1\\ &\leqslant 2n.\end{aligned}$$
It is not difficult to see that
$$\frac{1}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{1}{2n}.$$
From this equality, we obtain that
$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{|S|}{2n}.$$
Since $|S| \geqslant k$, we have
$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{k}{2n}.$$
Since S is an arbitrary vertex cut of G, it follows from the definition of $\tau(G)$ that $\tau \geqslant \dfrac{k}{2n}$.

Therefore, we have $\tau \geqslant \dfrac{k}{2n}$. □

Theorem 1.4. *Let G be a connected graph of order n such that $\tau(G)=\tau$ and $t(G)=t$. Then $\tau \leqslant t$.*

Proof. The proof is very similar to Theorem 1.3. Clearly, we know
$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \leqslant \frac{|S|}{\omega(G-S)}.$$
From the definition of $t(G)$, we know that $|S| \geqslant t\omega(G-S)$. By combining these two inequalities, we obtain
$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \leqslant t.$$
By the definition of $\tau(G)$, we have $\tau \leqslant t$. □

Theorem 1.5. *Let G be a connected graph of order n such that $\alpha(G)=\alpha$, $k(G)=k$, $r(G)=r$, $s(G)=s$ and $\tau(G)=\tau$. Then $\tau \geqslant \dfrac{2k}{3n+s-2r+2}$.*

Proof. Let S be a vertex cut of G. From the definition of $r(G)$, we know that $r \geqslant \omega(G-S)-|S|-C_{\max}(G-S)$. Similarly, from the definition of $s(G)$, we have $\omega(G-S)-|S| \leqslant s$. At the same, combining this with the fact that $\omega(G-S)+|S| \leqslant n$ we have that $\omega(G-S) \leqslant \dfrac{n+s}{2}$.

Since $|S| \leqslant \alpha$, we have
$$\omega(G-S)+|S| \leqslant \frac{n+s}{2}+\alpha$$
and since $r \geqslant \omega(G-S)-|S|-C_{\max}(G-S)$, it implies
$$\omega(G-S)-C_{\max}(G-S) \leqslant r+|S|$$
$$\omega(G-S)-C_{\max}(G-S) \leqslant r+\alpha.$$

Removing this form the last inequality side by side, we obtain

$$|S| + C_{\max}(G-S) \leqslant \frac{n+s}{2} - r.$$

On the other hand, from the definition of absolute value, we know that

$$|n - (|S| + C_{\max}(G-S))| \leqslant |n| + |(|S| + C_{\max}(G-S))|,$$

where both n and $(|S| + C_{\max}(G-S))$ are positive integers. Therefore, we get

$$|n - (|S| + C_{\max}(G-S))| \leqslant n + |S| + C_{\max}(G-S).$$

Combining this with the $|S| + C_{\max}(G-S) \leqslant \frac{n+s}{2} - r$, we get

$$|n - (|S| + C_{\max}(G-S))| \leqslant n + \frac{n+s}{2} - r.$$

It is easy to see that

$$|n - (|S| + C_{\max}(G-S))| + 1 \leqslant n + \frac{n+s}{2} - r + 1$$

$$|n - (|S| + C_{\max}(G-S))| + 1 \leqslant \frac{3n+s-2r+2}{2}$$

and

$$\frac{1}{|n - (|S| + C_{\max}(G-S))| + 1} \geqslant \frac{2}{3n+s-2r+2}.$$

Since S is an arbitrary vertex cut of G, we obtain

$$\frac{|S|}{|n - (|S| + C_{\max}(G-S))| + 1} \geqslant \frac{2|S|}{3n+s-2r+2}.$$

For every $S \subset V(G)$ and $|S| \geqslant k$, we obtain

$$\frac{|S|}{|n - (|S| + C_{\max}(G-S))| + 1} \geqslant \frac{2k}{3n+s-2r+2}.$$

It follows from the definition of $\tau(G)$ that $\tau \geqslant \frac{2k}{3n+s-2r+2}$. □

Theorem 1.6. *Let G be a connected graph of order n such that $\tau(G) = \tau, r(G) = r$, $I(G) = I$ and $\alpha(G) = \alpha$. Then $\tau \leqslant \frac{\alpha}{r+I}$.*

Proof. Let S be a vertex cut of G. Then, from the definition of absolute value, we know that

$$|n - (|S| + C_{\max}(G-S))| \geqslant |n| - |(|S| + C_{\max}(G-S))|$$

where n and $(|S| + C_{\max}(G-S))$ are positive integer. It is easy to check that

$$|n - (|S| + C_{\max}(G-S))| \geqslant n - (|S| + C_{\max}(G-S))$$

$$|n - (|S| + C_{\max}(G-S))| + 1 \geqslant n + 1 - (|S| + C_{\max}(G-S)).$$

From the definition of $I(G)$, we know that $|S| + C_{\max}(G-S) \geqslant I$. Therefore, we have

$$\omega(G-S) \leqslant n + 1 - I.$$

Hence, we obtain

$$\begin{aligned}|n-(|S|+C_{\max}(G-S))|+1 &\geqslant \omega(G-S)+I-(|S|+C_{\max}(G-S))\\ &\geqslant \omega(G-S)-(|S|+C_{\max}(G-S))+I.\end{aligned}$$

The maximum possible value for $|n-(|S|+C_{\max}(G-S))|+1$ gives the minimum possible value for $\tau(G)$. Hence, we have

$$\begin{aligned}\max\{|n-(|S|+C_{\max}(G-S))|+1\} &\geqslant \max\{\omega(G-S)-(|S|+C_{\max}(G-S))\}+\max\{I\}\\ &\geqslant r+I.\end{aligned}$$

It is easy to see that

$$\frac{1}{\max\{|n-(|S|+C_{\max}(G-S))|+1\}} \leqslant \frac{1}{r+I}.$$

Since S is an arbitrary vertex cut of G, we obtain

$$\frac{|S|}{\max\{|n-(|S|+C_{\max}(G-S))|+1\}} \leqslant \frac{|S|}{r+I}.$$

For every $S \subset V(G)$, $|S| \leqslant \alpha$. Then it is easy to see that

$$\frac{|S|}{\max\{|n-(|S|+C_{\max}(G-S))|+1\}} \leqslant \frac{\alpha}{r+I}.$$

It follows from the definition of $\tau(G)$ that $\tau \leqslant \dfrac{\alpha}{r+I}$. □

Theorem 1.7. *Let G be a connected graph of order n such that $\tau(G)=\tau, k(G)=k,\ \alpha(G)=\alpha$, $\delta(G)=\delta$ and $b(G)=b$. Then $\tau \leqslant \dfrac{\alpha}{(n-\delta)b-(k-1)}$.*

Proof. Let S be a vertex cut of G. We know the facts that $|S| \geqslant k$ and $b \leqslant \dfrac{n-1}{n-\delta}$.

It is easy to check that,

$$\begin{aligned} b &\leqslant \frac{n-1}{n-\delta}\\ (n-\delta)b &\leqslant n-1\\ n &\geqslant (n-\delta)b+1.\end{aligned}$$

Since $C_{\max}(G-S) \geqslant 1$, we have

$$\begin{aligned}|S|+C_{\max}(G-S) &\geqslant k+C_{\max}(G-S)\\ |S|+C_{\max}(G-S) &\geqslant k+1.\end{aligned}$$

Removing this from the last equality side by side, we obtain

$$\begin{aligned}n-(|S|+C_{\max}(G-S)) &\geqslant (n-\delta)\,b+1-k-1\\ n-(|S|+C_{\max}(G-S)) &\geqslant (n-\delta)\,b-k.\end{aligned}$$

Then from the definition of absolute value, we know that

$$|n-(|S|+C_{\max}(G-S))| \ \geqslant\ |n|-\ ||S|+C_{\max}(G-S)|\,,$$

where n and $(|S|+C_{\max}(G-S))$ are positive integers. Hence,

$$|n-(|S|+C_{\max}(G-S))| \geqslant n-(|S|+C_{\max}(G-S)).$$

Combining this with the last inequality, we have

$$|n-(|S|+C_{\max}(G-S))| \geqslant (n-\delta)b-k.$$

It is easy to see that

$$\begin{aligned} |n-(|S|+C_{\max}(G-S))| +1 &\geqslant (n-\delta)b-k+1 \\ |n-(|S|+C_{\max}(G-S))| +1 &\geqslant (n-\delta)b-(k-1) \end{aligned}$$

and

$$\frac{1}{|n-(|S|+C_{\max}(G-S))| +1} \leqslant \frac{1}{(n-\delta)b-(k-1)}.$$

Since S is an arbitrary vertex cut of G, we obtain

$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))| +1} \leqslant \frac{|S|}{(n-\delta)b-(k-1)}.$$

For every $S \subset V(G)$ and $|S| \leqslant \alpha$, we have

$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))| +1} \leqslant \frac{\alpha}{(n-\delta)b-(k-1)}.$$

It follows from the definition of $\tau(G)$ that $\tau \leqslant \dfrac{\alpha}{(n-\delta)b-(k-1)}$. □

Connectivity and binding number are computable in polynomial time. But, covering number is not computable in polynomial time. Thus, new relationship that does not involve covering number may be useful. Using the inequality $\alpha(G) \leqslant n-1$ (see [6]), where n is order of given graph G, we get the following Result 1.8. It is clear that we have upper bound which is computable in polynomial time.

Result 1.8. *Let G be a connected graph of order n such that $\tau(G)=\tau, k(G)=k, \delta(G)=\delta$ and $b(G)=b$. Then $\tau \leqslant \frac{n-1}{(n-\delta)b-(k-1)}$.*

Theorem 1.9. *Let G be a connected graph of order n such that $\tau(G)=\tau, k(G)=k$ and $T(G)=T$. Then $\tau \geqslant \dfrac{(T+1)k}{2T(n-1)+3(n+1)}$.*

Proof. Let S be a vertex cut of G. We know the facts that $|n-(|S|+C_{\max}(G-S))| \leqslant |n|+||S|+C_{\max}(G-S)|$ and $|S|+C_{\max}(G-S) \leqslant n+1-\omega(G-S)$. Combining these two inequalities, we obtain

$$|n-(|S|+C_{\max}(G-S))| \leqslant 2n+1-\omega(G-S).$$

It is easy to check that

$$\begin{aligned} |n-(|S|+C_{\max}(G-S))| &\leqslant 2n+1-\omega(G-S)+\omega(G-S)-\omega(G-S) \\ &= 2n+1+\omega(G-S)-2\omega(G-S) \end{aligned}$$

and

$$|n-(|S|+C_{\max}(G-S))|+1 \leqslant 2n+2+\omega(G-S)-2\omega(G-S).$$

Then from the definition of $T(G)$, we know that $T \leqslant \frac{|S|+C_{\max}(G-S)}{\omega(G-S)}$. Combining this with the fact that $|S| + C_{\max}(G-S) \leqslant n+1-\omega(G-S)$, we obtain $\omega \leqslant \frac{n+1}{T+1}$. And also we know that $\omega(G-S) \geqslant 2$.

Hence, we have

$$\begin{aligned}|n-(|S|+C_{\max}(G-S))|+1 &\leqslant 2n+2+\frac{n+1}{T+1}-4\\ &= 2(n-1)+\frac{n+1}{T+1}\end{aligned}$$

and

$$\frac{1}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{T+1}{2T(n-1)+3(n+1)}.$$

Since S is an arbitrary vertex cut of G, we obtain

$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{(T+1)\,|S|}{2T(n-1)+3(n+1)}.$$

For every $S \subset V(G)$ and $|S| \geqslant k$, we have

$$\frac{|S|}{|n-(|S|+C_{\max}(G-S))|+1} \geqslant \frac{(T+1)k}{2T(n-1)+3(n+1)}.$$

It follows from the definition of $\tau(G)$ that $\tau \geqslant \dfrac{(T+1)k}{2T(n-1)+3(n+1)}$. □

Theorem 1.10. *Let G be a connected graph of order n such that $\tau(G)=\tau$, $r(G)=r$ and $\alpha(G)=\alpha$ Then $\tau \leqslant \dfrac{\alpha}{r+\alpha+1}$.*

Proof. Let S be a vertex cut of G. We know the facts that

$$|n-(|S|+C_{\max}(G-S))| \geqslant |n|-||S|+C_{\max}(G-S)|$$

and $\omega(G-S)+|S| \leqslant n$. Combining these two inequalities, we obtain

$$|n-(|S|+C_{\max}(G-S))| \geqslant \omega(G-S)-C_{\max}(G-S)$$

and

$$\begin{aligned}|n-(|S|+C_{\max}(G-S))|+1 &\geqslant \omega(G-S)-C_{\max}(G-S)+1\\ &\geqslant \omega(G-S)-C_{\max}(G-S)+|S|-|S|+1.\end{aligned}$$

The maximum possible value for $|n-(|S|+C_{\max}(G-S))|+1$ gives the minimum possible value for $\tau(G)$ and $|S| \leqslant \alpha$. Hence, we have

$$\max\{|n-(|S|+C_{\max}(G-S))|+1\} \geqslant \max\{\omega(G-S)-C_{\max}(G-S)-|S|\}+\max\{\alpha+1\} = r+\alpha+1.$$

It is easy to see that

$$\frac{1}{\max\{|n-(|S|+C_{\max}(G-S))|+1\}} \leqslant \frac{1}{r+\alpha+1}.$$

Since S is an arbitrary vertex cut of G and $|S| \leqslant \alpha$, we get

$$\frac{|S|}{\max\{|n-(|S|+C_{\max}(G-S))|+1\}} \leqslant \frac{\alpha}{r+\alpha+1}.$$

It follows from the definition of $\tau(G)$ that $\tau \leqslant \dfrac{\alpha}{r+\alpha+1}$. □

Theorem 1.11. *Let G be a noncomplete connected graph of order n such that $\tau(G) = \tau, I(G) = I$ and $k(G) = k$. Then* $\tau \geqslant \dfrac{k}{n - I + 1}$.

Proof. Let S be a vertex cut of G. Using the fact $|S| + C_{\max}(G - S) \leqslant n + 1 - \omega(G - S)$, we have

$$\begin{aligned} n - (|S| + C_{\max}(G - S)) &\geqslant n - n - 1 + \omega(G - S) \\ &\geqslant \omega(G - S) - 1. \end{aligned}$$

Since $\omega(G - S) \geqslant 2$, we obtain

$$n - (|S| + C_{\max}(G - S)) \geqslant 1.$$

This implies that $n - (|S| + C_{\max}(G - S))$ is positive. Then from the definition of $I(G)$ we know that $|S| + C_{\max}(G - S) \geqslant I$. It is easy to see that

$$n - (|S| + C_{\max}(G - S)) \leqslant n - I$$

where $n - I$ is positive. When we get the absolute value at both sides of the inequality, we obtain

$$\begin{aligned} |n - (|S| + C_{\max}(G - S))| &\leqslant |n - I| \\ &\leqslant n - I \end{aligned}$$

and

$$|n - (|S| + C_{\max}(G - S))| + 1 \leqslant n - I + 1$$

$$\frac{1}{|n - (|S| + C_{\max}(G - S))| + 1} \geqslant \frac{1}{n - I + 1}.$$

Since S is an arbitrary vertex cut of G, we have

$$\frac{|S|}{|n - (|S| + C_{\max}(G - S))| + 1} \geqslant \frac{|S|}{n - I + 1}.$$

Since $|S| \geqslant k$, we get

$$\frac{|S|}{|n - (|S| + C_{\max}(G - S))| + 1} \geqslant \frac{k}{n - I + 1}.$$

It follows from the definition of $\tau(G)$ that $\tau \geqslant \dfrac{k}{n - I + 1}$. □

2. Conclusion

The vulnerability of a communication can be measured by the vertex attack tolerance (VAT) of the graph describing the network. The vertex attack tolerance is the only measure that fully captures both the major bottlenecks of a network and the resulting component size distribution upon targeted node attacks. In this article, we have established the relationships between the vertex attack tolerance and some other vulnerability parameters, *i.e.* the connectivity, toughness, binding, integrity, tenacity, scattering number and rupture degree.

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