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THE CONNECTIVITY AND NATURE DIAGNOSABILITY OF EXPANDED k-ARY n-CUBES *

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Abstract. Connectivity and Diagnosability play an important role in measuring the fault tolerance of interconnection networks. As a topology structure of interconnection networks, the expanded k-ary n-cube XQ_n^k has many good properties. In this paper, we prove that (1) the connectivity of XQ_n^k is $4n$; (2) the nature connectivity of XQ_n^k is $8n-4$; (3) the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n-3$ for $n \geq 2$.

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1. Introduction

Many multiprocessor systems have interconnection networks (networks for short) as underlying topologies and a network is usually represented by a graph where nodes represent processors and links represent communication links between processors. We use graphs and networks interchangeably. For the system, study of the topological properties of its network is important. Furthermore, some processors may fail in the system, so processor fault identification plays an important role for reliable computing. The first step to deal with faults is to identify the faulty processors from the fault-free ones. The identification process is called the diagnosis of the system. A system G is said to be t-diagnosable if all faulty processors can be identified without replacement, provided that the number of faults presented does not exceed t. The diagnosability $t(G)$ of G is the maximum value of t such that G is t-diagnosable [6, 8, 12]. For a t-diagnosable system, Dahbura and Masson [6] proposed an algorithm with time complex $O(n^{2.5})$, which can effectively identify the set of faulty processors.

Several diagnosis models (*e.g.*, Preparata, Metze, and Chien's (PMC) model [18], Barsi, Grandoni, and Maestrini's (BGM) model [2], and Maeng and Malek's (MM) model [14]) have been proposed to investigate the diagnosability of multiprocessor systems. In particular, two of the proposed models, the PMC model and MM model, are well known and widely used. In the PMC model, the diagnosis of the system is achieved through two linked processors testing each other. In the MM model, to diagnose a system, a node sends the same task to two of its neighbor vertices, and then compares their responses. For this reason, the MM model is also said to be the comparison model. Sengupta and Dahbura [6] proposed a special case of the MM model, called the MM* model, in which each node must test its any pair of adjacent nodes. Numerous studies have been investigated under the PMC model and MM model or MM* model, see [5, 8, 12, 13, 16, 26].

In the traditional measurement of a system-level diagnosability for the multiprocessor system, one generally assumes that any subset of processors may simultaneously fail. If all the neighbor vertices of some node v

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are faulty simultaneously, it is impossible to determine whether v is faulty or fault-free. As a consequence, the diagnosability of a system is less than its minimum node degree. However, in a large-scale multiprocessor system, we can safely assume that all neighbor vertices of any node do not fail at the same time. Based on this assumption, Lai *et al.* [12] introduced the restricted diagnosability of the multiprocessor system called the conditional diagnosability of the system. They consider the situation that any fault set cannot contain all the neighbor vertices of any vertex in a system. Since the probability that the all neighbors of a fault node fail and create faults is more to the probability that the all neighbors of a fault-free node fail and create faults in the system, we consider the situation that no fault set can contain all the neighbors of any fault-free node in the system, which is called the nature diagnosability of the system. In 2012, Peng *et al.* [16] proposed a measure for fault diagnosis of the system, namely, the g-good-neighbor diagnosability of the system (which is also called the g-good-neighbor conditional diagnosability), which requires that every fault-free node contains at least g fault-free neighbors. In [16], they studied the g-good-neighbor diagnosability of the n-dimensional hypercube under the PMC model. In [21], Wang and Han studied the g-good-neighbor diagnosability of the n-dimensional hypercube under MM* model. Yuan *et al.* [26,27] studied that the g-good-neighbor diagnosability of the k-ary n-cube ($k \geq 3$) under the PMC model and MM* model. The Cayley graph $C\varGamma_n$ generated by the transposition tree $\varGamma_n$ has recently received considerable attention. In [19, 20], Wang *et al.* studied the g-good-neighbor diagnosability of $C\varGamma_n$ under the PMC model and MM* model for $g = 1, 2$. In [19], Wang *et al.* proved that the nature diagnosability of the system is less than or equal to the conditional diagnosability of the system. Therefore, the nature diagnosability of the system is nature and one important study topic. The n-dimensional bubble-sort star graph BS_n has many good properties. In 2016, Wang *et al.* [23] studied the 2-good-neighbor connectivity and 2-good-neighbor diagnosability of BS_n. In 2015, Zhang *et al.* [28] proposed a new measure for fault diagnosis of the system, namely, the g-extra diagnosability, which restrains that every fault-free component has at least $(g + 1)$ fault-free nodes. In [28], they studied the g-extra diagnosability of the n-dimensional hypercube under the PMC model and MM* model. In 2016, Wang *et al.* [22] studied the 2-extra diagnosability of BS_n under the PMC model and MM* model. In 2017, Wang and Yang [24] studied the 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model and MM* model.

The k-ary n-cube has many desirable properties, such as ease of implementation of algorithms and ability to reduce message latency by exploiting communication locality found in many parallel applications [4, 7]. Therefore, a number of distributed-memory parallel systems (also known as multicomputers) have been built with a k-ary n-cube forming the underlying topology, such as the Cray T3D [11], the J-machine [15], the iWarp [17] and the IBM Blue Gene [1]. In 2011, Xiang and Stewart [25] proposed the augmented k-ary n-cube. In 2016, Zhao and Wang [29] studied the nature diagnosability of augmented 3-ary n-cubes, and Hao and Wang [9] studied the nature diagnosability of augmented k-ary n-cubes for $k \geq 4$. In this paper, we extend the k-ary n-cube and define an expanded k-ary n-cube XQ_n^k. The connectivity and diagnosability of XQ_n^k have been studied in this paper. We prove that (1) the connectivity of XQ_n^k is $4n$ and XQ_n^k is tightly $4n$ super connected; (2) the nature connectivity of XQ_n^k is $8n - 4$; (3) the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n - 3$ for $n \geq 2$.

2. Preliminaries

In this section, some definitions and notations needed for our discussion, the expanded k-ary n-cube, the PMC model and MM* model are introduced.

2.1. Definitions and Notations

A multiprocessor system is modeled as an undirected simple graph $G = (V, E)$, whose vertices (nodes) represent processors and edges (links) represent communication links. Given a nonempty vertex subset V' of V, the induced subgraph by V' in G, denoted by $G[V']$, is a graph, whose vertex set is V' and the edge set is the set of all the edges of G with both endpoints in V'. The degree $d_G(v)$ of a vertex v is the number of edges incident with v. We denote by $\delta(G)$ the minimum degrees of vertices of G. For any vertex v, we define the neighborhood $N_G(v)$ of v in G to be the set of vertices adjacent to v. u is called a neighbor or a neighbor vertex of v for $u \in N_G(v)$. Let $S \subseteq V$. We use $N_G(S)$ to denote the set $\cup_{v\in S} N_G(v)\backslash S$. For neighborhoods and degrees, we will usually omit the subscript for the graph when no confusion arises. Let F_1 and F_2 be two distinct subsets of V for $G = (V, E)$. Define the symmetric difference $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. A graph G is said to be k-regular if for any vertex v, $d_G(v) = k$. A set of edges $M \subseteq E(G)$ is called a matching if they are independent. A matching is said to be perfect if it covers all points of G. Let $G = (V, E)$ be a connected graph. The connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or only one vertex left. A fault set $F \subseteq V$ is called a nature faulty set if $|N(v) \cap (V\backslash F)| \geq 1$ for every vertex v in $V\backslash F$. A nature cut of G is a nature faulty set F such that $G - F$ is disconnected. The minimum cardinality of nature cuts is said to be the nature connectivity of G, denoted by $\kappa^*(G)$. For graph-theoretical terminology and notation not defined here we follow [3].

2.2. The PMC model and the MM^* model

Under the PMC model [26], to diagnose a system G, two adjacent nodes in G are capable to perform tests on each other. For two adjacent nodes u and v in $V(G)$, the test performed by u on v is represented by the ordered pair (u, v). The outcome of a test (u, v) is 1 (respectively, 0) if u evaluate v as faulty (respectively, fault-free). In the PMC model, we usually assume that the testing result is reliable (respectively, unreliable) if the node u is fault-free(respectively, faulty). A test assignment T for a system G is a collection of tests for every adjacent pair of vertices. It can be modeled as a directed testing graph $T = (V(G), L)$, where $(u, v) \in L$ implies that u and v are adjacent in G. The collection of all test results for a test assignment T is called a syndrome. Formally, a syndrome is a function $\sigma : L \mapsto \{0, 1\}$.

The set of all faulty processors in the system is called a faulty set. This can be any subset of $V(G)$. For a given syndrome σ, a subset of vertices $F \subseteq V(G)$ is said to be consistent with σ if syndrome σ can be produced from the situation that, for any $(u, v) \in L$ such that $u \in V \setminus F$, $\sigma(u, v) = 1$ if and only if $v \in F$. This means that F is a possible set of faulty processors. Since a test outcome produced by a faulty processor is unreliable, a given set F of faulty vertices may produce a lot of different syndromes. On the other hand, different fault sets may produce the same syndrome. Let $\sigma(F)$ denote the set of all syndromes which F is consistent with.

Under the PMC model, two distinct sets F_1 and F_2 in $V(G)$ are said to be indistinguishable if $\sigma(F_1)\cap\sigma(F_2) \neq \emptyset$, otherwise, F_1 and F_2 are said to be distinguishable. Besides, we say (F_1, F_2) is an indistinguishable pair if $\sigma(F_1) \cap \sigma(F_2) \neq \emptyset$; else, (F_1, F_2) is a distinguishable pair.

Using the MM model [26], the diagnosis is carried out by sending the same testing task to a pair of processors and comparing their responses. Under the MM model, we always assume the output of a comparison performed by a faulty processor is unreliable. The comparison scheme of a system $G = (V, E)$ is modeled as a multigraph, denoted by $M = (V(G), L)$, where L is the labeled-edge set. A labeled edge $(u, v)_w \in L$ represents a comparison in which two vertices u and v are compared by a vertex w, which implies $uw, vw \in E(G)$. The collection of all comparison results in $M = (V(G), L)$ is called the syndrome, denoted by σ^*, of the diagnosis. If the comparison $(u, v)_w$ disagrees, then $\sigma^*((u, v)_w) = 1$, otherwise, $\sigma^*((u, v)_w) = 0$. Hence, a syndrome is a function from L to $\{0, 1\}$. The MM* model is a special case of the MM model and each node of G must test its any pair of adjacent nodes, *i.e.*, if $uw, vw \in E(G)$, then $(u, v)_w \in L$.

Similarly to the PMC model, we can define a subset of vertices $F \subseteq V(G)$ is consistent with a given syndrome $\sigma*$ and two distinct sets F_1 and F_2 in $V(G)$ are indistinguishable (respectively, distinguishable) under the MM* model.

A system $G = (V, E)$ is nature t-diagnosable if F_1 and F_2 are distinguishable, for each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$. The nature diagnosability $t_n(G)$ of G is the maximum value of t such that G is nature t-diagnosable.

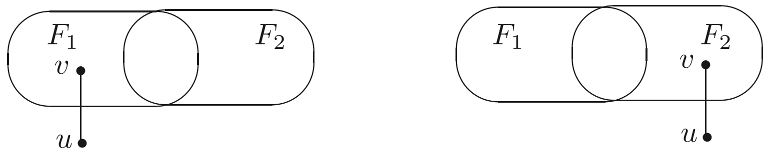

FIGURE 1. Illustration of a distinguishable pair (F_1, F_2) under the PMC model.

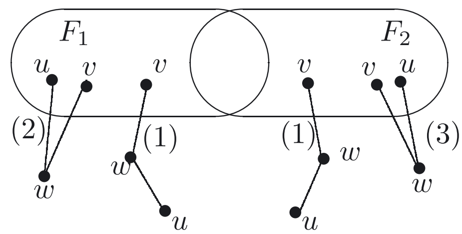

FIGURE 2. Illustration of a distinguishable pair (F_1, F_2) under the MM* model.

Before discussing the nature diagnosability of the expanded k-ary n-cube XQ_n^k under the PMC and MM* model, we first give existing results.

Theorem 2.1 [26]**.** *A system $G = (V, E)$ is nature t-diagnosable under the PMC model if and only if there is an edge $uv \in E$ with $u \in V \backslash (F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ (See Fig. 1).*

Theorem 2.2 [6, 26]**.** *A system $G = (V, E)$ is nature t-diagnosable under the MM^* model if and only if each distinct pair of nature faulty subsets F_1 and F_2 of V with $|F_1| \leq t$ and $|F_2| \leq t$ satisfies one of the following conditions.*

(1) *There are two vertices $u, w \in V \setminus (F_1 \cup F_2)$ and there is a vertex $v \in F_1 \triangle F_2$ such that $uw \in E$ and $vw \in E$.*
(2) *There are two vertices $u, v \in F_1 \setminus F_2$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$.*
(3) *There are two vertices $u, v \in F_2 \setminus F_1$ and there is a vertex $w \in V \setminus (F_1 \cup F_2)$ such that $uw \in E$ and $vw \in E$ (See Fig. 2).*

2.3. The expanded k-ary n-cube

The expanded k-ary n-cube, denoted by XQ_n^k ($n \geq 1$ and even $k \geq 6$), is a graph consisting of k^n vertices $\{u_0u_1 \ldots u_{n-1} : 0 \leq u_i \leq k-1, 0 \leq i \leq n-1\}$. Two vertices $u = u_0u_1 \ldots u_{n-1}$ and $v = v_0v_1 \ldots v_{n-1}$ are adjacent if and only if there exists an integer $j \in \{0, 1, \ldots, n-1\}$ such that $u_j = v_j + g \pmod{k}$ and $u_i = v_i$, for $i \in \{0, 1, \ldots, n-1\} \setminus \{j\}$ and $g \in \{1, -1, 2, -2\}$. For clarity of presentation, we omit writing "(mod k)" in similar expressions for the remainder of the paper. For terminology and notation not defined here we follow [10]. The expanded k-ary 1-cube XQ_1^k is depicted in Figure 3.

We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots,$ $XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u = u_0u_1 \ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0, 1, \ldots, k-1\}$. Let $u \in V(XQ[i])$. Then $N(u) \setminus V(XQ[i])$ is said to be outside neighbor vertices of u.

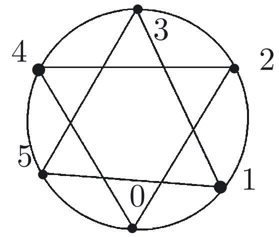

FIGURE 3. (a) The expanded k-ary 1-cube XQ_1^k.

Proposition 2.3. *Each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$.*

Proof. Note that the vertex set of XQ_{n-1}^k is $\{u_0u_1 \dots u_{n-2} : 0 \leq u_i \leq k-1, 0 \leq i \leq n-2\}$ and the vertex set of $XQ[i]$ is $\{u_0u_1 \dots u_{n-2}i : 0 \leq u_j \leq k-1, 0 \leq j \leq n-2, i \in \{0, 1, \dots, k-1\}\}$. Therefore, $|\{u_0u_1 \dots u_{n-2} : 0 \leq u_i \leq k-1, 0 \leq i \leq n-2\}| = |\{u_0u_1 \dots u_{n-2}i : 0 \leq u_j \leq k-1, 0 \leq j \leq n-2, i \in \{0, 1, \dots, k-1\}\}|$. Now define a mapping from $V(XQ_{n-1}^k)$ to $V(XQ[i])$ given by

$$\varphi : \quad u_0u_1u_2 \dots u_{n-2} \to u_0u_1 \dots u_{n-2}i.$$

It is clear that φ is bijective. Let $u = u_0u_1u_2 \dots u_{n-2}$, $v = v_0v_1v_2 \dots v_{n-2}$, and $uv \in E(XQ_{n-1}^k)$. Then, the definition of XQ_{n-1}^k, there exists an integer $j \in \{0, 1, \dots, n-2\}$ such that $v_j = u_j + g \pmod k$ and $u_i = v_i$, for $i \in \{0, 1, \dots, n-2\} \setminus \{j\}$, where $g \in \{1, -1, 2, -2\}$. Therefore, $\varphi(v) = v_0v_1v_2 \dots v_{n-2}i = u_0u_1 \dots u_{j-1}, u_j + g, u_{j+1} \dots u_{n-2}i$. Note that $\varphi(u) = u_0u_1 \dots u_{j-1}, u_j, u_{j+1} \dots u_{n-2}i$. Thus, $\varphi(u)\varphi(v) \in E(XQ[i])$.

Let $\varphi(u) = u_0u_1 \dots u_{j-1}, u_j, u_{j+1} \dots u_{n-2}i$, $\varphi(v) = v_0v_1v_2 \dots v_{n-2}i$ and $\varphi(u)\varphi(v) \in E(XQ[i])$. Then there exists an integer $j \in \{0, 1, \dots, n-2\}$ such that $v_j = u_j + g \pmod k$ and $u_i = v_i$, for $i \in \{0, 1, \dots, n-2\} \setminus \{j\}$, where $g \in \{1, -1, 2, -2\}$, *i.e.*, $\varphi(v) = v_0v_1v_2 \dots v_{n-2}i = u_0u_1 \dots u_{j-1}, u_j+g, u_{j+1} \dots u_{n-2}i$. Therefore, $\varphi^{-1}(v) = v_0v_1v_2 \dots v_{n-2} = u_0u_1 \dots u_{j-1}, u_j + g, u_{j+1} \dots u_{n-2}$. Note that $\varphi^{-1}(u) = u_0u_1 \dots u_{j-1}, u_j,$
$u_{j+1} \dots u_{n-2}$. Thus, $uv = \varphi^{-1}(u)\varphi^{-1}(v) \in E(XQ_{n-1}^k)$. □

Let Q be a finite group, and let S be a spanning set of Q such that S does not contain the identity element. The directed Cayley graph $Cay(S, Q)$ is defined as follows: its vertex set is Q, its arc set is $\{(g, gs) : g \in Q, s \in S\}$. If for every $s \in S$ we also have $s^{-1} \in S$, then each of the arc set of $Cay(S, Q)$ has parallel edges going in different directions. If we replace two arc of parallel edges going in different directions in $Cay(S, Q)$ with an edge, then we obtain an undirected graph called the undirected Cayley graph. Every Cayley graph in this paper is an undirected Cayley graph.

Let $(Z_k)^n$ denote the n-fold Cartesian product of the group $(Z_k, \oplus_k)$, where $Z_k = \{0, 1, \dots, k-1\}$ and where k denotes addition modulo k. Let $x = (x_0, x_1, \dots, x_{n-1}) \in (Z_k)^n$. Then $x^{-1} = (k - x_0, k - x_1, \dots, k - x_{n-1})$.

Theorem 2.4. *Let $n \geq 1$ and even $k \geq 6$. The expanded k-ary n-cube XQ_n^k is the Cayley graph $Cay(S, (Z_k)^n)$, where the spanning set S is $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1), (k-1, 0, 0, \dots, 0), (0, k-1, 0, \dots, 0), \dots, (0, \dots, 0, k-1), (2, 0, \dots, 0), (0, 2, 0, \dots, 0), \dots, (0, \dots, 0, 2), (k-2, 0, \dots, 0), (0, k-2, 0, \dots, 0), \dots, (0, \dots, 0, k-2)\}$.*

Proof. Note that $V(XQ_n^k) = (Z_k)^n$. Now define a mapping from $V(XQ_n^k)$ to $(Z_k)^n$ given by

$$\varphi : \quad u_1u_2u_3 \dots u_{n-1} \to u_1u_2 \dots u_{n-1}.$$

Then φ is bijective. Let $uv \in E(XQ_n^k)$. Then, the definition of XQ_n^k, there exists an integer $j \in \{0, 1, \dots, n-1\}$ such that $v_j = u_j + g \pmod k$ and $u_i = v_i$, for $i \in \{0, 1, \dots, n-1\} \setminus \{j\}$, where $g \in \{1, -1, 2, -2\}$. Note that $k - 1 \equiv -1 \pmod k$ and $k - 2 \equiv -2 \pmod k$. Let $s = (0, \dots, 0, 0 + g, 0, \dots, 0)$, and let $0 + g$ be the j position in the s. Then $s \in S$. Note that $\varphi(u)\varphi(v) = uv$. Therefore, $v = u + s$ and hence $\varphi(u)\varphi(v) \in E(Cay(S, (Z_k)^n))$.

Let $\varphi(u)\varphi(v) \in E(Cay(S,(Z_k)^n))$. Then, the definition of $Cay(S,(Z_k)^n)$, there exists an $s \in S$ such that $\varphi(v) = \varphi(u) + s$. Note that $\varphi(u) = u$ and $\varphi(v) = v$. Therefore, $v = \varphi(v) = \varphi(u) + s = u + s$. Note that $\varphi^{-1}(u)\varphi^{-1}(v) = uv$ and $v = u + s$. Let $s = (0, \dots, 0, 0 + g, 0, \dots, 0)$, and let $0 + g$ be the j position in the s. Then $v_j = u_j + g \pmod k$ and $u_i = v_i$, for $i \in \{0, 1, \dots, n-1\} \setminus \{j\}$. Note that $k - 1 \equiv -1 \pmod k$ and $k - 2 \equiv -2 \pmod k$. Therefore, $g \in \{1, -1, 2, -2\}$ and hence $uv \in E(XQ_n^k)$. □

Note that XQ_n^k is a special Cayley graph. Therefore, XQ_n^k has the following properties.

The automorphism group of a graph G is transitive if there exists an automorphism φ to any pair u, v of vertices in G such that $\varphi(u) = v$. In this case, G is called vertex transitive. The following proposition is clear.

Proposition 2.5. *XQ_n^k is $4n$-regular, vertex transitive.*

The girth is the length of a shortest cycle in a graph G. The following proposition is clear.

Proposition 2.6. *The girth of XQ_n^k is 3.*

Proposition 2.7. *Let $u \in V(XQ[i])$. Then four outside neighbor vertices of u are in four different $XQ[j]'s$.*

Proof. Let $u = u_0u_1 \dots u_{n-2}i$. Then $u \in V(XQ[i])$, $u_0u_1 \dots u_{n-2}i + 1 \in V(XQ[i+1])$, $u_0u_1 \dots u_{n-2}i - 1 \in V(XQ[i-1])$, $u_0u_1 \dots u_{n-2}i + 2 \in V(XQ[i+2])$ and $u_0u_1 \dots u_{n-2}i - 2 \in V(XQ[i-2])$. □

Proposition 2.8. *Let XQ_1^k be the expanded k-ary 1-cube.*

(1) *If $k = 6$ and two vertices u, v are adjacent, then there are at most two common neighbor vertices of these two vertices,* i.e., $|N(u) \cap N(v)| \le 2$. *If $k = 6$ and two vertices u, v are not adjacent, then there are at most four common neighbor vertices of these two vertices,* i.e., $|N(u) \cap N(v)| \le 4$.
(2) *If $k \ge 8$, then there are at most two common neighbor vertices of two vertices u, v,* i.e., $|N(u) \cap N(v)| \le 2$.

Proof. Let $u, v \in V(XQ_1^k)$. Suppose that $k = 6$. Then $XQ_1^k = XQ_1^6$. By Proposition 2.5, without loss of generality, we suppose that $u = 0$. Note that $N(0) = \{1, 2, 4, 5\}$ and $N(3) = \{1, 2, 4, 5\}$. Note that two vertices $0, 3$ are not adjacent and $N(0) \cap N(3) = \{1, 2, 4, 5\}$. Therefore, there are at most four common neighbor vertices of these two vertices, *i.e.*, $|N(u) \cap N(v)| \le 4$. From Figures 3a and 3b (geometry) is symmetrical about the axis 03. Therefore, we consider only edges 01 and 02 for adjacent two vertices. Note that $N(0) = \{1, 2, 4, 5\}$ and $N(1) = \{0, 2, 3, 5\}$. Therefore, $N(0) \cap N(1) = \{2, 5\}$. $N(0) = \{1, 2, 4, 5\}$ and $N(2) = \{0, 1, 3, 4\}$. Therefore, $N(0) \cap N(2) = \{1, 4\}$. Thus, for adjacent two vertices u, v, there are at most two common neighbor vertices of these two vertices, *i.e.*, $|N(u) \cap N(v)| \le 2$.

Suppose that $k \ge 8$. By Proposition 2.5, we suppose that $u = 0$. From Figure 3b, Figure 3b (geometry) is symmetrical about the axis $0\frac{k}{2}$. Therefore, we consider only two vertices: $u = 0$ and $v \in \{1, 2, \dots, \frac{k}{2}\}$. Note that $N(0) = \{1, 2, k-2, k-1\}$, $N(1) = \{0, 2, 3, k-1\}$ and $N(2) = \{0, 1, 3, 4\}$. Therefore, $N(0) \cap N(1) = \{2, k-1\}$ and $N(0) \cap N(2) = \{1\}$. Thus, for adjacent two vertices u, v, there are at most two common neighbor vertices of these two vertices, *i.e.*, $|N(u) \cap N(v)| \le 2$. Now consider two vertices: $u = 0$ and $v \in \{3, 4, \dots, \frac{k}{2}\}$. Let $v = 3$. Note that $N(3) = \{1, 2, 4, 5\}$. Therefore, $N(0) \cap N(3) = \{1, 2\}$. Note that $N(4) = \{2, 3, 5, 6\}$. Therefore, $N(0) \cap N(4) = \{2, 6\}$ when $k = 8$ and $N(0) \cap N(4) = \{2\}$ when $k \ge 10$. Let $v \in \{5, 6, \dots, \frac{k}{2}\}$ and $x \in N(v)$. Then $3 \le x \le k - 3$. Therefore, $N(0) \cap N(x) = \emptyset$. Thus, there are at most two common neighbor vertices of these two vertices u, v, *i.e.*, $|N(u) \cap N(v)| \le 2$. □

Proposition 2.9. *Let XQ_n^k be the expanded k-ary n-cube.*

(1) *If $k = 6$ and two vertices u, v are adjacent, then there are at most two common neighbor vertices of these two vertices,* i.e., $|N(u) \cap N(v)| \le 2$. *If $k = 6$ and two vertices u, v are not adjacent, then there are at most four common neighbor vertices of these two vertices,* i.e., $|N(u) \cap N(v)| \le 4$.
(2) *If $k \ge 8$, then there are at most two common neighbor vertices of two vertices u, v,* i.e., $|N(u) \cap N(v)| \le 2$.

Proof. We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots,$ $XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1\ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0,1,\ldots,k-1\}$. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \le i \le k-1$. Let $u, v \in V(XQ_n^k)$. By Proposition 2.5, without loss of generality, we suppose that $u = \underbrace{00\ldots0}_{n}$. Then $u \in V(XQ[0])$.

Suppose that $k=6$. When $n=1$, the result holds by Proposition 2.8. We proceed by induction on n ($n \ge 2$). Our induction hypothesis is the following.

(a) If two vertices u, v are adjacent, then there are at most two common neighbor vertices of these two vertices, *i.e.*, $|N(u) \cap N(v)| \le 2$ in XQ_{n-1}^6.
(b) If two vertices u, v are not adjacent, then there are at most four common neighbor vertices of these two vertices, *i.e.*, $|N(u) \cap N(v)| \le 4$ in XQ_{n-1}^6.

Let $v \in V(XQ[0])$. By the induction hypothesis, (a) if two vertices u, v are adjacent, $|N(u) \cap N(v)| \le 2$ in $XQ[0]$; (b) if two vertices u, v are not adjacent, $|N(u) \cap N(v)| \le 4$ in $XQ[0]$. By Proposition 2.7, $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) = \emptyset$ for $i \in \{1,2,\ldots,5\}$. Therefore, $|N(u) \cap N(v)| \le 2$ for (a) and $|N(u) \cap N(v)| \le 4$ for (b) in this case.

Suppose that $v \in V(XQ[i])$ for $i \in \{1,2,\ldots,5\}$. If $v \in \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\}$, then, by the induction hypothesis, (a) if two vertices u, v are adjacent, $|N(u) \cap N(v)| \le 2$; (b) if two vertices u, v are not adjacent, $|N(u) \cap N(v)| \le 4$. Note that $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) \setminus \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\} = \emptyset$ for $i \in \{0,1,2,\ldots,5\}$. Therefore, $|N(u) \cap N(v)| \le 2$ or $|N(u) \cap N(v)| \le 4$ in this case. Let $v \in V(XQ[i]) \setminus \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \underbrace{0\ldots0}_{n-1}3, \underbrace{0\ldots0}_{n-1}4, \underbrace{0\ldots0}_{n-1}5\}$ for $i \in \{1,2,3,4,5\}$. Since $|N(u) \cap V(XQ[i])| \le 1$ for $i \in \{1,2,3,4,5\}$, $|N(v) \cap V(XQ[0])| \le 1$ and $(N(u) \cap V(XQ[j])) \cap (N(v) \cap V(XQ[j])) = \emptyset$ for $i \ne j$, $|N(u) \cap N(v)| \le 2$ holds.

Suppose that $k \ge 8$. When $n=1$, the result holds by Proposition 2.8. We proceed by induction on n. Our induction hypothesis is that $|N(u) \cap N(v)| \le 2$ for two vertices u, v in XQ_{n-1}^k. Let $v \in V(XQ[0])$. By the induction hypothesis, $|N(u) \cap N(v)| \le 2$ for two vertices u, v in $XQ[0]$. By Proposition 2.7, $(N(u) \cap V(XQ[i])) \cap (N(v) \cap V(XQ[i])) = \emptyset$ for $i \in \{1,2,\ldots,k-1\}$. Therefore, $|N(u) \cap N(v)| \le 2$ in this case.

Suppose that $v \in V(XQ[i])$ for $i \in \{1,2,\ldots,k-2,k-1\}$. If $v \in \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}(k-1)\}$, then $|N(u) \cap N(v)| \le 2$ by Propositions 2.7 and 2.8. Let $v \in V(XQ[i]) \setminus \{\underbrace{0\ldots0}_{n-1}1, \underbrace{0\ldots0}_{n-1}2, \ldots, \underbrace{0\ldots0}_{n-1}(k-1)\}$. Note that $|N(u) \cap V(XQ[i])| \le 1$, $|N(v) \cap V(XQ[0])| \le 1$ and $(N(u) \cap V(XQ[j])) \cap (N(v) \cap V(XQ[j])) = \emptyset$ for $i \ne j$. Therefore, there are at most two common neighbor vertices of two vertices u, v, *i.e.*, $|N(u) \cap N(v)| \le 2$. $\square$

3. The connectivity of the expanded k-ary n-cube

In the process of the proof of the nature diagnosability of the expanded k-ary n-cube XQ_n^k, we use the nature connectivity of XQ_n^k. Therefore, in this section, we shall show the connectivity and nature connectivity of XQ_n^k.

Proposition 3.1. *The connectivity* $\kappa(XQ_1^k) = 4$.

Proof. By Menger's Theorem, a graph XQ_1^k has connectivity $\kappa(XQ_1^k) = 4$ if and only if, given any two distinct vertices of $V(XQ_1^k)$, there are 4 vertex-disjoint paths joining them. By Theorem 2.4, it is sufficient to show that, for $u = 0$ and a distinct vertex v of $V(XQ_1^k)$, there are 4 vertex-disjoint paths joining u and v. By the symmetry, we will prove that, for $u = 0$ and one $v \in \{1, 2, \ldots, \frac{k}{2}\}$, there are 4 vertex-disjoint paths joining u

and v. Let an odd $i \in \{2, 3, \ldots, \frac{k}{2}\}$. We have that four vertex-disjoint paths: $0, 1, 3, 5, \ldots, i$; $0, 2, 4, \ldots, i-1, i$; $0, k-1, k-3, k-5, \ldots, i$ and $0, k-2, k-4, \ldots, i+1, i$. When $i = 1$, we have that four vertex-disjoint paths: $0, 1$; $0, k-1, 1$; $0, 2, 1$ and $0, k-2, k-4, \ldots, 4, 3, 1$. Let an even $i \in \{1, 2, 3, \ldots, \frac{k}{2}\}$. We have that four vertex-disjoint paths: $0, 1, 3, \ldots, i-1, i$; $0, 2, 4, \ldots, i$; $0, k-1, k-3, k-5, \ldots, i+1, i$ and $0, k-2, k-4, \ldots, i$. □

Proposition 3.2. *The connectivity* $\kappa(XQ_2^k) = 8$.

Proof. Note $\kappa(XQ_2^k) \leq \delta(XQ_2^k) = 8$. We prove this statement by contradiction. Suppose that $F \subseteq V(XQ_2^k)$ with $|F| \leq 7$ is a cut of XQ_2^k. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ_1^k for $0 \leq i \leq k-1$. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0, 1, 2, \ldots, k-1\}$.

Suppose that $|F_i| = \max\{|F_i| : 0 \leq i \leq k-1\}$. Note that the vertex set of $XQ[i]$ is $\{u_0 i : 0 \leq u_0 \leq k-1, i \in \{1, \ldots, k-1\}\}$ and the vertex set of $XQ[0]$ is $\{u_0 0 : 0 \leq u_0 \leq k-1\}$. Now define a mapping from $V(XQ_2^k)$ to $V(XQ_2^k)$ given by

$$\varphi: \quad u_0 u_1 \to u_0(u_1 - i).$$

Then $\varphi(u_0 i) = u_0 0$.

Claim 1. φ is an automorphism of XQ_2^k.

It is clear that φ is bijective. Let $u = u_0 u_1$, $v = v_0 v_1$, and $uv \in E(XQ_2^k)$. Then, the definition of XQ_2^k, $v_0 = u_0 + g \pmod k$ and $v_1 = u_1$, or $v_0 = u_0$, $v_1 = u_1 + g \pmod k$, where $g \in \{1, -1, 2, -2\}$. Suppose, first, that $v_0 = u_0 + g \pmod k$ and $v_1 = u_1$. Note $\varphi(u) = u_0, u_1 - i$ and $\varphi(v) = \varphi(u_0 + g, u_1) = u_0 + g, u_1 - i$. Suppose, second, that $v_0 = u_0$, $v_1 = u_1 + g \pmod k$. Note $\varphi(u) = u_0, u_1 - i$ and $\varphi(v) = \varphi(u_0, u_1 + g) = u_0, u_1 + g - i$. Therefore, $\varphi(u)\varphi(v) \in E(XQ_2^k)$ by the definition of XQ_2^k.

Let $\varphi(u) = u_0, u_1 - i$, $\varphi(v) = v_0, v_1 - i$ and $\varphi(u)\varphi(v) \in E(XQ_2^k)$. Then, the definition of XQ_2^k, $v_0 = u_0 + g \pmod k$ and $v_1 - i = u_1 - i$, or $v_0 = u_0$, $v_1 - i = u_1 - i + g \pmod k$, where $g \in \{1, -1, 2, -2\}$. Suppose, first, that $v_0 = u_0 + g \pmod k$ and $v_1 - i = u_1 - i$. Then $\varphi^{-1}(u) = u_0 u_1$ and $\varphi^{-1}(v) = u_0 + g, u_1$. Suppose, second, that $v_0 = u_0$, $v_1 - i = u_1 - i + g \pmod k$. Then $\varphi^{-1}(u) = u_0 u_1$ and $\varphi^{-1}(v) = u_0, u_1 + g$. Therefore, $uv = \varphi^{-1}(u)\varphi^{-1}(v) \in E(XQ_{n-1}^k)$ by the definition of XQ_2^k. Therefore, φ is an automorphism.

Claim 2. Let φ be the above. If $F \subseteq V(XQ_2^k)$ is a cut of XQ_2^k, then $\varphi(F)$ is also a cut of XQ_2^k. In particular, $\varphi(F_i) \subseteq V(XQ[0])$ and $|\varphi(F_i)| = |F_i|$.

Since φ is bijective, $|\varphi(F)| = |F|$ and $|\varphi(F_i)| = |F_i|$. Let $B_1, \ldots, B_k$ $(k \geq 2)$ be the components of $XQ_2^k - F$. Then $[V(B_i), V(B_j)] = \emptyset$ for $1 \leq i, j \leq k$ and $i \neq j$. Let $b_i \in V(B_i)$ and $b_j \in V(B_j)$. Then b_i is not adjacent to b_j. Since φ is an automorphism, $\varphi(b_i)$ is not adjacent to $\varphi(b_j)$. Therefore, $[\varphi(V(B_i)), \varphi(V(B_j))] = \emptyset$ for $1 \leq i, j \leq k$ and $i \neq j$, and hence $\varphi(F)$ is also a cut of XQ_2^k. Let $f \in F_i$. Then $f = u_0 i$ for $0 \leq u_0 \leq k-1$. Therefore, $\varphi(f) = u_0 0 \in V(XQ[0])$ and hence $\varphi(F_i) \subseteq V(XQ[0])$.

By Claim 2, without loss of generality, we suppose that $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| = 1$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are six F_i's such that $|F_i| = 1$ for $i \in \{1, 2, \ldots, k-1\}$ and $k \geq 8$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 2. $|F_0| = 2$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are at most five F_i's such that $1 \leq |F_i| \leq 2$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 3. $|F_0| = 3$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, there are at most four F_i's such that $1 \leq |F_i| \leq 3$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Without loss of generality, we suppose that $|F_1| = 3$. Then $|F_{k-1}| \leq 1$. Since there is a complete matching between $XQ[0]$ and $XQ[k-1]$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 4. $|F_0| = 4$.

In this case, there are at most three F_i's such that $1 \leq |F_i| \leq 3$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_1| + |F_2| + \ldots + |F_{k-1}| = 3$, by Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 5. $|F_0| = 5$.

In this case, there are at most two F_i's such that $1 \leq |F_i| \leq 2$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_1| + |F_2| + \ldots + |F_{k-1}| = 2$, by Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 6. $|F_0| = 6$.

In this case, there is one F_i's such that $|F_i| = 1$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 3.1, $XQ[i] - F_i$ is connected. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_1| + |F_2| + \ldots + |F_{k-1}| = 1$, by Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 7. $|F_0| = 7$.

In this case, $|F_1| = |F_2| = \ldots = |F_{k-1}| = 0$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_1| + |F_2| + \ldots + |F_{k-1}| = 0$, by Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

By Cases 1−7, The connectivity XQ_2^k is 8. □

Theorem 3.3. *Let XQ_n^k be the expanded k-ary n-cube with $n \geq 1$ and even $k \geq 6$, Then the connectivity $\kappa(XQ_n^k) = 4n$.*

Proof. We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots,$ $XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u = u_0 u_1 \ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0, 1, \ldots, k-1\}$. When $n = 1$ and $n = 2$, the result holds by Propositions 3.1 and 3.2. We proceed by induction on n. Our induction hypothesis is $\kappa(XQ_{n-1}^k) = 4n-4$ when $n \geq 3$. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ_{n-1}^k for $0 \leq i \leq k-1$. We will prove $\kappa(XQ_n^k) = 4n$. Suppose that $F \subseteq V(XQ_n^k)$ is a minimum cut of XQ_n^k. Since $\kappa(XQ_n^k) \leq \delta(XQ_n^k) = 4n$, $|F| \leq 4n$ holds. It is sufficient to show that $XQ_n^k - F$ is connected for $|F| \leq 4n-1$. We prove this statement by contradiction. Suppose that $F \subseteq V(XQ_n^k)$ with $|F| \leq 4n-1$ is a cut of XQ_n^k. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0, 1, 2, \ldots, k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| \leq 4n-5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-5$. By the induction hypothesis, $XQ[i] - F_i$ is connected. Since $k^{n-1} > 4n-5+(4n-5) = 8n-10$ and there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 2. $4n-4 \leq |F_0| \leq 4n-1$.

In this case, there are at most three F_i's such that $1 \leq |F_i| \leq 3$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

By Cases 1 and 2, The connectivity XQ_n^k is $4n$. □

Remarks on Theorem 3.3. First, the connectivity of the expanded k-ary n-cube XQ_n^k is maximum. Second, by Menger's Theorem, any two distinct vertices of XQ_n^k, there are $4n$ vertex-disjoint paths joining them. Having a high connectivity is a desirable property of any interconnection network as it provides fault-tolerance with regard to message routing, allows for hot-spots to be avoided, and allows large messages to be split up into smaller ones and routed in parallel along vertex-disjoint paths.

A connected graph G is super connected if every minimum cut F of G isolates one vertex. If, in addition, $G - F$ has two components, one of which is an isolated vertex, then G is tightly $|F|$ super connected.

Theorem 3.4. *Let XQ_n^k be the expanded k-ary n-cube with $n \geq 1$ and even $k \geq 6$, Then XQ_n^k is tightly $4n$ super connected.*

Proof. Let $F \subseteq V(XQ_n^k)$ with $|F| = 4n$ be any minimum cut of XQ_n^k. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0, 1, 2, \ldots, k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| \leq 4n - 5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n - 5$. By Theorem 3.3, $XQ[i] - F_i$ is connected. Since $k^{n-1} > 4n - 5 + (4n - 5) = 8n - 10$ and there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 2. $|F_0| = 4n - 4$.

Suppose that there is only one F_i such that $|F_i| \neq 0$. Then $|F_i| = 4$. Without loss of generality, we suppose that $|F_1| = 4$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{2, 3, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[2] - F_3) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Since $|F_{k-1}| = 0$ (or $|F_2| = 0$) and there is a complete matching between $XQ[0]$ and $XQ[k-1]$ (or $XQ[0]$ and $XQ[2]$), $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Suppose that there are two F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 3$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Suppose that there are three F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 2$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Suppose that there are four F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 1$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Let $XQ[0] - F_0$ be connected. Since $k^{n-1} > 4n - 4 + 1 = 4n - 3$ and there is a complete matching between $XQ[0]$ and $XQ[1]$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k. Let $XQ[0] - F_0$ be disconnected and let $B_1, \ldots, B_k$ $(k \geq 2)$ be the components of $XQ[0] - F_0$. If $k \geq 3$, then, by Proposition 2.7, $(N(V(B1)) \cup N(V(B_2))) \cap (V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1}))| \geq 8$. If $|V(B_r)| \geq 2$ $(1 \leq r \leq k-1)$, then, by Proposition 2.7, $|N(V(B_1) \cap (V(XQ[1] - F_1) \cup \ldots \cup (V(XQ[k-1] - F_{k-1}))| \geq 8$. Combining this with $|F_1| + \ldots + |F_{k-1}| = 4$, we have that $XQ[0] - F_0$ has two components, one of which is an isolated vertex v. Since $k^{n-1} > 4n - 4 + 1 + 1 = 4n - 2$ and there is a complete matching between $XQ[0]$ and $XQ[1]$, $XQ_n^k[V(XQ[0] - F_0 - v) \cup V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Therefore, $XQ_n^k - F$ has two components, one of which is an isolated vertex.

Case 3. $4n - 3 \leq |F_0| \leq 4n$.

In this case, there are at most three F_i's such that $1 \leq |F_i| \leq 3$. By Proposition 3.1, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

By Cases 1–3, XQ_n^k is tightly $4n$ super connected. □

Proposition 3.5. *Let XQ_2^k be the expanded k-ary 2-cube with even $k \geq 6$, and let $F \subseteq V(XQ_2^k)$ with $|F| \leq 11$. If $XQ_2^k - F$ is disconnected, then $XQ_2^k - F$ has two components, one of which is an isolated vertex.*

Proof. We can partition XQ_2^k into k disjoint subgraphs $XQ_2^k[0]$, $XQ_2^k[1], \ldots, XQ_2^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1 \in V(XQ_2^k)$ has a fixed integer i in the last position u_1 for $i \in \{0, 1, \ldots, k-1\}$. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ_1^k for $0 \leq i \leq k-1$. By Theorem 3.3, $\kappa(XQ[i]) = 4$. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0, 1, 2, \ldots, k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. We consider the following cases.

Case 1. $|F_0| \leq 3$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 3$. By Theorem 3.3, $XQ[i] - F$ is connected.

Suppose that $|F_0| \leq 2$. Then $|F_i| \leq 2$ for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, k-2\}$, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Suppose that $|F_0| = 3$. Then $|F_i| \leq 3$ for $i \in \{1, 2, \ldots, k-1\}$. If $|F_i| \leq 2$ for $i \in \{1, 2, \ldots, k-1\}$, then $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. If $k \geq 8$, then $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. Therefore, let $k = 6$ and there be F_i's for $i \in \{1, 2, 3, 4, 5\}$ such that $|F_i| = 3$. Since $|F_1| + \ldots + |F_5| \leq 8$, there are at most two F_i's such that $|F_i| = 3$. Suppose that there is one F_i such that $|F_i| = 3$. Without loss of generality, let that $|F_1| = 3$. Then $|F_5| \leq 2$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, 4\}$, $Q_2^6[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[5] - F_5)]$ is connected. Since there is a complete matching between $XQ[0]$ and $XQ[5]$, $Q_2^6 - F$ is connected, a contradiction to that F is a cut of Q_2^6. Suppose that there are two F_i such that $|F_i| = 3$. Without loss of generality, let that $|F_1| = 3$ and $|F_5| = 3$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, 4\}$, $Q_2^6[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[5] - F_5)]$ is connected. Since there is a complete matching between $XQ[0]$ and $XQ[2]$, $Q_2^6 - F$ is connected, a contradiction to that F is a cut of Q_2^6.

Case 2. $|F_0| = 4$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4$. Since $|F_1| + \ldots + |F_5| \leq 7$, there is at most one F_i such that $|F_i| = 4$ for $i \in \{1, 2 \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4$. Then $|F_2| + \ldots + |F_{k-1}| \leq 3$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{2, 3, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0, 1, \ldots, 4\}$, $XQ_2^k[V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Theorem 3.4, $XQ[i] - F_i$ is connected or $XQ[i] - F_i$ has two components, one of which is an isolated vertex v_i for $i \in \{0, 1\}$. Let $XQ[i] - F_i$ be connected for $i \in \{1, 2\}$. Then $|V(XQ[i] - F_i)| \geq 2$ for $i \in \{1, 2\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. Without loss of generality, suppose that $XQ[1] - F_1$ has two components, one of which is an isolated vertex and $XQ[0] - F_0$ is connected. Since $|V(XQ[0] - F_0)| \geq 2$ and $|F_2| + \ldots + |F_{k-1}| \leq 3$, by Proposition 2.7, $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Therefore, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex. Then $XQ[i] - F_i$ is disconnected for $i \in \{1, 2\}$. Suppose that $k = 6$. Then $XQ[i] - F_i$ has two components, two of which are isolated vertices for $i \in \{1, 2\}$. Since $|F_2| + \ldots + |F_5| \leq 3$, by Theorem 3.4, $XQ_2^6[V(XQ[i] - F_i) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[5] - F_5)]$ is connected, or $XQ_2^6[V(XQ[i] - F_i) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[5] - F_5)]$ has two components, one of which is an isolated vertex v_i for $i \in \{0, 1\}$. Note that $|N(v_0) \cap N(v_1)| \leq 2$. Since $|N(v_0) \cap N(v_1)| \leq 2$ and $|F_2| + \ldots + |F_5| \leq 3$, $XQ_2^6 - F$ is connected, or $XQ_2^6 - F$ has two components, one of which is an isolated vertex. Suppose that $k \geq 8$. Since $|V(XQ[0] - F_0)| \geq 3$ and $|F_2| + \ldots + |F_{k-1}| \leq 3$, $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected, or $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. If $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected, then $XQ_2^k - F$ is connected, or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Then $XQ_2^k[V(XQ[0] - F_0) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. Since $|V(XQ[1] - F_1)| \geq 3$, $XQ_2^k[V(XQ[1] - F_1) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected, or $XQ_2^k[V(XQ[1] - F_1) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex. Suppose that $XQ_2^k[V(XQ[i] - F_i) \cup V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ has two components, one of which is an isolated vertex v_i for $i \in \{0,1\}$. By Proposition 2.9, $|N(v_0) \cap N(v_1)| \leq 2$. Since $|N(v_0) \cap N(v_1)| \leq 2$ and $|F_2| + \ldots + |F_{k-1}| \leq 3$, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Suppose that there are at most three F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 3$ for $i \in \{2,3,\ldots,k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{2,3,\ldots,k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[2] - F_2) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 3. $|F_0| = 5$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 11 - 5 = 6$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 5$ for $i \in \{1,2,\ldots,k-1\}$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ (or $XQ[i]$ and $XQ[i+2]$), for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_2| + \ldots + |F_5| \leq 6$, by Proposition 2.7, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \ldots + |F_{k-1}| \leq 6$, there are at most three F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Note that there is at most one F_i such that $|F_i| = 5$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 5$. Since $|F_1| + \ldots + |F_{k-1}| \leq 6$, there are at most two F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 4. $|F_0| = 6$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 11 - 6 = 5$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_2| + \ldots + |F_5| \leq 5$, by Proposition 2.7, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \ldots + |F_{k-1}| \leq 5$, there are at most two F_i's such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Note that there is at most one F_i such that $|F_i| = 5$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 5$. Since $|F_1| + \ldots + |F_{k-1}| \leq 5$, there are at most one F_i such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 5. $|F_0| = 7$.

In this case, $k \geq 8$ and $|F_1| + \ldots + |F_5| \leq 4$. Suppose that $|F_i| \leq 3$ for $i \in \{1,2,\ldots,k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1,2\ldots,k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0,1,\ldots,k-2\}$, $XQ_2^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_2| + \ldots + |F_5| \leq 4$, by Proposition 2.7, $XQ_2^k - F$ is connected, or $XQ_2^k - F$ has two components, one of which is an isolated vertex.

Note that there is at most one F_i such that $|F_i| = 4$ for $i \in \{1,2,\ldots,k-1\}$. Without loss of generality, let that $|F_1| = 4$. Since $|F_1| + \ldots + |F_{k-1}| \leq 4$, there are at most one F_i such that $|F_i| \neq 0$ for $i \in \{1,2,\ldots,k-1\}$. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k.

Case 6. $8 \leq |F_0| \leq 11$.

In this case, $|F_1| + \ldots + |F_5| \leq 3$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$ for $i \in \{0,1,\ldots,k-2\}$, $Q_2^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 2.7, $XQ_2^k - F$ is connected, a contradiction to that F is a cut of XQ_2^k. □

Proposition 3.6. *Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$, and let $F \subseteq V(XQ_n^k)$ with $|F| \leq 8n-5$. If $XQ_n^k - F$ is disconnected, then $XQ_n^k - F$ has two components, one of which is an isolated vertex.*

Proof. We can partition XQ_n^k into k disjoint subgraphs $XQ_n^k[0]$, $XQ_n^k[1], \ldots,$ $XQ_n^k[k-1]$ (abbreviated as $XQ[0]$, $XQ[1]$, $\ldots$, $XQ[k-1]$, if there is no ambiguity), where every vertex $u_0u_1\ldots u_{n-1} \in V(XQ_n^k)$ has a fixed integer i in the last position u_{n-1} for $i \in \{0, 1, \ldots, k-1\}$. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ^k_{n-1} for $0 \leq i \leq k-1$. Let $F \subseteq V(XQ^k_n)$ with $|F| \leq 8n-5$ and let $XQ^k_n - F$ is disconnected. Let $F_i = F \cap V(XQ[i])$ for $i \in \{0, 1, 2, \ldots, k-1\}$ with $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$. When $n = 2$, the result holds by Propositions 3.5. We proceed by induction on n. Our induction hypothesis is that $XQ^k_{n-1} - F$ has two components, one of which is an isolated vertex for $|F| \leq 8n-13$ and $n \geq 3$ if $XQ^k_{n-1} - F$ is disconnected. By Proposition 2.3, each $XQ[i]$ is isomorphic to XQ^k_{n-1} for $0 \leq i \leq k-1$. We consider the following cases.

Case 1. $|F_0| \leq 4n-5$.

Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-5$ for $i \in \{1, 2, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{0, 1, \ldots, k-1\}$. Since $k^{n-1} > 4n-5+(4n-5) = 8n-10$ and there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ^k_n - F$ is connected, a contradiction to that F is a cut of XQ^k_n.

Case 2. $|F_0| = 4n-4$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 8n-5-(4n-4) = 4n-1$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-4$ for $i \in \{1, 2, \ldots, k-1\}$. Therefore, there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1, 2, \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$.

Suppose that there are four F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 1$ for $i \in \{2, 3, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{2, 3, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ^k_n[V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Theorem 3.4, $XQ[i] - F_i$ is connected or $XQ[i] - F_i$ has two components, one of which is an isolated vertex v_i for $i \in \{0, 1\}$. Let $XQ[i] - F_i$ be connected for $i \in \{1, 2\}$. Note that $k^{n-1} - (4n-4) > 2$ and hence $|V(XQ[i]-F_i)| \geq 2$. By Proposition 2.7, $XQ^k_n - F$ is connected, a contradiction to that F is a cut of XQ^k_n. Without loss of generality, suppose that $XQ[1] - F_1$ has two components, one of which is an isolated vertex and $XQ[0] - F_0$ is connected. Since $|V(XQ[0]-F_0)| \geq 2$ and $|F_2| + \ldots + |F_{k-1}| = 3$, by Proposition 2.7, $XQ^k_n[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Therefore, $XQ^k_n - F$ is connected, or $XQ^k_n - F$ has two components, one of which is an isolated vertex. Then $XQ[i] - F_i$ be disconnected for $i \in \{1, 2\}$. Since $|V(XQ[0]-F_0)| \geq 3$ and $|F_2| + \ldots + |F_{k-1}| = 3$, $XQ^k_n[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected, or $XQ^k_n[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex. If $XQ^k_n[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected, then $XQ^k_n - F$ is connected, or $XQ^k_n - F$ has two components, one of which is an isolated vertex. Then $XQ^k_n[V(XQ[0]-F_0) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex v_0. Since $|V(XQ[1]-F_1)| \geq 3$, $XQ^k_n[V(XQ[1]-F_1) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected, or $XQ^k_n[V(XQ[1]-F_1) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex. Suppose that $XQ^k_n[V(XQ[i]-F_i) \cup V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ has two components, one of which is an isolated vertex v_i for $i \in \{0, 1\}$. By Proposition 2.9, $|N(v_0) \cap N(v_1)| \leq 2$. Since $|N(v_0) \cap N(v_1)| \leq 2$ and $|F_2| + \ldots + |F_{k-1}| \leq 3$, $XQ^k_n - F$ is connected, or $XQ^k_n - F$ has two components, one of which is an isolated vertex.

Suppose that there are three F_i's such that $|F_i| \neq 0$. Then $|F_i| \leq 2$ for $i \in \{2, 3, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{2, 3, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ^k_n[V(XQ[2]-F_2) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. By Proposition 2.7, $XQ^k_n - F$ is connected, a contradiction to that F is a cut of XQ^k_n.

Case 3. $|F_0| = 4n-3$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 8n-5-(4n-3) = 4n-2$. Since $|F_0| = \max\{|F_i| : 0 \leq i \leq k-1\}$, $|F_i| \leq 4n-3$ for $i \in \{1, 2, \ldots, k-1\}$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1, 2, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1, 2 \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ^k_n[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-3 \leq 8n-13$, $XQ[0] - F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-3+4n-4+1 = 8n-6$, $XQ^k_n - F$ is connected, or has two components, one of which is an isolated.

Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1, 2, \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_1|+\ldots+|F_{k-1}| \leq 4n-2$, there are three F_i's such that $|F_i| \neq 0$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Note that there is at most one F_i such that $|F_i| = 4n-3$ for $i \in \{1, 2, \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4n-3$. Since $|F_1|+\ldots+|F_{k-1}| \leq 4n-2$, there are two F_i's such that $|F_i| \neq 0$ for $i \in \{1, 2, \ldots, k-1\}$. By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 4. $|F_0| = 4n - 2$.

In this case, $|F_1|+\ldots+|F_{k-1}| \leq 8n-5-(4n-2) = 4n-3$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1, 2, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1, 2 \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-2 \leq 8n-13$, $XQ[0] - F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-2+4n-4+1 = 8n-5$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex. Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1, 2, \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_2| + \ldots + |F_{k-1}| \leq 1$, By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 5. $|F_0| = 4n - 1$.

In this case, $|F_1|+\ldots+|F_{k-1}| \leq 8n-5-(4n-1) = 4n-4$. Suppose that $|F_i| \leq 4n-5$ for $i \in \{1, 2, \ldots, k-1\}$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1, 2 \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Since $|F_0| = 4n-1 \leq 8n-13$, $XQ[0] - F_0$ has two components, one of which is an isolated vertex v_0 by the induction hypothesis. Since $k^{n-1} > 4n-1+4n-4+1 = 8n-4$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex. Note that there is at most one F_i such that $|F_i| = 4n-4$ for $i \in \{1, 2, \ldots, k-1\}$. Without loss of generality, let that $|F_1| = 4n-4$. Since $|F_2| + \ldots + |F_{k-1}| = 0$, By Proposition 2.7, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k.

Case 6. $4n \leq |F_0| \leq 8n - 13$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 8n - 5 - 4n = 4n - 5$. By Theorem 3.3, $XQ[i] - F$ is connected for $i \in \{1, 2 \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k[V(XQ[1] - F_1) \cup \ldots \cup V(XQ[k-1] - F_{k-1})]$ is connected. Suppose that $XQ[0]$ is connected. Since $k^{n-1} > 8n-5$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k. Then $XQ[0]$ is disconnected. By the induction hypothesis, $XQ[0] - F_0$ has two components, one of which is an isolated vertex. Since $k^{n-1} > 8n-5+1 = 8n-4$, $XQ_n^k - F$ is connected, or has two components, one of which is an isolated vertex.

Case 7. $8n - 12 \leq |F_0| \leq 8n - 5$.

In this case, $|F_1| + \ldots + |F_{k-1}| \leq 7$. Since $n \geq 3$, $\kappa(XQ[i]) = 4(n-1) \geq 8$ holds for $i \in \{1, 2, \ldots, k-1\}$ by Theorem 3.3. By Theorem 3.3, $XQ[i] - F_i$ is connected for $i \in \{1, 2, \ldots, k-1\}$. Since there is a complete matching between $XQ[i]$ and $XQ[i+1]$, for $i \in \{0, 1, \ldots, k-2\}$, $XQ_n^k[V(XQ[1]-F_1) \cup \ldots \cup V(XQ[k-1]-F_{k-1})]$ is connected. Suppose that $XQ[0] - F_0$ is connected. Since $k^{n-1} > 8n-5$ and there is a complete matching between $XQ[0]$ and $XQ[1]$, $XQ_n^k - F$ is connected, a contradiction to that F is a cut of XQ_n^k. Then $XQ[0] - F_0$ is disconnected. Let $B_1, \ldots, B_k$ ($k \geq 2$) be the components of $XQ[0] - F_0$. If $k \geq 3$, then, by Proposition 2.7, $|N(V(B_1) \cup V(B_2)) \cap (V(XQ[1]) \cup \ldots \cup V(XQ[k-1]))| \geq 8$. If $|V(B_j)| \geq 2$, then, by Proposition 2.7, $|N(V(B_j)) \cap (V(XQ[1]) \cup \ldots \cup V(XQ[k-1]))| \geq 8$ ($1 \leq j \leq k$). Combining this with $|F_1| + \ldots + |F_{k-1}| \leq 7$, we have that $XQ_n^k - F$ is connected or $XQ_n^k - F$ has two components, one of which is an isolated vertex. □

Lemma 3.7. *Let* $A = \{\underbrace{0\ldots0}_{n}, 1\underbrace{0\ldots0}_{n-1}\}$*. If* $F_1 = N_{XQ_n^k}(A)$*,* $F_2 = A \cup N_{XQ_n^k}(A)$*, then* $|F_1| = 8n-4$*,* $|F_2| = 8n-2$*,* $\delta(XQ_n^k - F_1) \geq 1$*, and* $\delta(XQ_n^k - F_2) \geq 1$ *(*$n \geq 2$ *or* $n = 1$ *and* $k \geq 8$*) (See Fig. 2).*

Proof. By $A = \{\underbrace{0\ldots0}_{n}, 1\underbrace{0\ldots0}_{n-1}\}$, we have $XQ_n^k[A] = K_2$. From calculating, we have $|F_1| = |N_{XQ_n^k}(A)| = 8n-4$ and $|F_2| = |A| + |F_1| = 8n-2$ by Proposition 2.6. Suppose $n = 1$ and $k \geq 8$. From Figure 3b, $XQ_1^k - F_2$

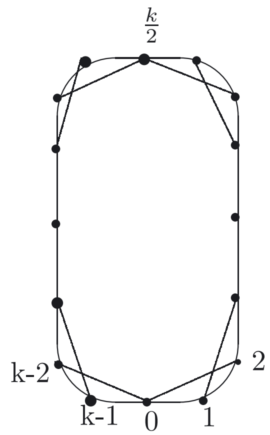

FIGURE 3. (b) An illustration about the proof of Lemma 3.7.

is connected. Therefore, $\delta(XQ_1^k - F_1) \geq 1$ and $\delta(XQ_1^k - F_2) \geq 1$. Let $n \geq 2$, $k \geq 8$ and $x \in V(XQ_n^k) \setminus F_2$. By Proposition 2.9, $|N_{XQ_n^k}(x) \cap F_2| \leq 4$. Therefore, $\delta(XQ_n^k - F_2) \geq 4n - 4 \geq 1$. Let $n \geq 3$, $k = 6$ and $x \in V(XQ_n^k) \setminus F_2$. By Proposition 2.9, $|N_{XQ_n^k}(x) \cap F_2| \leq 8$. Therefore, $\delta(XQ_n^k - F_2) \geq 4n - 8 \geq 1$.

Let $n = 2$, $k = 6$ and $x \in V(XQ_2^6) \setminus F_2$. Then $V(XQ[0]) - F_2 = \emptyset$. Suppose that $x \in V(XQ[i]) \setminus F_2$ for $i \in \{1, 2, \ldots, 5\}$. Let $u = 00$ and $v = 10$. If $x \in \{01, 02, 03, 04, 05\}$, then $x = 03$. Note $|N(x) \cap N(v)| = 0$ and hence $|N_{XQ_2^6}(x) \cap F_2| \leq 4$ in this case. Let $x \in V(XQ[i]) \setminus \{01, 02, 03, 04, 05\}$ for $i \in \{1, 2, 3, 4, 5\}$. Since $|N(u) \cap V(XQ[i])| \leq 1$ for $i \in \{1, 2, 3, 4, 5\}$, $|N(x) \cap V(XQ[0])| \leq 1$, $|N(u) \cap N(x)| \leq 2$ holds. Similarly, $|N(v) \cap N(x)| \leq 2$. Therefore, $|N_{XQ_2^6}(x) \cap F_2| \leq 4$ and hence $\delta(XQ_2^6 - F_2) \geq 4 \times 2 - 4 \geq 1$. Note that $XQ_2^6 - F_1$ has two parts $XQ_2^6 - F_2$ and $XQ_2^6[A] = K_2$. Note that $\delta(XQ_2^6[A]) = 1$. Therefore, $\delta(XQ_2^6 - F_1) \geq 1$. □

Theorem 3.8. *Let XQ_n^k be the expanded k-ary n-cube with $n \geq 1$ and even $k \geq 6$, Then the nature connectivity of XQ_n^k is $8n - 4$, i.e., $\kappa^*(XQ_n^k) = 8n - 4$.*

Proof. Let $A = \{\underbrace{0 \ldots 0}_{n}, 1\underbrace{0 \ldots 0}_{n-1}\}$ in Lemma 3.7. Then $|N(A)| = 8n - 4$. Since $N(A)$ is a nature cut of XQ_n^k, $\kappa^*(XQ_n^k) \leq 8n - 4$ holds.

By Proposition 3.6, if $F \subseteq V(XQ_n^k)$ with $|F| \leq 8n - 5$, then $XQ_n^k - F$ is connected or $XQ_n^k - F$ has two components, one of which is an isolated vertex. Therefore, if F is a nature cut of XQ_n^k, then $|F| \geq 8n - 4$. Combining this with $\kappa^*(XQ_n^k) \leq 8n - 4$, we have that $\kappa^*(XQ_n^k) = 8n - 4$. □

4. THE NATURE DIAGNOSABILITY OF THE EXPANDED k-ARY n-CUBE UNDER THE PMC MODEL

In this section, we shall show the nature diagnosability of the he expanded k-ary n-cube under the PMC model.

Lemma 4.1. *Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is less than or equal to $8n - 3$, i.e., $t_n(XQ_n^k) \leq 8n - 3$.*

Proof. Let A be defined in Lemma 3.7, and let $F_1 = N_{XQ_n^k}(A)$, $F_2 = A \cup N_{XQ_n^k}(A)$. By Lemma 3.7, $|F_1| = 8n - 4$, $|F_2| = 8n - 2$, $\delta(XQ_n^k - F_1) \geq 1$ and $\delta(XQ_n^k - F_2) \geq 1$. Therefore, F_1 and F_2 are both nature faulty sets of XQ_n^k with $|F_1| = 8n - 4$ and $|F_2| = 8n - 2$. Since $A = F_1 \triangle F_2$ and $N_{XQ_n^k}(A) = F_1 \subset F_2$, there is no edge of XQ_n^k between $V(XQ_n^k)\backslash(F_1 \cup F_2)$ and $F_1 \triangle F_2$. By Theorem 2.1, we can deduce that XQ_n^k is not nature $(8n-2)$-diagnosable under the PMC model. Hence, by the definition of the nature diagnosability, we conclude that the nature diagnosability of XQ_n^k is less than $8n-2$, *i.e.*, $t_n(XQ_n^k) \leq 8n-3$. □

Lemma 4.2. *Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is more than or equal to $8n-3$,* i.e.*, $t_n(XQ_n^k) \geq 8n-3$.*

Proof. By the definition of the nature diagnosability, it is sufficient to show that XQ_n^k is nature $(8n-3)$-diagnosable. By Theorem 2.1, to prove XQ_n^k is nature $(8n-3)$-diagnosable, it is equivalent to prove that there is an edge $uv \in E(XQ_n^k)$ with $u \in V(XQ_n^k)\backslash(F_1 \cup F_2)$ and $v \in F_1 \triangle F_2$ for each distinct pair of nature faulty subsets F_1 and F_2 of $V(XQ_n^k)$ with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$.

We prove this statement by contradiction. Suppose that there are two distinct nature faulty subsets F_1 and F_2 of $V(XQ_n^k)$ with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$, but the vertex set pair (F_1, F_2) is not satisfied with the condition in Theorem 2.1, *i.e.*, there are no edges between $V(XQ_n^k)\backslash(F_1 \cup F_2)$ and $F_1 \triangle F_2$. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Suppose $V(XQ_n^k) = F_1 \cup F_2$. By the definition of XQ_n^k, $|F_1 \cup F_2| = k^n$. It is obvious that $k^n > 16n-6$ for $n \geq 2$. Since $n \geq 5$, we have that $k^n = |V(XQ_n^k)| = |F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq |F_1| + |F_2| \leq 2(8n-3) = 16n-6$, a contradiction. Therefore, $V(XQ_n^k) \neq F_1 \cup F_2$.

Since there are no edges between $V(XQ_n^k)\backslash(F_1 \cup F_2)$ and $F_1 \triangle F_2$, and F_1 is a nature faulty set, $XQ_n^k - F_1$ has two parts $XQ_n^k - F_1 - F_2$ and $XQ_n^k[F_2 \backslash F_1]$ (for convenience). Thus, $\delta(XQ_n^k - F_1 - F_2) \geq 1$ and $\delta(XQ_n^k[F_2 \backslash F_1]) \geq 1$. Similarly, $\delta(XQ_n^k[F_1 \setminus F_2]) \geq 1$ when $F_1 \setminus F_2 \neq \emptyset$. Therefore, $F_1 \cap F_2$ is also a nature faulty set. When $F_1 \setminus F_2 = \emptyset$, $F_1 \cap F_2 = F_1$ is also a nature faulty set. Since there are no edges between $V(XQ_n^k - F_1 - F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut. By Theorem 3.8, $|F_1 \cap F_2| \geq 8n-4$. Note that $|F_2 \backslash F_1| \geq 2$. Therefore, $|F_2| = |F_2 \backslash F_1| + |F_1 \cap F_2| \geq 2 + 8n - 4 = 8n-2$, which contradicts with that $|F_2| \leq 8n-3$. So XQ_n^k is nature $(8n-3)$-diagnosable. By the definition of $t_n(XQ_n^k)$, $t_n(XQ_n^k) \geq 8n-3$. □

Combining Lemmas 4.1 and 4.2, we have the following theorem.

Theorem 4.3. *Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the PMC model is $8n-3$.*

5. The nature diagnosability of the expanded k-ary n-cube XQ_n^k under the MM* model

In this section, we shall show the nature diagnosability of the he expanded k-ary n-cube under the MM* model.

Lemma 5.1. *Let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the MM* model is less than or equal to $8n-3$,* i.e.*, $t_n(XQ_n^k) \leq 8n-3$.*

Proof. Let A, F_1 and F_2 be defined in Lemma 3.7 (See Fig. 2). By Lemma 3.7, $|F_1| = 8n-4$, $|F_2| = 8n-2$, $\delta(XQ_n^k - F_1) \geq 1$ and $\delta(XQ_n^k - F_2) \geq 1$. So both F_1 and F_2 are nature faulty sets. By the definitions of F_1 and F_2, $F_1 \triangle F_2 = A$. Note $F_1 \setminus F_2 = \emptyset$, $F_2 \setminus F_1 = A$ and $(V(XQ_n^k) \setminus (F_1 \cup F_2)) \cap A = \emptyset$. Therefore, both F_1 and F_2 are not satisfied with any one condition in Theorem 2.2, and XQ_n^k is not nature $(8n-2)$-diagnosable. Hence, $t_n(XQ_n^k) \leq 8n-3$. □

Lemma 5.2. *Let $n \geq 2$ and let XQ_n^k be the expanded k-ary n-cube with even $k \geq 6$. Then the nature diagnosability of XQ_n^k under the MM* model is more than or equal to $8n-3$,* i.e.*, $t_n(XQ_n^k) \geq 8n-3$.*

Proof. By the definition of nature diagnosability, it is sufficient to show that XQ_n^k is nature $(8n-3)$-diagnosable. By Theorem 2.2, suppose, on the contrary, that there are two distinct nature faulty subsets F_1 and F_2 of XQ_n^k with $|F_1| \leq 8n-3$ and $|F_2| \leq 8n-3$, but the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 2.2. Without loss of generality, assume that $F_2 \setminus F_1 \neq \emptyset$. Similarly to the discussion on $V(XQ_n^k) \neq F_1 \cup F_2$ in Lemma 4.2, we can deduce $V(XQ_n^k) \neq F_1 \cup F_2$. Therefore, $V(XQ_n^k) \neq F_1 \cup F_2$.

Claim 1. $XQ_n^k - F_1 - F_2$ has no isolated vertex.

Suppose, on the contrary, that $XQ_n^k - F_1 - F_2$ has at least one isolated vertex w. Since F_1 is a nature faulty set, there is a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 2.2, there is at most one vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Thus, there is just a vertex $u \in F_2 \setminus F_1$ such that u is adjacent to w. Assume $F_1 \setminus F_2 = \emptyset$. Then $F_1 \subseteq F_2$. Since F_2 is a nature faulty set, $XQ_n^k - F_2 = XQ_n^k - F_1 - F_2$ has no isolated vertex, a contradiction. Therefore, let $F_1 \setminus F_2 \neq \emptyset$ as follows. Similarly, we can deduce that there is just a vertex $v \in F_1 \setminus F_2$ such that v is adjacent to w. Let $W \subseteq V(XQ_n^k) \setminus (F_1 \cup F_2)$ be the set of isolated vertices in $XQ_n^k[V(XQ_n^k) \setminus (F_1 \cup F_2)]$, and let H be the subgraph induced by the vertex set $V(XQ_n^k) \setminus (F_1 \cup F_2 \cup W)$. Then for any $w \in W$, there are $(4n-2)$ neighbors in $F_1 \cap F_2$. Since $|F_2| \leq 8n-3$, we have $\sum_{w \in W} |N_{XQ_n^k[(F_1 \cap F_2) \cup W]}(w)| = |W|(4n-2) \leq \sum_{v \in F_1 \cap F_2} d_{XQ_n^k}(v) \leq |F_1 \cap F_2|(4n-2) \leq (|F_2|-1)(4n-2) \leq (8n-4)(4n-2) = 32n^2 - 32n + 8$. It follows that $|W| \leq \frac{32n^2-32n+8}{4n-2} \leq 8n-4$. Note $|F_1 \cup F_2| = |F_1| + |F_2| - |F_1 \cap F_2| \leq 2(8n-3) - (4n-2) = 12n-4$. Suppose $V(H) = \emptyset$. Then $k^n = |V(XQ_n^k)| = |F_1 \cup F_2| + |W| \leq 12n - 4 + 8n - 4 = 20n - 8$. This is a contradiction to $n \geq 2$. So $V(H) \neq \emptyset$. Since the vertex set pair (F_1, F_2) is not satisfied with the condition (1) of Theorem 2.2, and any vertex of $V(H)$ is not isolated in H, we induce that there is no edge between $V(H)$ and $F_1 \triangle F_2$. Thus, $F_1 \cap F_2$ is a vertex cut of XQ_n^k and $\delta(XQ_n^k - (F_1 \cap F_2)) \geq 1$, *i.e.*, $F_1 \cap F_2$ is a nature cut of XQ_n^k. By Theorem 3.8, $|F_1 \cap F_2| \geq 8n-4$. Because $|F_1| \leq 8n-3$, $|F_2| \leq 8n-3$, and neither $F_1 \setminus F_2$ nor $F_2 \setminus F_1$ is empty, we have $|F_1 \setminus F_2| = |F_2 \setminus F_1| = 1$. Let $F_1 \setminus F_2 = \{v_1\}$ and $F_2 \setminus F_1 = \{v_2\}$. Then for any vertex $w \in W$, w are adjacent to v_1 and v_2. According to Proposition 2.9, there are at most two common neighbors for any pair of vertices in XQ_n^k when $k \geq 8$, it follows that there are at most two isolated vertices in $XQ_n^k - F_1 - F_2$, *i.e.*, $|W| \leq 2$.

Suppose that there is exactly one isolated vertex v in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, v_1\})| \leq 1$ and $|[N_{XQ_n^k}(v_1) \setminus \{v\}] \cap [N_{XQ_n^k}(v_2) \setminus \{v\}]| \leq 1$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, v_1\}| = (4n-2) + (4n-2) + (4n-2) - 3 = 12n - 9$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 12n - 9 = 12n - 8 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n-3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(w) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. By Proposition 2.9, there are at most two common neighbors for any pair of vertices in XQ_n^k. Thus, it follows that $|(N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| = 0$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(w) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\}| = (4n-2) + (4n-2) + (4n-3) + (4n-3) - 1 - 1 - 1 - 1 = 16n - 14$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 16n - 14 = 16n - 13 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n-3$.

Suppose that $k = 6$, and v_1 and v_2 are adjacent. By Proposition 2.9, $|N(v_1) \cap N(v_2)| \leq 2$. Therefore, $|W| \leq 2$.

Suppose that there is exactly one isolated vertex v in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, v_1\})| \leq 1$ and $|[N_{XQ_n^k}(v_1) \setminus \{v\}] \cap [N_{XQ_n^k}(v_2) \setminus \{v\}]| \leq 1$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, v_1\}| = (4n-2) + (4n-2) + (4n-2) - 3 = 12n - 9$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 12n - 9 = 12n - 8 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n-3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(w) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\})| \leq 1$

and $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| \leq 1$. By Proposition 2.9, there are at most two common neighbors for any pair of vertices in XQ_n^k $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, it follows that $|(N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\})| = 0$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(w) \setminus \{v_1, v_2\})| = 0$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(w) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, w, v_2\}| + |N_{XQ_n^k}(v_2) \setminus \{v, w, v_1\}| = (4n-2) + (4n-2) + (4n-3) + (4n-3) - 1 - 1 - 1 - 1 = 16n - 14$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 16n - 14 = 16n - 13 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n - 3$.

Suppose that $k = 6$, and v_1 and v_2 are not adjacent. By Proposition 2.9, $|N(v_1) \cap N(v_2)| \leq 4$ and hence $|W| \leq 4$. If $|N(v_1) \cap N(v_2)| = 4$, then $v_1, v_2 \in V(XQ[i])$. From Figure 3, $XQ_1^6[N(v_1) \cap N(v_2)]$ is connected. Therefore, $|W| \leq 3$. Since $|N(v_1) \cap N(v_2)| \neq 3$, $|W| \leq 2$ holds.

Suppose that there is exactly one isolated vertex v in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v\})| \leq 2$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v\})| \leq 2$ and $|[N_{XQ_n^k}(v_1) \setminus \{v\}] \cap [N_{XQ_n^k}(v_2) \setminus \{v\}]| \leq 3$. Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v\}| + |N_{XQ_n^k}(v_2) \setminus \{v\}| = (4n-2) + (4n-1) + (4n-1) - 2 - 2 - 3 = 12n - 11$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 12n - 11 = 12n - 10 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n - 3$.

Suppose that there are exactly two isolated vertices v and w in $XQ_n^k - F_1 - F_2$. Let v_1 and v_2 be adjacent to v and w, respectively. Then $N_{XQ_n^k}(v) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(w) \setminus \{v_1, v_2\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_1) \setminus \{v, w\} \subseteq F_1 \cap F_2$, $N_{XQ_n^k}(v_2) \setminus \{v, w\} \subseteq F_1 \cap F_2$, $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w\})| \leq 2$ and $|(N_{XQ_n^k}(v) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w\})| \leq 2$. $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_1) \setminus \{v, w\})| \leq 2$ and $|(N_{XQ_n^k}(w) \setminus \{v_1, v_2\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w\})| \leq 2$. By Proposition 2.9, there are at most four common neighbors for any pair of vertices in XQ_n^k. Thus, it follows that $|(N_{XQ_n^k}(v_1) \setminus \{v, w\}) \cap (N_{XQ_n^k}(v_2) \setminus \{v, w\})| \leq 2$.

Thus, $|F_1 \cap F_2| \geq |N_{XQ_n^k}(v) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(w) \setminus \{v_1, v_2\}| + |N_{XQ_n^k}(v_1) \setminus \{v, w\}| + |N_{XQ_n^k}(v_2) \setminus \{v, w\}| = (4n-2) + (4n-2) + (4n-2) + (4n-2) - 2 - 2 - 2 - 2 - 2 = 16n - 18$. It follows that $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 1 + 16n - 18 = 16n - 17 > 8n - 3$ $(n \geq 2)$, which contradicts $|F_2| \leq 8n - 3$.

The proof of Claim 1 is complete.

Let $u \in V(XQ_n^k) \setminus (F_1 \cup F_2)$. By Claim 1, u has at least one neighbor in $XQ_n^k - F_1 - F_2$. Since the vertex set pair (F_1, F_2) is not satisfied with any one condition in Theorem 2.2, by the condition (1) of Theorem 2.2, for any pair of adjacent vertices $u, w \in V(XQ_n^k) \setminus (F_1 \cup F_2)$, there is no vertex $v \in F_1 \triangle F_2$ such that $uw \in E(XQ_n^k)$ and $vw \in E(XQ_n^k)$. It follows that u has no neighbor in $F_1 \triangle F_2$. By the arbitrariness of u, there is no edge between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$. Since $F_2 \setminus F_1 \neq \emptyset$ and F_1 is a nature faulty set, $\delta_{XQ_n^k}([F_2 \setminus F_1]) \geq 1$ and hence $|F_2 \setminus F_1| \geq 2$. Since both F_1 and F_2 are nature faulty sets, and there is no edge between $V(XQ_n^k) \setminus (F_1 \cup F_2)$ and $F_1 \triangle F_2$, $F_1 \cap F_2$ is a nature cut of XQ_n^k. By Theorem 3.8, we have $|F_1 \cap F_2| \geq 8n - 4$. Therefore, $|F_2| = |F_2 \setminus F_1| + |F_1 \cap F_2| \geq 2 + (8n - 4) = 8n - 2$, which contradicts $|F_2| \leq 8n - 3$. Therefore, XQ_n^k is nature $(8n-3)$-diagnosable and $t_n(XQ_n^k) \geq 8n - 3$. The proof is complete. □

Combining Lemmas 5.1 and 5.2, we have the following theorem.

Theorem 5.3. *Let $n \geq 2$. Then the nature diagnosability of the expanded k-ary n-cube XQ_n^k under the MM^* model is $8n - 3$.*

6. Conclusions

In this paper, we investigate the problem of the nature diagnosability of the expanded k-ary n-cube XQ_n^k under the PMC model and MM*. It is proved that the nature diagnosability of XQ_n^k under the PMC model and MM* model is $8n - 3$ for $n \geq 2$. The work will help engineers to develop more different measures of the nature diagnosability based on application environment, network topology, network reliability, and statistics related to fault patterns.

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References

[1] N.R. Adiga, M.A. Blumrich, D. Chen, P. Coteus, A. Gara, M.E. Giampapa, P. Heidelberger, S. Singh, B.D. Steinmacher−Burow, T. Takken, M. Tsao and P. Vranas, Blue Gene/L torus interconnection network. *IBM J. Res. Dev.* **49** (2005) 265–276.

[2] F. Barsi, F. Grandoni and P. Maestrini, A theory of diagnosability of digital systems. *IEEE Trans. Comput.* **25** (1976) 585–593.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory. Springer, New York (2007).

[4] Bella Bose, Bob Broeg, Younggeun Kwon and Yaagoub Ashir, Lee distance and topological properties of k-ary n-cubes. *IEEE Trans. Comput.* **44** (1995) 1021–1030.

[5] Nai-Wen Chang and Sun−Yuan Hsieh, Structural properties and conditional diagnosability of star graphs by using the PMC model, *IEEE Trans. Parallel Distrib. Syst.* **25** (2014) 3002–3011.

[6] A.T. Dahbura and G.M. Masson, An $O(n^{2.5})$ fault identification algorithm for diagnosable systems, *IEEE Trans. Comput.* **33** (1984) 486–492.

[7] Kh. Day and A.-E. Al-Ayyoub and F. diameter of k-ary n-cube networks. *IEEE Trans. Parallel and Distrib. Syst.* **8** (1997) 903–907.

[8] J. Fan, Diagnosability of crossed cubes under the comparison diagnosis model. *IEEE Trans. Parallel Distrib. Syst.* **13** (2002) 1099–1104.

[9] Y. Hao and Shiying Wang, The 1-good-neighbor diagnosibility of augmented k-ary n-cubes. *Adv. Appl. Math.* **5** (2016) 762–772.

[10] Th.W. Hungerford, Algebra. Springer-Verlag, New York (1974).

[11] R.E. Kessler and J.L. Schwarzmeier, Cray T3D: a new dimension for Cray research, in *Proc. 38th IEEE Comput. Soc. Inter. Confer., Spring, San Francisco* (1993) 176–182.

[12] P.-Lien Lai, J.M. Tan, Ch.-P. Chang and Lih−Hsing Hsu, Conditional diagnosability measures for large multiprocessor systems. *IEEE Trans. Comput.* **54** (2005) 165–175.

[13] Cheng−Kuan Lin, Jimmy J.M. Tan, Lih−Hsing Hsu, Eddie Cheng and László Lipták, Conditional diagnosability of Cayley graphs generated by transposition trees under the comparison diagnosis model. *J. Interconnection Netw.* **9** (2008) 83–97.

[14] J. Maeng and M. Malek, A comparison connection assignment for self-diagnosis of multiprocessor systems, in *Proc. 11th Inter. Symp. Fault-Tolerant Comput.* (1981) 173–175.

[15] M. Noakes and W.J. Dally, System design of the J-machine, in: Proceedings of the sixth MIT conference on Advanced research in VLSI. MIT Press, Cambridge (1990) 179–194.

[16] Shao−Lun Peng, Cheng−Kuan Lin, Ji.J.M. Tan and Lih−Hsing Hsu, The g-good-neighbor conditional diagnosability of hypercube under PMC model. *Appl. Math. Comput.* **218** (2012) 10406–10412.

[17] C. Peterson, J. Sutton and P. Wiley, iWarp: a 100-MOPS VLIW microprocessor for multicomputers. *IEEE Micro* **11** (1991) 26–37.

[18] F.P. Preparata, G. Metze and R.T. Chien, On the connection assignment problem of diagnosable systems. *IEEE Trans. Comput.* **EC-16** (1967) 848–854.

[19] Mujiangshan Wang, Yubao Guo and Shiying Wang, The 1-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the PMC model and MM* model. *Int. J. Comput. Math.* **94** (2017) 620–631.

[20] Mujiangshan Wang, Yuqing Lin and Shiying Wang, The 2-good-neighbor diagnosability of Cayley graphs generated by transposition trees under the PMC model and MM* model. *Theoret. Comput. Sci.* **628** (2016) 92–100.

[21] Shiying Wang and Weiping Han, The g-good-neighbor conditional diagnosability of n-dimensional hypercubes under the MM* model. *Inf. Process. Lett.* **116** (2016) 574–577.

[22] Shiying Wang, Zhenhua Wang and Mujiangshan Wang, The 2-extra connectivity and 2-extra diagnosability of bubble-sort star graph networks. *Comput. J.* **59** (2016) 1839–1856.

[23] Shiying Wang, Zhenhua Wang and Mujiangshan Wang, The 2-good-neighbor connectivity and 2-good-neighbor diagnosability of bubble-sort star graph networks. *Discrete Appl. Math.* **217** (2017) 691–706.

[24] Shiying Wang and Yuxing Yang, The 2-good-neighbor (2-extra) diagnosability of alternating group graph networks under the PMC model and MM* model. *Appl. Math. Comput.* **305** (2017) 241–250.

[25] Yonghong Xiang and Iain A. Stewart, Augmented k-ary n-cubes. *Inf. Sci.* **181** (2011) 239–256.

[26] Jun Yuan, Aixia Liu, Xue Ma, Xiuli Liu, Xiao Qin and Jifu Zhang, The g-good-neighbor conditional diagnosability of k-ary n-cubes under the PMC model and MM* model. *IEEE Trans. Parallel Distrib. Syst.* **26** (2015) 1165–1177.

[27] Jun Yuan, Aixia Liu, Xiao Qin, Jifu Zhang and Jing Li, g-Good-neighbor conditional diagnosability measures for 3-ary n-cube networks. *Theoret. Comput. Sci.* **622** (2016) 144–162.

[28] Shurong Zhang and Weihua Yang, The g-extra conditional diagnosability and sequential t/k-diagnosability of hypercubes. *Inter. J. Comput. Math.* **93** (2016) 482–497.

[29] Nan Zhao and Shiying Wang, The 1-good-neighbor diagnosability of augmented 3-ary n-cubes. *Adv. Appl. Math.* **5** (2016) 754–761.

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