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A BETTER BOUND OF RANDOMIZED ALGORITHMS FOR THE MULTISLOPE SKI-RENTAL PROBLEM *

Maolin Hu[1] and Weijun Xu[2,**]

Abstract. The multislope ski-rental problem is an extension of the classical ski-rental problem, where the player has several lease options in addition to the pure rent and buy options. For the additive general model, Lotker, Patt-Shamir and Rawitz [in: *SIAM J. Discr. Math.* **26** (2012) 718–736] obtained a randomized algorithm with the competitive ratio bounded by $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$. However, obtaining a better bound on the competitive factor as a function of the slopes parameters remains an open problem in their paper. In this paper, we study randomized algorithm for the additive multislope ski rental problem, and extend the competitive ratio bound $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$ proposed by Lotker *et al.* to $\frac{\mathrm{e}}{\mathrm{e}-1+r_k/r_0}$.

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1. Introduction

Research on online rental (rent-or-buy) problems began in 1992 when Karp [7] put forward the classical "ski rental" model, which is very famous in the field of theoretical computer sciences. A person plans to go ski, but he/she doesn't know the exact number of days to ski. As a result, he/she has to consider about renting or buying skis. To rent skis, he/she needs to pay 1 per day; to buy the skis, the cost is s ($s > 1$) and this will free him/her from renting any longer. The dilemma is: which costs less, to rent, to buy, or to rent for the first few days, then to buy? Karp proved that the best choice is to rent skis in the first $s-1$ days, and to buy the skis in the sth day if the skier continues to ski. The competitive ratio in this case is $2 - \frac{1}{s}$. This problem in continuous time case is discussed by Karlin *et al.* [5, 6] as following. In the buy option, a one-time cost is incurred, and thereafter usage is free of charge. In the rent option, the cost is proportional to usage time, and there is no one-time cost. The deterministic competitive ratio is 2. In the randomized model, the algorithm chooses a random time to switch from the rent to the buy option (the adversary is assumed to know the algorithm but not the actual outcomes of random experiments). The optimal randomized online algorithm for this classical ski rental or buy problem has a competitive ratio of $\frac{\mathrm{e}}{\mathrm{e}-1} \approx 1.582$. The algorithm is to select the rent and the buy according to

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[1] School of Mathematical Science, Huaiyin Normal University, Huaian, Jiangsu 223300, P.R. China. humaolin2000@163.com

[2] School of Business Administration, South China University of Technology, Guangzhou 510641, P.R. China. xuwj@scut.edu.cn
** Corresponding author: Weijun Xu

the probability density function $p_0(t)$ and $p_1(t)$, respectively, for $t \geq 0$ if the game has not ended yet, where

$$p_0(t) = \begin{cases} \frac{\mathrm{e}-\mathrm{e}^t}{\mathrm{e}-1} & 0 \leq t \leq 1 \\ 0, & t \geq 1. \end{cases} \qquad p_1(t) = \begin{cases} \frac{\mathrm{e}^t-1}{\mathrm{e}-1} & 0 \leq t \leq 1 \\ 1, & t \geq 1. \end{cases}$$

A strict generalization of the two-slope discrete bahncard problem was studied by Fleischer [4]. Furthermore, Lotker *et al.* [8] investigated the continuous version of two-slope ski rental without pure buy. In their model, there are two options: in option 1 the buy cost is 0 and the rental rate is 1 for each time unit, and in option 2 the buy cost is $(1-a)$ and the rental rate is a for each time unit, where $0 \leq a < 1$. Lotker *et al.* [8] obtained a randomized algorithm with optimal expected competitive ratio of $\mathrm{e}/(\mathrm{e}-1+a)$. Lotker *et al.* [9] also considered a multislope ski rental problem. In the additive model, for the case of pure buy option $r_k = 0$, they obtained an randomized algorithm with competitive ratio of $\frac{\mathrm{e}}{\mathrm{e}-1}$. However, for $r_k > 0$, they showed that the competitive ratio is bounded by $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$. How to obtain a better bound on the competitive factor as a function of the slopes parameters remains an open problem in their paper. In this paper, we study randomized algorithms for the additive multislope ski rental problem, and extend the competitive ratio bound $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$ proposed by Lotker *et al.* [9] to the ratio $\frac{\mathrm{e}}{\mathrm{e}-1+r_k/r_0}$.

2. Problem statement and preliminaries

In this section we formalize the additive version of the multislope ski rental problem. The treatment and notation of multislope ski rental problem in this paper will follow that of [9]. A $k+1$-slope ski rental problem is defined by a set of $k+1$ slopes, and for each slope $i \in \{0, 1, \ldots, k\}$ there is a buying cost b_i and a renting cost rate r_i. Without loss of generality (see the justification in Sect. 2 in [9]), we may assume $b_0 = 0 < b_1 < \ldots < b_k$ and $r_0 > r_1 > \ldots > r_k \geq 0$. If holding the resource under slope i for t time units, then the user is charged $b_i + r_i t$ cost units. In the additive case, buying costs are cumulative, namely to move from slope i to slope j we need only pay the difference in buying prices $b_j - b_i$. The slope can be represented by a line: the ith slope corresponds to the line $c = b_i + r_i t$. Similar to Lotker's work [9], Figure 1 shows geometrical interpretation of a multislope ski rental instance with k + 1 slopes, where $b_0 = 0$. s_i is the time where slopes $i-1$ and i intersect, and satisfies $s_0 = 0, s_i = \frac{b_i - b_{i-1}}{r_{i-1} - r_i}, i = 1, 2, \ldots, k$.

An adversary gets to choose how long time t the game will last. As offline player, she (he) knows the exact time t. Therefore, for any given time $t > 0$, the offline optimal algorithm is to select the slope i when $s_i < t \leq s_{i+1}$ $(i = 0, 1, 2, \ldots, k-1)$, and select the slope k when $t > s_k$. More formally, the optimal offline cost at time t is

$$C_{OPT}(t) = \begin{cases} b_i + r_i t, & s_i < t \leq s_{i+1}, i = 0, 1, 2, \ldots, k-1; \\ b_k + r_k t, & t > s_k. \end{cases}$$

As online player, he does not know the exact time t, and the task is to minimize total cost until the game is over. Therefore, in order to make the online cost as small as possible, as the time t given by the offline adversary increasing, an deterministic (online) algorithms has to transfer from one slope to another slope. If the game ends at time t, the optimal solution is to select the slope with the least cost at time t. The thick line in Figure 1 denotes the optimal cost for any given t. We assume that slope transitions can be only forward, *i.e.*, the only transitions allowed are of the type $i \to j$ for $j > i$. In fact, this assumption does not restrict generality in the additive model.

A randomized (online) algorithm can be described using a probability distribution over the family of deterministic algorithms. However, for our purpose a more general formalism used by Lotker *et al.* [9] is adopted in this paper to describe randomized algorithms. For all times t, we specify a probability distribution over the set of $k+1$ slopes that determines the actual cost paid by any online algorithm. In [9], a randomized profile (or simply a profile) is specified by a vector $p(t) = \big(p_0(t), p_1(t), \ldots, p_k(t)\big)$ of $k+1$ functions, where $p_i(t)$ is the probability of being in slope i at time t. The correctness requirement of a profile is $\sum_{i=0}^{k} p_i(t) = 1$ for all $t \geq 0$.

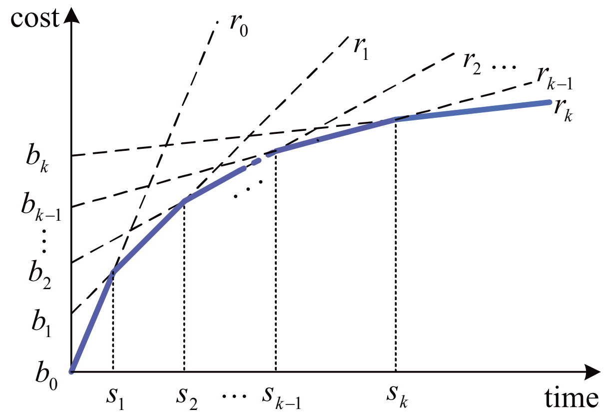

FIGURE 1. $k+1$ slopes ski rental problem.

Lotker *et al.* [9] verified that any randomized online algorithm is related to some profile. Therefore, looking to find an randomized algorithm is equivalent to a profile. The performance of a profile is defined by its total accrued cost, which consists of expected cost of $k+1$ slopes. Given a randomized profile p, the expected rental cost at time $t>0$ is defined as following

$$C_p(t)=\sum_{i=0}^{k} r_i\int_0^t p_i(\tau)\mathrm{d}\tau+\sum_{i=0}^{k} b_i p_i(t).$$

As is customary, we shall be interested in the competitive ratio of a profile p for the $k+1$ slopes rental problem, defined to be

$$R_k=\sup\left\{C_p(t)/C_{OPT}(t)|t>0\right\}.$$

3. A RANDOMIZED ALGORITHMS AND ITS COMPETITIVE RATIO

In this section, we firstly show a randomized algorithm of the $k+1$-slope ski-rental problem that is a generalization of the optimal algorithm in [8] for the 2-slope case, then discuss its competitive ratio.

Theorem 3.1. *For the $k+1$-slope ski rental problem, the following profile p*

$$p_0(t)=\begin{cases}\frac{\mathrm{e}-\mathrm{e}^{\frac{t}{s_1}}+\frac{r_k}{r_0}}{\mathrm{e}-1+\frac{r_k}{r_0}}, & t\le s_1;\\ \frac{\frac{r_k}{r_0}}{\mathrm{e}-1+\frac{r_k}{r_0}}, & t\ge s_1;\end{cases}$$

$$p_j(t)=\begin{cases}\frac{\mathrm{e}^{\frac{t}{s_j}}-\mathrm{e}^{\frac{t}{s_{j+1}}}}{\mathrm{e}-1+\frac{r_k}{r_0}}, & t\le s_j;\\ \frac{\mathrm{e}-\mathrm{e}^{\frac{t}{s_{j+1}}}}{\mathrm{e}-1+\frac{r_k}{r_0}}, & s_j\le t\le s_{j+1};\\ 0, & t\ge s_{j+1}.\end{cases}\qquad j=1,2,\dots k-1$$

$$p_k(t)=\begin{cases}\frac{\mathrm{e}^{\frac{t}{s_k}}-1}{\mathrm{e}-1+\frac{r_k}{r_0}}, & t\le s_k;\\ \frac{\mathrm{e}-1}{\mathrm{e}-1+\frac{r_k}{r_0}}, & t\ge s_k.\end{cases}$$

is corresponding to a randomized online algorithm with competitive ratio

$$R_k = \frac{\mathrm{e}}{\mathrm{e} - 1 + \frac{r_k}{r_0}},$$

where $p_j(t)$ is the probability function on slope j $(j = 0, 1, 2, \dots k)$.

Proof. Firstly, it is easy to show that for any $t \geq 0$, each $p_j(t)\,(j = 0, 1, 2, \dots k)$ belonging to profile p is a nonnegative continuous function and satisfies $\sum_{j=0}^{k} p_j(t) = 1$. Therefore, the profile p is well defined. Next, we prove the competitive ratio for the profile p is $\mathrm{e}/(\mathrm{e} - 1 + r_k/r_0)$. For simplicity, E is short for $\mathrm{e} - 1 + r_k/r_0$.

(i) When $0 < t \leq s_1$, for arbitrary $0 \leq \tau \leq t$, each $p_j(\tau)$ $(j = 0, 1, 2, \dots k)$ belonging to profile p is as follows:

$$p_0(\tau) = \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_1}} + \frac{r_k}{r_0}}{E}, \quad p_j(\tau) = \frac{\mathrm{e}^{\frac{\tau}{s_i}} - \mathrm{e}^{\frac{\tau}{s_{i+1}}}}{E} \quad (j = 1, 2, \dots k-1), \quad p_k(\tau) = \frac{\mathrm{e}^{\frac{\tau}{s_k}} - 1}{E}.$$

From $b_j - b_{j-1} = s_j(r_{j-1} - r_j), j = 1, 2, 3, \dots, k, b_0 = 0$, we have

$$\begin{aligned}\sum_{j=0}^{k} r_j \int_0^t p_j(\tau)\mathrm{d}\tau &= r_0 \int_0^t \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_1}} + \frac{r_k}{r_0}}{E}\mathrm{d}\tau + \sum_{j=1}^{k-1} r_j \int_0^t \frac{\mathrm{e}^{\frac{\tau}{s_j}} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{E}\mathrm{d}\tau + r_k \int_0^t \frac{\mathrm{e}^{\frac{\tau}{s_k}} - 1}{E}\mathrm{d}\tau \\ &= \frac{1}{E}\left[r_0\ et\ + \sum_{j=1}^{k} s_j(r_{j-1} - r_j)\left(1 - \mathrm{e}^{\frac{t}{s_j}}\right)\right] \\ &= \frac{1}{E}\left[r_0\ et\ + b_k - \sum_{j=1}^{k}(b_j - b_{j-1})\mathrm{e}^{\frac{t}{s_j}}\right].\end{aligned}$$

$$\sum_{j=0}^{k} b_j p_j(t) = \sum_{j=1}^{k-1} b_j \frac{\mathrm{e}^{\frac{t}{s_j}} - \mathrm{e}^{\frac{t}{s_{i+1}}}}{E} + b_k \frac{\mathrm{e}^{\frac{t}{s_k}} - 1}{E} = \frac{1}{E}\left[\sum_{j=1}^{k}(b_j - b_{j-1})\mathrm{e}^{\frac{t}{s_j}} - b_k\right].$$

Thus, the total expected cost of profile p is

$$C_p(0 < t \leq s_1) = \sum_{j=0}^{k} r_j \int_0^t p_j(\tau)\mathrm{d}\tau + \sum_{j=0}^{k} b_j p_j(t) = \frac{r_0\ et}{E}\cdot$$

Because of $0 < t \leq s_1$, the optimal offline cost $C_{OPT}(0 < t \leq s_1) = r_0 t$. Therefore we have

$$R_k(0 < t \leq s_1) = \frac{C(0 < t \leq s_1)}{r_0 t} = \frac{e}{E}. \tag{3.1}$$

(ii) When $s_i \leq t \leq s_{i+1}$, for arbitrary $0 \leq \tau \leq t$ and positive integer $1 \leq i \leq k-1$, each $p_j(\tau)(j = 0, 1, 2, \dots k)$ belonging to profile p is as follows:

$$p_0(\tau) = \begin{cases} \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_1}} + \frac{r_k}{r_0}}{E}, & \tau \leq s_1; \\ \frac{\frac{r_k}{r_0}}{E}, & s_1 \leq \tau \leq t; \end{cases} \qquad p_j(\tau) = \begin{cases} \frac{\mathrm{e}^{\frac{\tau}{s_j}} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{E}, & \tau \leq s_j; \\ \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{E}, & s_j \leq \tau \leq s_{j+1}; \\ 0, & s_{j+1} \leq \tau \leq t; \end{cases} \quad 1 \leq j \leq i-1.$$

$$p_i(\tau) = \begin{cases} \frac{\mathrm{e}^{\frac{\tau}{s_i}} - \mathrm{e}^{\frac{\tau}{s_{i+1}}}}{E}, & \tau \leq s_i; \\ \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_{i+1}}}}{E}, & s_i \leq \tau \leq t; \end{cases} \qquad p_j(\tau) = \frac{\mathrm{e}^{\frac{\tau}{s_j}} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{E}, \tau \leq t, i+1 \leq j \leq k-1.$$

$$p_k(\tau) = \frac{\mathrm{e}^{\frac{\tau}{s_k}} - 1}{E}, \tau \leq t.$$

Hence, we have

$$r_0\int_0^t p_0(\tau)\mathrm{d}\tau = \frac{r_0}{E}\left[\int_0^{s_1}\left(\mathrm{e}-\mathrm{e}^{\frac{\tau}{s_1}}+\frac{r_k}{r_0}\right)\mathrm{d}\tau+\int_{s_1}^t\frac{r_k}{r_0}\mathrm{d}\tau\right]=\frac{(s_1r_0+r_kt)}{E}, \tag{3.2}$$

$$\begin{aligned} r_j\int_0^t p_j(\tau)\mathrm{d}\tau &= \frac{r_j}{E}\left[\int_0^{s_j}(\mathrm{e}^{\frac{\tau}{s_j}}-\mathrm{e}^{\frac{\tau}{s_{j+1}}})\mathrm{d}\tau+\int_{s_j}^{s_{j+1}}(\mathrm{e}-\mathrm{e}^{\frac{\tau}{s_{j+1}}})\mathrm{d}\tau+\int_{s_{j+1}}^t 0\mathrm{d}\tau\right]\\ &= \frac{r_j(s_{j+1}-s_j)}{E},\quad j=1,2,\ldots,i-1; \end{aligned} \tag{3.3}$$

$$\begin{aligned} r_i\int_0^t p_i(\tau)\mathrm{d}\tau &= \frac{r_i}{E}\left[\int_0^{s_i}\left(\mathrm{e}^{\frac{\tau}{s_i}}-\mathrm{e}^{\frac{\tau}{s_{i+1}}}\right)\mathrm{d}\tau+\int_{s_i}^t(\mathrm{e}-\mathrm{e}^{\frac{\tau}{s_{i+1}}})\mathrm{d}\tau\right]\\ &= \frac{r_i(s_{i+1}-s_i)}{E}+\frac{r_i(et-s_{i+1}\mathrm{e}^{\frac{t}{s_{i+1}}})}{E}; \end{aligned}$$

$$r_j\int_0^t p_j(\tau)\mathrm{d}\tau = \frac{r_j}{E}\int_0^t(\mathrm{e}^{\frac{\tau}{s_j}}-\mathrm{e}^{\frac{\tau}{s_{j+1}}})\mathrm{d}\tau=\frac{r_j(s_{j+1}-s_j)}{E}+\frac{r_j(s_j\mathrm{e}^{\frac{t}{s_j}}-s_{j+1}\mathrm{e}^{\frac{t}{s_{j+1}}})}{E},$$

where $j=i+1,\, i+2,\ldots,k-1$;

$$r_k\int_0^t p_k(\tau)\mathrm{d}\tau=\frac{r_k}{E}\int_0^t(\mathrm{e}^{\frac{\tau}{s_k}}-1)\mathrm{d}\tau=\frac{r_ks_k\mathrm{e}^{\frac{t}{s_k}}}{E}-\frac{r_ks_k+r_kt}{E}\cdot$$

Using $b_j-b_{j-1}=s_j(r_{j-1}-r_j), j=1,2,3,\ldots,k, b_0=0$, we obtain

$$\begin{aligned} \sum_{j=0}^k r_j\int_0^t p_j(\tau)\mathrm{d}\tau &= \frac{1}{E}\Bigg[(s_1r_0+r_kt)+\sum_{j=1}^{k-1}r_j(s_{j+1}-s_j)+r_i\left(et\ -s_{i+1}\mathrm{e}^{\frac{t}{s_{i+1}}}\right)\\ &\quad+\sum_{j=i+1}^{k-1}r_j\left(s_j\mathrm{e}^{\frac{t}{s_j}}-s_{j+1}\mathrm{e}^{\frac{t}{s_{j+1}}}\right)+r_ks_k\mathrm{e}^{\frac{t}{s_k}}-r_ks_k-r_kt\Bigg]\\ &= \frac{1}{E}\left[r_i\ et\ +b_k-\sum_{j=i+1}^k(b_j-b_{j-1})\mathrm{e}^{\frac{t}{s_j}}\right]. \end{aligned}$$

In addition, we also obtain

$$\begin{aligned} \sum_{j=0}^k b_jp_j(t) &= \sum_{j=0}^{i-1}0+b_i\frac{\mathrm{e}-\mathrm{e}^{\frac{t}{s_{i+1}}}}{E}+\sum_{j=i+1}^{k-1}b_j\frac{\mathrm{e}^{\frac{t}{s_j}}-\mathrm{e}^{\frac{t}{s_{j+1}}}}{E}+b_k\frac{\mathrm{e}^{\frac{t}{s_k}}-1}{E}\\ &= \frac{1}{E}\left[b_ie-b_k+\sum_{j=i+1}^k(b_j-b_{j-1})\mathrm{e}^{\frac{t}{s_j}}\right]. \end{aligned}$$

Hence, the total expected cost of profile p is

$$C_p(s_i\le t\le s_{i+1})=\sum_{i=0}^k r_i\int_0^t p_i(\tau)\mathrm{d}\tau+\sum_{i=0}^k b_ip_i(t)=\frac{\mathrm{e}(b_i+r_it)}{E}.$$

Because of $s_i<t\le s_{i+1}$, the optimal offline cost $C_{OPT}(s_i\le t\le s_{i+1})=b_i+r_it$. Therefore we have

$$R_k(s_i\le t\le s_{i+1})=\frac{C(s_i\le t\le s_{i+1})}{b_i+r_it}=\frac{\frac{\mathrm{e}(b_i+r_it)}{E}}{b_i+r_it}=\frac{\mathrm{e}}{E}. \tag{3.4}$$

(iii) When $t \geq s_k$, for arbitrary $0 \leq \tau \leq t$, each $p_j(\tau)$ $(j = 0, 1, 2, \ldots k)$ belonging to profile p is as follows:

$$p_0(t) = \begin{cases} \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_1}} + \frac{r_k}{r_0}}{E}, \tau \leq s_1; \\ \frac{\frac{r_k}{r_0}}{E}, \qquad s_1 \leq \tau \leq t. \end{cases}$$

$$p_j(t) = \begin{cases} \frac{\mathrm{e}^{\frac{\tau}{s_j}} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{\mathrm{e}}, \tau \leq s_j; \\ \frac{\mathrm{e} - \mathrm{e}^{\frac{\tau}{s_{j+1}}}}{E}, \quad s_j \leq \tau \leq s_{j+1}; \\ 0, \qquad s_{j+1} \leq \tau \leq t. \end{cases} \qquad j = 1, 2, \ldots k-1.$$

$$p_k(t) = \begin{cases} \frac{\mathrm{e}^{\frac{\tau}{s_k}} - 1}{E}, \tau \leq s_k; \\ \frac{\mathrm{e} - 1}{E}, \quad s_k \leq \tau \leq t. \end{cases}$$

Based on the analogy of (3.2) and (3.3), we have

$$r_0 \int_0^t p_0(\tau) \mathrm{d}\tau = \frac{(s_1 r_0 + r_k t)}{E},$$
$$r_j \int_0^t p_j(\tau) \mathrm{d}\tau = \frac{r_j(s_{j+1} - s_j)}{E}, \qquad j = 1, 2, \ldots, k-1\cdot$$

However,

$$r_k \int_0^t p_k(\tau) \mathrm{d}\tau = \frac{r_k}{E} \left[\int_0^{s_k} \left(\mathrm{e}^{\frac{\tau}{s_k}} - 1 \right) \mathrm{d}\tau + \int_{s_k}^t (e-1) \mathrm{d}\tau \right] = \frac{r_k \ et \ - r_k s_k - r_k t}{E}\cdot$$

Thus,

$$\sum_{j=0}^k r_j \int_0^t p_j(\tau) \mathrm{d}\tau = \frac{1}{E} \left[(s_1 r_0 + r_k t) + \sum_{j=1}^{k-1} r_j (s_{j+1} - s_j) r_k \ et \ - r_k s_k - r_k t \right] = \frac{b_k + r_k \ et}{E}\cdot$$

Due to $t \geq s_k$, $p_1(t) = p_2(t) = \ldots = p_{k-1}(t) = 0$, but $p_k(t) = \frac{\mathrm{e}-1}{E}$, thus,

$$\sum_{j=0}^k b_j p_j(t) = \frac{b_k \mathrm{e} - b_k}{E}\cdot$$

Therefore, the total expected cost of profile p is

$$C_p(t \geq s_k) = \sum_{i=0}^k r_i \int_0^t p_i(\tau) \mathrm{d}\tau + \sum_{i=0}^k b_i p_i(t) = \frac{\mathrm{e}(b_k + r_k t)}{E}\cdot$$

Because of $t \geq s_k$, the optimal offline cost is $b_k + r_k t$. Hence, we have

$$R_k(t \geq s_k) = \frac{\mathrm{e}}{E}\cdot \tag{3.5}$$

From (3.1), (3.4) and (3.5), we have

$$R_k = \sup \left\{ R_k(0 < t \leq s_1), R_k(s_i \leq t \leq s_{i+1}), R_k(t \geq s_k) \right\} = \frac{\mathrm{e}}{\mathrm{e} - 1 + \frac{r_k}{r_0}}\cdot \qquad \square$$

Corollary 3.2. *For the $k+1$-slope ski rental problem, when $r_k = 0$, the profile p of randomized online algorithm is*

$$p_0(t) = \begin{cases} \frac{\mathrm{e}-\mathrm{e}^{\frac{t}{s_1}}}{\mathrm{e}-1}, & t \leq s_1; \\ 0, & t \geq s_1; \end{cases}$$

$$p_i(t) = \begin{cases} \frac{\mathrm{e}^{\frac{t}{s_i}}-\mathrm{e}^{\frac{t}{s_{i+1}}}}{\mathrm{e}-1}, & t \leq s_i; \\ \frac{\mathrm{e}-\mathrm{e}^{\frac{t}{s_{i+1}}}}{\mathrm{e}-1}, & s_i \leq t \leq s_{i+1}; \\ 0, & t \geq s_{i+1}; \end{cases} \qquad i = 1, 2, \ldots k-1.$$

$$p_k(t) = \begin{cases} \frac{\mathrm{e}^{\frac{t}{s_k}}-1}{\mathrm{e}-1}, & t \leq s_k; \\ 1, & t \geq s_k; \end{cases}$$

and the competitive ratio is

$$R_k = \frac{\mathrm{e}}{\mathrm{e}-1}.$$

Corollary 3.2 shows that when $r_k = 0$, the randomized algorithm of ours and its competitive ratio $\frac{\mathrm{e}}{\mathrm{e}-1}$ happen to be same as the ones given by Lotker *et al.* [9].

Corollary 3.3. *When $k = 1$, $r_0 = 1$, and $r_1 = 0$, the profile p of the randomized online algorithm is as follows:*

$$p_0(t) = \begin{cases} \frac{\mathrm{e}-\mathrm{e}^{\frac{t}{b_1}}}{\mathrm{e}-1}, & 0 \leq t \leq b_1; \\ 0, & t \geq b_1; \end{cases}$$

$$p_1(t) = \begin{cases} \frac{\mathrm{e}^{\frac{t}{b_1}}-1}{\mathrm{e}-1}, & 0 \leq t \leq b_1; \\ 1, & t \geq b_1; \end{cases}$$

and its competitive ratio is $\frac{\mathrm{e}}{\mathrm{e}-1}$.

Corollary 3.3 indicates that when $k = 1$, $r_0 = 1$, and $r_1 = 0$, the randomized algorithm of ours and its competitive ratio is optimal.

4. Conclusion

We have studied randomized algorithms for additive multislope ski rental problem. For the case of pure buy option $r_k = 0$, Lotker *et al.* [9] obtained a randomized algorithm with competitive ratio of $\frac{\mathrm{e}}{\mathrm{e}-1}$. However, for $r_k > 0$, they only showed that the competitive ratio is bounded by $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$. In this paper, we extend the above competitive ratio bound $\frac{\mathrm{e}-r_k/r_0}{\mathrm{e}-1}$ proposed by Lotker *et al.* [9] to the ratio $\frac{\mathrm{e}}{\mathrm{e}-1+r_k/r_0}$. When $k = 1$, it is implied that the competitive ratio of the 2-slope case is $\frac{\mathrm{e}}{\mathrm{e}-1+r_1/r_0}$, which is in consistence with previously reports [8,9].

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