# STATE HYPERSTRUCTURES OF TREE AUTOMATA BASED ON LATTICE-VALUED LOGIC

# Maryam Ghorani\*

Abstract. In this paper, an association is organized between the theory of tree automata on one hand and the hyperstructures on the other hand, over complete residuated lattices. To this end, the concept of order of the states of a complete residuated lattice-valued tree automaton (simply  $\mathscr{L}$ -valued tree automaton) is introduced along with several equivalence relations in the set of the states of an  $\mathscr{L}$ -valued tree automaton. We obtain two main results from this study: one of the relations can lead to the creation of Kleene's theorem for  $\mathscr{L}$ -valued tree automata, and the other one leads to the creation of a minimal  $\mathscr{L}$ -valued tree automaton that accepts the same language as the given one.

Mathematics Subject Classification. 68Q70, 68Q45, 20N25, 20N20

Received December 14, 2016. Accepted May 14, 2018.

## 1. INTRODUCTION

The mathematical formulation of a fuzzy automaton was introduced by Wee and Fu [50] and Santos [48]. Subsequently, the fundamentals of fuzzy language theory were established by Lee and Zadeh [27], and by Thomason and Marinos [49]. Thereafter, many other authors have contributed to this field (see [15, 31, 40, 45]). Fuzzy automata have many significant applications such as in learning systems, fuzzy discrete event systems and neural networks [32, 40–42, 46]. Due to the importance of residuated lattice valued logic, Qiu [43, 44] established a fundamental framework of automata theory based on complete residuated lattice-valued logic, which generalized some of the results obtained in fuzzy finite automata studied in [40, 45]. In recent years, several other authors have studied lattice-valued automata (to get more information refer to references [28, 30, 47, 51–53]).

The notion of fuzzy tree automata has been considered by numerous authors and researchers. Inagaki and Fukumura [16] investigated fuzzy tree automaton as a special case of weighted tree automaton that accepts formal tree series over a complete semiring. Mordeson and Malik [40] defined a fuzzy tree automaton as an acceptor of a fuzzy dendrolanguage. Esik and Liu [13] studied fuzzy tree automata with membership in a distributive lattice and, after defining a fuzzy recognizable tree language, they derived a Kleene's theorem for fuzzy tree automata. Bozapalidis and Bozapalidoy [3] showed that linear tree homomorphisms preserve syntactic recognizability. More results on fuzzy tree automata can be found in [24, 26, 39] and references therein. Since, fuzzy tree automata take values in the unit interval [0, 1], to enhance the processing ability of fuzzy tree automata, Ghorani and Zahedi [20–22], Ghorani *et al.* [23] and Ghorani [18, 19] extended the membership value

Keywords and phrases: Tree automaton, lattice-valued logic, Kleene's theorem, hypergroup, minimization.

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran.

<sup>\*</sup> Corresponding author: Ghorani@shahroodut.ac.ir

to a more general algebraic structure and considered tree automata based on complete residuated lattice-valued logic.

Fuzzy sets introduced by Zadeh in 1965 [54], and hyperstructures introduced by Marty in 1934 [33], are now used globally both on the theoretical point of view and for their many applications. The relation between fuzzy sets and hyperstructures have been already considered by Corsini [7], Corsini and Leoreanu [8], Ameri and Zahedi [1] and others [6, 9-12, 55].

Using the tools and methods of hyperstructure theory, Massouros [37] obtained a new proof of the famous Kleene's theorem for automata theory, which states that "a subset of the set of words is acceptable by an automaton if and only if it is defined by a regular expression." He also defined various hyperoperations on the set of states of deterministic and non-deterministic acceptors. Moreover, Massouros and Mittas [38] described constructions of some hyperstructures on the set of words formed from the given input alphabet and on the set of states of the corresponding automata. Some other important results were obtained by Massouros [35, 36], and by Chvalina and Chvalinova [4].

Now, in this paper we extend the concept of fuzzy hyperstructures to complete residuated lattice-valued hyperstructures (or  $\mathscr{L}$ -valued hyperstructures). Also, we present some connections of tree automata theory and languages theory with hyperstructures theory over complete residuated lattices. Using some hyperoperations, we prove the Kleene's theorem for  $\mathscr{L}$ -valued tree automata. Kleene's theorem gives the basic algebraic operations on the languages recognized by  $\mathscr{L}$ -valued tree automata and establishes the equivalence between  $\mathscr{L}$ -valued tree automata and regular expressions. Moreover, by introducing a new  $\mathscr{L}$ -valued hypergroup, we provide a minimization algorithm for  $\mathscr{L}$ -valued tree automata.

The outline of the paper is as follows: in Section 2, we provide some preliminaries and basic definitions. In Section 3, we present some definitions on  $\mathscr{L}$ -valued hyperstructures and by defining an  $\mathscr{L}$ -valued hyperoperation, we prove the Kleene's theorem for  $\mathscr{L}$ -valued tree automata. In Section 4, by introducing a new  $\mathscr{L}$ -valued hypergroup, we obtain a minimization algorithm for  $\mathscr{L}$ -valued tree automata. Conclusions are given in Section 5.

## 2. Preliminaries and basic definitions

In this section, we review some definitions and concepts concerning complete residuated lattices and ( $\mathscr{L}$ -valued) tree automata. For more details, the reader is referred to [2, 5, 13, 17, 20, 23].

### 2.1. Complete residuated lattice

**Definition 2.1** ([2]). A complete residuated lattice is a 5-tuple  $l = \langle \mathcal{L}, +, \cdot, \odot, \rho \rangle$  where:

- (i)  $\langle \mathcal{L}, +, \cdot \rangle$  is a complete lattice with the least and the greatest elements 0 and 1, respectively;
- (ii)  $\odot$  and  $\rho$  are two binary operations on  $\mathscr{L}$  such that  $\odot$  is isotone and  $\langle \mathscr{L}, \odot, 1 \rangle$  is a commutative monoid, and  $\rho$  is antitone in the first and isotone in the second variable; that is, for any  $a_1, a_2, b \in \mathscr{L}$  if  $a_1 \leq a_2$ then  $a_1 \odot b \leq a_2 \odot b$ ,  $b \odot a_1 \leq b \odot a_2$ ,  $a_2\rho b \leq a_1\rho b$  and  $b\rho a_1 \leq b\rho a_2$ ;
- (iii) for all  $a, b, c \in \mathcal{L}$ ,  $a \odot b \leq c$  if and only if  $a \leq b\rho c$ .

**Example 2.2** ([20]). Let  $\mathscr{L} = [0,1] \subseteq \mathbb{R}$ . Then  $l = \langle \mathscr{L}, \lor, \land, \odot, \rho \rangle$  is a complete residuated lattice, where  $a \odot b = \max(0, a + b - 1), a\rho b = \min(1, 1 - a + b)$  for any  $a, b \in [0,1]$ , and  $\lor$  and  $\land$  are the symbols of the truth-valued lattice, representing max and min, respectively.

**Example 2.3** ([19]). Let  $\mathscr{L} = [-4, 0] \subseteq \mathbb{R}$  with natural ordering and  $a, b \in \mathscr{L}$ . Define

$$a \odot b = \begin{cases} a+b-ab, & \text{if } a, b \in Q, \\ a+b, & \text{if } a \text{ or } b \in Q^c, \end{cases}$$

TABLE 1. The operation  $\rho$  in Example 2.4.

ρ	0	a	b	с	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	1	1	$\mathbf{c}$	1
$\mathbf{c}$	0	b	b	1	1
1	0	a	b	с	1

$$a\rho b = \begin{cases} \min\{0, \frac{b-a}{1-a}\}, & \text{if } a, b \in Q, \\ \min\{0, b-a\}, & \text{if } a \text{ or } b \in Q^c, \end{cases}$$

where, Q and  $Q^c$  show the set of rational and irrational numbers, respectively. It is obvious that  $\langle \mathcal{L}, \vee, \wedge \rangle$  is a complete lattice ( $\vee$  and  $\wedge$  are max and min, respectively) and condition (ii) of Definition 2.1 holds. However, condition (iii) of Definition 2.1 does not hold because  $-1 \odot -1 = -3 \leq -\sqrt{5}$ , but  $-1 > -1\rho - \sqrt{5} = -\sqrt{5} + 1$ . Hence,  $l = \langle \mathcal{L}, \vee, \wedge, \odot, \rho \rangle$  is not a complete residuated lattice.

**Example 2.4** ([19]). Suppose that  $l = \langle \mathcal{L}, \lor, \land, \odot, \rho \rangle$  where  $\mathcal{L} = \{0, a, b, c, 1\}$  with 0 < a < b, c < 1, and b and c are incomparable,  $\odot = \land$  and  $\rho$  is defined as Table 1. It is easy to check that conditions (i) and (iii) of Definition 2.1 hold. However condition (ii) of Definition 2.1 does not hold, because  $a \leq c$  but  $b\rho a > b\rho c$ .

**Remark 2.5.** In this paper, by an  $\mathscr{L}$ -valued logic we mean the complete residuated lattice-valued logic; that is, the set of truth values is  $\mathscr{L}$ , which possesses nullary connective a ( $a \in \mathscr{L}$ ), an addition binary connective & as well as usual connectives  $\lor$ ,  $\land$  and implication  $\rightarrow$ . In addition, in  $\mathscr{L}$ -valued logic, the only designated truth value is 1; in other words, a formula  $\varphi$  written by  $\models^{l} \varphi$  is valid, if and only if  $[\varphi] = 1$  for any interpretation, where  $[\varphi]$  stands for the truth value of  $\varphi$ . In this paper, the truth valuation rules of predicate logical and set theoretical formulas are displayed as follows:

(i) 
$$[a] = a(a \in \mathscr{L}), \quad [\varphi \lor \psi] = [\phi] + [\psi], \quad [\varphi \land \psi] = [\varphi] \cdot [\psi],$$
$$[\varphi \to \psi] = [\varphi]\rho[\psi], \quad [\varphi \& \psi] = [\varphi] \odot [\psi].$$

(ii) If X is the universe then  $\mathscr{L}^X$  denotes the class of all  $\mathscr{L}$ -valued subsets of X, where by an  $\mathscr{L}$ -valued subset of X we mean a mapping from X to  $\mathscr{L}$ . Now, let  $A \in \mathscr{L}^X$ . Then,

$$[(\exists x)\varphi(x)] = \sum_{x \in X} [\varphi(x)], \quad [(\forall x)\varphi(x)] = \prod_{x \in X} [\varphi(x)], \quad [x \in A] = A(x)$$

In addition, the following derived formulas will be used:

 $\begin{array}{ll} \text{(a)} & \neg \varphi = ^{def} \varphi \to 0, \; \varphi \leftrightarrow \psi = ^{def} (\varphi \to \psi) \wedge (\psi \to \varphi), \\ \text{(b)} & A \subseteq B = ^{def} (\forall x) ((x \in A) \to (x \in B)), \; A \equiv B = ^{def} (A \subseteq B) \wedge (B \subseteq A). \end{array}$ 

**Remark 2.6.** We utilize  $\sum_{i=1}^{n} [\varphi_i]$  and  $\prod_{i=1}^{n} [\varphi_i]$  instead of  $[\varphi_1] + [\varphi_2] + \cdots + [\varphi_n]$  and  $[\varphi_1] \cdot [\varphi_2] \cdots \cdot [\varphi_n]$ , respectively. Also, if  $A \in \mathscr{L}^X$ , then we can write A as follows:

$$A = \{ (x, A(x)) \mid x \in X, A(x) \in \mathscr{L} \}.$$

l is said to be a chain if  $\mathscr{L}$  is a chain set. In this paper, we assume that the complete residuated lattice l is also a chain, *i.e.* all pairs of elements are comparable.

We emphasis that  $\mathscr{L}$  is the set of truth values,  $\langle \mathscr{L}, +, \cdot \rangle$  is a complete lattice and  $\mathscr{L}$  with four operations  $(+, \cdot, \odot, \rho)$ , satisfying the conditions of Definition 2.1, is a complete residuated lattice, denoted by  $l = \langle \mathscr{L}, +, \cdot, \odot, \rho \rangle$ . We confuse the expressions *l*-valued and  $\mathscr{L}$ -valued, that is by the expression  $\mathscr{L}$ -valued we mean complete residuated lattice-valued.

## **2.2.** Trees and ( $\mathscr{L}$ -valued) tree automata

We denote by  $\mathcal{W}$  the set of positive integers and by  $\mathcal{W}^*$  the set of finite strings over  $\mathcal{W}$ . The empty string is denoted by  $\varepsilon$ . Let  $P \subseteq \mathcal{W}^*$ . Then  $\bar{P} = \{p \in \mathcal{W}^* \mid \exists p' \in \mathcal{W}^*, pp' \in P\}$  consists of all the prefixes of all the strings in P. For example if  $P = \{123\}$ , then  $\bar{P} = \{\varepsilon, 1, 12, 123\}$ . In general  $P \subseteq \bar{P}$ . P is said to be prefix-closed if  $P = \bar{P}$ .

**Definition 2.7** ([5]). A ranked alphabet is a couple (F, Arity) where F is a finite set and Arity is a mapping from F into  $W \cup \{0\}$ . The arity of a symbol  $f \in F$  is Arity(f). The set of symbols of arity n is denoted by  $F_n$ . Here, we use parenthesis and commas for a short declaration of symbols with arity. For instance, f(,) is a short declaration for a binary symbol f. The set  $\mathcal{T}(F)$  of trees, over the ranked alphabet F, is the smallest set defined by:

 $-F_0 \subseteq \mathcal{T}(F),$ - if  $p \ge 1, f \in F_p$  and  $t_1, \ldots, t_p \in \mathcal{T}(F)$ , then  $f(t_1, \ldots, t_p) \in \mathcal{T}(F)$ .

**Example 2.8.** Let  $F = \{a, f(,)\}$ . Hear f is a binary symbol and a is a constant (*i.e.* Arity(f) = 2 and Arity(a) = 0). Then we have  $\mathcal{T}(F) = \{a, f(a, a), f(a, f(a, a)), \ldots\}$ . Trees can be represented in a graphical way. For instance, the tree f(a, f(a, a)) is represented by:



Also a tree  $t \in \mathcal{T}(F)$  can be defined as a partial function  $t : \mathcal{W}^* \to F$  with domain pos(t), satisfying the following properties:

(i)  $t(\varepsilon) = Head(t)$ , where Head(t) is the root symbol of t.

- (ii) pos(t) is non-empty and prefix-closed.
- (iii)  $\forall p \in pos(t), \text{ if } t(p) \in F_n, n \ge 1, \text{ then } \{j | pj \in pos(t)\} = \{1, \dots, n\}.$
- (iv)  $\forall p \in pos(t)$ , if  $t(p) \in F_0$ , then  $\{j | pj \in pos(t)\} = \emptyset$ .

Each element in pos(t) is called a position.

A subtree  $t|_p$  of a tree  $t \in \mathcal{T}(F)$  at position p is defined as follows:

- $pos(t|_p) = \{j \mid pj \in pos(t)\},\$
- $\forall q \in pos(t|_p), \ t|_p(q) = t(pq).$

**Definition 2.9** ([5]). The height of a tree t, denoted by Height(t), is inductively defined by:

- $Height(t) = 0 \text{ if } t \in X,$
- Height(t) = 1 if  $t \in F_0$ ,
- $Height(t) = 1 + \max\left\{Height(t_i) \mid i \in \{1, \dots, n\}\right\}$  if  $Head(t) \in F_n$ .

**Example 2.10.** Let  $F = \{a, b, g(), f(, ), h(, ,)\}$ . Consider the tree t as follows:



The root symbol of t is h (*i.e.* Head(t) = h); the set of positions of t is  $pos(t) = \{\varepsilon, 1, 2, 3, 11, 12, 31\}$ ; the set of prefixes of all the strings in pos(t) is  $\{\varepsilon, 1, 2, 3, 11, 12, 31\}$ , therefore pos(t) is prefix-closed; Height(t) = 3;  $t|_1 = f(a, b)$ ;  $t|_2 = a$ ;  $t|_3 = g(b)$ ;  $t|_{11} = a$ ;  $t|_{12} = b$ ;  $t|_{31} = b$ .

**Definition 2.11.** ([20]) A tuple  $\mathcal{A} = (Q, F, Q_f, \delta)$  is called a complete residuated lattice valued tree automaton (for short, an  $\mathscr{L}$ -valued tree automaton), where

- (i) Q is a set of states.
- (ii) F is a ranked alphabet.
- (iii)  $Q_f: Q \to \mathscr{L}$  is an  $\mathscr{L}$ -valued subset of Q over  $\mathscr{L}$  and is called the set of  $\mathscr{L}$ -valued final states.
- (vi) For each  $n \ge 0$ ,  $\delta_n$  is an  $\mathcal{L}$ -valued mapping from  $Q^n \times F_n \times Q$  to  $\mathscr{L}$ .

**Example 2.12.** Let  $F = \{a, g(), f(,)\}$  and  $\mathscr{L} = [0, 1] \subseteq \mathbb{R}$ . Consider  $\mathscr{A} = (Q, F, Q_f, \delta)$  with components:  $Q = \{q_0, q_1\}, Q_f(q_0) = 0.5, Q_f(q_1) = 1$  and

$$\delta_0(a,q_0) = 0.5, \quad \delta_1(q_0,g,q_0) = 0.9, \quad \delta_1(q_0,g,q_1) = 0.6, \quad \delta_2((q_0,q_0),f,q_1) = 0.3$$

It is obvious that  $\mathcal{A}$  is an  $\mathcal{L}$ -valued tree automaton.

A deterministic  $\mathscr{L}$ -valued tree automaton is an  $\mathscr{L}$ -valued tree automaton  $\mathscr{A}_d = (Q_d, F, Q_{f_d}, \delta_d)$ , where for each  $\sigma \in F_n$  and  $q_1, \ldots, q_n \in Q_d$  there exists at most one  $q \in Q_d$  such that  $\delta_d((q_1, \ldots, q_n), \sigma, q) > 0$ . An  $\mathscr{L}$ -valued tree automaton  $\mathscr{A} = (Q, F, Q_f, \delta)$  is said to be complete, if there exists at least one rule  $\delta((q_1, \ldots, q_n), \sigma, q) > 0$ , for all  $n \ge 0$ ,  $\sigma \in F_n$  and  $q_1, \ldots, q_n \in Q$ .

**Remark 2.13.** An operation of the  $\mathscr{L}$ -valued tree automaton can be seen as a movement from  $Q^n$  to Q, according to the ranked input letter and the family of  $\mathcal{L}$ -valued subsets  $\delta = (\delta_n)_{n \geq 0}$ . We will usually write  $\delta$  for  $\delta_n$ .

In the following example we give an  $\mathscr{L}$ -valued tree automaton that is deterministic and complete.

**Example 2.14.** Let  $F = \{a, g(), f(,)\}$  and  $\mathscr{L} = [0, 1] \subseteq \mathbb{R}$ . Consider the  $\mathscr{L}$ -valued tree automaton  $\mathscr{A} = (Q, F, Q_f, \delta)$  as:  $Q = \{q_0, q_1, q_2\}, Q_f(q_0) = 0.3, Q_f(q_1) = 0.9, Q_f(q_2) = 0.4$  and

$\delta(a, q_0) = 0.5,$	$\delta(q_2, g, q_1) = 0.9,$	$\delta(q_0, g, q_1) = 0.6,$	$\delta(q_1, g, q_1) = 0.8,$
$\delta\bigl((q_0,q_0),f,q_2\bigr) =$	0.4, $\delta((q_0, q_1), f$	$\delta(q_1, q_2) = 0.9, \qquad \delta((q_1, q_2)) = 0.9, \qquad$	$(q_0), f, q_2) = 0.3,$
$\delta\bigl((q_1,q_1),f,q_2\bigr) =$	0.4, $\delta((q_0, q_2), f$	$\delta(q_1, q_2) = 0.2, \qquad \delta((q_1, \delta))$	$(q_2), f, q_2) = 0.5,$
$\delta\bigl((q_2,q_0),f,q_2\bigr) =$	$\delta(q_2, q_1), f$	$\delta(q_2) = 0.1, \qquad \delta((q_2,$	$(q_2), f, q_2) = 0.9.$

It is obvious that the  $\mathscr{L}$ -valued tree automaton  $\mathscr{A}$  is deterministic and complete.

# 3. Kleene's theorem for $\mathscr{L}$ -valued tree automata

The study of fuzzy hyperstructures is an interesting research area of fuzzy sets theory and many works have been done in this area ([7, 10–12, 55]). In this section, we extend the concept of fuzzy hyperstructures to  $\mathscr{L}$ -valued hyperstructures. Moreover, we present some connections of tree automaton theory and languages theory with hyperstructures theory over complete residuated lattices.

**Definition 3.1.** The  $\mathscr{L}$ -valued response mapping  $r_{\mathscr{A}} : \mathcal{T}(F) \times Q \to \mathscr{L}$ , by induction on  $t \in \mathcal{T}(F)$ , is defined as follows:

(i) If  $t = \sigma \in F_0$ , then  $r_{\mathcal{A}}(t,q) = \delta(t,q)$ ,  $\forall q \in Q$ , (ii) If  $t = \sigma(t_1, \ldots, t_n)$ ,  $\forall t_1, \ldots, t_n \in \mathcal{T}(F)$  and  $\sigma \in F_n$ , then

$$r_{\mathcal{A}}(t,q) = \sum_{(q_1,\ldots,q_n)\in Q^n} \left(\delta((q_1,\ldots,q_n),\sigma,q)\cdot\prod_{i=1}^n r_{\mathcal{A}}(t_i,q_i)\right).$$

**Remark 3.2.** Let  $t \in \mathcal{T}(F)$  and  $q \in Q$ . If  $r_{\mathcal{R}}(t,q) > 0$ , then we say that the tree t leads to q.

The  $\mathscr{L}$ -valued behaviour  $B_{\mathscr{A}}(q)$  from Q and the  $\mathscr{L}$ -valued behaviour  $B_{\mathscr{A}}(Q_f)$  of  $\mathscr{A}$ , are defined as follows, respectively:

$$B_{\mathcal{A}}(q) = \Big\{ \big(t, B_{\mathcal{A}}(q)(t)\big) \mid t \in \mathcal{T}(F) \Big\},\$$

where

$$B_{\mathcal{A}}(q)(t) = r_{\mathcal{A}}(t,q) \odot Q_f(q),$$

and

$$B_{\mathcal{A}}(Q_f) = \Big\{ \big(t, B_{\mathcal{A}}(Q_f)(t)\big) \mid t \in \mathcal{T}(F) \Big\},\$$

where

$$B_{\mathcal{A}}(Q_f)(t) = \sum_{q \in Q} B_{\mathcal{A}}(q)(t).$$

Also, L is said to be  $\mathscr{L}$ -valued acceptable if there exists an  $\mathscr{L}$ -valued tree automaton  $\mathscr{A}$  such that  $L = B_{\mathscr{A}}(Q_f)$ .

Now, we apply the preceding descriptions in the following example.

**Example 3.3.** Suppose that  $l = \langle \mathscr{L}, \lor, \land, \odot, \rho \rangle$  is a complete residuated lattice where  $\mathscr{L} = [0, 1] \subseteq \mathbb{R}$ ,  $a \odot b = \max(0, a + b - 1)$  and  $a\rho b = \min(1, 1 - a + b)$  for any  $a, b \in [0, 1]$ . Also, let  $\mathscr{A} = (Q, F, Q_f, \delta)$  be defined as follows:  $F = \{a, g(), f(,)\}, Q = \{q_0, q_1, q_2\}, Q_f(q_0) = 0.6, Q_f(q_1) = 1, Q_f(q_2) = 0.4$  and  $\delta(a, q_0) = 0.8, \delta(q_0, g, q_1) = 1, \delta(q_1, g, q_1) = 0.5, \delta((q_1, q_1), f, q_2) = 0.9, \delta(q_1, g, q_2) = 0.9$ . We have:

$$\begin{array}{ll} B_{\mathcal{A}}(q_0)(a) = 0.8 \odot 0.6 = 0.4, & B_{\mathcal{A}}(q_1) \left(g^n(a)\right) = 0.5 \odot 1 = 0.5, & \forall n \ge 2, \\ B_{\mathcal{A}}(q_1) \left(g(a)\right) = 0.8 \odot 1 = 0.8, & B_{\mathcal{A}}(q_2) \left(g^n(a)\right) = 0.5 \odot 0.4 = 0, & \forall n \ge 3, \\ B_{\mathcal{A}}(q_2) \left(g^2(a)\right) = 0.8 \odot 0.4 = 0.2, & B_{\mathcal{A}}(q_2) \left(f(g(a), g(a))\right) = 0.8 \odot 0.4 = 0.2, \end{array}$$

$$B_{\mathcal{A}}(q_2) \left( f(g^n(a), g^m(a)) \right) = \begin{cases} 0.2 & n = m = 1, \\ 0 & o.w. \end{cases}$$

$$B_{\mathcal{A}}(q_0) = \{(a, 0.4)\}, \ B_{\mathcal{A}}(q_1) = \{(g(a), 0.8), (g^n(a), 0.5) \mid n \ge 2\}, \ B_{\mathcal{A}}(q_2) = \{(g^2(a), 0.2), (f(g(a), g(a)), 0.2)\}.$$

Therefore

$$B_{\mathcal{A}}(Q_f)(a) = 0.4, \quad B_{\mathcal{A}}(Q_f)(g(a)) = 0.8, \quad B_{\mathcal{A}}(Q_f)(g^2(a)) = 0.5 \lor 0.2 = 0.5,$$

$$B_{\mathcal{A}}(Q_f)(g^n(a)) = 0.5, \ \forall n \ge 3, \quad B_{\mathcal{A}}(Q_f)(f(g(a), g(a))) = 0.2$$

and

$$B_{\mathcal{A}}(Q_f) = \Big\{ (a, 0.4), \big(g(a), 0.8\big), \big(g^n(a), 0.5\big), \big(f(g(a), g(a)), 0.2\big) \mid n \ge 2 \Big\}.$$

**Definition 3.4.** Let  $\mathcal{A} = (Q, F, Q_f, \delta)$  be an  $\mathscr{L}$ -valued tree automaton. A state  $q \in Q$  is accessible if there exists a tree  $t \in \mathcal{T}(F)$  such that  $r_{\mathcal{A}}(t,q) > 0$ . An  $\mathscr{L}$ -valued tree automaton is called reduced if all of its states are accessible.

The  $\mathscr{L}$ -valued tree automaton  $\mathscr{A}$  given in Example 3.3 is reduced, since all of the states are accessible.

**Definition 3.5.** Let  $\mathcal{A}$  be an  $\mathcal{L}$ -valued tree automaton over the ranked alphabet F, and  $t \in \mathcal{T}(F)$ . An  $\mathcal{L}$ -valued run r of  $\mathcal{A}$  on t is a mapping  $r : pos(t) \times Q \to \mathscr{L}$  compatible with  $\delta$ , *i.e.*,  $\forall p, p_i \in pos(t)$ , if  $t(p) = \sigma \in F_n$ , r(p,q) > 0 and  $r(p_i,q_i) > 0$ ,  $\forall i \in \{1,\ldots,n\}$ , then  $\delta((q_1,\ldots,q_n),\sigma,q) > 0$ .

Assume that  $\mathcal{K} = \{\Box_1, \ldots, \Box_n\}$  is a set of constants and  $t \in \mathcal{T}(F \cup \mathcal{K})$ . Let  $L_1, \ldots, L_n, \mu \in \mathscr{L}^{\mathcal{T}(F \cup \mathcal{K})}$  where  $\mu(t) > 0$  and  $\mu(\Box_i) = 1, \forall i, 1 \le i \le n$ .

Then the  $\mathscr{L}$ -valued tree substitution of  $(\Box_1, \mu(\Box_1)), \ldots, (\Box_n, \mu(\Box_n))$  by  $L_1, \ldots, L_n$  in  $(t, \mu(t))$ , denoted by  $(t, \mu(t)) \left\{ (\Box_1, \mu(\Box_1)) \leftarrow L_1, \ldots, (\Box_n, \mu(\Box_n)) \leftarrow L_n \right\}$ , is defined by the following identities:

(i) 
$$(\Box_i, \mu(\Box_i)) \left\{ (\Box_1, \mu(\Box_1)) \leftarrow L_1, \dots, (\Box_n, \mu(\Box_n)) \leftarrow L_n \right\} = L_i \text{ for } i = 1, \dots, n$$

(ii)  $(a,\mu(a))\left\{\left(\Box_1,\mu(\Box_1)\right)\leftarrow L_1,\ldots,\left(\Box_n,\mu(\Box_n)\right)\leftarrow L_n\right\}=\left\{\left(a,\mu(a)\right)\right\}$  for all  $a\in F$  such that arity of a is 0 and  $a\neq\Box_1,\ldots,a\neq\Box_n$ ,

(iii) 
$$(f(s_1,\ldots,s_m),\mu(f(s_1,\ldots,s_m)))\left\{\left(\Box_1,\mu(\Box_1)\right)\leftarrow L_1,\ldots,\left(\Box_n,\mu(\Box_n)\right)\leftarrow L_n\right\}$$
  
=  $\left\{(f(t_1,\ldots,t_m),\mu(f(t_1,\ldots,t_m)))\mid (t_i,\mu(t_i))\in (s_i,\mu(s_i))\left\{(\Box_1,\mu(\Box_1))\leftarrow L_1,\ldots,(\Box_n,\mu(\Box_n))\leftarrow L_n\right\}\right\}$ , and  $\mu(t_i)=\mu(s_i)\cdot\prod_{j=1}^n\mu(\alpha_j),\ 1\leq i\leq m$  such that

$$(t_i, \mu(t_i)) = (s_i, \mu(s_i)) \{ (\Box_j, \mu(\Box_j)) \leftarrow (\alpha_j, \mu(\alpha_j)) \mid \forall j, \ 1 \le j \le n \}, \text{and} \ (\alpha_j, \mu(\alpha_j)) \in L_j \}$$

also  $\mu(f(t_1,\ldots,t_m)) = \mu(f(s_1,\ldots,s_m)) \cdot \prod_{i=1}^m \mu(t_i)$  and  $s_i \neq \Box_i, 1 \le i \le m$ .

(iv) 
$$(f(\Box_1,\ldots,\Box_n),\mu(f(\Box_1,\ldots,\Box_n)))$$
  $\{(\Box_1,\mu(\Box_1)) \leftarrow L_1,\ldots,(\Box_n,\mu(\Box_n)) \leftarrow L_n\} = \{(f(t_1,\ldots,t_n),\mu(t_1,\ldots,t_n))) \mid (t_i,\mu(t_i)) \in L_i, 1 \le i \le n\}$ , where  $\mu(f(t_1,\ldots,t_n)) = \mu(f(\Box_1,\ldots,\Box_n)) \cdot \prod_{i=1}^n \mu(t_i)$ .

Let M and L be  $\mathscr{L}$ -valued subsets of  $\mathcal{T}(F \cup \mathfrak{K})$ . The concatenation of M to L is defined as  $L_{\Box}M = \bigcup_{(t,\mu(t)) \in L} \left\{ (t,\mu(t)) \left\{ (\Box,\mu(\Box)) \leftarrow M \right\} \right\}$  and Kleene-star  $L^{*,\Box}$  is defined as  $L^{*,\Box} = \bigcup_{n \geq 0} L^{n,\Box}$ , where

(i)  $L^{0,\square} = \{(\square, \mu(\square))\},\$ (ii)  $L^{n+1,\square} = L^{n,\square} \cup L_{.\square}L^{n,\square}.$ 

**Example 3.6.** Let  $F = \{a, g(), f(,)\}$  and  $\mathcal{K} = \{\Box_1, \Box_2\}$ . Also let  $t = f(\Box_1, f(\Box_1, \Box_2)), \mu(t) = 0.6, \mu(f(\Box_1, \Box_2)) = 0.8, \mu(\Box_1) = \mu(\Box_2) = 1$  and  $L_1 = \{(a, 0.7), (g(a), 0.2)\}$ . Then

$$(t, 0.6)\{(\Box_1, 1) \leftarrow L_1\} = \left\{ \left( f(a, f(a, \Box_2)), 0.6 \right), \left( f(a, f(g(a), \Box_2)), 0.2 \right), \\ \left( f(g(a), f(g(a), \Box_2)), 0.2 \right), \left( f(g(a), f(a, \Box_2)), 0.2 \right) \right\}.$$

**Example 3.7.** Let  $F = \{a, b, g(), f(,)\}, L = \{(a, 0.7), (f(a, \Box), 0.5)\}$  and  $M = \{(b, 0.4), (f(g(a), \Box), 0.3)\}$ . Then

$$L \cdot \square M = \{(a, 0.7), (f(a, b), 0.4), (f(a, f(g(a), \square)), 0.3)\}$$

and

$$L^{*,\Box} = \{(\Box, \mu(\Box))\} \cup \{(a, 0.7), (f(a, \Box), 0.5)\} \cup \{(a, 0.7), (f(a, a), 0.5), (f(a, f(a, \Box)), 0.5)\} \cup \dots$$

Let  $\mathcal{H}$  be a non-void crisp set (*i.e.* an element is either a member of  $\mathcal{H}$  or not). An  $\mathscr{L}$ -valued hyperoperation "o" on  $\mathcal{H}$  is an  $\mathscr{L}$ -valued subset of  $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$  over  $\mathscr{L}$ , *i.e.* a mapping from  $\mathcal{H} \times \mathcal{H} \times \mathcal{H}$  to  $\mathscr{L}$ . If o is an  $\mathscr{L}$ -valued hyperoperation, then we write (aob)(x) instead of o(a, b, x) and the notation (aob)(x) means the truth value of aob at the element x.

Let  $\mathcal{A}, \mathcal{B} \in \mathscr{L}^{\mathcal{H}}$  and  $o \in \mathscr{L}^{\mathcal{H} \times \mathcal{H} \times \mathcal{H}}$ . Then we define

$$\mathcal{A}o\mathcal{B}(x) = {}^{def} \sum_{a,b\in\mathcal{H}} \left( \mathcal{A}(a) \odot \mathcal{B}(b) \odot aob(x) \right), \quad \forall x \in \mathcal{H}.$$

**Definition 3.8.** Let  $o \in \mathscr{L}^{\mathcal{H} \times \mathcal{H} \times \mathcal{H}}$ . Then  $\mathcal{HG1} \in \mathscr{L}^{\mathscr{L}^{\mathcal{H} \times \mathcal{H} \times \mathcal{H}}}$  is an  $\mathscr{L}$ -valued set of semihypergroups if:

$$\mathcal{HG1}(o) =^{def} \left[ \left( (aob)oc \right) \equiv \left( ao(boc) \right) \right], \ \forall a, b, c \in \mathcal{H},$$

where

$$(aob)oc = \left\{ \left( x, ((aob)oc)(x) \right) \mid x \in \mathcal{H} \text{ and } \left( (aob)oc \right)(x) \in \mathscr{L} \right\}$$

$$((aob)oc)(x) = \sum_{y \in \mathcal{H}} ((aob)(y) \odot (yoc)(x)),$$

$$ao(boc) = \Big\{ \big(x, (ao(boc))(x)\big) \mid x \in \mathcal{H} \text{ and } (ao(boc))(x) \in \mathscr{L} \Big\},\$$

$$(ao(boc))(x) = \sum_{y \in \mathcal{H}} ((aoy)(x) \odot (boc)(y)).$$

Intuitively,  $\left[ \left( (aob)oc \right) \equiv \left( ao(boc) \right) \right]$  stands for the truth value of  $\left( (aob)oc \right) \equiv \left( ao(boc) \right)$ ,

**Remark 3.9.** It is obvious that  $\mathcal{HG1}(o) \in \mathscr{L}$ . Note that  $\mathcal{HG1}(o)$  stands for the truth value of the proposition that "o" is an  $\mathscr{L}$ -valued semihypergroup.

**Example 3.10.** Let  $l = \langle \mathscr{L}, +, \cdot, \odot, \rho \rangle$  be a complete residuated lattice where  $\mathscr{L} = [0, 1] \subseteq \mathbb{R}$  and operators  $+, \cdot, \odot$  and  $\rho$  are defined as Example 2.2. Moreover, let  $\mathcal{H} = \{a, b\}$  and the  $\mathscr{L}$ -valued hyperoperation "o" on  $\mathcal{H}$  be as follows:

$$(aoa)(a) = 0.1, (aoa)(b) = 0.2, (bob)(a) = 0.4, (bob)(b) = 0.5, (aob)(a) = 0.3, (aob)(b) = 0.2, (boa)(a) = 0.2, (boa)(b) = 0.2.$$

Utilizing Definition 3.8 and items (i) and (ii) b in Remark 2.5, we obtain  $\mathcal{HG1}(o)$ .  $\forall a, b, c \in \mathcal{H}$ , we can obtain:

$$\begin{bmatrix} ((aob)oc) \equiv (ao(boc)) \end{bmatrix} = \prod_{x \in \mathcal{H}} \left( \sum_{y \in \mathcal{H}} ((aob)(y) \odot (yoc)(x)) \rho \sum_{r \in \mathcal{H}} ((aor)(x) \odot (boc)(r)) \right) \\ \cdot \sum_{r \in \mathcal{H}} ((aor)(x) \odot (boc)(r)) \rho \sum_{y \in \mathcal{H}} ((aob)(y) \odot (yoc)(x)) = 1.$$

Therefore  $\mathcal{HG1}(o) = 1$ .

**Definition 3.11.** Let  $\mathcal{HG1}(o) > 0$ . Then  $\mathcal{HG} \in \mathscr{L}^{\mathscr{H} \times \mathcal{H} \times \mathcal{H}}$  is an  $\mathscr{L}$ -valued set of hypergroups if:

$$\mathcal{H}\mathcal{G}(o) =^{def} [\mathcal{H}oa \equiv ao\mathcal{H}] \land [\mathcal{H}oa \equiv \mathcal{H}] \land [ao\mathcal{H} \equiv \mathcal{H}] \forall a \in \mathcal{H},$$

where

$$\mathcal{H}oa = \left\{ \left(x, \mathcal{H}oa(x)\right) \mid x \in \mathcal{H} \text{ and } \mathcal{H}oa(x) \in \mathscr{L} \right\},\$$
$$ao\mathcal{H} = \left\{ \left(x, ao\mathcal{H}(x)\right) \mid x \in \mathcal{H} \text{ and } ao\mathcal{H}(x) \in \mathscr{L} \right\},\$$
$$\mathcal{H}oa(x) = \sum_{y \in \mathcal{H}} \left(\mathcal{H}(y) \odot yoa(x)\right),\$$
$$ao\mathcal{H}(x) = \sum_{y \in \mathcal{H}} \left(aoy(x) \odot \mathcal{H}(y)\right),\$$

and

$$\mathcal{H}(y) = \begin{cases} 1 & y \in \mathcal{H}, \\ 0 & y \notin \mathcal{H}. \end{cases}$$

**Remark 3.12.** It is obvious that  $\mathcal{HG}(o) \in \mathscr{L}$ . If  $\mathcal{HGI}(o) > 0$ , then  $\mathcal{HG}(o)$  stands for the truth value of the proposition that "o" is an  $\mathscr{L}$ -valued hypergroup.

Let  $\mathcal{HG}(o) > 0$ . Then  $\mathcal{B}-\mathcal{HG} \in \mathscr{L}^{\mathscr{H} \times \mathcal{H} \times \mathcal{H}}$  is an  $\mathscr{L}$ -valued set of *B*-hypergroups, if the following conditions hold:  $\forall a, b \in \mathcal{H}$ 

- aob(a) > 0,
- aob(b) > 0,
- $aob(x) = 0, \forall x \in \mathcal{H}, x \neq a, b.$

If the above conditions hold then  $\mathcal{B}-\mathcal{H}\mathcal{G}(o) = \mathcal{H}\mathcal{G}(o)$ , otherwise  $\mathcal{B}-\mathcal{H}\mathcal{G}(o) = 0$ . Note that  $\mathcal{B}-\mathcal{H}\mathcal{G}(o)$  is the truth value of the proposition that "o" is an  $\mathscr{L}$ -valued B-hypergroup,

**Example 3.13.** Consider Example 3.10. We obtain  $\mathcal{HG}(o)$ .

$$\mathcal{H}oa(a) = \sum_{y \in \mathcal{H}} \left( \mathcal{H}(y) \odot yoa(a) \right) = \left( \mathcal{H}(a) \odot (aoa)(a) \right) + \left( \mathcal{H}(b) \odot (boa)(a) \right) = (1 \odot 0.1) + (1 \odot 0.2) = 0.2.$$

By a similar method, we can see that  $\mathcal{H}oa(b) = 0.2$ ,  $\mathcal{H}ob(a) = 0.4$ ,  $\mathcal{H}ob(b) = 0.5$ ,  $ao\mathcal{H}(a) = 0.3$ ,  $ao\mathcal{H}(b) = 0.2$ ,  $bo\mathcal{H}(a) = 0.4$  and  $bo\mathcal{H}(b) = 0.5$ .

Now, we show that  $\forall a \in \mathcal{H}, \ [\mathcal{H}oa \equiv ao\mathcal{H}]:$ 

$$\prod_{x \in \mathcal{H}} (\mathcal{H}oa(x)\rho ao\mathcal{H}(x)) \cdot (ao\mathcal{H}(x)\rho\mathcal{H}oa(x)) = (\mathcal{H}oa(a)\rho ao\mathcal{H}(a)) \cdot (ao\mathcal{H}(a)\rho\mathcal{H}oa(a)) \cdot (\mathcal{H}oa(b)\rho ao\mathcal{H}(b)) \\ \cdot (ao\mathcal{H}(b)\rho\mathcal{H}oa(b)) \cdot (\mathcal{H}ob(a)\rho bo\mathcal{H}(a)) \cdot (bo\mathcal{H}(a)\rho\mathcal{H}ob(a)) \\ \cdot (\mathcal{H}ob(b)\rho bo\mathcal{H}(b)) \cdot (bo\mathcal{H}(b)\rho\mathcal{H}ob(b)) \\ = 0.9.$$

Therefore

$$\forall a \in \mathcal{H}, \ [\mathcal{H}oa \equiv ao\mathcal{H}] = 0.9.$$

Moreover, we show that  $\forall a \in \mathcal{H}, \ [\mathcal{H}oa \equiv \mathcal{H}] :$ 

$$\prod_{x \in \mathcal{H}} (\mathcal{H}oa(x)\rho\mathcal{H}(x)) \cdot (\mathcal{H}(x)\rho\mathcal{H}oa(x)) = (\mathcal{H}oa(a)\rho\mathcal{H}(a)) \cdot (\mathcal{H}(a)\rho\mathcal{H}oa(a)) \cdot (\mathcal{H}oa(b)\rho\mathcal{H}(b)) \cdot (\mathcal{H}(b)\rho\mathcal{H}oa(b)) \\ \cdot (\mathcal{H}ob(a)\rho\mathcal{H}(a)) \cdot (\mathcal{H}(a)\rho\mathcal{H}ob(a)) \cdot (\mathcal{H}ob(b)\rho\mathcal{H}(b)) \cdot (\mathcal{H}(b)\rho\mathcal{H}ob(b)) \\ = 0.2.$$

Therefore

$$\forall a \in \mathcal{H}, \ [\mathcal{H}oa \equiv \mathcal{H}] = 0.2.$$

By a similar method, we can show that  $\forall a \in \mathcal{H}, [ao\mathcal{H} \equiv \mathcal{H}] = 0.2$ . Utilizing Definition 3.11 we have

$$\mathcal{HG}(o) = 0.9 \cdot 0.2 \cdot 0.2 = 0.2.$$

**Definition 3.14.**  $\mathcal{PR} \in \mathscr{L}^{\mathscr{L}^{\mathcal{H} \times \mathcal{H} \times \mathcal{H}}}$  is an  $\mathscr{L}$ -valued set of partially reversibles if:

$$\mathcal{PR}(o) =^{def} xoy(w)\rho\big(\sum_{x' \in s(x)} wox'(y) + \sum_{y' \in s(y)} woy'(x)\big)$$

where  $\forall a \in \mathcal{H}$ ,  $s(a) = \{b \in \mathcal{H} \mid aob(e) > 0 \& boa(e) > 0\}$  and e is the neutral element (*i.e.*,  $\forall a \in \mathcal{H}$ ,  $aoe \equiv eoa \equiv a$ ).

### Definition 3.15. Let

 $\mathcal{HR} = \{ ((\boxplus, \boxtimes), \mathcal{HR}(\boxplus, \boxtimes)) \mid \boxtimes \text{ is an } \mathscr{L}\text{-valued operation and } \boxplus \text{ is an } \mathscr{L}\text{-valued hyperoperation on } \mathcal{H} \}.$ 

Then  $\mathcal{HR}$  is an  $\mathscr{L}$ -valued set of hyperringoids if:

- (i)  $\mathcal{H}G(\boxplus) > 0$
- (ii)  $\forall a, b, c \in \mathcal{H}$ ,  $(a \boxtimes b) \boxtimes c \equiv a \boxtimes (b \boxtimes c)$ ;
- (iii)  $\forall a, b, c \in \mathcal{H}, a \boxtimes (b \boxplus c) \equiv (a \boxtimes b) \boxplus (a \boxtimes c);$
- (iv)  $\forall a, b, c \in \mathcal{H}$ ,  $(a \boxplus b) \boxtimes c \equiv (a \boxtimes c) \boxplus (b \boxtimes c)$ .

If all of the above conditions hold, then  $\mathcal{HR}(\boxplus, \boxtimes)$  is truth value the proposition that " $(\boxplus, \boxtimes)$ " is an  $\mathscr{L}$ -valued hyperringoid.

An  $\mathscr{L}$ -valued hyperringoid with  $\mathcal{B}-\mathcal{H}\mathcal{G}(\boxplus) > 0$  is called An  $\mathscr{L}$ -valued *B*-hyperringoid.

**Example 3.16.** It is easy to see that  $\mathcal{HR}(+, \Box) > 0$  and  $\mathcal{B}-\mathcal{HR}(+, \Box) > 0$  where,  $\Box$  is an  $\mathscr{L}$ -valued operation and + is an  $\mathscr{L}$ -valued hyperoperation on  $\mathcal{T}(F \cup \mathcal{K})$ .

**Definition 3.17.** Let  $\mathcal{HR}(\boxplus,\boxtimes) > 0$ . Then  $\mathscr{L}$ -valued rational subsets of  $\mathcal{H}$  are:

- (i) finite  $\mathscr{L}$ -valued subsets of  $\mathcal{H}$ ,
- (ii) finite  $\mathscr{L}$ -valued hyperoperation " $\boxplus$ " and finite  $\mathscr{L}$ -valued operation " $\boxtimes$ " of  $\mathscr{L}$ -valued rational subsets,
- (iii) series of the form  $\sum_{n=1}^{\infty} (\prod_{i=1}^{n} \mathcal{A})$ , where  $\mathcal{A}$  is an  $\mathscr{L}$ -valued rational subset and the notations  $\sum$  and  $\prod$  denote the  $\mathscr{L}$ -valued hyperoperation " $\mathbb{H}$ " and  $\mathscr{L}$ -valued operation " $\boxtimes$ ", respectively.

**Remark 3.18.** Note that if we set  $\mathscr{L} = [0, 1]$ , then the preceding definitions of  $\mathscr{L}$ -valued hyperstructures reduce to their fuzzy counterparts [25, 34].

From the 4-tuple defining the  $\mathscr{L}$ -valued tree automaton, the set of the states Q can receive the structure of an  $\mathscr{L}$ -valued hypergroup through the proper definition of certain hyperoperations. In this way,  $\mathscr{L}$ -valued hypergroups can be attached to the  $\mathscr{L}$ -valued automaton and describe its structure and operation. Here, we consider some  $\mathscr{L}$ -valued hypergroups as:

- a) The attached  $\mathscr{L}$ -valued order hypergroup,
- b) The attached  $\mathscr{L}$ -valued behaviour hypergroup.

Using the attached  $\mathscr{L}$ -valued order hypergroup, we can prove the Kleene's theorem and the attached  $\mathscr{L}$ -valued behaviour hypergroup can lead to the creation of the minimal  $\mathscr{L}$ -valued tree automaton that has the same behaviour with the given  $\mathscr{L}$ -valued tree automaton. Now, we shall introduce some new  $\mathscr{L}$ -valued hyperstructures and will show how these structures can be connected to the theory of  $\mathscr{L}$ -valued tree automata.

**Definition 3.19.** The order of a state  $q \in Q$ , denoted by *ord* q, is the minimum of the height of trees that lead to q.

One or more unreachable states possibly exist, *i.e.*, there is no tree that leads to these states. These states have no influence in the operation of the  $\mathscr{L}$ -valued tree automaton and therefore their order is not necessary to be defined. Let  $\mathcal{R} = \{ ord \ q \ | \ q \in Q \}$  and  $f \in \mathscr{L}^{\mathcal{R}}$  be such that  $f(ord \ q) \leq f(ord \ p)$  if and only if  $ord \ q \leq ord \ p$ .

Then the  $\mathscr{L}$ -valued equivalence class of q is defined as:

$$C_q = \left\{ \left(p, C_q(p)\right) \mid p \in Q \right\}, \text{ where } C_q(p) = \left(f(ord \ q) \ \rho \ f(ord \ p)\right) \odot \left(f(ord \ p) \ \rho \ f(ord \ q)\right).$$

According to the definition of the  $\mathscr{L}$ -valued hyperoperation, different types of hypergroups can be introduced in Q, with the use of the notion of order. Here, through introducing the notion of the order of a state, an  $\mathscr{L}$ -valued hypergroup of the  $\mathscr{L}$ -valued tree automaton appears which is called the attached  $\mathscr{L}$ -valued order hypergroup.

This  $\mathscr{L}$ -valued hypergroup is needed for the proof of Kleene's theorem along with using the tools and methods derived from the theory of hyperstructures.

**Theorem 3.20.** The set of states of an  $\mathscr{L}$ -valued tree automaton, endowed with the following  $\mathscr{L}$ -valued hyperoperation is an  $\mathscr{L}$ -valued hypergroup.

$$(q \star q)(p) = \begin{cases} 0 & Q_f(q) > 0 \text{and} Q_f(p) = 0\\ \sum_{ord \ z \le ord \ q} C_z(p) & o.w. \end{cases}$$
(3.1)

and

$$(q \star p)(r) = (q \star q)(r) + (p \star p)(r), \ \forall q, p, r \in Q.$$

*Proof.* Since  $C_z$  is reflexive and transitive, for any  $q, q', p \in Q$ , we have

(1)  $(q \star q)(q) = C_q(q) = 1,$ (2)  $(q \star q)(q') \ge (q \star q)(p) \odot (p \star p)(q').$ 

From (1) and (2), we have

$$(q \star q)(q') \ge \sum_{p \in Q} \left( (q \star q)(p) \odot (p \star p)(q') \right) \ge (q \star q)(q) \odot (q \star q)(q') = (q \star q)(q'),$$

i.e.,

$$(q \star q)(q') = \sum_{p \in Q} \Big( (q \star q)(p) \odot (p \star p)(q') \Big).$$

Now we show that  $\mathcal{HG1}(\star) > 0$ . For any  $q, q', p \in Q$  and  $s \in Q$ , we have

$$\begin{split} \big((q \star q') \star p\big)(s) &= \sum_{r \in Q} \left( (q \star q')(r) \odot (r \star p)(s) \right) = \sum_{r \in Q} \left( (q \star q')(r) \odot ((r \star r)(s) + (p \star p)(s)) \right) \\ &= \sum_{r \in Q} \left( \left( (q \star q')(r) \odot (r \star r)(s) \right) + \left( (q \star q')(r) \odot (p \star p)(s) \right) \right) \\ &= \sum_{r \in Q} \left( (q \star q')(r) \odot (r \star r)(s) \right) + \sum_{r \in Q} \left( (q \star q')(r) \odot (p \star p)(s) \right) \\ &= \sum_{r \in Q} \left( \left( (q \star q)(r) \odot (r \star r)(s) \right) + \left( (q' \star q')(r) \odot (r \star r)(s) \right) \right) + (p \star p)(s) \\ &= (q \star q)(s) + (q' \star q')(s) + (p \star p)(s). \end{split}$$

Similarly, we obtain

$$\sum_{k \in Q} \left( (q' \star p)(k) \odot (q \star k)(s) \right) = (q \star q)(s) + (q' \star q')(s) + (p \star p)(s).$$

Thus,  $\sum_{r \in Q} \left( (q \star q')(r) \odot (r \star p)(s) \right) = \sum_{k \in Q} \left( (q' \star p)(k) \odot (q \star k)(s) \right)$  and  $\mathcal{H}\mathcal{G}\mathfrak{l}(\star) = 1$ . Now, we prove that  $\mathcal{H}\mathcal{G}(\star) > 0$ .  $\forall q, r \in Q$  we have

$$(q \star Q)(r) = \sum_{q' \in Q} \left( (q \star q')(r) \odot Q(q') \right) = \sum_{q' \in Q} \left( (q \star q)(r) + (q' \star q')(r) \right) = (r \star r)(r) = 1.$$

Also,

$$(Q \star q)(r) = \sum_{q' \in Q} \left( Q(q') \odot (q' \star q)(r) \right) = \sum_{q' \in Q} \left( (q' \star q')(r) + (q \star q)(r) \right) = (r \star r)(r) = 1.$$

So,  $(q \star Q)(r) = Q(r) = (Q \star q)(r)$ . Therefore,  $\mathcal{HG}(\star) = 1$  and the proof is completed.

The  $\mathscr{L}$ -valued hypergroup defined in (3.1) is named attached  $\mathscr{L}$ -valued order hypergroup.

Next, we will use the  $\mathscr{L}$ -valued hypergroup (3.1) to prove the Kleene's theorem that states "an  $\mathscr{L}$ -valued subset of  $\mathcal{T}(F)$  is behaviour of an  $\mathscr{L}$ -valued tree automaton, if and only if it is  $\mathscr{L}$ -valued rational." Note that the  $\mathscr{L}$ -valued hypergroup (3.1) is directly related to  $\mathcal{T}(F)$ , since we can define:

$$\lambda(q \star q) = \Big\{ \big(t, \lambda(q \star q)(t)\big) \mid t \in \mathcal{T}(F) \Big\},\$$

where

$$\lambda(q \star q)(t) = \sum_{p \in Q, (q \star q)(p) > 0} r_{\mathcal{A}}(t, p)$$

To prove the Kleene's theorem, first we must provide some lemmas:

**Lemma 3.21.** The  $\mathscr{L}$ -valued subset  $\lambda(q \star q) \in \mathscr{L}^{\mathcal{T}(F)}$  is  $\mathscr{L}$ -valued rational.

*Proof.* For each  $1 \leq i, j \leq |Q|$  and  $\mathcal{K} \subseteq Q$ , we define  $T(i, j, \mathcal{K})$  as the  $\mathscr{L}$ -valued set of trees  $t \in \mathcal{T}(F \cup \mathcal{K})$  such that there is an  $\mathscr{L}$ -valued run r of  $\mathcal{A}$  on t satisfying the following properties:

$$\begin{array}{l} -r(\varepsilon,q_i) > 0;\\ -\sum_{i=1}^{j} r(p,q_i) > 0 \ \text{ for all } p \neq \varepsilon, \ t(p) \in F_n. \end{array}$$

In the other words,  $T(i, j, \mathcal{K})(t) > 0$  if we can reach  $q_i$  at the root by using only states in  $\{q_1, \ldots, q_j\}$  when we assume that the leaves are states of  $\mathcal{K}$ .

Obviously  $\lambda(q \star q) = \bigcup_i \{T(i, |Q|, \emptyset) \mid (q \star q)(q_i) > 0\}$ , that is  $\lambda(q \star q)(t) = \sum_i T(i, |Q|, \emptyset)(t)$ , where  $(q \star q)(q_i) > 0$ . Now, we are going to show the rationality of  $T(i, j, \mathcal{K})$  by induction on j.

- Base case j = 0. The set  $T(i, 0, \mathcal{K})$  is the  $\mathscr{L}$ -valued set of trees t where the root is labelled by  $q_i$ , the leaves are labelled by  $\mathcal{K}$  and no internal node is labelled by some q. Therefore, there exist  $a_1, \ldots, a_n, a \in F_0$  such that t = a or  $t = f(a_1, \ldots, a_n)$ , also  $T(i, 0, \mathcal{K})(t) = r_{\mathcal{A}}(t, q_i)$ . Hence  $T(i, 0, \mathcal{K}) \in \mathscr{L}^{T(F \cup \mathcal{K})}$  is finite and is  $\mathscr{L}$ -valued rational.
- Induction case. Assume that for any  $\mathcal{K} \subseteq Q$ ,  $0 \leq j' < j$  and  $1 \leq i \leq |Q|$ , the set  $T(i, j', \mathcal{K})$  is  $\mathscr{L}$ -valued rational. Let us claim that the following equality holds:

$$T(i, j, \mathcal{K})(t) = T(i, j - 1, \mathcal{K})(t) + \left[T(i, j - 1, \mathcal{K} \cup \{q_j\}) \cdot_{q_j} T(j, j - 1, \mathcal{K} \cup \{q_j\})^{*, q_j} \cdot_{q_j} T(j, j - 1, \mathcal{K})\right](t).$$

By induction hypothesis, each set in the right-hand side of the equality defining  $T(i, j, \mathcal{K})$  is  $\mathscr{L}$ -valued rational. This yields the desired result. Now we show that our claim holds. The inclusion is shown as follows: Let  $T(i, j, \mathcal{K})(t) > 0$  and n be the number of occurrences of  $q_j$ . If  $n \neq 0$ , then there exist some subtrees in T(j, j - 1)

 $1, \mathcal{K} \cup \{q_j\}$ ) and  $T(j, j-1, \mathcal{K})$ . Also since the root of t is labelled by  $q_i$ , there is a subtree in  $T(i, j-1, \mathcal{K} \cup \{q_j\})$ . Applying the concatenation operation for these trees and iterating this process, we reach to  $(T(i, j-1, \mathcal{K} \cup \{q_j\}), q_j T(j, j-1, \mathcal{K}))(t) > 0$ . Also if n = 0, then  $T(i, j-1, \mathcal{K})(t) > 0$ . Therefore, this yields the desired result.

The converse inclusion is shown by induction on the number of occurrences of  $q_j$ .

- Base case. Let  $(T(i, j-1, \mathcal{K} \cup \{q_j\})) \cdot q_j T(j, j-1, \mathcal{K} \cup \{q_j\})^{*, q_j} \cdot q_j T(j, j-1, \mathcal{K}))(t) > 0$  and the number of occurrences of  $q_j$  be one. Then,  $T(i, j, \mathcal{K})(t) > 0$ .
- Induction case. Let the following property hold for n.

$$\Big(T(i,j-1,\mathcal{K}\cup\{q_j\})\cdot_{q_j}T(j,j-1,\mathcal{K}\cup\{q_j\})^{n,q_j}\cdot_{q_j}T(j,j-1,\mathcal{K})\Big)(t) \le T(i,j,k)(t).$$

It is obvious that

Also, by definition we have  $T(i, j - 1, \mathcal{K})(t) \leq T(i, j, \mathcal{K})(t)$ . Therefore,  $T(i, j, \mathcal{K})$  and also  $\lambda(q \star q)$  are  $\mathscr{L}$ -valued rational subsets of  $\mathcal{T}(F \cup \mathcal{K})$  where  $(q \star q)(q_i) > 0$ .

It is obvious that if  $Q_f$  is the set of  $\mathcal{L}$ -valued final states of  $\mathcal{A}$ , then  $\bigcup_{q \in Q, Q_f(q) > 0} \lambda(q \star q)$  is the behaviour of  $\mathscr{L}$ -valued tree automaton  $\mathcal{A}$ , therefore we have the following theorem:

**Theorem 3.22.** The behaviour of an  $\mathscr{L}$ -valued tree automaton is  $\mathscr{L}$ -valued rational subset of  $\mathcal{T}(F)$ .

From the above theorem the direct part of Kleene's theorem is derived. Now, we are going to consider the other part of Kleene's theorem.

Let  $\mathcal{Y}$  be a finite set. Then,  $\mathcal{Y}$  is a set of  $\mathscr{L}$ -valued ranked hyperoperators over arbitrary set  $\mathcal{M}$ , if there exists an external ranked hyperoperation from  $\mathcal{Y} \times \mathcal{M}^n \times \mathcal{M}$  to  $\mathscr{L}$ , where  $n \geq 0$ . Now, let  $\mathcal{Y}$  be a set of  $\mathscr{L}$ -valued ranked hyperoperators over  $\mathcal{M}$  and  $\mathcal{HR}(\boxplus, \boxtimes) > 0$ , where  $\boxtimes$  is an  $\mathscr{L}$ -valued operation and  $\boxplus$  is an  $\mathscr{L}$ -valued hyperoperation on  $\mathcal{T}(\mathcal{Y})$ . Then, an  $\mathscr{L}$ -valued subset  $\mathcal{R}$  of  $\mathcal{T}(\mathcal{Y})$  will be named the set of  $\mathscr{L}$ -valued behaviours from  $\mathcal{M}$ , if there exists  $\mathcal{F} \in \mathscr{L}^{\mathcal{M}}$  such that  $\left[\phi(\mathcal{F})(t)\rho\mathcal{R}(t)\right] \odot \left[\mathcal{R}(t)\rho\phi(\mathcal{F})(t)\right] > 0, \forall t \in \mathcal{T}(\mathcal{Y})$  where  $\phi(\mathcal{F}) \in \mathscr{L}^{\mathcal{T}(\mathcal{Y})}$ .

**Lemma 3.23.** Let  $\mathcal{A} = (Q, F, Q_f, \delta)$  be an  $\mathscr{L}$ -valued tree automaton. For a given  $Q_f \in \mathscr{L}^Q$ , all the  $\mathscr{L}$ -valued behaviours can be found.

*Proof.* Let  $Q = \{q_j \mid j = 1, ..., n\}$  be a finite set with |Q| = n and

$$X_j = \left\{ \left(t, r_{\mathcal{A}}(t, q_j)\right) \mid t \in \mathcal{T}(F) \text{ and } r_{\mathcal{A}}(t, q_j) \in \mathscr{L} \right\}, \ j = 1, \dots, n.$$

Obviously,  $X_1, \ldots, X_n$  are the  $\mathscr{L}$ -valued behaviours from Q and  $\mathscr{B}-\mathcal{HR}(+, \square) > 0$  where,  $\square$  is an  $\mathscr{L}$ -valued operation and + is an  $\mathscr{L}$ -valued hyperoperation on  $\mathcal{T}(F)$ . Also let

$$B_j = \begin{cases} 1 & Q_f(q_j) > 0\\ 0 & o.w. \end{cases}$$

$$A_{ij} = \left\{ \left(t, T(i, j-1, \mathcal{K} \cup \{q_j\})(t)\right) \mid t \in \mathcal{T}(F) \text{ and } T(i, j-1, \mathcal{K} \cup \{q_j\})(t) \in \mathscr{L} \right\},\$$

where  $\mathcal{K} \subseteq Q$  and  $1 \leq i, j \leq n$ . For a given  $Q_f$ , the set of  $\mathscr{L}$ -valued behaviours from Q is the solution of the following  $n \times n$  system:

$$X_{1}(t) = (A_{11}.q_{1}X_{1})(t) + (A_{12}.q_{2}X_{2})(t) + \dots + (A_{1n}.q_{n}X_{n})(t) + B_{1}$$
  

$$X_{2}(t) = (A_{21}.q_{1}X_{1})(t) + (A_{22}.q_{2}X_{2})(t) + \dots + (A_{2n}.q_{n}X_{n})(t) + B_{1}$$
  

$$\vdots$$
  

$$X_{n}(t) = (A_{n1}.q_{1}X_{1})(t) + (A_{n2}.q_{2}X_{2})(t) + \dots + (A_{nn}.q_{n}X_{n})(t) + B_{1}$$

By solving this system, according to the theory developed in [35], we get  $X_1, \ldots, X_n$  which are all the  $\mathscr{L}$ -valued behaviours from Q. Therefore, the lemma is proved.

Lemma 3.23 guarantees the existence of all the  $\mathcal{L}$ -valued behaviours from Q. Therefore, in the following two lemmas the existence of  $\mathscr{L}$ -valued behaviours  $L_1$  from  $Q_1$  and  $L_2$  from  $Q_2$  is guranteed.

**Lemma 3.24.** Let F be a set of  $\mathscr{L}$ -valued ranked hyperoperators over  $Q_1$  and  $Q_2$ . If  $L_1$  and  $L_2$  are the sets of  $\mathscr{L}$ -valued behaviours from  $Q_1$  and  $Q_2$ , respectively, then there exists a set Q with hyperoperators from F such that  $L_1 + L_2$  is also a set of  $\mathscr{L}$ -valued behaviours from Q.

Proof. By hypothesis, there exist  $Q_{f_1} \in \mathscr{L}^{Q_1}$  and  $Q_{f_2} \in \mathscr{L}^{Q_2}$  such that  $\left(\phi_1(Q_{f_1})(t_1)\rho L_1(t_1)\right) \odot \left(L_1(t_1)\rho\phi_1 > (Q_{f_1})(t_1)\right) > 0$  and  $\left(\phi_2(Q_{f_2})(t_2)\rho L_2(t_2)\right) \odot \left(L_2(t_2)\rho\phi_2(Q_{f_2})(t_2)\right) > 0, \forall t_1, t_2 \in \mathcal{T}(F)$ . Now, we define  $Q = Q_1 \cup Q_2, \ Q_f = Q_{f_1} \cup Q_{f_2}$  and  $\phi(Q_f) = \phi_1(Q_{f_1}) \cup \phi_2(Q_{f_2})$  where  $\phi(Q_f) \in \mathscr{L}^{\mathcal{T}(F)}$ . Therefore we obtain the desired result.

Also, it is easy to prove the following lemma.

**Lemma 3.25.** Let F be a set of  $\mathscr{L}$ -valued ranked hyperoperators over  $Q_1$  and  $Q_2$ . If  $L_1$  and  $L_2$  are the sets of  $\mathscr{L}$ -valued behaviours from  $Q_1$  and  $Q_2$ , respectively, then there exists a set Q such that  $L_{1:\square}L_2$  is also a set of  $\mathscr{L}$ -valued behaviours from Q.

From the Lemma 3.25 we have the following corollary:

### Corollary 3.26. Let L be

the set of  $\mathscr{L}$ -valued behaviours from Q. Then  $L^{n,\Box}$  is also a set of  $\mathscr{L}$ -valued behaviours from Q.

Utilizing the Lemmas 3.24 and 3.25, Corollary 3.26 and this fact that every finite set of trees defines the  $\mathscr{L}$ -valued behaviour of an  $\mathscr{L}$ -valued tree automaton, we have the other part of Kleene's theorem that is as follows:

**Theorem 3.27.** Every  $\mathscr{L}$ -valued rational subset of  $\mathcal{T}(F)$  is  $\mathscr{L}$ -valued behaviour of an  $\mathscr{L}$ -valued tree automaton.

From Theorems 3.22 and 3.27, we have the Kleene's theorem as follows:

**Theorem 3.28** (Kleene's Theorem). An  $\mathscr{L}$ -valued subset of  $\mathcal{T}(F)$  is the behaviour of an  $\mathscr{L}$ -valued tree automaton, if and only if it is  $\mathscr{L}$ -valued rational.

**Remark 3.29.** Using the tools and methods of hyperstructure theory, Massouros [37] obtained a new proof of the famous Kleene's theorem for automata theory. Let us recall that if we let  $\mathscr{L} = \{0, 1\}$  then an  $\mathscr{L}$ -valued tree automaton reduces to a tree automaton. Moreover, we know that the words over a finite alphabet can be

viewed as unary terms, therefore automata become special tree automata. Hence the obtained results in this section, extend the corresponding ones in automata theory [37].

**Remark 3.30.** If  $\mathscr{L} = \{0, 1\}$ , Theorem 3.28 is just Kleene's theorem for tree automata in [5, 17]. If we consider  $\mathscr{L} = [0, 1] \subseteq \mathbb{R}$  and maximum arity of trees equals to one, then we obtain the Kleene's theorem for fuzzy automata [40]. The Kleene's theorem for  $\mathscr{L}$ -valued automaton [29], fuzzy tree automata [13] and weighted tree automata over semirings [14] be seen as special cases of Theorem 3.28.

# 4. Minimal $\mathscr{L}$ -valued tree automata

In this section, we define an  $\mathscr{L}$ -valued hypergroup that can lead to the creation of the minimal  $\mathscr{L}$ -valued tree automaton that accepts the same language as the initial one. Let  $\mathscr{A} = (Q, F, \{q_f\}, \delta)$  be an  $\mathscr{L}$ -valued tree automaton where  $q_f$  is the only final state of the  $\mathscr{L}$ -valued tree automaton  $\mathscr{A}$ .

We define  $[q] = {}^{def} \left\{ \left(q', [q](q')\right) \mid q' \in Q \right\}$  where

$$[q](q') =^{def} \prod_{(q_1,\dots,q_{i-1},q_{i+1},\dots,q_n) \in Q^{n-1}} \prod_{f \in F_n} \left[ \left( \delta \left( (q_1,\dots,q_{i-1},q,q_{i+1},\dots,q_n),f,p \right) \rho \right) \right) \right]$$

$$\delta((q_1, \ldots, q_{i-1}, q', q_{i+1}, \ldots, q_m), f, p')) \odot \left(\delta((q_1, \ldots, q_{i-1}, q', q_{i+1}, \ldots, q_n), f, p')\rho\right)$$

$$\delta((q_1,\ldots,q_{i-1},q,q_{i+1},\ldots,q_m),f,p)) \odot [p](p')$$

If [q](q') > 0, then we say that the states q and q' have the same  $\mathscr{L}$ -valued behaviour. Now, we introduce an  $\mathscr{L}$ -valued hyperoperation "o" on Q as follows:

$$(q_1 o q_2)(q) = \begin{cases} [q_1](q) + [q_2](q) & [q_1](q) \neq [q_2](q) \\ [q_1](q) + [q_f](q) & [q_1](q) = [q_2](q), q_1 \neq q_2 \\ [q_1](q) & [q_1](q) = [q_2](q), q_1 = q_2, \end{cases}$$
(4.1)

where  $q_f$  is the only final state of the  $\mathscr{L}$ -valued tree automaton or the conventional final state. It is obvious that  $\mathcal{H}_{\mathcal{G}}(o) > 0$ . Also it can be easily shown that  $\mathcal{P}_{\mathcal{R}}(o) > 0$ . Thus we have the following theorem:

**Theorem 4.1.** The  $\mathscr{L}$ -valued hyperoperation "o" defined in (4.1), is an  $\mathscr{L}$ -valued partially reversible hypergroup.

The  $\mathscr{L}$ -valued partially reversible hypergroup defined in (4.1), is named attached  $\mathscr{L}$ -valued behaviour hypergroup.

**Remark 4.2.** The notion of the behaviour is directly relevant to the creation of a minimum  $\mathscr{L}$ -valued tree automaton which accepts the same language as the initial one. Therefore, if in an  $\mathscr{L}$ -valued tree automaton there exist two states of the same behaviour, it makes no difference in the process of reaching the final state, whether we use the first one or the other. So, if the behaviour hypergroup is  $\mathscr{L}$ -valued partially reversible, then we can construct an  $\mathscr{L}$ -valued hypergroup and an  $\mathscr{L}$ -valued tree automaton which has this new hypergroup as behaviour hypergroup that has less states than the original one, but it accepts exactly the same language as the initial one.

In the following theorem, the existence of the minimal form of an  $\mathscr{L}$ -valued tree automaton is shown.

**Theorem 4.3.** Let  $\mathcal{A} = (Q, F, \{q_f\}, \delta)$  be an  $\mathscr{L}$ -valued tree automaton and  $\mathcal{PR}(o) > 0$ , where  $o \in \mathscr{L}^{Q \times Q \times Q}$ . Then, there exists  $Q' \subseteq Q$  such that  $\mathcal{HG}(o) > 0$ , where  $o \in \mathscr{L}^{Q' \times Q' \times Q'}$ . Also, there exists  $\mathcal{A}' = (Q', F, Q'_f, \delta')$  such that  $B_{\mathcal{A}}(Q) = B_{\mathcal{A}'}(Q')$ .

*Proof.* Let  $\mathcal{PR}(o) > 0$ . Then, there exist  $q_1, q_2 \in Q$  such that  $(q_1 o q_1)(q) = (q_2 o q_2)(q)$ ,  $\forall q \in Q$ , where  $(q_1 o q_2)(q_f) > 0$ . Since  $[q_1] = [q_2]$ ,  $B_{\mathcal{A}}(q_1) = B_{\mathcal{A}}(q_2)$ . Therefore, we can omit one of the states. Thus, we consider  $Q' = Q \setminus \{q_1 \in Q \mid \exists q_2 \in Q, [q_1] = [q_2]\}$  such that there exists an  $\mathscr{L}$ -valued hyperoperation "o" on Q' as

$$(qop)(q') = [q](q') + [p](q').$$

Obviously,  $\mathcal{H}\mathcal{G}(o) > 0$ . Also, there exists an  $\mathscr{L}$ -valued tree automaton corresponding to the hypergroup as  $\mathcal{A}_{min} = (Q_{min}, F, \{q_f\}, \delta_{min})$  where  $Q_{min} = Q'$  and  $\delta_{min}((q_1, \ldots, q_n), q_t, \sigma) = \sum \{\delta((p_1, \ldots, p_n), p_t, \sigma) : [q_t](p_t) > 0, [q_i](p_i) \neq 0, i = 1, \ldots, n\}$ .

In Algorithm 4.4, we provide a minimization algorithm for  $\mathscr{L}$ -valued hypergroup defined in (4.1). Utilizing Algorithm 4.4, we obtain a new  $\mathscr{L}$ -valued hypergroup on Q' such that  $Q' \subset Q$  and both show the same behaviour.

Algorithm 4.4. A procedure for obtaining minimal  $\mathscr{L}$ -valued hypergroup.

 $\begin{array}{lll} \textbf{Input:} & \text{The } \mathscr{L}\text{-valued behaviour hypergroup given in (4.1);} \\ & \text{Set } Q' = Q; \\ & \text{For all } q \in Q \text{ obtain } qoq; \\ & \text{If } qoq = pop \text{ for all } q, p \in Q \text{ such that } q \neq p, \text{ then} \\ & \text{Set } Q' = Q \setminus \{p\}; \\ \textbf{Output:} & \text{The minimal } \mathscr{L}\text{-valued behaviour hypergroup corresponding to input;} \end{array}$ 

Next, we will illustrate the Algorithm 4.4 by an example.

**Example 4.5.** Let  $\mathscr{L} = [0,1] \subseteq \mathbb{R}$  and  $\mathscr{A} = (Q, F, Q_f, \delta)$  be a deterministic, complete and reduced  $\mathscr{L}$ -valued tree automaton where:

$$Q = \{q_0, q_1, q_2, q_3\}, F = \{a, b, g(,)\}, Q_f(q_0) = 0.2, Q_f(q_1) = 0.4, Q_f(q_2) = Q_f(q_3) = 0$$

and  $\delta$  be as follows:

$$\begin{split} \delta(a,q_0) &= 1, & \delta(b,q_1) = 1, & \delta\big((q_3,q_3),g,q_1\big) = 0.2. \\ \delta\big((q_0,q_0),g,q_2\big) &= 0.3, & \delta\big((q_0,q_1),g,q_2\big) = 0.3, & \delta\big((q_0,q_2),g,q_3\big) = 0.8, \\ \delta\big((q_0,q_3),g,q_2\big) &= 0.3, & \delta\big((q_1,q_0),g,q_2\big) = 0.3, & \delta\big((q_1,q_1),g,q_2\big) = 0.3, \\ \delta\big((q_1,q_2),g,q_3\big) &= 0.8, & \delta\big((q_1,q_3),g,q_2\big) = 0.3, & \delta\big((q_2,q_0),g,q_3\big) = 0.2, \\ \delta\big((q_2,q_1),g,q_3\big) &= 0.2, & \delta\big((q_2,q_2),g,q_1\big) = 0.1, & \delta\big((q_2,q_3),g,q_0\big) = 0.1, \\ \delta\big((q_3,q_0),g,q_1\big) &= 0.1, & \delta\big((q_3,q_1),g,q_1\big) = 0.1, & \delta\big((q_3,q_2),g,q_0\big) = 0.1, \end{split}$$

Also we consider  $\mathscr{L}$ -valued hypergroup defined in (4.1). Then we have:

$$q_0 o q_0 = \{(q_0, 1), (q_1, 1)\}, \qquad q_2 o q_2 = \{(q_2, 1)\}, q_1 o q_1 = \{(q_0, 1), (q_1, 1)\}, \qquad q_3 o q_3 = \{(q_3, 1)\}.$$

Therefore, by Algorithm 4.4 we have:

$$q_1 o q_2 = \{(q_1, 1), (q_2, 1)\}, \qquad q_1 o q_3 = \{(q_1, 1), (q_3, 1)\}, \qquad q_2 o q_3 = \{(q_2, 1), (q_3, 1)\}.$$

Thus,  $\mathscr{L}$ -valued tree automaton which has this new hypergroup as  $\mathscr{L}$ -valued behaviour hypergroup is as:  $\mathcal{A}_{min} = (Q_{min}, F, Q_{f_{min}}, \delta_{min})$  where,

$$Q_{min} = \{q_1, q_2, q_3\}, F = \{a, b, g(,)\}, Q_{f_{min}}(q_1) = 0.4$$

and  $\delta_{min}$  is as follows:

$$\begin{split} \delta(a,q_1) &= 1, & \delta(b,q_1) = 1, & \delta((q_3,q_3),g,q_1) = 0.2, \\ \delta((q_1,q_1),g,q_2) &= 0.3, & \delta((q_1,q_2),g,q_3) = 0.8, & \delta((q_1,q_3),g,q_2) = 0.3, \\ \delta((q_2,q_1),g,q_3) &= 0.2, & \delta((q_2,q_2),g,q_1) = 0.1, & \delta((q_2,q_3),g,q_1) = 0.1, \\ \delta((q_3,q_1),g,q_1) &= 0.1, & \delta((q_3,q_2),g,q_1) = 0.1. \end{split}$$

Obviously,  $\mathscr{L}$ -valued tree automata  $\mathscr{A}$  and  $\mathscr{A}_{min}$  have the same behaviour.

## 5. Conclusions

This paper presented two types of  $\mathscr{L}$ -valued hyperstructures that are used for solving the main problems in the theory of languages and tree automata over complete residuated lattices. Through defining  $\mathscr{L}$ -valued hyperoperations, we obtained the following attached hypergroups:

- a) The attached  $\mathscr{L}$ -valued order hypergroup,
- b) The attached  $\mathscr{L}$ -valued behaviour hypergroup.

Using the attached  $\mathscr{L}$ -valued order hypergroup, we proved the Kleene's theorem for  $\mathscr{L}$ -valued tree automata, and the attached  $\mathscr{L}$ -valued behaviour hypergroup led to the creation of a minimal  $\mathscr{L}$ -valued tree automaton that has the same behaviour as the given  $\mathscr{L}$ -valued tree automaton.

Acknowledgements. The author would like to express her gratitude to the handling editor and anonymous referees for their helpful comments and suggestions that improved the quality of the paper.

### References

- [1] R. Ameri and M.M. Zahedi, Hypergroup and join space induced by a fuzzy subset. Pure Math. Appl. 8 (1997) 155-168.
- [2] G. Birkhof, Lattice Theory. American Mathematical Society, Providence (1984).
- [3] S. Bozapalidis and O.L. Bozapalidoy, Fuzzy tree language recognizability. Fuzzy Sets Syst. 161 (2010) 716–734.
- [4] J. Chvalina and L. Chvalinova, State hypergroup of automata. Acta Mat. et. Inf. Univ. Ostraviensis 4 (1996) 105-120.
- [5] H. Comon, M. Dauchet, R. Gilleron, F. Jacquemard, D. Lugiez, C. Loding, S. Tison and M. Tommasi, Tree Automata: technigues and Applications. Available at http://tata.gforge.inria.fr (2007).
- [6] P. Corsini, Prolegomena of Hypergroup Theory. Edited by Aviani (1993).
- [7] P. Corsini, Join spaces, power sets, fuzzy sets, in Proc. of the Fifth International Congress of Algebraic Hyperstructures and Their Applications (Iasi, 1993). Hadronic Press, Palm Harbor (1994) 45–52.
- [8] P. Corsini and V. Leoreanu, Applications of Hyperstructure Theory. Advances in Mathematics. Kluwer Academic Publishers, Dordrecht (2003).
- [9] B. Davvaz, Fuzzy H<sub>v</sub>-groups. Fuzzy Sets Syst. **101** (1999) 191–195.
- [10] B. Davvaz, Interval-valued fuzzy subhypergroups. Korean J. Comput. Appl. Math 6 (1999) 197–202.
- [11] B. Davvaz, On fuzzy relations and fuzzy subhypergroups. Pure Math. Appl. 11 (2000) 51–58.
- [12] W.A. Dudek, J. Zhan and B. Davvaz, On intuitionistic (S, T)-fuzzy hypergroups. Soft Comput. 12 (2008) 1229–1238.
- [13] Z. Esik and G. Liu, Fuzzy tree automata. Fuzzy Sets Syst. 158 (2007) 1450-1460.
- [14] Z. Esik and W. Kuich, Formal tree series, in: Weighted Automata: Theory and Applications. J. Automat. Lang. Comb. 8 (2002) 219–285.

- [15] J. Ignjatovic, M. Ciric and S. Bogdanovic, Determinization of fuzzy automata with membership values in complete residuated lattices. Inf. Sci. 178 (2008) 164–180.
- [16] Y. Inagaki and T. Fukumura, On the description of fuzzy meaning of context-free language, Fuzzy sets and their applications to cognitive and decision processes, in Proc. of the US-Japan Seminar, University of California, Berkeley, CA, 1974. Academic Press, New York (1975) 301–328.
- [17] F. Gécseg and M. Steinby, Tree Automata. Akademiai Kiado, Budapest, 1984.
- [18] M. Ghorani, On characterization of fuzzy tree pushdown automata. To appear in: Soft Comput. DOI: 10.1007/s00500-017-2829-7 (2017).
- [19] M. Ghorani, Tree automata based on complete residuated lattice-valued logic: reduction algorithm and decision problems. To appear in: Iran. J. Fuzzy Syst. DOI: 10.22111/ijfs.2017.3392 (2017).
- [20] M. Ghorani and M.M. Zahedi, Characterization of complete residuated lattice-valued finite tree automata. Fuzzy Sets Syst. 199 (2012) 28–46.
- [21] M. Ghorani and M.M. Zahedi, Alternating regular tree grammars in the framework of lattice-valued logic. Iran. J. Fuzzy Syst. 13 (2016) 71–94.
- [22] M. Ghorani and M.M. Zahedi, Coding tree languages based on lattice-valued logic. Soft Comput. 21 (2017) 3815–3825.
- [23] M. Ghorani, M.M. Zahedi and R. Ameri, Algebraic properties of complete residuated lattice-valued tree automata. Soft Computing 16 (2012) 1723–1732.
- [24] M. Gori and A. Petrosino, Encoding non-deterministic fuzzy tree automata into recursive neural networks. IEEE Trans. Neur. Net. 156 (2004) 1435–1449.
- [25] M. Horry and M.M. Zahedi, Hypergroups and general fuzzy automata. Iran. J. Fuzzy Syst. 6 (2009) 61-74.
- [26] E.T. Lee, Fuzzy tree automata and syntactic pattern recognition. IEEE Trans. Pattern Anal. Mach. Intell. PAMI-4 (1982) 445–449.
- [27] E.T. Lee and L.A. Zadeh, Note on fuzzy languages. Inf. Sci. 1 (1969) 421-434.
- [28] H.X. Lei and Y. Li, Minimization of states in automata theory based on finite lattice-ordered monoids. Inf. Sci. 177 (2007) 1413–1421.
- [29] Y.M. Li, Finite automata theory with membership values in lattices. Inf. Sci. 181 (2011) 1003–1017.
- [30] Y.M. Li and W. Pedrycz, Minimization of lattice finite automata and its application to the decomposition of lattice languages. *Fuzzy Sets Syst.* 158 (2007) 1423–1436.
- [31] L. Li and D. Qiu, On the state minimization of fuzzy automata. IEEE Trans. Fuzzy Syst. 32 (2015) 434–443.
- [32] F. Lin and H. Ying, Modeling and control of fuzzy discrete event systems. IEEE Trans. Syst. Man Cybern. B 32 (2002) 408–415.
- [33] F. Marty, Sur une generalisation delanotion degroupe. Huitieme Congr. math. Scan. Stockholm (1934) 45-49.
- [34] Ch.G. Massouros and G.G. Massouros, On certain fundamental properties of hypergroups and fuzzy hypergroups- Mimic fuzzy hypergroups. Int. J. Risk Theory 2 (2012) 71–82.
- [35] G.G. Massouros, Automata, Languages and Hypercompositional Structures. Doctoral Thesis, National Technical University of Athens (1993).
- [36] G.G. Massouros, An automaton during its operation, in Proc. of the 5th International Congress on Algebraic Hyperstructures and Applications, HadronicPress (1994) 267–276.
- [37] G.G. Massouros, Automata and hypermoduloids, in Proc. of the 5th International Congress on Algebraic Hyperstructures and Applications (Iasi 1993), Hadronic Press (1994) 251–266.
- [38] G.G. Massouros and J. Mittas, Languages, automata and hypercompositional hyperstructures and applications, in Proc. of the 4th International Congress on Algebraic Hyperstructures and Applications (Xanthi 1990), World Scientific (1991) 137–147.
- [39] S. Moghari and M.M. Zahedi, Similarity-based minimization of fuzzy tree automata. J. Appl. Math. Comput. 55 (2016) 417–436.
- [40] J.N. Mordeson and D.S. Malik, Fuzzy Automata and Languages: theory and Applications. Chapman & Hall CRC, London, Boca Raton (2002).
- [41] C.W. Omlin, K.K. Thornber and C.L. Giles, Fuzzy finite-state automata can be deterministically encoded in recurrent neural networks. *IEEE Trans. Fuzzy Syst.* 5 (1998) 76–89.
- [42] W. Pedrycz and A. Gacek, Learning of fuzzy automata. Int. J. Comput. Intell. Appl. 1 (2001) 19–33.
- [43] D.W. Qiu, Automata theory based on completed residuated lattice-valued logic (I). Sci. China (Series F) 44 (2001) 419–429.
- [44] D.W. Qiu, Automata theory based on completed residuated lattice-valued logic (II). Sci. China (Series F) 45 (2002) 442-452.
- [45] D.W. Qiu, Characterizations of fuzzy finite automata. Fuzzy Sets Syst. 141 (2004) 391-414.
- [46] D.W. Qiu, Supervisory control of fuzzy discrete event systems: a formal approach. IEEE Trans. Sys. Man Cybern.-Part B 35 (2005) 72–88.
- [47] D.W. Qiu, Pumping lemma in automata theory based on complete residuated lattice-valued logic: a note. Fuzzy Sets Syst. 157 (2006) 2128–38.
- [48] E.S. Santos, Maximin automata. Inform. Control 12 (1968) 367-377.
- [49] M.G. Thomason and P.N. Marinos, Deterministic acceptors of regular fuzzy languages. IEEE Trans. Sys. Man Cybern. 4 (1974) 228–230.
- [50] W.G. Wee and K.S. Fu, A formulation of fuzzy automata and its application as a model of learning systems. *IEEE Trans. Sys. Man Cybern.* 5 (1969) 215–223.

- [51] L. Wu and D.W. Qiu, Automata theory based on completed residuated lattice-valued logic: reduction and minimization. Fuzzy Sets Syst. 161 (2010) 1635–1656.
- [52] H.Y. Xing, D.W. Qiu and F.C. Liu, Automata theory based on complete residuated lattice-valued logic: pushdown automata. *Fuzzy Sets Syst.* 160 (2009) 1125–1140.
- [53] H.Y. Xing, D.W. Qiu, F.C. Liu and Z.J. Fan, Equivalence in automata theory based on complete residuated lattice-valued logic. Fuzzy Sets Syst. 158 (2007) 1407–1422.
- [54] L.A. Zadeh, Fuzzy sets. Inf. Control 8 (1965) 338–353.
- [55] M. Zahedi, M. Bolurian and A. Hasankhani, On polygroups and fuzzy subpolygroups. J. Fuzzy Math. 3 (1995) 1-15.

42