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EPISTURMIAN MORPHISMS AND A GALOIS THEOREM ON CONTINUED FRACTIONS

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Abstract. We associate with a word w on a finite alphabet A an episturmian (or Arnoux-Rauzy) morphism and a palindrome. We study their relations with the similar ones for the reversal of w. Then when $|A| = 2$ we deduce, using the Sturmian words that are the fixed points of the two morphisms, a proof of a Galois theorem on purely periodic continued fractions whose periods are the reversal of each other.

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Introduction

Sturmian words on a two-letter alphabet have been actively studied, at least since the fundamental paper of Morse and Hedlund [15]. For a survey, see [13]. These words have a deep relation with simple continued fractions. Among the generalizations of Sturmian words to any finite alphabet, the one known as Arnoux-Rauzy sequences [2, 16–18] or, with even slightly more generality, as episturmian words [3, 8, 11, 12] leads to many properties extending those of Sturmian words. In particular the continued fractions point of view has a transposition for episturmian words [20]. In this paper the famous Lagrange theorem relating periodic continued fractions to quadratic numbers was extended to a "multidimensional" continued fraction algorithm (see also [19]). However this one has not the same efficiency as the classical one for approximating irrationals by rationals. Evariste Galois

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found, when a student at Lycée Louis le Grand[1], a theorem relating the values of two purely periodic (simple) continued fractions whose periods are reversal of each other [9]. In the same paper Galois also gives a characterization of quadratic numbers with a purely periodic continued fraction expansion. Both results are used in [1] for characterizing the so-called Sturm numbers (see also [13], Th. 2.3.26).

Here we study relations involving some (finite) episturmian words and episturmian morphisms. More precisely we associate with a (finite) word w a palindrome $Pal(w)$ and a morphism μ_w. Such palindromes (which are the bispecial Arnoux-Rauzy words of [14]) play an important role in the theory of (infinite) episturmian words [8], in particular in the Sturmian case where they are called central words [13].

In Section 2 we establish some relations between $Pal(w)$, μ_w, $Pal(\tilde{w})$, $\mu_{\tilde{w}}$, where $\tilde{w}$ is the reversal of w. These relations are explained by the fact that the incidence matrices of μ_w and $\mu_{\tilde{w}}$ are similar.

In Section 3, applying this to a 2-letter alphabet, we prove the above-mentioned Galois theorem by considering the standard episturmian words which are the fixed points of μ_w and $\mu_{\tilde{w}}$.

It seems to us that this confirms the interest of the multidimensional continued fraction algorithm in the above sense. However it remains much work to do on the subject, in particular in relation with the generalized intercept introduced in [11] and the generalized Ostrowski numeration systems [4, 12].

1. Preliminaries

The alphabet A, $|A| \geq 2$, is finite and will be kept throughout the paper, ε is the empty word, A^* is the set of (finite) words and A^ω the set of (right) infinite words on A.

The length of $v = x_1 \cdots x_n$, $x_i \in A$, is $|v| = n$ and the number of occurences of letter x in v is $|v|_x$. If $|v|_x = 0$, then v is *x-free*. The *reversal* of v is $\tilde{v} = x_n \cdots x_1$ and v is a *palindrome* if $\tilde{v} = v$.

If $v \in A^*$ the *(Parikh) vector* of v is the $|A| \times 1$ vector of the $|v|_x$, $x \in A$, and if φ is an (endo)morphism of A^* its *incidence matrix* is the $|A| \times |A|$ matrix $M = (m_{xy})$ whose columns are the vectors of the $|\varphi(y)|$, $y \in A$, *i.e.*, $m_{xy} = |\varphi(y)|_x$.

For $u \in A^*$ its *(right) palindromic closure* is the shortest palindrome $u^{(+)}$ having u as a prefix. We have $u^{(+)} = uv^{-1}u$ with v the longest palindromic suffix of u (when $h = fg$ we sometimes write $f = hg^{-1}$ and $g = f^{-1}h$).

Now we say some words about (infinite) episturmian words, limiting ourselves to what is really needed here. Let $\Delta \in A^\omega$, $\Delta = x_1 x_2 \cdots x_i \cdots$, $x_i \in A$. The infinite word having the sequence of prefixes $u_1 = \varepsilon$, $u_2 = x_1$, ..., $u_{i+1} = (u_i x_i)^{(+)}$, ... is the *standard episturmian word directed* by Δ (also called characteristic Arnoux-Rauzy

[1] The author also studied at Lycée Louis le Grand but without discovering any theorem.

sequence if Δ contains infinitely many occurrences of each of its letters). In particular for $|A| = 2$ Arnoux-Rauzy sequences are the Sturmian words, while episturmian words also include the "periodic Sturmian words", *i.e.*, the infinite words given by cutting sequences with rational slope.

Now let $w = x_1 \cdots x_n$, $x_i \in A$, then using the successive letters of w construct as previously the sequence $u_1 = \varepsilon, \ldots, u_{n+1} = (u_n x_n)^{(+)}$ of all palindromic prefixes of u_{n+1}. We write $u_{n+1} = Pal(w)$ and say that w *directs* $Pal(w)$. Let $x \in A$. If w is x-free then $Pal(wx) = Pal(w)xPal(w)$. If x occurs in w write $w = w_1 x w_2$ with w_2 x-free. Then the longest palindromic prefix of $Pal(w)$ followed by x in $Pal(w)$ is $Pal(w_1)$ whence easily

$$Pal(wx) = Pal(w)Pal(w_1)^{-1}Pal(w). \tag{1}$$

Consider for any letter a the morphism ψ_a such that $\psi_a(a) = a$ and $\psi_a(b) = ab$ for any letter $b \neq a$. The ψ_a generate by composition the monoid of *pure standard episturmian morphisms* [11], Section 2.1.

For any $w = x_1 \cdots x_n$ as previously we write $\mu_w = \psi_{x_1} \circ \psi_{x_2} \cdots \circ \psi_{x_n}$. In particular μ_ε is the identity and, for $x \in A$, $\mu_x = \psi_x$. We also denote by M_w the incidence matrix of μ_w.

The relations between w, μ_w, M_w are one-to-one and indeed are isomorphisms between A^*, the monoid of pure standard episturmian morphisms and the monoid of the M_w.

For continued fractions see [5] for instance.

2. Words and matrices relations

The first lemma recalls and proves for sake of completeness some relations appearing with different notations in [8, 11]. This allows to prove the curious relation $|Pal(w)| = |Pal(\tilde{w})|$ and some other ones. Then we give an interpretation in terms of the matrices M_w and $M_{\tilde{w}}$.

Lemma 2.1.

1) For any palindrome p and letter x, $\mu_x(p)x$ is a palindrome.

2) For any $w \in A^*$, $x \in A$ we have

$$Pal(xw) = \mu_x\big(Pal(w)\big)x. \tag{2}$$

3) For any $v, w \in A^*$,

$$Pal(vw) = \mu_v\big(Pal(w)\big)Pal(v). \tag{3}$$

Proof.

Part 1) follows from the easy fact that, for $u \in A^*$, $x \in A$, $\mu_x(\tilde{u})x = x\widetilde{\mu_x(u)}$.

Part 2) is proved by induction on $|w|$. If $|w| = 0$ the assertion is trivial. Otherwise let $w = vy$, $y \in A$. If y occurs in v write $v = v_1 y v_2$ with v_2 y-free. Then by equation (1) $Pal(w) = Pal(v)Pal(v_1)^{-1}Pal(v)$ whence

$$\mu_x\big(Pal(w)\big)x = \mu_x\big(Pal(v)\big)xx^{-1}\mu_x\big(Pal(v_1)\big)^{-1}\mu_x\big(Pal(v)\big)x.$$

As $|v| < |w|$ and $|v_1| < |w|$ we get by induction hypothesis

$$\mu_x\big(Pal(w)\big)x = Pal(xv)Pal(xv_1)^{-1}Pal(xv).$$

As $xv = xv_1yv_2$ and $|v_2|_y = 0$, equation (1) shows that the right member is $Pal(xw)$.

If on the contrary $|v|_y = 0$ then $Pal(w) = Pal(v)yPal(v)$ whence

$$\mu_x\big(Pal(w)\big)x = \mu_x\big(Pal(v)\big)xx^{-1}\mu_x(y)\mu_x\big(Pal(v)\big)x = Pal(xv)x^{-1}\mu_x(y)Pal(xv).$$

Thus if $y \neq x$ then $|w|_y = 0$, whence $\mu_x\big(Pal(w)\big)x = Pal(xv)yPal(xv) = Pal(xvy) = Pal(xw)$. If $y = x$ then similarly $\mu_x\big(Pal(w)\big)x = Pal(xv)Pal(xv) = Pal(xvx) = Pal(xw)$.

For 3) the proof is by induction on $|v|$. The assertion is trivial for $|v| = 0$. Otherwise let $v = yv_1$. Using part 2) of the lemma and the induction hypothesis we successively get

$$\begin{aligned} Pal(vw) = Pal(yv_1w) = \mu_y\big(Pal(v_1w)\big)y = \mu_y\Big(\mu_{v_1}\big(Pal(w)\big)Pal(v_1)\Big)y \\ = \mu_{yv_1}\big(Pal(w)\big)\mu_y\big(Pal(v_1)\big)y = \mu_v\big(Pal(w)\big)Pal(v). \quad \square \end{aligned}$$

Lemma 2.2. *For $w \in A^*$ if $x \in A$ occurs in w write $w = w_1xw_2$ with w_1 x-free. Then*

$$|Pal(w)|_x = |Pal(w_2)| + 1. \tag{4}$$

Proof. By Lemma 2.1 $Pal(w) = \mu_{w_1}\big(Pal(xw_2)\big)Pal(w_1)$ whence as w_1 is x-free

$$\begin{aligned} |Pal(w)|_x = |\mu_{w_1}\big(Pal(xw_2)\big)|_x = |Pal(xw_2)|_x \\ = |\mu_x\big(Pal(w_2)\big)x|_x = |Pal(w_2)| + 1. \quad \square \end{aligned}$$

We deduce a rather curious relation between the palindromes directed by w and $\tilde{w}$.

Theorem 2.3. *For any $w \in A^*$, $Pal(w)$ and $Pal(\tilde{w})$ have the same length.*

Proof. The proof is by induction on $|w|$. This is trivial for $|w| = 0$. Otherwise set $w = vx$. If v is x-free then $Pal(w) = Pal(v)xPal(v)$ whence $|Pal(w)| = 2|Pal(v)| + 1$. Also, by equation (2), $Pal(\tilde{w}) = Pal(x\tilde{v}) = \mu_x\big(Pal(\tilde{v})\big)x$. As $Pal(\tilde{v})$ is x-free, $|\mu_x\big(Pal(\tilde{v})\big)| = 2|Pal(\tilde{v})|$. Thus using the induction hypothesis $|Pal(\tilde{w})| = 2|Pal(\tilde{v})| + 1 = 2|Pal(v)| + 1 = |Pal(w)|$.

Otherwise x occurs in v. Write $v = v_1xv_2$ with v_2 x-free. Then by equation (1) $Pal(w) = Pal(vx) = Pal(v)Pal(v_1)^{-1}Pal(v)$ whence $|Pal(w)| = 2|Pal(v)| - |Pal(v_1)|$.

Also $Pal(\tilde{w}) = Pal(x\tilde{v}) = \mu_x\big(Pal(\tilde{v})\big)x$ whence $|Pal(\tilde{w})| = 2|Pal(\tilde{v})| - |Pal(\tilde{v})|_x + 1$. But Lemma 2.2 applied to $\tilde{v} = \tilde{v}_2 x \tilde{v}_1$ gives $|Pal(\tilde{v})|_x = |Pal(\tilde{v}_1)| + 1$. Thus using induction hypothesis,

$$|Pal(\tilde{w})| = 2|Pal(\tilde{v})| - |Pal(\tilde{v}_1)| = 2|Pal(v)| - |Pal(v_1)| = |Pal(w)|. \qquad \square$$

Lemma 2.4. *Let $w \in A^*$, $y \in A$.*

1) If w is y-free then $\mu_w(y) = Pal(w)y$. Otherwise write $w = v_1 y v_2$, v_2 y-free, then $\mu_w(y) = Pal(w)Pal(v_1)^{-1}$.

2) With $x \in A$, if w is x-free or y-free then $|\mu_w(y)|_x = |Pal(w)|_x + |y|_x$. Otherwise write $w = v_1 y v_2 = w_1 x w_2$ with $|v_2|_y = |w_1|_x = 0$. Then

$$|\mu_w(y)|_x = |Pal(w)|_x - |Pal(v_1)|_x = |Pal(w)|_x - |Pal(\tilde{w}_2)|_y.$$

Proof. For 1), by equation (3), $Pal(wy) = \mu_w(y)Pal(w)$. If w is y-free then $Pal(wy) = Pal(w)yPal(w)$ whence $\mu_w(y) = Pal(w)y$. Otherwise, with $w = v_1 y v_2$, v_2 y-free, $Pal(wy) = Pal(w)Pal(v_1)^{-1}Pal(w)$ whence $\mu_w(y)$ as claimed.

For 2), if w is y-free then $|\mu_w(y)|_x = |Pal(w)y|_x = |Pal(w)|_x + |y|_x$. If w is x-free and $x \neq y$ then $|\mu_w(y)|_x = 0 = |Pal(w)|_x + |y|_x$.

Otherwise, write $w = v_1 y v_2 = w_1 x w_2$ with $|v_2|_y = |w_1|_x = 0$. Then $\mu_w(y) = Pal(w)Pal(v_1)^{-1}$ whence $|\mu_w(y)|_x = |Pal(w)|_x - |Pal(v_1)|_x$.

It remains to show that $|Pal(v_1)|_x = |Pal(\tilde{w}_2)|_y$. If $|v_1| \leq |w_1|$ then v_1 is x-free, w_2 is y-free, hence $|Pal(v_1)|_x = |Pal(\tilde{w}_2)|_y = 0$. Otherwise, $|v_1| > |w_1|$. Write $v_1 = w_1 x u$, $u \in A^*$. Then using Lemma 2.2, as v_2 is y-free and w_1 is x-free, $|Pal(v_1)|_x = |Pal(u)|+1$ and similarly $|Pal(\tilde{w}_2)|_y = |Pal(\tilde{u})|+1 = |Pal(u)|+1$. $\square$

Corollary 2.5. *The traces of M_w and $M_{\tilde{w}}$ are equal.*

Proof. For $x \in A$, if $|w|_x = 0$ then $|\mu_w(x)|_x = |\mu_{\tilde{w}}(x)|_x = 1$. Otherwise let $w = v_1 x v_2 = w_1 x w_2$ with $|v_2|_x = |w_1|_x = 0$. Then by Lemma 2.4 $|\mu_w(x)|_x = |Pal(w)|_x - |Pal(v_1)|_x = |Pal(w)|_x - |Pal(\tilde{w}_2)|_x$. Similarly $|\mu_{\tilde{w}}(x)|_x = |Pal(\tilde{w})|_x - |Pal(\tilde{w}_2)|_x$. Thus in both cases $|\mu_w(x)|_x - |\mu_{\tilde{w}}(x)|_x = |Pal(w)|_x - |Pal(\tilde{w})|_x$.

Summing over $x \in A$ we get $\mathrm{tr}(M_w) - \mathrm{tr}(M_{\tilde{w}}) = |Pal(w)| - |Pal(\tilde{w})| = 0$. $\square$

It is possible from this to show that M_w and $M_{\tilde{w}}$ have the same eigenvalues. Indeed M_{w^k} and $M_{\tilde{w}^k}$ have the same traces for any integer k. As the trace is the sum of the eigenvalues and as the eigenvalues of M_{w^k} and $M_{\tilde{w}^k}$ are the k-th powers of those of M_w and $M_{\tilde{w}}$ we get that the Newton sums of the eigenvalues of M_w and $M_{\tilde{w}}$ are the same and that these matrices have the same characteristic polynomial.

Theorem 2.6. *For $x \in A$ set $L_w(x) = \sum_{y \in A} |\mu_w(y)|_x$. Then*

$$L_w(x) = (|A| - 1)|Pal(w)|_x + 1. \tag{5}$$

Proof. If $|w|_x = 0$ then $L_w(x) = |\mu_w(x)|_x = 1 = (|A| - 1)|Pal(w)|_x + 1$. Otherwise let $w = w_1 x w_2$ with w_1 x-free. Then by Lemma 2.4, 2) $|\mu_w(y)|_x = |Pal(w)|_x - |Pal(\tilde{w}_2)|_y$. Summing over y we get $L_w(x) = |A||Pal(w)|_x - |Pal(\tilde{w}_2)|$.

As by Lemma 2.2 $|Pal(\tilde{w}_2)| = |Pal(w_2)| = |Pal(w)|_x - 1$ we get $L_w(x) = (|A|-1)|Pal(w)|_x + 1$. □

Corollary 2.7.

$$\sum_{y \in A} |\mu_w(y)| = (|A|-1)|Pal(w)| + |A|. \tag{6}$$

Proof. By summation of equation (5) over x. □

Remark 2.1. The $|\mu_w(y)|$ are periods of $Pal(w)$ and indeed formula (6) is the same as the one given in [7] for $|A| = 3$ and [10] for $|A| \geq 2$. More precisely the ordered set of the $|\mu_w(y)|$ is a "good" $|A|$-uple in the sense of these papers with its GCD equal to 1 and $Pal(w)$ is a word of maximal length having these periods $|\mu_w(y)|$ and not having period 1 (this is a particular case of the Fine and Wilf theorem for $|A|$ periods).

Proposition 2.8. *For any $x, y \in A$*

$$L_w(x) - L_{\tilde{w}}(y) = (|A|-1)(|\mu_w(y)|_x - |\mu_{\tilde{w}}(x)|_y).$$

Proof. By Theorem 2.6, $L_w(x) - L_{\tilde{w}}(y) = (|A|-1)(|Pal(w)|_x - |Pal(\tilde{w})|_y)$. If $|w|_x = 0$ or $|w|_y = 0$ then by Lemma 2.4, 2) $|\mu_w(y)|_x = |Pal(w)|_x + |y|_x$ and $|\mu_{\tilde{w}}|_y = |Pal(\tilde{w})|_y + |x|_y$ whence the result. Otherwise let $w = w_1 x w_2$ with $|w_1|_x = 0$. Then by Lemma 2.4, 2) $|\mu_w(y)|_x = |Pal(w)|_x - |Pal(\tilde{w}_2)|_y$ and $|\mu_{\tilde{w}}(x)|_y = |Pal(\tilde{w})|_y - |Pal(\tilde{w}_2)|_y$ whence the result. □

Example 2.1. With $A = \{1,2,3\}$, $w = 123^2 2^3$, $\tilde{w} = 2^3 3^2 21$,

$$Pal(w) = 12131213121213121312121312131212131213121$$

$$Pal(\tilde{w}) = 22232223222232223222122232223222232223222,$$

both of length 41, and

$$M_w = \begin{pmatrix} 21 & 5 & 17 \\ 12 & 3 & 10 \\ 8 & 2 & 7 \end{pmatrix} \qquad M_{\tilde{w}} = \begin{pmatrix} 1 & 1 & 1 \\ 16 & 23 & 26 \\ 4 & 6 & 7 \end{pmatrix}.$$

It is possible to write Proposition 2.8 as a matricial relation, but we will see now a more direct way.

Theorem 2.9. *Matrices M_w and $M_{\tilde{w}}$ are similar and are related by*

$$HM_w = (HM_{\tilde{w}})^{\mathrm{T}} \tag{7}$$

with $H = (h_{xy})$ given by $h_{xx} = 0$, $h_{xy} = 1$, $x, y \in A$, $x \neq y$.

Proof. For any letter x it is immediate to verify that $HM_x = (HM_x)^{\mathrm{T}}$ whence (7) by product. □

Remark 2.2. Let $K = (k_{xy})$ such that, $k_{xx} = 2 - |A|$, $k_{xy} = 1$, $x, y \in A$, $x \neq y$, then $HK = KH = (|A| - 1)\mathrm{I}$ whence

$$M_{\tilde{w}}^{\mathrm{T}} = (|A| - 1)^{-1} H M_w K.$$

It is possible to deduce from Theorem 2.9 some relations given above and even some other ones due to the particular form of H, for example $\det(M_w - M_{\tilde{w}}^{\mathrm{T}}) = 0$ for $|A| > 2$ and $\det(M_w - M_{\tilde{w}}) = 0$ for odd $|A|$.

Corollary 2.10. *The word w is a palindrome if and only if the matrix HM_w is symmetrical.*

Proof. By Theorem 2.9 HM_w is symmetrical if and only if $M_w = M_{\tilde{w}}$. Thus, in view of the bijection between w and M_w the proof is over. □

Remark 2.3. When A is a 2-letter alphabet, $\{1, 2\}$, [6] gives a number theoretical condition on the $|\mu_w(y)|$ equivalent to w is a palindrome, namely $|\mu_w(1)|^2 \equiv 1 \mod (|\mu_w(1)| + |\mu_w(2)|)$. It could be asked whether this condition can be extended to any finite alphabets.

3. The case $|A| = 2$ and a Galois theorem

From now on $A = \{1, 2\}$. Set $M_w = (m_{ij})$, $M_{\tilde{w}} = (m'_{ij})$, $1 \leq i, j \leq 2$. For short we also set $p_1 = m_{11} + m_{21}$, $p_2 = m_{12} + m_{22}$ and similarly $p'_1 = m'_{11} + m'_{21}$, $p'_2 = m'_{12} + m'_{22}$.

The matrix H of Theorem 2.9 becomes the permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, thus by this theorem $M_{\tilde{w}} = \begin{pmatrix} m_{22} & m_{12} \\ m_{21} & m_{11} \end{pmatrix}$.

Consider an infinite simple continued fraction, $[e_1, e_2, \ldots]$, $e_1 \geq 0$, $e_2, e_3, \ldots > 0$ and the standard Sturmian infinite word s directed by $\Delta = 1^{e_1} 2^{e_2} 1^{e_3} \cdots$. Let α_1, α_2 be the frequencies of $1, 2$ in s. It is well known [13] that $\alpha_2 = [0, e_1 + 1, e_2, e_3, \ldots]$.

Consider in particular an immediately periodic continued fraction, $[\overline{e_1, e_2, \ldots, e_d}]$ (thus $e_1 > 0$, now) and without loss of generality suppose d is even. Then s is directed by $\Delta = w^\omega$ where $w = 1^{e_1} 2^{e_2} \cdots 2^{e_d}$, *i.e.*, s is the fixed point of μ_w. Also consider $\tilde{w} = 2^{e_d} 1^{e_{d-1}} \cdots 1^{e_1}$ and s' directed by $\tilde{w}^\omega$, *i.e.*, the fixed point of $\mu_{\tilde{w}}$.

The above-mentioned Galois Theorem is as follows.

Theorem 3.1. *With $\theta = [\overline{e_1, e_2, \ldots, e_d}]$ and $\theta' = [\overline{e_d, e_{d-1}, \ldots, e_1}]$, the algebraic conjugate of the (quadratic by Lagrange's Theorem) number θ is $-1/\theta'$.*

Let us verify that. Set $\theta^* = -1/\theta'$. As said above the frequency of 2 in s is $\alpha_2 = [0, e_1 + 1, \overline{e_2, e_3, \ldots, e_d, e_1}]$ and the frequency of 1 in s' is $\alpha'_1 = [0, e_d + 1, \overline{e_{d-1}, e_{d-2}, \ldots, e_1, e_d}]$.

It follows

$$\alpha_2 = \frac{1}{\theta + 1} \qquad \alpha'_1 = \frac{1}{\theta' + 1} = \frac{\theta^*}{\theta^* - 1}. \tag{8}$$

Now the vector $(\alpha_1, \alpha_2)^{\mathrm{T}}$ is an eigenvector of M_w corresponding to the dominating eigenvalue ξ of M_w (*i.e.*, ξ is the greater root of $\xi^2 - (m_{11}+m_{22})\xi + 1$). Calculation of this vector gives

$$\alpha_1 = \frac{\xi - p_2}{p_1 - p_2} \qquad \alpha_2 = \frac{p_1 - \xi}{p_1 - p_2} \tag{9}$$

and similarly for frequencies α'_1, α'_2 of $1, 2$ in s'. Then using (8) we get

$$\theta = \frac{\xi - p_2}{p_1 - \xi} \qquad \theta^* = \frac{p'_2 - \xi}{p'_1 - \xi}\cdot \tag{10}$$

In order to verify that θ and θ^* are conjugate it suffices to verify that $\theta\theta^*$ and $\theta + \theta^*$ are rational and this is easy using $\xi^2 = (m_{11} + m_{22})\xi - 1$ and $\det(M_w) = m_{11}m_{22} - m_{12}m_{21} = 1$.

Remark 3.1. This proof is by far less direct than that of Galois but raises the question of possible generalization to finite alphabets and multidimensional continued fractions.

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