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DOMAIN DECOMPOSITION METHOD FOR CRACK PROBLEMS WITH NONPENETRATION CONDITION

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Abstract. The work deals with an iteration method for numerical solving the equilibrium problem of two-dimensional elastic body with a crack under the nonpenetration condition. The method is based on the domain decomposition and Uzawa's algorithm. To construct an algorithm, the domain is partitioned into two subdomains whose common boundary contains the crack. In each subdomain the linear problems are solved. We use Lagrangian multipliers to couple the solutions and provide the nonpenetration condition on the crack.

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1. Introduction

There are different approaches to model cracks in solids. The classical models are characterized by linear boundary conditions imposed at the crack faces [7, 12, 27]. It is known that such models have shortcoming because there can be situations when the crack faces penetrate each other.

It is natural to impose such boundary conditions which exclude mutual penetration of crack faces. The book [19] and parers [20–22, 30, 32] contain results for crack models with the non-penetration conditions for a wide class of elasticity problems. This theory is characterized by unilateral constraint conditions, and it leads to free boundary value problems.

In the present paper, the equilibrium problem of the two-dimensional elastic body with a crack is considered. The inequality type boundary conditions are imposed on the crack faces [19]. We assume a clamping condition at the part of the external boundary. The body is in equilibrium under the action of a given surface traction on the other part of the external boundary.

We use the domain decomposition method [31], based on the saddle-point theory, to construct the iteration algorithm of seeking the solution of equilibrium problem. To this end, the domain is partitioned into two subdomains in such a way that the crack is at the common boundary of subdomains. In each subdomain the linear problem of the elasticity theory is solved. Lagrangian multipliers are used for "gluing" the solutions and providing the nonpenetration conditions. The iteration algorithm is based on Uzawa's method of solution of

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variational inequalities [10, 17]. The convergence of the algorithm is proved. Numerical experiments illustrate the performance of the algorithm.

The domain decomposition method is widely used for numerical solution of many problems of mathematical physics (see, *e.g.*, [3, 24, 28, 29]). The application of the domain decomposition method to the solution of contact problems can be found in [4, 5, 8, 13, 25].

There are not so many works devoted to the numerical solution of crack problems with the nonpenetration condition. In [33], a model problem for deforming an ideal elastoplastic body with a crack was investigated. For discretized problem the Uzawa algorithm was applied, but the convergence of the algorithm to the solution of continuous problem was not proved. The iterative algorithm for the solving of a crack problem based on penalty method was realized in [26]. In this case, by increase of a penalty parameter the stiffness matrix becomes ill-conditioned. In [15, 16] this approach was improved by using a primal-active set method. Numerical tests showed that the primal-active set strategy determines the exact solution of the discretized model in few iterations.

2. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, $\Gamma_N \cap \Gamma_D = \emptyset$ and $meas\, \Gamma_D > 0$. Let $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections such that $\overline{\Gamma}_c \cap \overline{\Gamma}_D = \emptyset$. Suppose that Ω is partitioned into two subdomains Ω_1 and Ω_2 with Lipschitz boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Suppose that $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$. Let us consider that $\Gamma_c \subset \Sigma$; denote $\Gamma_g = \Sigma \setminus \overline{\Gamma}_c$. We choose the unit normal vector ν to Σ in such a way that ν is the external normal vector to Ω_1. Denote by τ a unit tangent vector on Σ; denote by n an external normal unit vector to Ω. Finally, suppose that $\Omega_c = \Omega \setminus \overline{\Gamma}_c$ is the domain with a crack.

Let u be a two-component vector of displacements defined in the domain Ω_c, *i.e.*, $u(x) : \Omega_c \to \mathbb{R}^2$; let $\sigma = \{\sigma_{ij}\}_{i,j=1}^2$ and $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1}^2$ be the stress and the strain tensors which are related by linear Hooke's law:

$$\sigma_{ij}(u) = c_{ijkl}\varepsilon_{kl}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2. \tag{2.1}$$

The coefficients c_{ijkl}, $i,j,k,l = 1,2$, satisfy the following conditions:

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad a^{-1}\,\xi_{ij}\xi_{ij} \le c_{ijkl}\xi_{ij}\xi_{kl} \le a\,\xi_{ij}\xi_{ij} \quad \forall \xi_{ij} = \xi_{ji} \tag{2.2}$$

for some constant $a > 0$. We use the Einstein summation convention: repeated indices i,j,k,l are summed from 1 to 2.

Now, we consider the following mixed boundary value problem: for given $f \in L_2(\Gamma_N)^2$, find u satisfying

$$-\mathrm{div}\,\sigma(u) = 0 \quad \text{in} \quad \Omega_c, \tag{2.3}$$

$$u = 0 \quad \text{on} \quad \Gamma_D, \tag{2.4}$$

$$\sigma(u)n = f \quad \text{on} \quad \Gamma_N,$$

$$[u]\cdot\nu \ge 0,\ [\sigma_\nu(u)] = 0,\ \sigma_\nu(u)([u]\cdot\nu) = 0 \quad \text{on} \quad \Gamma_c, \tag{2.5}$$

$$\sigma_\nu(u) \le 0, \quad \sigma_\tau(u) = 0 \quad \text{on} \quad \Gamma_c^+ \cup \Gamma_c^-. \tag{2.6}$$

Here Γ_c^α is a edge of the crack Γ_c belonging to the boundary of the subdomain Ω_α, $\alpha = 1,2$; $[v] = v|_{\Gamma_c^2} - v|_{\Gamma_c^1}$ is a jump of the function v on Γ_c, $v|_{\Gamma_c^\alpha}$ is a trace of v on Γ_c^α; $\sigma_\nu(u) = (\sigma(u)\nu)\cdot\nu$ and $\sigma_\tau(u) = (\sigma(u)\nu)\cdot\tau$ are normal and tangent components of the surface traction on Γ_c, respectively.

Equations (2.3) and boundary conditions (2.4)−(2.6) define the displacements of the body containing the crack Γ_c and being in an equilibrium under applied surface traction f on Γ_N. Conditions (2.5)−(2.6) provide the nonpenetration of crack faces Γ_c^1 and Γ_c^2 into each other.

Let us give a variational form of the problem (2.3). To do this, we define the functional space

$$\mathcal{V} = \{v \in H^1(\Omega_c)^2 \ \ | \ \ v = 0 \ \text{ a.e. on } \ \Gamma_D\};$$

the set of admissible displacements

$$K_c = \{v \in \mathcal{V} \ \ | \ \ [v] \cdot \nu \geq 0 \ \text{ a.e. on } \ \Gamma_c\}.$$

Next, we define the energy functional

$$\Pi(v) = \frac{1}{2} \int\limits_{\Omega_c} \sigma(v) : \varepsilon(v) \mathrm{d}x - \int\limits_{\Gamma_N} f \cdot v \mathrm{d}s,$$

where $\sigma(v) : \varepsilon(v) = \sigma_{ij}(v)\varepsilon_{ij}(v)$. The boundary value problem (2.3)−(2.5) can be formulated as the following minimization problem: find a function $u \in K_c$ such that

$$\Pi(u) = \inf_{v \in K_c} \Pi(v). \tag{2.7}$$

It is known (see, *e.g.*, [19], Thm. 1.30, p. 62) that there exists a unique solution $u \in K_c$ of the problem (2.7), which satisfies equation (2.3) and boundary conditions (2.4)−(2.6) in a weak sense.

3. Domain decomposition

In this section, by using domain decomposition, let us rewrite problem (2.7) in an equivalent form. For this purpose, we define the following functional spaces

$$\mathcal{V}^\alpha = \{v^\alpha \in H^1(\Omega_\alpha)^2 \ \ | \ \ v^\alpha = 0 \ \text{ a.e. on } \ \partial\Omega_\alpha \cap \Gamma_D\}, \quad \alpha = 1, 2,$$

and a set $K_{gc} \subset \mathcal{V}^1 \times \mathcal{V}^2$, where

$$K_{gc} = \{(v^1, v^2) \in \mathcal{V}^1 \times \mathcal{V}^2 \ \ | \ \ (v^2 - v^1) \cdot \nu = 0, \ (v^2 - v^1) \cdot \tau = 0 \text{ a.e. on } \Gamma_g, (v^2 - v^1) \cdot \nu \geq 0 \text{ a.e. on } \Gamma_c\}.$$

We assume that $meas\,(\partial\Omega_\alpha \cup \Gamma_D) > 0$, $\alpha = 1, 2$. Therefore, due to Korn's inequality and (2.2) (see, *e.g.* [9], Thm. 3.1, p. 115), the norm in space $\mathcal{V}^\alpha$ can be defined as follows

$$\|v^\alpha\|^2_{\mathcal{V}^\alpha} = \int\limits_{\Omega_\alpha} \sigma(v^\alpha) : \varepsilon(v^\alpha) \mathrm{d}x, \quad v^\alpha \in \mathcal{V}^\alpha,$$

which is equivalent to standard norm for $\mathcal{V}^\alpha$, $\alpha = 1, 2$.

By using the decomposition of domain Ω into Ω_1 and Ω_2, we represent the energy functionals $\Pi(v)$ as sum of two functional defined on subdomains Ω_1 and Ω_2, *i.e.*,

$$\Pi(v) = \Pi_1(v^1) + \Pi_2(v^2),$$

where

$$\Pi_\alpha(v^\alpha) = \frac{1}{2} \int\limits_{\Omega_\alpha} \sigma(v^\alpha) : \varepsilon(v^\alpha) \mathrm{d}x - \int\limits_{\Gamma_N \cap \partial\Omega_\alpha} f \cdot v^\alpha \mathrm{d}s,$$

$$v^\alpha = v|_{\Omega_\alpha} \in \mathcal{V}^\alpha, \quad \alpha = 1, 2.$$

Consider the following minimization problem: find a pair $(u^1, u^2) \in K_{gc}$ such that

$$\Pi_1(u^1) + \Pi_2(u^2) = \inf_{(v^1, v^2) \in K_{gc}} \left(\Pi_1(v^1) + \Pi_2(v^2)\right). \tag{3.1}$$

Theorem 3.1. *Problem* (3.1) *has a unique solution* $(u^1, u^2) \in K_{gc}$. *Moreover,*

$$u^\alpha = u|_{\Omega_\alpha}, \quad \alpha = 1, 2, \tag{3.2}$$

where u *is the solution of problem* (2.7).

Proof. The existence and uniqueness of the solution of problem (3.1) follows from the theory of calculus of variations (see, *e.g.*, [10], Prop. 1.2, p. 35). Next, conditions

$$(v^2 - v^1) \cdot \nu = 0, \quad (v^2 - v^1) \cdot \tau = 0 \quad \text{a.e. on} \quad \Gamma_g$$

are equivalent to condition

$$v^2 - v^1 = 0 \quad \text{a.e. on} \quad \Gamma_g.$$

Hence, equalities (3.2) follow from the fact that inclusion $(v^1, v^2) \in K_{gc}$ holds iff the function

$$v(x) = \begin{cases} v^1(x), & \text{if } x \in \Omega_1, \\ v^2(x), & \text{if } x \in \Omega_2 \end{cases}$$

belongs to the set K_c. The theorem is proved. □

Remark 3.2. Due to Gateaux's differentiability of the functionals Π_α, $\alpha = 1, 2$, problem (3.1) is equivalent the following variational inequality (see, *e.g.*, [10], Proposition 2.1, p. 38):

$$\int\limits_{\Omega_1} \sigma(u^1) : \varepsilon(v^1 - u^1)\mathrm{d}x + \int\limits_{\Omega_2} \sigma(u^2) : \varepsilon(v^2 - u^2)\mathrm{d}x$$
$$\geq \int\limits_{\Gamma_N \cap \partial\Omega_1} f \cdot (v^1 - u^1)\mathrm{d}s + \int\limits_{\Gamma_N \cap \partial\Omega_2} f \cdot (v^2 - u^2)\mathrm{d}s \quad \forall (v^1, v^2) \in K_{gc}, \tag{3.3}$$

which, in particular, implies the identity

$$\int\limits_{\Omega_1} \sigma(u^1) : \varepsilon(u^1)\mathrm{d}x + \int\limits_{\Omega_2} \sigma(u^2) : \varepsilon(u^2)\mathrm{d}x = \int\limits_{\Gamma_N \cap \partial\Omega_1} f \cdot u^1 \mathrm{d}s + \int\limits_{\Gamma_N \cap \partial\Omega_2} f \cdot u^2 \mathrm{d}s. \tag{3.4}$$

4. Mixed variational formulation

We associate the Lagrange function with problem (3.1). To this end, let us define

$$\Lambda^c = \{\lambda^c \in L_2(\Gamma_c) \;\mid\; \lambda^c \geq 0 \text{ a.e. on } \Gamma_c\},$$

$$\Lambda^\nu = L_2(\Gamma_g), \quad \Lambda^\tau = L_2(\Gamma_g),$$

$$\Lambda = \Lambda^c \times \Lambda^\nu \times \Lambda^\tau.$$

For $\lambda = (\lambda^c, \lambda^\nu, \lambda^\tau) \in \Lambda$ we introduce the Lagrange function for problem (2.7) as follows

$$L(v^1, v^2, \lambda) = \Pi_1(v^1) + \Pi_2(v^2)$$
$$+ \int\limits_{\Gamma_c} \lambda^c (v^1 - v^2) \cdot \nu \mathrm{d}s + \int\limits_{\Gamma_g} \lambda^\nu (v^1 - v^2) \cdot \nu \mathrm{d}s + \int\limits_{\Gamma_g} \lambda^\tau (v^1 - v^2) \cdot \tau \mathrm{d}s.$$

The following equality is valid

$$\sup_{\lambda\in\Lambda} L(v^1, v^2, \lambda) = \begin{cases} \Pi_1(v^1) + \Pi_2(v^2), & \text{if } (v^1, v^2) \in K_{gc}, \\ +\infty & \text{else.} \end{cases}$$

It follows that problem (3.1) takes the following form: find a pair $(u^1, u^2) \in \mathcal{V}^1 \times \mathcal{V}^2$ such that

$$\Pi_1(u^1) + \Pi_2(u^2) = \inf_{(v^1,v^2)\in\mathcal{V}^1\times\mathcal{V}^2} \sup_{\lambda\in\Lambda} L(v^1, v^2, \lambda). \tag{4.1}$$

Therefore, it is reasonable to connect the problem of seeking the saddle point of the Lagrangian L with problem (4.1). Unfortunately, due to fact that the trace operator from $H^1(\Omega_\alpha)$, $\alpha = 1, 2$, in $L_2(\Gamma_g)$ and $L_2(\Gamma_c)$ is not surjective, the well-known theorems do not guarantee the existence of the saddle point of the Lagrangian L. Moreover, the solution u of problem (2.7) and, respectively, functions u^1 and u^2 can have a singularities in the crack tips of order $\sqrt{r}$, where r is a distance to the crack tip (see, *e.g.*, [12, 20], Sect. 4.6, p. 148). Therefore, in this case there may not exist Lagrange multipliers belonging to the space $L_2(\Gamma_g)$.

In what follows, we consider the family of problems of seeking the saddle points, which depends on parameter $p > 0$ and approximates problem (3.1). To this end, we use the approach applied in [6] (Chap. 5) for investigation an elasto-plastic torsion problem.

Let $p > 0$; suppose that

$$U_p^\alpha = \{v^\alpha \in \mathcal{V}^\alpha | \|v^\alpha\|_{\mathcal{V}^\alpha} \le p\}, \quad \alpha = 1, 2,$$

$$\Lambda_p^c = \{\lambda^c \in L_2(\Gamma_c) | 0 \le \lambda^c \le p \text{ a.e. on } \Gamma_c\},$$

$$\Lambda_p^\nu = \{\lambda^\nu \in L_2(\Gamma_g) | -p \le \lambda^\nu \le p \text{ a.e. on } \Gamma_g\},$$

$$\Lambda_p^\tau = \{\lambda^\tau \in L_2(\Gamma_g) | -p \le \lambda^\tau \le p \text{ a.e. on } \Gamma_g\},$$

$$\Lambda_p = \Lambda_p^c \times \Lambda_p^\nu \times \Lambda_p^\tau.$$

Consider the following family of saddle point problems depending on parameter p: find functions $(u_p^1, u_p^2, \mu_p) \in U_p^1 \times U_p^2 \times \Lambda_p$ such that

$$L(u_p^1, u_p^2, \lambda) \le L(u_p^1, u_p^2, \mu_p) \le L(v^1, v^2, \mu^p) \quad \forall (v^1, v^2, \lambda) \in U_p^1 \times U_p^2 \times \Lambda_p. \tag{4.2}$$

By virtue of the fact that the sets U_p^1, U_p^2, Λ_p are convex, closed and bounded in corresponding Banach spaces and the Lagrangian L is convex and lower semicontinuous with respect to (v^1, v^2) and concave and upper semicontinuous with respect to λ, for all $p > 0$ problem (4.2) has the solution. Moreover, the pair (u_p^1, u_p^2) is uniquely defined (see, *e.g.*, [10], Proposition 2.1, p. 171 and Lem. 1.2, p. 188).

Inequalities (4.2) are equivalent to the following variational inequalities

$$\begin{aligned}&\int\limits_{\Omega_1}\sigma(u_p^1):\varepsilon(v^1-u_p^1)\mathrm{d}x+\int\limits_{\Omega_2}\sigma(u_p^2):\varepsilon(v^2-u_p^2)\mathrm{d}x\\&+\int\limits_{\Gamma_c}\mu_p^c(v^1-v^2-(u_p^1-u_p^2))\cdot\nu)\mathrm{d}s+\int\limits_{\Gamma_g}\mu_p^\nu(v^1-v^2-(u_p^1-u_p^2))\cdot\nu\mathrm{d}s\\&+\int\limits_{\Gamma_g}\mu_p^\tau(v^1-v^2-(u_p^1-u_p^2))\cdot\tau\mathrm{d}s\\\geq&\int\limits_{\Gamma_N\cap\partial\Omega_1}f\cdot(v^1-u_p^1)\mathrm{d}s+\int\limits_{\Gamma_N\cap\partial\Omega_2}f\cdot(v^2-u_p^2)\mathrm{d}s,\ \forall(v^1,v^2)\in U_p^1\times U_p^2,\end{aligned}\tag{4.3}$$

$$\int\limits_{\Gamma_c}\lambda^c(u_p^1-u_p^2)\mathrm{d}s\leq\int\limits_{\Gamma_c}\mu_p^c(u_p^1-u_p^2)\mathrm{d}s\quad\forall\lambda^c\in\Lambda_p^c,\tag{4.4}$$

$$\int\limits_{\Gamma_g}\lambda^\gamma(u_p^1-u_p^2)\cdot\gamma\mathrm{d}s\leq\int\limits_{\Gamma_g}\mu_p^\gamma(u_p^1-u_p^2)\cdot\gamma\mathrm{d}s\quad\forall\lambda^\gamma\in\Lambda_p^\gamma,\quad\gamma=\nu,\tau.\tag{4.5}$$

Theorem 4.1. *Let (u^1,u^2) be the solution of problem (3.1) and (u_p^1,u_p^2,μ_p) be the solution of problem (4.2); then as $p\to\infty$*

$$u_p^\alpha\to u^\alpha\quad\text{strongly in}\quad\mathcal{V}^\alpha,\quad\alpha=1,2.\tag{4.6}$$

Proof. Substitution of $v^\alpha=0$, $\alpha=1,2$ into (4.3) yields

$$\begin{aligned}\|u_p^1\|_{\mathcal{V}^1}^2+\|u_p^2\|_{\mathcal{V}^2}^2+\int\limits_{\Gamma_c}\mu_p^c(u_p^1-u_p^2)\cdot\nu\mathrm{d}s+\int\limits_{\Gamma_g}\mu_p^\nu(u_p^1-u_p^2)\cdot\nu\mathrm{d}s\\+\int\limits_{\Gamma_g}\mu_p^\tau(u_p^1-u_p^2)\cdot\tau\mathrm{d}s\leq\int\limits_{\Gamma_N\cap\partial\Omega_1}f\cdot u_p^1\mathrm{d}s+\int\limits_{\Gamma_N\cap\partial\Omega_2}f\cdot u_p^2\mathrm{d}s.\end{aligned}\tag{4.7}$$

Substituting $\lambda=0$ into (4.4) and (4.5), respectively, we obtain

$$\int\limits_{\Gamma_c}\mu_p^c(u_p^1-u_p^2)\cdot\nu\mathrm{d}s\geq0,\tag{4.8}$$

$$\int\limits_{\Gamma_g}\mu_p^\gamma(u_p^1-u_p^2)\cdot\gamma\mathrm{d}s\geq0,\quad\gamma=\nu,\tau.\tag{4.9}$$

Hence, due to Hölder's inequality and the boundedness of the trace operator, from (4.7) we have

$$\|u_p^1\|_{\mathcal{V}^1}^2+\|u_p^2\|_{\mathcal{V}^2}^2\leq\int\limits_{\Gamma_N\cap\partial\Omega_1}f\cdot u_p^1\mathrm{d}s+\int\limits_{\Gamma_N\cap\partial\Omega_2}f\cdot u_p^2\mathrm{d}s\leq K_1\|f\|_{L_2(\partial\Omega_1)}\|u_p^1\|_{\mathcal{V}^1}+K_2\|f\|_{L_2(\partial\Omega_2)}\|u_p^2\|_{\mathcal{V}^2},$$

where K_α is the norm of the trace operator $T_\alpha^{\Gamma_N}:\mathcal{V}^\alpha\to H^{1/2}(\Gamma_N)$, $\alpha=1,2$. Therefore, we get estimates

$$\|u_p^\alpha\|_{\mathcal{V}^\alpha}\leq K\quad\forall p>0,\ \alpha=1,2,\tag{4.10}$$

where $K=\sqrt{K_1^2\|f\|_{L_2(\partial\Omega_1)}^2+K_2^2\|f\|_{L_2(\partial\Omega_2)}^2}$.

Since sequences $\{u_p^1\}$ and $\{u_p^2\}$ are bounded, there exist subsequences $\{u_{\tilde p}^1\}$ and $\{u_{\tilde p}^2\}$ and functions $\tilde u^1 \in \mathcal{V}^1$, $\tilde u^2 \in \mathcal{V}^2$ such that

$$u_{\tilde p}^\alpha \to \tilde u^\alpha \quad \text{weakly in} \quad \mathcal{V}^\alpha,\ \alpha = 1, 2,$$

as $\tilde p \to \infty$.

Show that $(\tilde u^1, \tilde u^2) \in K_{gc}$. To this end, first note that from inequality (4.4) it follows that

$$\mu_p^c(x) = \begin{cases} 0, & \text{if } \ (u_p^1(x) - u_p^2(x)) \cdot \nu(x) < 0, \\ p, & \text{if } \ (u_p^1(x) - u_p^2(x)) \cdot \nu(x) > 0. \end{cases} \tag{4.11}$$

Moreover, from (4.5) we get for $\gamma = \nu, \tau$

$$\mu_p^\gamma(x) = \begin{cases} -p, & \text{if } \ (u_p^1(x) - u_p^2(x)) \cdot \gamma < 0, \\ p, & \text{if } \ (u_p^1(x) - u_p^2(x)) \cdot \gamma > 0. \end{cases} \tag{4.12}$$

Introduce notation

$$I_p^c = \int\limits_{\Gamma_c} ((u_p^1 - u_p^2) \cdot \nu)^+ \mathrm{d}s,$$

$$I_p^\gamma = \int\limits_{\Gamma_g} ((u_p^1 - u_p^2) \cdot \gamma)^- \mathrm{d}s + \int\limits_{\Gamma_g} ((u_p^1 - u_p^2) \cdot \gamma)^+ \mathrm{d}s, \quad \gamma = \nu, \tau,$$

where

$$v^+(x) = \begin{cases} v(x), & \text{if } \ v(x) \geq 0, \\ 0, & \text{if } \ v(x) < 0, \end{cases}$$

$v^-(x) = v^+(x) - v(x)$.

Note that (4.11) and (4.12) imply

$$\int\limits_{\Gamma_c} \mu_p^c (u_p^1 - u_p^2) \mathrm{d}s = p I_p^c,$$

$$\int\limits_{\Gamma_g} \mu_p^\gamma \left(u_p^1 - u_p^2\right) \cdot \gamma \mathrm{d}s = p I_p^\gamma, \quad \gamma = \nu, \tau.$$

By using these formulas, we can rewrite (4.7) in the following form:

$$0 \leq \|u_p^1\|^2_{\mathcal{V}^1} + \|u_p^2\|^2_{\mathcal{V}^2} + p(I_p^c + I_p^\nu + I_p^\tau) \leq \int\limits_{\Gamma_N \cap \partial\Omega_1} f \cdot u_p^1 \mathrm{d}s + \int\limits_{\Gamma_N \cap \partial\Omega_2} f \cdot u_p^2 \mathrm{d}s$$

Due to (4.10), we have the following estimate

$$0 \leq p(I_p^c + I_p^\nu + I_p^\tau) \leq M, \tag{4.13}$$

where $M = K\|f\|_{L_2(\Gamma_N)}$. The last implies

$$\lim_{p \to \infty} (I_p^c + I_p^\nu + I_p^\tau) = 0.$$

We take an arbitrary nonnegative function $\phi \in C_0^\infty(\Gamma_c)$. We have

$$\int\limits_{\Gamma_c} \phi(\tilde u^1 - \tilde u^2) \cdot \nu \mathrm{d}s = \lim_{\tilde p \to \infty} \int\limits_{\Gamma_c} \phi(u_{\tilde p}^1 - u_{\tilde p}^2) \cdot \nu \mathrm{d}s \leq (\max_{x \in \Gamma_c} \phi(x)) \lim_{\tilde p \to \infty} \int\limits_{\Gamma_c} ((u_{\tilde p}^1 - u_{\tilde p}^2) \cdot \nu)^+ \mathrm{d}s = 0,$$

and, consequently, $(\tilde u^2 - \tilde u^1) \cdot \nu \geq 0$ a.e. on Γ_c.

Let $\psi \in C_0^\infty(\Gamma_g)$; then we have

$$\int_{\Gamma_g} \psi(\tilde{u}^1 - \tilde{u}^2)\cdot\nu \mathrm{d}s = \lim_{\tilde{p}\to\infty} \int_{\Gamma_g} \psi(u^1_{\tilde{p}} - u^2_{\tilde{p}})\cdot\nu \mathrm{d}s.$$

The following inequalities are valid:

$$-\left(\max_{x\in\Gamma_g}\psi(x)\right)\int_{\Gamma_g}\left((u^1_{\tilde{p}} - u^2_{\tilde{p}})\cdot\nu\right)^-\mathrm{d}s \le \int_{\Gamma_g}\psi(u^1_{\tilde{p}} - u^2_{\tilde{p}})\cdot\nu\mathrm{d}s \le \left(\max_{x\in\Gamma_g}\psi(x)\right)\int_{\Gamma_g}\left((u^1_{\tilde{p}} - u^2_{\tilde{p}})\cdot\nu\right)^+\mathrm{d}s$$

Passing to the limit as $\tilde{p}\to\infty$, we obtain

$$\int_{\Gamma_c}\phi(\tilde{u}^1 - \tilde{u}^2)\cdot\nu\mathrm{d}s = 0.$$

It means that $(\tilde{u}^2 - \tilde{u}^1)\cdot\nu = 0$ a.e. on Γ_g. Similarly, it is shown that $(\tilde{u}^2 - \tilde{u}^1)\cdot\tau = 0$ a.e. on Γ_g.

Thus, the pair $(\tilde{u}^1, \tilde{u}^2)$ belongs to the set K_{gc}.

Now we show that $(\tilde{u}^1, \tilde{u}^2)$ coincides with (u^1, u^2). Let us take an arbitrary $(v^1, v^2) \in K_{gc}$. Then for all $\tilde{p} \ge \max\{\|v^1\|_{\mathcal{V}^1}, \|v^2\|_{\mathcal{V}^2}\}$ the inclusion $(v^1, v^2) \in U^1_{\tilde{p}} \times U^2_{\tilde{p}}$ is valid. By virtue of (4.8) and (4.9), from inequality (4.3) for pair $(v^1, v^2) \in K_{gc} \cap (U^1_{\tilde{p}} \times U^2_{\tilde{p}})$ we get

$$\int_{\Omega_1}\sigma(u^1_{\tilde{p}}) : \varepsilon(v^1 - u^1_{\tilde{p}})\mathrm{d}x + \int_{\Omega_2}\sigma(u^2_{\tilde{p}}) : \varepsilon(v^2 - u^2_{\tilde{p}})\mathrm{d}x \ge \int_{\Gamma_N\cap\partial\Omega_1} f\cdot(v^1 - u^1_{\tilde{p}})\mathrm{d}s + \int_{\Gamma_N\cap\partial\Omega_2} f\cdot(v^2 - u^2_{\tilde{p}})\mathrm{d}s. \quad (4.14)$$

By virtue of weak lower semicontinuity of the norm, it is possible to pass to the limit in (4.14) as $\tilde{p}\to\infty$. As a result, we have inequality

$$\int_{\Omega_1}\sigma(\tilde{u}^1) : \varepsilon(v^1 - \tilde{u}^1)\mathrm{d}x + \int_{\Omega_2}\sigma(\tilde{u}^2) : \varepsilon(v^2 - \tilde{u}^2)\mathrm{d}x \ge \int_{\Gamma_N\cap\partial\Omega_1} f\cdot(v^1 - \tilde{u}^1)\mathrm{d}s + \int_{\Gamma_N\cap\partial\Omega_2} f\cdot(v^2 - \tilde{u}^2)\mathrm{d}s, \quad (4.15)$$

that is valid for all $(v^1, v^2) \in K_{gc}$ (due to arbitrariness of the choice). Variational inequality (4.15) coincides with (3.3), which uniquely defines the pair (u^1, u^2). Thus, we conclude that (u^1_p, u^2_p) weakly converges to (u^1, u^2) as $p\to\infty$ in $\mathcal{V}^1\times\mathcal{V}^2$.

Finally, we show that (4.6) is valid. Taking into account (4.8) and (4.9), from (4.7) we get

$$\|u^1_p\|^2_{\mathcal{V}^1} + \|u^2_p\|^2_{\mathcal{V}^2} \le \int_{\Gamma_N\cap\partial\Omega_1} f\cdot u^1_p\mathrm{d}s + \int_{\Gamma_N\cap\partial\Omega_2} f\cdot u^2_p\mathrm{d}s.$$

By virtue of weak lower semicontinuity of the norm, after passing to the limit as $p\to\infty$ the last inequality yields

$$\|u^1\|^2_{\mathcal{V}^1} + \|u^2\|^2_{\mathcal{V}^2} \le \liminf_{p\to\infty}\left(\|u^1_p\|^2_{\mathcal{V}^1} + \|u^2_p\|^2_{\mathcal{V}^2}\right) \le \limsup_{p\to\infty}\left(\|u^1_p\|^2_{\mathcal{V}^1} + \|u^2_p\|^2_{\mathcal{V}^2}\right) \le \int_{\Gamma_N\cap\partial\Omega_1} f\cdot u^1\mathrm{d}s + \int_{\Gamma_N\cap\partial\Omega_2} f\cdot u^2\mathrm{d}s.$$

It follows from (3.4) that

$$\int_{\Gamma_N\cap\partial\Omega_1} f\cdot u^1\mathrm{d}s + \int_{\Gamma_N\cap\partial\Omega_2} f\cdot u^2\mathrm{d}s = \|u^1\|^2_{\mathcal{V}^1} + \|u^2\|^2_{\mathcal{V}^2}.$$

Therefore, we get

$$\|u_p^1\|^2_{\mathcal{V}^1} + \|u_p^2\|^2_{\mathcal{V}^2} \to \|u^1\|^2_{\mathcal{V}^1} + \|u^2\|^2_{\mathcal{V}^2}$$

as $p \to \infty$. Since (u_p^1, u_p^2) converges weakly to (u_1, u_2) and the norm of (u_p^1, u_p^2) converges to the norm of (u_1, u_2), the (u_p^1, u_p^2) converges strongly to (u_1, u_2) in $\mathcal{V}^1 \times \mathcal{V}^2$ as $p \to \infty$ (see, *e.g.*, [34], Thm. 8, p. 124). The theorem is proved. □

Now we show that for all "sufficiently" great p the set of saddle points (u_p^1, u_p^2, μ_p) of the Lagrangian L on the set $U_p^1 \times U_p^2 \times \Lambda_p$ coincides with the set of saddle points of the same Lagrangian on $\mathcal{V}^1 \times \mathcal{V}^2 \times \Lambda_p$. Namely, the following theorem is valid.

Theorem 4.2. *For all $p > K$, where K is the constant in (4.10), (u_p^1, u_p^2, μ_p) is the saddle point of the Lagrangian L on the set $U_p^1 \times U_p^2 \times \Lambda_p$ iff it satisfies (4.4), (4.5) and the following relations:*

$$\int\limits_{\Omega_\alpha} \sigma(u_p^\alpha) : \varepsilon(v^\alpha)\mathrm{d}x + (-1)^{\alpha+1} \int\limits_{\Gamma_c} \mu_p^c v^\alpha \cdot \nu \mathrm{d}s + (-1)^{\alpha+1} \int\limits_{\Gamma_g} \mu_p^\nu v^\alpha \cdot \nu \mathrm{d}s$$
$$+ (-1)^{\alpha+1} \int\limits_{\Gamma_g} \mu_p^\tau v^\alpha \cdot \tau \mathrm{d}s = \int\limits_{\Gamma_N \cap \partial\Omega_\alpha} f \cdot v^\alpha \mathrm{d}s \quad \forall v^\alpha \in \mathcal{V}^\alpha, \quad \alpha = 1, 2. \quad (4.16)$$

Proof. To proof the theorem, it is sufficient to show that variational inequality (4.3) is valid for all functions $(v^1, v^2) \in \mathcal{V}^1 \times \mathcal{V}^2$. Indeed, in this case we can substitute $(\pm v^1 + u_p^1, u_p^2)$ and $(u_p^1, \pm v^2 + u_p^2)$ as test functions in order to obtain (4.16).

Let $p > K$; due to (4.10), functions (u_p^1, u_p^2) belong to $U_K^1 \times U_K^2 \subset U_p^1 \times U_p^2$. Fix the number $\delta \in (0, (p-K)/2)$. Then the open set

$$U(u_p^1, u_p^2, \delta) = \{(v^1, v^2) \in \mathcal{V}^1 \times \mathcal{V}^2 \;\;|\;\; \|v^1 - u_p^1\|_{\mathcal{V}^1} < \delta, \; \|v^2 - u_p^2\|_{\mathcal{V}^2} < \delta\}$$

is contained in $U_p^1 \times U_p^2$.

We take arbitrary $(v^1, v^2) \in \mathcal{V}^1 \times \mathcal{V}^2$ and find $\beta > 0$ such that $\beta\|v^\alpha - u_p^\alpha\|_{\mathcal{V}^\alpha} < \delta$, $\alpha = 1, 2$. Then the pair $(u_p^1 + \beta(v^1 - u_p^1), u_p^2 + \beta(v^2 - u_p^2))$ belongs to the set $U(u_p^1, u_p^2, \delta) \subset U_p^1 \times U_p^2$. Hence, it can be substitute into (4.3) as a test function. After calculations, we get the variational inequality (4.3) which is valid for all functions $(v^1, v^2) \in \mathcal{V}^1 \times \mathcal{V}^2$. The theorem is proved. □

Thus, we obtain that nonlinear problem (2.7) defined in the domain Ω is approximated by two linear problems, defined in Ω_1 and Ω_2 and connected by Lagrange multipliers μ_p^c, μ_p^ν and μ_p^τ.

5. Iteration process for approximating problem

The goal of this section is to construct iteration Uzawa-type algorithm for problem (4.4), (4.5) and (4.16). We consider that $p > K$, where K is the constant in (4.10). By $P_{\Lambda_p^c}$ we denote the projection operator on the set Λ_p^c in $L_2(\Gamma_c)$; by $P_{\Lambda_p^\gamma}$ we denote the projection operator on the set Λ_p^γ in $L_2(\Gamma_g)$, $\gamma = \nu, \tau$. It is easy to check that

$$P_{\Lambda_p^c} v(x) = \begin{cases} 0, & \text{if } \; v(x) \le 0, \\ v(x), & \text{if } \; 0 < v(x) < p, \\ p, & \text{if } \; v(x) \ge p, \end{cases}$$

$$P_{\Lambda_p^\nu} v(x) = P_{\Lambda_p^\tau} v(x) = \begin{cases} -p, & \text{if } \; v(x) \le -p, \\ v(x), & \text{if } \; -p < v(x) < p, \\ p, & \text{if } \; v(x) \ge p, \end{cases}$$

Inequalities (4.4) and (4.5) are equivalent to the fact that for any real number $\theta > 0$ functions $\mu_p^c \in \Lambda_p^c$, $\mu_p^\gamma \in \Lambda_p^\gamma$ are fixed points of operators

$$P_{\Lambda_p^c}\left(\lambda^c + \theta(u_p^1 - u_p^2)\cdot\nu\right) : \Lambda_p^c \to \Lambda_p^c,$$

$$P_{\Lambda_p^\gamma}\left(\lambda^\gamma + \theta(u_p^1 - u_p^2)\cdot\gamma\right) : \Lambda_p^\gamma \to \Lambda_p^\gamma$$

$\gamma = \nu, \tau$, respectively, *i.e.* (see [10], Example on p. 39)

$$\mu_p^c = P_{\Lambda_p^c}\left(\mu_p^c + \theta(u_p^1 - u_p^2)\cdot\nu\right), \tag{5.1}$$

$$\mu_p^\gamma = P_{\Lambda_p^\gamma}\left(\mu_p^\gamma + \theta(u_p^1 - u_p^2)\cdot\gamma\right), \quad \gamma = \nu, \tau. \tag{5.2}$$

Basing on equalities (5.1) and (5.2), construct a convergent iteration algorithm to find the saddle point of Lagrangian L on the set $\mathcal{V}^1 \times \mathcal{V}^2 \times \Lambda_p$.

Algorithm 1.

i. Choose $\mu^{c,0} \in \Lambda_p^c$, $\mu^{\nu,0} \in \Lambda_p^\nu$ and $\mu^{\tau,0} \in \Lambda_p^\tau$. Set $k = 0$.

ii. For each $k \geq 0$ we find $u^{1,k}$ and $u^{2,k}$ as solutions of the following variational equalities:

$$\int\limits_{\Omega_\alpha} \sigma(u^{\alpha,k}) : \varepsilon(v^\alpha)\mathrm{d}x + (-1)^{\alpha+1}\int\limits_{\Gamma_c} \mu^{c,k} v^\alpha \cdot \nu \mathrm{d}s + (-1)^{\alpha+1}\int\limits_{\Gamma_g} \mu^{\nu,k} v^\alpha \cdot \nu \mathrm{d}s$$
$$+ (-1)^{\alpha+1}\int\limits_{\Gamma_g} \mu^{\tau,k} v^\alpha \cdot \tau \mathrm{d}s = \int\limits_{\Gamma_N \cap \partial\Omega_\alpha} f \cdot v^\alpha \mathrm{d}s \quad \forall v^\alpha \in \mathcal{V}^\alpha, \ \alpha = 1, 2. \tag{5.3}$$

iii. Set

$$\mu^{c,k+1} = P_{\Lambda_p^c}(\mu^{c,k} + \theta(u^{1,k} - u^{2,k})\cdot\nu), \tag{5.4}$$

$$\mu^{\gamma,k+1} = P_{\Lambda_p^\gamma}(\mu^{\gamma,k} + \theta(u^{1,k} - u^{2,k})\cdot\gamma), \quad \gamma = \nu, \tau. \tag{5.5}$$

iv. Stop or $k = k + 1$, goto (ii).

Theorem 5.1. *There exists θ^* such that for all $\theta \in (0, \theta^*)$ the sequences $\{u^{\alpha,k}\}$ converge to u_p^α strongly in $\mathcal{V}^\alpha$, $\alpha = 1, 2$.*

Proof. Convergence of Algorithm 1 follows from the general theorem of Uzawa's algorithm convergence (see, *e.g.*, [10], Prop. 1.1, p. 189 or [17], Thm. 4.49, p. 118). Let us check the conditions of proposition in [10]. The sets Λ_p and U_p^α, $\alpha = 1, 2$, are non-empty closed convex sets. Moreover, Λ_p is bounded. The function $\lambda \to L(v^1, v^2, \lambda)$ is affine continuous one.

Due to boundedness of linear trace operator (see, *e.g.*, [11], Thm. 1, p. 258), the following inequality

$$\|v^\alpha - w^\alpha\|_{L_2(\Gamma_c)} \leq G\|v^\alpha - w^\alpha\|_{\mathcal{V}^\alpha}, \quad \forall v^\alpha, w^\alpha \in \mathcal{V}^\alpha,$$

holds, where the constant G depends only Ω_α, $\alpha = 1, 2$. The similar inequality takes place for $L_2(\Gamma_g)$.

Finally, by virtue of Gâteaux-differentiability of the functionals Π_α, $\alpha = 1, 2$, and Korn's inequality, we can apply Proposition 1.1 in [10], p. 189. The theorem is proved. □

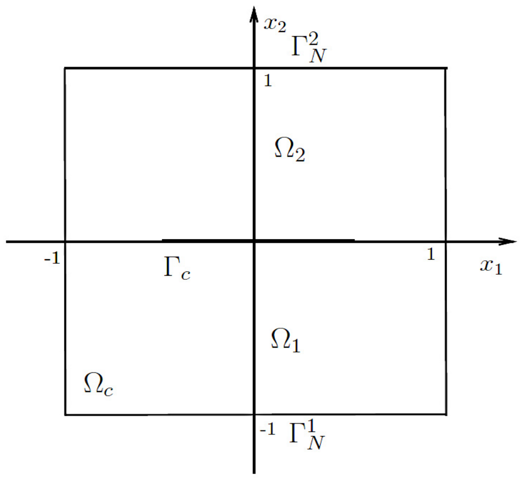

FIGURE 1. Domain Ω_c with the crack Γ_c.

Remark 5.2. If for each $k \geq 0$ we set $\mu^{c,k} = 0$, Algorithm 1 converges to solution of the problem which approximates (as $p \to \infty$) the equilibrium problem of elastic body with a crack whose faces are free of traction, *i.e.*,

$$\sigma_\nu(u) = 0, \quad \sigma_\tau(u) = 0 \quad \text{a.e. on} \quad \Gamma_c.$$

6. NUMERICAL EXPERIMENTS

Algorithm 1 was realized by using FreeFEM++ ([14]).

Let us consider 2D-Lamé problem with a rectilinear crack. We choose Ω as the square (see Fig. 1)

$$\Omega = (-1, 1) \times (-1, 1),$$

which decomposed into two subdomains

$$\Omega_1 = (-1, 1) \times (-1, 0), \quad \Omega_2 = (-1, 1) \times (0, 1)$$

with a common boundary

$$\Sigma = (-1, 1) \times \{0\}.$$

Let $\Gamma_c = (-1/2, 1/2) \times \{0\}$ be a crack; then $\Gamma_g = \Sigma \setminus \overline{\Gamma}_c = ((-1, -1/2) \cup (1/2, 1)) \times \{0\}$ is a part of common boundary of subdomains Ω_1 and Ω_2, where "gluing" occurs. The body is fixed on $\Gamma_D = (\{-1\} \cup \{1\}) \times (-1, 1)$. By $\Gamma_N^1 = (-1, 1) \times \{-1\}$ and $\Gamma_N^2 = (-1, 1) \times \{1\}$ we denote the lower boundary and the upper boundary of square, respectively.

The plane-stress Lamé model of an isotropic solid is given in term of the stress tensor (see, *e.g.*, [23], Example 6.5, p. 156)

$$\sigma_{11}(u) = (2\mu + \lambda)\varepsilon_{11}(u) + \lambda\varepsilon_{22}(u), \quad \sigma_{12}(u) = \sigma_{21}(u) = 2\mu\varepsilon_{12}(u),$$

$$\sigma_{22}(u) = \lambda\varepsilon_{11}(u) + (2\mu + \lambda)\varepsilon_{22}(u),$$

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{2\nu\mu}{1-2\nu},$$

where the strain tensor is given in (2.1). We take the following values of material parameters (see [18])

$$\nu = 0.3, \quad E = 6.9 \times 10^4 \,\text{mPa}.$$

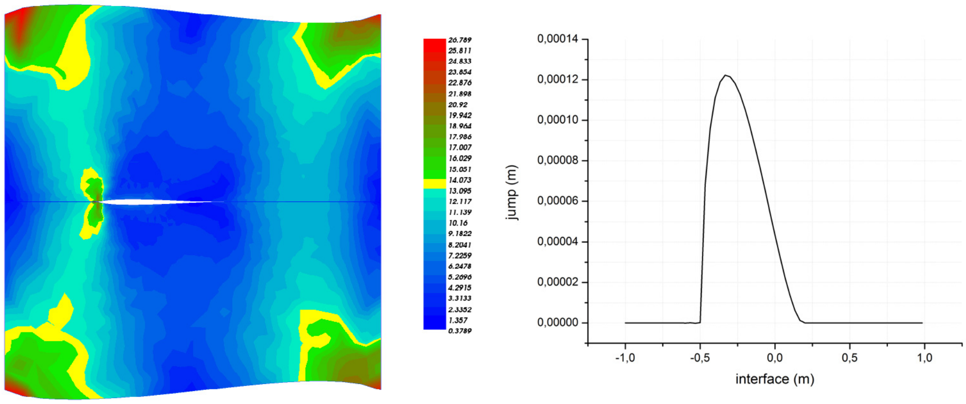

FIGURE 2. Partial closing of the of the crack faces.

We assume that in all numerical experiments below $\theta = 2500$, $p = 10^7$. The stopping criterion is

$$\max\left(\frac{\|u^{1,k} - u^{1,k-1}\|_{\mathcal{V}^1}}{\|u^{1,k}\|_{\mathcal{V}^1}}, \frac{\|u^{2,k} - u^{2,k-1}\|_{\mathcal{V}^2}}{\|u^{2,k}\|_{\mathcal{V}^2}}\right) < 10^{-6}.$$

The spaces $\mathcal{V}_i$, $i = 1, 2$, are approximated by finite-element spaces consisting of piecewise linear functions – Lagrange $P1$-elements (see, *e.g.*, [1], Def. 6.3.5, p. 176).

Example 1 (Partial closing of the of the crack faces)**.** At Γ_N^1 and Γ_N^2 we impose loading by the following traction forces: $f = 10^{-3}\mu x$ on Γ_N^1 and $f = -10^{-3}\mu x$ on Γ_N^2. Such loading provides the closing of the crack faces in the vicinity of its right tip and opening mode in the vicinity of the left one.

Let N be the number of nodes lying on the interface Σ; M be the number of nodes lying on the external boundary $\partial\Omega$. In this example, we shall investigate dependence of Algorithm 1 on various value N and M. In the Table 1 we report minimal and maximal mesh sizes $h^{\alpha}_{\min}$ and $h^{\alpha}_{\max}$, the number of nodes $Nodes^{\alpha}$ and triangles $Triangle^{\alpha}$ in the subdomain Ω_{α}, $\alpha = 1, 2$; the number of iterations *iter*.

From the data of Table 1 we can conclude that the number of iterations practically does not depend on mesh size.

On the left in Figure 2 the domain Ω_c after deformation in Lagrange coordinates $x + 300u(x)$ with an amplification factor and Von Mises stresses are presented. We can see the singularities near the left tip of the crack and its absence at the right tip. This is consistent with theoretical results (see, *e.g.*, [2]). On the right in Figure 2 the jump of normal displacements is shown.

As mentioned in the introduction, for the crack problems with linear boundary conditions imposed on cracks it is possible to get the mutual penetration of crack faces. On Figure 3 the solution of linear problem with the same loading is represented. The mesh corresponds to $N = 48$, $M = 80$; the number of iteration is 2613. Resulting deformations and Von Mises stresses are depicted on the left; the graph of the jump of the normal displacements is given on the right. We see that the mutual penetration of crack faces occurs.

Example 2 (Opening mode)**.** At the boundaries Γ_N^1 and Γ_N^2 we impose loading by the following traction forces: $f = -10^{-3}\mu$ on Γ_N^1 and $f = 10^{-3}\mu$ on Γ_N^2. Such loading provides an opening mode of the crack.

The resulting deformations in Lagrange coordinates $x + 30u(x)$ with an amplification factor 30 and Von Mises stresses are depicted on the left in Figure 4; the jump of normal displacements along the interface Σ is shown on the left. In this example, the number of iteration is 4508 for $N = 48$, $M = 80$.

TABLE 1. Dependence on the mesh for the partial closing case.

N	M	$h^1_{\min}$	$h^1_{\max}$	$h^2_{\min}$	$h^2_{\max}$	$Nodes^1$	$Triangle^1$	$Nodes^2$	$Triangle^2$	$iter$
12	32	0,092	0,365	0,099	0,365	111	180	116	190	2370
24	48	0,041	0,242	0,044	0,242	289	504	286	498	2868
48	80	0,021	0,157	0,02	0,157	921	1704	911	1684	2659
96	144	0,01	0,086	0,01	0,086	3121	5976	3133	6000	2692
128	192	0,0075	0,062	0,0078	0,062	5493	10 632	5525	10 696	2463

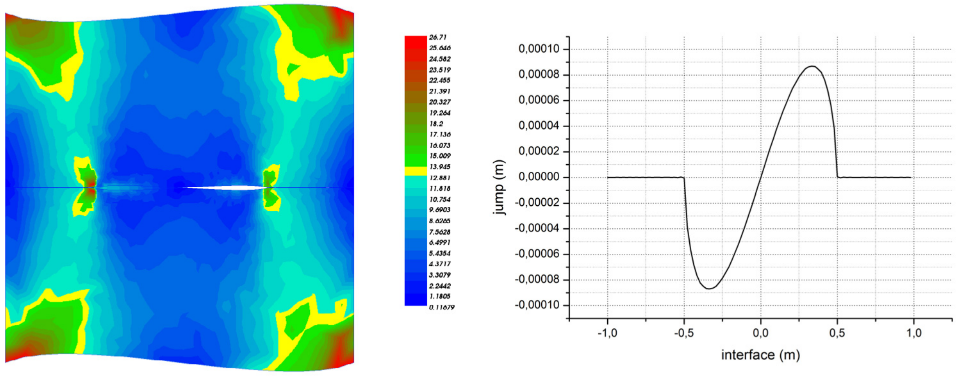

FIGURE 3. Solution of linear problem.

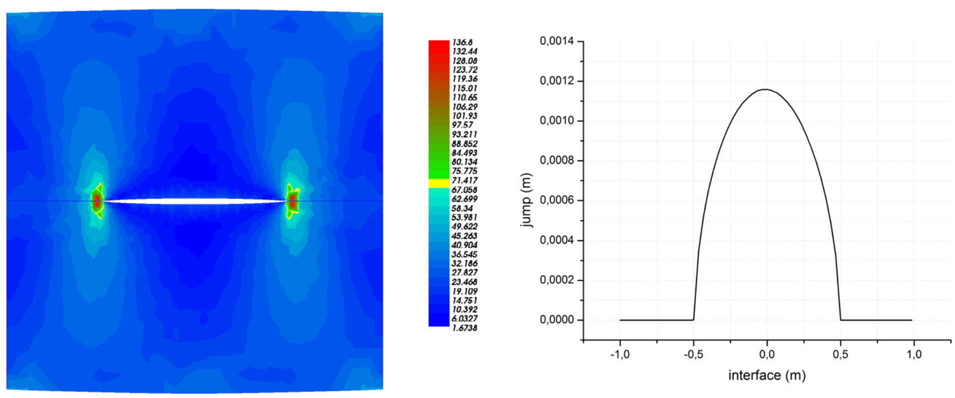

FIGURE 4. Opening mode.

Example 3 (Solution for curvilinear crack)**.** In this example we investigate the performance of Algorithm 1 for a curvilinear crack. Let us consider the following configuration of the domain. We again choose Ω as the square with the crack

$$\Gamma_c = \{(x, y) \;\mid\; y = 1/10 \sin(2\pi x),\; x \in (-1/2, 1/2)\}.$$

TABLE 2. Parameters of mesh for curvilinear crack.

N	M	$h^1_{\min}$	$h^1_{\max}$	$h^2_{\min}$	$h^2_{\max}$	Nodes[1]	Triangle[1]	Nodes[2]	Triangle[2]
48	80	0,019	0,153	0,02	0,157	903	1668	905	1672

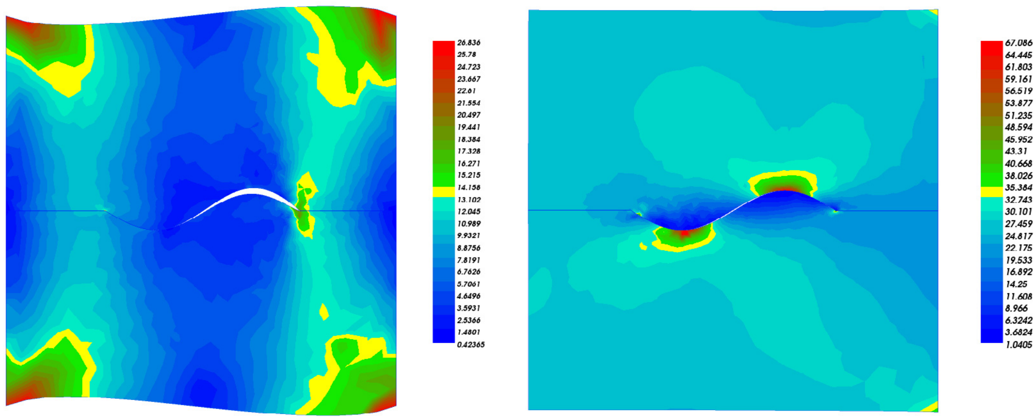

FIGURE 5. Curvilinear crack.

Let $\Gamma_g = \{(x, y) \mid y = 0,\ x \in (-1, -1/2) \cup (1/2, 1)\}$ and $\Sigma = \Gamma_c \cup \Gamma_g$. We shall consider two cases of loadings:

Case 1. We assume that the body is fix on right and left sides of the square Ω. We impose loading by the following traction forces: $f = 10^{-3}\mu x$ on the bottom side $(-1, 1) \times \{-1\}$ and $f = -10^{-3}\mu x$ the top side $(-1, 1) \times \{1\}$ of the square Ω.

Case 2. We assume that the body is fix on right side of the square Ω. Constant traction forces $f = 10^{-3}\mu$ acts on the left sides of the square Ω.

In Table 2 Parameters of mesh for both cases are given.

Resulting deformations and Von Mises stresses are depicted in Figure 5: on the left – for Case 1 (an amplification factor is equal to 300, $iter = 3535$), on the right – for Case 2 (an amplification factor is equal to 50, $iter = 4238$).

7. CONCLUSION

The work presents the iteration algorithm of solving the equilibrium problem of two-dimensional elastic body with a crack under the nonpenetration condition. To construct the algorithm, we used the technique of domain decomposition and Uzawa's algorithm of numerical solving of problems with unilateral constraint. The suggested algorithm has advantages such as the simplicity of its realization, parallelization of computing processes, possibility of using nonconforming discretization. However, the algorithm has the disadvantage which consists in low convergence velocity. For example, the primal-dual active set strategy determines the exact solution of the discretized model in a few iterations [15, 16]; the iterative process of penalty iteration method convergences exponentially to the solution of the penalized problem [26]. To accelerate of the Algorithm 1 we can use, for example, the augmented Lagrangian which is combination of penalty method and Lagrange multiplier approach (for notion of augmented Lagrangian see, *e.g.*, [17]).

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