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A MIXED FORMULATION OF THE TIKHONOV REGULARIZATION AND ITS APPLICATION TO INVERSE PDE PROBLEMS

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Abstract. This paper is dedicated to a new way of presenting the Tikhonov regularization in the form of a mixed formulation. Such formulation is well adapted to the regularization of linear ill-posed partial differential equations because when it comes to discretization, the mixed formulation enables us to use some standard finite elements. As an application of our theory, we consider an inverse obstacle problem in an acoustic waveguide. In order to solve it we use the so-called "exterior approach", which couples the mixed formulation of Tikhonov regularization and a level set method. Some 2d numerical experiments show the feasibility of our approach.

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1. Introduction

In this paper we introduce a mixed formulation of the standard well-known Tikhonov regularization in a general Hilbert setting. More precisely, let us consider three Hilbert spaces V, M and H, equipped with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_M$ and $(\cdot,\cdot)_H$ and corresponding norms $||\cdot||_V$, $||\cdot||_M$ and $||\cdot||_H$. Let us denote $A\colon V \to H$ a continuous onto operator. For some $f \in H$, we consider the affine space $V_f = \{u \in V,\ Au = f\}$. The corresponding vector space is denoted V_0, equipped with the norm $||\cdot||_V$. Lastly, we consider a continuous operator $B\colon V \longrightarrow M$ such that its restriction $B\colon V_0 \longrightarrow M$ is injective but not onto. A usual ill-posed problem consists in solving, for data $f \in H$ and $L \in M$, the following problem:

$$\text{Find} \quad u \in V_f \quad \text{such that} \quad Bu = L. \tag{1.1}$$

A typical example of such ill-posed problem is the Cauchy problem for the Laplace equation. It consists, for a sufficiently smooth domain Ω of $\mathbb{R}^d$ and sufficiently smooth data on a subpart Γ_0 of the boundary $\partial\Omega$, to find

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a function u in Ω such that

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \Gamma_0 \\ \partial_\nu u = g & \text{on} \quad \Gamma_0, \end{cases} \tag{1.2}$$

where ν is the outward unit normal to Ω. We will see that such problem (1.2), after it has been defined in a rigorous mathematical framework, actually fits the abstract framework (1.1). One of the more classical way to address the abstract problem (1.1) is to introduce the Tikhonov regularization (see for example [23, 24]), which consists in solving, for $\varepsilon > 0$, the well-posed variational problem:

$$\text{Find} \quad u_\varepsilon \in V_f \quad \text{such that} \quad (Bu_\varepsilon, Bv)_M + \varepsilon(u_\varepsilon, v)_V = (L, Bv)_M, \quad \forall v \in V_0. \tag{1.3}$$

The idea of mixed formulation is simply to introduce the new unknown $\lambda_\varepsilon = Bu_\varepsilon - L$, so that the problem (1.3) is obviously equivalent to the variational system:

$$\text{Find} \quad (u_\varepsilon, \lambda_\varepsilon) \in V_f \times M \quad \text{such that} \quad \begin{cases} \varepsilon(u_\varepsilon, v)_V + (Bv, \lambda_\varepsilon)_M = 0, & \forall v \in V_0 \\ (Bu_\varepsilon, \mu)_M - (\lambda_\varepsilon, \mu)_M = (L, \mu)_M, & \forall \mu \in M. \end{cases} \tag{1.4}$$

The terminology "mixed" refers to the introduction of an additional unknown which transforms the initial Tikhonov regularized problem into a system of two coupled problems of two unknowns, following the ideas developed in [10] in the context of partial differential equations. The most useful application of such mixed formulation of Tikhonov regularization seems to be the numerical resolution of linear ill-posed partial differential equations like the Cauchy problem for the Laplace equation (1.2), because the mixed formulation is directly in the form of a weak formulation which can be discretized with the help of classical finite elements. Our paper also offers a connection with the old concept of quasi-reversibility in the sense that our mixed formulation of Tikhonov regularization can also be seen as a mixed formulation of quasi-reversibility. The quasi-reversibility method was first introduced by R. Lattès and J.-L. Lions in [26] to regularize some linear ill-posed PDE problems and later studied and applied by M.V. Klibanov and several collaborators (see for example [17, 25]). From the numerical point of view, the main drawback of these first formulations of quasi-reversibility was the fact that the order of the regularized problem is twice the order of the initial ill-posed problem, which requires the use of cumbersome finite elements (for example Hermite finite elements instead of Lagrange ones). Our mixed formulation of quasi-reversibility allows us to use some standard Lagrange finite elements. The present paper unifies within a single abstract framework all the previous results presented in [2, 4] and [7] for the Laplace, heat and wave equations. It should be noted that a second family of mixed formulation of quasi-reversibility was introduced in [19] and generalized in [18]. Such second family will not be discussed in the present paper.

As an illustration of possible application of our mixed formulation of Tikhonov regularization to solve inverse PDE problems, we consider an inverse obstacle problem for an acoustic waveguide in the time harmonic regime. More precisely, the objective is to identify a sound soft obstacle from a single pair of Cauchy data on the boundary of the waveguide. When many pairs of Cauchy data are known, which amounts to measure the scattered field at many receivers for many point sources, then we can compute a measurement matrix. In this case an efficient sampling type method like the Linear Sampling Method or the Factorization Method can be used, as done in [1, 8, 15, 29]. When only a single pair of Cauchy data is given, the measurement matrix degenerates into a single column and both the LSM and the FM are not applicable any longer. In that context, sampling methods of similar nature like the Convex Scattering support [9] or the Direct Sampling Method [27] can however be applied to the acoustic waveguide. In this paper we propose to apply an alternative iterative method called the "exterior approach". It couples a mixed formulation of quasi-reversibility as discussed before and a level set method. Such approach was introduced first in [5] for the Laplacian and then applied in [7] for the heat equation and [6] for the Stokes system. Our inverse obstacle problem is both non-linear and ill-posed: the non-linearity stems from the fact that we handle a geometric inverse problem while the ill-posedness stems from the nature of the boundary conditions. The "exterior approach" consists in fixing separately the problems caused by the ill-posedness and the non-linearity. More precisely, for a given obstacle, the mixed formulation of quasi-reversibility enables us to update the solution while for a given solution, the level set method enables us to update the obstacle. Note that the level set method that we use does not rely on a traditional eikonal equation (see for example [11]) but on a simple Poisson equation, which enables us to base our computations of the regularized solution and of the level set function on a single finite element mesh. Let us remark that, in the present article as in [1], and contrary to [8, 9, 15, 27, 29], the data are supported by the boundary of the waveguide, which is realistic in the framework of Non Destructive Testing.

Our paper is organized as follows. An abstract framework presenting two different mixed formulations of Tikhonov regularization constitutes Section 2. A natural one is presented first, but in view of taking noisy data into account in a better way, a relaxed one is then introduced. A typical and simple example of application of this abstract theory, that is the Cauchy problem for the Laplace equation, is presented in Section 3. Section 4 is dedicated to our inverse obstacle problem in an acoustic waveguide and the "exterior approach" to solve it. Some numerical experiments are shown in Section 5, which in particular compares our two different mixed formulations. Lastly, another application of our theory, that is the backward heat equation, is exposed in appendix.

2. The abstract framework

2.1. A general ill-posed problem

Let us consider the Hilbert spaces V, M and H already defined in the introduction, as well as the continuous onto operator $A\colon V \to H$ and the corresponding affine space $V_f = \{u \in V,\ Au = f\}$ for $f \in H$. For a continuous bilinear form b on $V \times M$ and a continuous linear form ℓ on M, let us consider the weak formulation: find $u \in V_f$ such that for all $\mu \in M$,

$$b(u,\mu) = \ell(\mu). \tag{2.1}$$

The bilinear form b is said to satisfy the inf–sup property on $V_0 \times M$ if

Assumption 2.1. There exists $\alpha > 0$ such that

$$\inf_{\substack{u\in V_0\\ u\neq 0}} \sup_{\substack{\mu\in M\\ \mu\neq 0}} \frac{b(u,\mu)}{||u||_V ||\mu||_M} \geq \alpha.$$

The bilinear form b is said to satisfy the solvability property on $V_0 \times M$ if

Assumption 2.2. For all $\mu \in M$,

$$\forall u \in V_0, \quad b(u,\mu) = 0 \Longrightarrow \mu = 0.$$

Lastly, b is said to satisfy the uniqueness property on $V_0 \times M$ if

Assumption 2.3. For all $u \in V_0$,

$$\forall \mu \in M, \quad b(u,\mu) = 0 \Longrightarrow u = 0.$$

Clearly, Assumption 2.1 implies Assumption 2.3, the converse implication is false. Besides, it is well-known from the Brezzi-Nečas-Babuška theorem (see for example [20]) that problem (2.1) is well-posed if and only if both conditions 2.1 and 2.2 are satisfied. In what follows, it is assumed that the bilinear form b does not satisfy the inf–sup condition 2.1, which from the Brezzi-Nečas-Babuška theorem implies that the problem (2.1) for a given ℓ is in general ill-posed.

2.2. A mixed formulation of Tikhonov regularization

A first regularized formulation of ill-posed problem (2.1) is the following: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_f \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\left\{ \begin{array}{l} \varepsilon(u_\varepsilon, v)_V + b(v, \lambda_\varepsilon) = 0 \\ b(u_\varepsilon, \mu) - (\lambda_\varepsilon, \mu)_M = \ell(\mu). \end{array} \right. \tag{2.2}$$

We have the following theorem.

Theorem 2.4. *For any $f \in H$ and $\ell \in M'$, the problem (2.2) has a unique solution. For some $f \in H$ and $\ell \in M'$ such that (2.1) has at least one solution, then the solution $(u_\varepsilon, \lambda_\varepsilon) \in V_f \times M$ satisfies $(u_\varepsilon, \lambda_\varepsilon) \to (u_m, 0)$ in $V \times M$ when $\varepsilon \to 0$, where u_m is the unique solution to the minimization problem*

$$\inf_{v \in K} ||v||_V, \quad K := \{v \in V_f,\ b(v, \mu) = \ell(\mu),\ \forall \mu \in M\}. \tag{2.3}$$

In particular, if the uniqueness condition 2.3 is satisfied, then $u_m = u$, where u is the unique solution to problem (2.1).

Proof. Let us introduce some $U \in V$ such that $AU = f$, which exists since A is onto, and let us set $\hat{u}_\varepsilon = u_\varepsilon - U$, so that problem (2.2) is equivalent to: find $(\hat{u}_\varepsilon, \lambda_\varepsilon) \in V_0 \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\left\{ \begin{array}{l} \varepsilon(\hat{u}_\varepsilon, v)_V + b(v, \lambda_\varepsilon) = -\varepsilon(U, v)_V \\ b(\hat{u}_\varepsilon, \mu) - (\lambda_\varepsilon, \mu)_M = \ell(\mu) - b(U, \mu), \end{array} \right. \tag{2.4}$$

which is itself equivalent to: find $(\hat{u}_\varepsilon, \lambda_\varepsilon) \in V_0 \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$A_\varepsilon((\hat{u}_\varepsilon, \lambda_\varepsilon); (v, \mu)) = L_\varepsilon((v, \mu)),$$

where the bilinear form A_ε and the linear form L_ε are given on $V_0 \times M$ by

$$A_\varepsilon((u, \lambda); (v, \mu)) = \varepsilon(u, v)_V + b(v, \lambda) - b(u, \mu) + (\lambda, \mu)_M$$

and

$$L_\varepsilon((v, \mu)) = -\varepsilon(U, v)_V - \ell(\mu) + b(U, \mu).$$

Since for $(u, \lambda) \in V_0 \times M$,

$$A_\varepsilon((u, \lambda); (u, \lambda)) \geq \varepsilon||u||_V^2 + ||\lambda||_M^2,$$

A_ε is coercive on $V_0 \times M$, which implies from the Lax-Milgram lemma that the problem (2.2) is well-posed for all $\varepsilon > 0$.

Now let us assume that $f \in H$ and $\ell \in M'$ are such that (2.1) has at least one solution. In this case, the set K is a non empty, convex and closed subset of V, while the mapping $u \mapsto ||u||_V^2$ is continuous and strictly convex. Then by standard results on minimization problems, the problem (2.3) has a unique solution u_m, which in particular satisfies $u_m \in V_f$ and

$$b(u_m, \mu) = \ell(\mu), \ \forall \mu \in M. \tag{2.5}$$

By subtracting (2.5) to the second equation of (2.2), we obtain that for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \varepsilon(u_\varepsilon, v)_V + b(v, \lambda_\varepsilon) = 0 \\ b(u_\varepsilon - u_m, \mu) - (\lambda_\varepsilon, \mu)_M = 0. \end{cases}$$

Choosing $v = u_\varepsilon - u_m \in V_0$, $\mu = \lambda_\varepsilon \in M$ and subtracting the two obtained equations we end up with

$$\varepsilon(u_\varepsilon, u_\varepsilon - u_m)_V + ||\lambda_\varepsilon||_M^2 = 0.$$

This in particular implies that

$$||u_\varepsilon||_V \leq ||u_m||_V, \ ||\lambda_\varepsilon||_M \leq \sqrt{\varepsilon}||u_m||_V.$$

The second inequality directly implies that $\lambda_\varepsilon \to 0$ in M when $\varepsilon \to 0$. From the first inequality, there exists some subsequence of u_ε, still denoted u_ε, such that $u_\varepsilon \rightharpoonup w$ in V for some $w \in V$. Since the set $\{v \in V_f, \ ||v||_V \leq ||u_m||_V\}$ is convex and closed, it is weakly closed, that is $w \in V_f$ and $||w||_V \leq ||u_m||_V$. Moreover, by passing to the limit in the second equation of (2.2), we obtain that for all $\mu \in M$, $b(w, \mu) = \ell(\mu)$. Since the minimization problem (2.3) has a unique solution, we conclude that $w = u_m$. We lastly remark that

$$||u_\varepsilon - u_m||_V^2 \leq -(u_m, u_\varepsilon - u_m)_V,$$

so that weak convergence in V implies strong convergence in V. By a standard contradiction argument, all the sequence u_ε (not only a subsequence), converges to u_m in V. □

Remark 2.5. That the problem (2.2) is well-posed for any $f \in H$ and $\ell \in M'$ means that it actually regularizes the ill-posed problem (2.1). It in particular applies to noisy data (f^δ, ℓ^δ) if they are smooth enough. However it may happen that contrary to the exact data f, the noisy data f^δ is not sufficiently smooth to belong to the range of operator A. This will be the case in the example of the Cauchy problem for the Laplace equation. One possibility is to build some regularized data from the initial one, for example at the numerical level. Another possibility is to modify our mixed formulation in order to tolerate less smooth data: this is the goal of the relaxed formulation that we present later on. Besides, we currently don't know how to choose the regularization parameter ε in (2.2), which should classically be chosen as a function of the amplitude of noise that corrupts the data (f, ℓ). This issue will also be corrected with the help of the relaxed formulation.

Now let us show the link between our regularized formulation (2.2) and the standard well-known Tikhonov regularization. More precisely, we can interpret (2.2) as a mixed formulation, in the sense of Brezzi-Fortin [10] for instance, of the Tikhonov regularization. Indeed, by the Riesz theorem, there exists a unique continuous operator $B\colon V \longrightarrow M$ and a unique $L \in M$ such that for all $u \in V$ and all $\mu \in M$,

$$(Bu, \mu)_M = b(u, \mu) \tag{2.6}$$

and

$$(L, \mu)_M = \ell(\mu). \tag{2.7}$$

Hence problem (2.1) is equivalent to find $u \in V_f$ such that $Bu = L$. The Tikhonov regularization of such ill-posed problem consists in solving, for $\varepsilon > 0$, the well-posed minimization problem

$$\inf_{v \in V_f} \left(||Bv - L||_M^2 + \varepsilon ||v||_V^2 \right). \tag{2.8}$$

The following proposition specifies the relationship between problems (2.2) and (2.8):

Proposition 2.6. *Let us denote by v_ε the unique solution to problem (2.8) and set $\mu_\varepsilon = Bv_\varepsilon - L$. Then $(v_\varepsilon, \mu_\varepsilon)$ coincides with the unique solution $(u_\varepsilon, \lambda_\varepsilon)$ to problem (2.2).*

Proof. Let us denote v_ε the solution to problem (2.8). Such solution is characterized by $v_\varepsilon \in V_f$ and

$$(Bv_\varepsilon - L, Bv)_M + \varepsilon (v_\varepsilon, v)_V = 0, \ \forall v \in V_0,$$

that is by setting $\mu_\varepsilon = Bv_\varepsilon - L \in M$, for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \varepsilon (v_\varepsilon, v)_V + (Bv, \mu_\varepsilon)_M = 0 \\ (Bv_\varepsilon, \mu)_M - (\mu_\varepsilon, \mu)_M = (L, \mu)_M, \end{cases}$$

that is $(v_\varepsilon, \mu_\varepsilon) \in V_f \times M$ solves problem (2.2) by using the definitions of B and L given by (2.6) and (2.7). We conclude that $(v_\varepsilon, \mu_\varepsilon) = (u_\varepsilon, \lambda_\varepsilon)$, which completes the proof. □

2.3. A relaxed mixed formulation of Tikhonov regularization

Let us now consider the case when the operator A: $V \to H$ is no more onto but has only a dense range. This framework is well-adapted to the case when the data $f \in H$ is corrupted by noise and hence is not sufficiently smooth to be in the range of A (see Rem. 2.5), as we will see in the example of the Cauchy problem for the Laplace equation. We additionally assume that condition 2.2 is satisfied, which implies that the operator B defined by (2.6) has a dense range too. A second regularized formulation of problem (2.1) is the following: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V \times M$ such that for all $(v, \mu) \in V \times M$,

$$\begin{cases} \varepsilon (u_\varepsilon, v)_V + \eta^2 (Au_\varepsilon, Av)_H + b(v, \lambda_\varepsilon) = \eta^2 (f, Av)_H \\ b(u_\varepsilon, \mu) - (\lambda_\varepsilon, \mu)_M = \ell(\mu). \end{cases} \tag{2.9}$$

Note that in problem (2.9), η is a fixed parameter that will be discussed later. We can prove the following result, which is the analogous of Theorem 2.4 for problem (2.9) instead of problem (2.2).

Theorem 2.7. *For any $f \in H$ and $\ell \in M'$, the problem (2.9) has a unique solution. For some $f \in H$ and $\ell \in M'$ such that (2.1) has at least one solution, the solution $(u_\varepsilon, \lambda_\varepsilon) \in V \times M$ satisfies $(u_\varepsilon, \lambda_\varepsilon) \to (u_m, 0)$ in $V \times M$ when $\varepsilon \to 0$, where u_m is the unique solution to the minimization problem (2.3). In particular, if the uniqueness condition 2.3 is satisfied, then $u_m = u$, where u is the unique solution to problem (2.1).*

Since the proof of Theorem 2.7 is very similar to that of Theorem 2.4, it is omitted. Additionally, in the same vein as Proposition 2.6, we have the following proposition:

Proposition 2.8. *Let us denote v_ε the unique solution to the minimization problem*

$$\inf_{v \in V} \left(\eta^2 ||Av - f||_H^2 + ||Bv - L||_M^2 + \varepsilon ||v||_V^2 \right)$$

and $\mu_\varepsilon = Bu_\varepsilon - L$. Then $(v_\varepsilon, \mu_\varepsilon)$ coincides with the unique solution $(u_\varepsilon, \lambda_\varepsilon)$ to problem (2.9).

Now let us introduce some noise on the data $\mathcal{L} = (f, L) \in H \times M$, namely we consider $\mathcal{L}^\delta = (f^\delta, L^\delta) \in H \times M$ such that for some δ_f, δ_ℓ, $\rho > 0$,

$$||f^\delta - f||_H \leq \delta_f, \quad ||L^\delta - L||_M \leq \rho\, \delta_\ell, \quad ||\mathcal{L}^\delta|| > \sqrt{\eta^2 \delta_f^2 + \rho^2 \delta_\ell^2}, \tag{2.10}$$

where

$$||\mathcal{L}|| = \sqrt{\eta^2 ||f||_H^2 + ||L||_M^2}. \tag{2.11}$$

Denoting

$$\varDelta = \sqrt{\eta^2 \delta_f^2 + \rho^2 \delta_\ell^2}, \tag{2.12}$$

it is readily seen from (2.10), (2.11) and (2.12) that

$$||\mathcal{L}^\delta - \mathcal{L}|| \leq \varDelta, \quad ||\mathcal{L}^\delta|| > \varDelta. \tag{2.13}$$

The parameter η is introduced in order to take into account the fact that f and L are different physical quantities. Let us recall that $L \in M$ is computed from $\ell \in M'$ thanks to (2.7). Given some noise on ℓ of given amplitude δ_ℓ, the parameter ρ is the ratio between the amplitude of the resulting noise on L and δ_ℓ. We will see later how these two parameters η and ρ can be chosen in practice. A classical strategy to choose the regularization parameter ε consists in the Morozov's discrepancy principle. It relies on the following well-known result (see for example [24] for a proof in the restricted case of a compact operator).

Lemma 2.9. *We consider two Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ and a continuous operator $\mathcal{A}\colon \mathcal{V} \longrightarrow \mathcal{H}$ which has a dense range. For $\varDelta > 0$, let us consider some data $\mathcal{L}^\varDelta \in \mathcal{H}$ such that $||\mathcal{L}^\varDelta||_\mathcal{H} > \varDelta$ and denote by $u_\varepsilon^\varDelta$ the unique minimizer in $\mathcal{V}$ of the Tikhonov functional $v \mapsto ||\mathcal{A}v - \mathcal{L}^\varDelta||_\mathcal{H}^2 + \varepsilon ||v||_\mathcal{V}^2$. There exists a unique $\varepsilon > 0$ such that $||\mathcal{A}u_\varepsilon^\varDelta - \mathcal{L}^\varDelta||_\mathcal{H} = \varDelta$.*

From Lemma 2.9, we obtain the following theorem.

Theorem 2.10. *Let us denote by $(u_\varepsilon^\delta, \lambda_\varepsilon^\delta)$ the solution to problem (2.9) for noisy data (f^δ, ℓ^δ) instead of exact data (f, ℓ). Assume that if L^δ and L are derived by (2.7) from ℓ^δ and ℓ, the noisy data (f^δ, L^δ) satisfy (2.10). Then there exists a unique $\varepsilon > 0$ such that the solution $(u_\varepsilon^\delta, \lambda_\varepsilon^\delta)$ satisfies*

$$\sqrt{\eta^2 ||Au_\varepsilon^\delta - f^\delta||_H^2 + ||\lambda_\varepsilon^\delta||_M^2} = \varDelta,$$

where $\varDelta$ is defined by (2.12).

Proof. We directly apply Lemma 2.9 for $\mathcal{V} = V$, $\mathcal{H} = H \times M$ equipped with the norm (2.11) and $\mathcal{A}$ defined for $v \in V$ by $\mathcal{A}v = (Av, Bv)$, in view of (2.13) and Proposition 2.8. Note that the operator $\mathcal{A}$ has a dense range since both operators A and B have a dense range. □

3. An example: the Cauchy problem for the Laplace equation

In our example, we will need the following functional spaces (see for example [28]). If $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and $\varGamma$ is an open subset of $\partial\Omega$, we denote by $H^{-1/2}(\varGamma)$ the set of restrictions to $\varGamma$ of distributions in $H^{-1/2}(\partial\Omega)$, while $\tilde{H}^{1/2}(\varGamma)$ is the subspace of functions in $H^{1/2}(\varGamma)$ which, once extended by 0 on the complementary part of $\partial\Omega$, belongs to $H^{1/2}(\partial\Omega)$. We recall that $\tilde{H}^{1/2}(\varGamma)$ is the dual space of $H^{-1/2}(\varGamma)$. Similarly, we denote by $H^{1/2}(\varGamma)$ the set of restrictions to $\varGamma$ of functions in $H^{1/2}(\partial\Omega)$, while $\tilde{H}^{-1/2}(\varGamma)$ is the subspace of distributions in $H^{-1/2}(\partial\Omega)$ which are supported by $\overline{\varGamma}$. We recall that $\tilde{H}^{-1/2}(\varGamma)$ is the dual space of $H^{1/2}(\varGamma)$.

The Cauchy problem for the Laplace equation is a simple and well-known example of ill-posed problem. This is why we first illustrate our approach on that example. Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d > 1$, the boundary $\partial\Omega$ of which is partitioned into two sets $\varGamma_0$ and $\varGamma_1$. More precisely, $\varGamma_0$ and $\varGamma_1$ are non empty open sets for the topology induced on $\partial\Omega$ from the topology on $\mathbb{R}^d$, $\partial\Omega = \overline{\varGamma_0} \cup \overline{\varGamma_1}$ and $\varGamma_0 \cap \varGamma_1 = \emptyset$. The Cauchy problem consists, for some data $(f, g) \in H^{1/2}(\varGamma_0) \times H^{-1/2}(\varGamma_0)$, in finding $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega \\ u = f & \text{on} \quad \varGamma_0 \\ \partial_\nu u = g & \text{on} \quad \varGamma_0, \end{cases} \tag{3.1}$$

where ν is the outward unit normal to Ω. The problem (3.1) is equivalent to a weak formulation of type (2.1).

Lemma 3.1. *For $(f, g) \in H^{1/2}(\varGamma_0) \times H^{-1/2}(\varGamma_0)$, the function $u \in H^1(\Omega)$ is a solution to problem (3.1) if and only if $u|_{\varGamma_0} = f$ and for all $\mu \in H^1(\Omega)$ with $\mu|_{\varGamma_1} = 0$,*

$$\int_\Omega \nabla u \cdot \nabla \mu \;\, \mathrm{d}x = \langle g, \mu|_{\varGamma_0} \rangle_{H^{-1/2}(\varGamma_0), \tilde{H}^{1/2}(\varGamma_0)}, \tag{3.2}$$

where the brackets stand for duality pairing between $H^{-1/2}(\varGamma_0)$ and $\tilde{H}^{1/2}(\varGamma_0)$.

Proof. First, let us assume that $u \in H^1(\Omega)$ and satisfies the weak formulation (3.2). We have $u = f$ on $\varGamma_0$ and by first choosing $\mu = \varphi \in \mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ denotes the space of infinitely smooth functions which are compactly supported in Ω, we obtain $\Delta u = 0$ in Ω in the distributional sense. By using a classical Green formula, we have for all $\mu \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla \mu \, \mathrm{d}x = -\int_\Omega \Delta u \, \mu \, \mathrm{d}x + \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}.$$

If in addition $\mu|_{\varGamma_1} = 0$ and using the fact that $\Delta u = 0$ in Ω, we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\varGamma_1} = 0$,

$$\int_\Omega \nabla u \cdot \nabla \mu \;\, \mathrm{d}x = \langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\varGamma_0), \tilde{H}^{1/2}(\varGamma_0)},$$

and by comparison with (3.2) we obtain that for all $\mu \in H^1(\Omega)$ with $\mu|_{\varGamma_1} = 0$,

$$\langle \partial_\nu u, \mu \rangle_{H^{-1/2}(\varGamma_0), \tilde{H}^{1/2}(\varGamma_0)} = \langle g, \mu \rangle_{H^{-1/2}(\varGamma_0), \tilde{H}^{1/2}(\varGamma_0)},$$

which implies that $\partial_\nu u = g$ in $H^{-1/2}(\varGamma_0)$. We conclude that u satisfies (3.1). Conversely, we would prove the same way that if $u \in H^1(\Omega)$ satisfies (3.1), then it satisfies (3.2). □

The weak formulation (3.2) is hence a particular instance of abstract problem (2.1) with $V = H^1(\Omega)$, $H = H^{1/2}(\varGamma_0)$, $M = \{\mu \in H^1(\Omega),\ \mu|_{\varGamma_1} = 0\}$, A: $H^1(\Omega) \to H^{1/2}(\varGamma_0)$ is the trace operator on $\varGamma_0$ (which is onto), $V_f = \{u \in H^1(\Omega),\ u|_{\varGamma_0} = f\}$ while for $(u, \mu) \in V \times M$,

$$b(u, \mu) = \int_\Omega \nabla u \cdot \nabla \mu \, \mathrm{d}x, \quad \ell(\mu) = \langle g, \mu|_{\varGamma_0} \rangle_{H^{-1/2}(\varGamma_0), \tilde{H}^{1/2}(\varGamma_0)}. \tag{3.3}$$

For this particular bilinear form b, only the two last conditions 2.2 and 2.3 are satisfied.

Proposition 3.2. *For the bilinear form b given by (3.3), the conditions 2.2 and 2.3 are satisfied while the condition 2.1 is not.*

Proof. We start by condition 2.2. For $\mu \in M = \{\mu \in H^1(\Omega),\ \mu|_{\Gamma_1} = 0\}$, let us assume that for all $u \in V_0 = \{u \in H^1(\Omega),\ u|_{\Gamma_0} = 0\}$,

$$\int_\Omega \nabla\mu \cdot \nabla u \,\mathrm{d}x = 0.$$

Choosing $u = \varphi \in \mathcal{D}(\Omega)$, we obtain that $\Delta\mu = 0$ in the distributional sense in Ω. The Green formula then gives that for all $u \in V_0$,

$$\langle \partial_\nu \mu, u\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)} = \langle \partial_\nu \mu, u\rangle_{H^{-1/2}(\Gamma_1),\tilde{H}^{1/2}(\Gamma_1)} = 0.$$

We conclude that $\mu \in H^1(\Omega)$ satisfies the homogeneous Cauchy problem

$$\begin{cases} \Delta\mu = 0 & \text{in} \quad \Omega \\ \mu = 0 & \text{on} \quad \Gamma_1 \\ \partial_\nu \mu = 0 & \text{on} \quad \Gamma_1, \end{cases}$$

so that $\mu = 0$ by the Holmgren's theorem. Similarly, condition 2.3 amounts to prove that if u solves the Cauchy problem (3.1) with $(f, g) = 0$, then $u = 0$. Besides, we know that the problem (3.1) is ill-posed (see for example [3]), which by contradiction proves from the Brezzi-Nečas-Babuška theorem that the inf–sup condition 2.1 is not satisfied. □

The mixed formulation (2.2) of the Tikhonov regularization can be applied, that is: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_f \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \varepsilon \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x + \int_\Omega \nabla v \cdot \nabla \lambda_\varepsilon \,\mathrm{d}x = 0 \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \mu \,\mathrm{d}x - \int_\Omega \nabla \lambda_\varepsilon \cdot \nabla \mu \,\mathrm{d}x = \langle g, \mu|_{\Gamma_0}\rangle_{H^{-1/2}(\Gamma_0),\tilde{H}^{1/2}(\Gamma_0)}. \end{cases} \tag{3.4}$$

Compared to the abstract formulation (2.2), in formulation (3.4) we have used the scalar product associated with the semi-norm in $H^1(\Omega)$ instead of the full norm in $H^1(\Omega)$, which is possible thanks to Poincaré's inequality. From Theorem 2.4 and since condition 2.3 is satisfied, the problem (3.4) is well-posed for any Cauchy data $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ and for (f, g) such that problem (3.1) has a (unique) solution u, we have $(u_\varepsilon, \lambda_\varepsilon) \to (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$. A mixed formulation such as (3.4) to regularize the Cauchy problem for the Laplace equation was first introduced in [4], where the discretization with a Finite Element Method is analyzed and numerical examples are shown. Note that [14] presents, in the particular case when $f = 0$, an analysis of nonconforming discretizations of a regularized formulation such as (3.4). However, it should be pointed out that while in the present work the ill-posed problem is regularized first and discretized afterwards, in [14] discretization takes place first and regularization afterwards.

Remark 3.3. As mentioned earlier in Remark 2.5, it seems difficult to choose the regularization parameter ε in (3.4). In addition, the natural setting for (f, g) in (3.4) is $H^{1/2}(\Gamma_0) \times \tilde{H}^{-1/2}(\Gamma_0)$, which is not appropriate for experimental noisy data that are rather expected to belong to the simpler space $L^2(\Gamma_0) \times L^2(\Gamma_0)$. And lastly, in the weak formulation problem (3.4), the Dirichlet data f is strongly imposed and the Neumann data g is weakly imposed, while both data are noisy and should be weakly imposed.

The issues listed in Remark 3.3 naturally lead us to consider the relaxed mixed formulation of Tikhonov regularization. Let us apply such relaxed formulation (2.9) to problem (3.1). To this aim we consider some

Cauchy data $(f,g) \in L^2(\mathit{\Gamma}_0) \times L^2(\mathit{\Gamma}_0)$ instead of $H^{1/2}(\mathit{\Gamma}_0) \times \tilde{H}^{-1/2}(\mathit{\Gamma}_0)$. In order to apply the regularized formulation (2.9), we now consider $V = H^1(\Omega)$, $H = L^2(\mathit{\Gamma}_0)$, $M = \{\mu \in H^1(\Omega),\ \mu|_{\mathit{\Gamma}_1} = 0\}$, A: $H^1(\Omega) \to L^2(\mathit{\Gamma}_0)$ is the trace operator on $\mathit{\Gamma}_0$ (which has a dense range), while the bilinear form b and the linear form ℓ are again given by (3.3), except that the duality bracket defining ℓ is now an integral since $g \in L^2(\mathit{\Gamma}_0)$. The regularized formulation (2.9) is in this case: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in H^1(\Omega) \times \{\mu \in H^1(\Omega),\ \mu|_{\mathit{\Gamma}_1} = 0\}$ such that for all $(v, \mu) \in H^1(\Omega) \times \{\mu \in H^1(\Omega),\ \mu|_{\mathit{\Gamma}_1} = 0\}$,

$$\left\{ \begin{array}{l} \varepsilon \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x + \eta^2 \int_{\mathit{\Gamma}_0} u_\varepsilon\, v \,\mathrm{d}s + \int_\Omega \nabla v \cdot \nabla \lambda_\varepsilon \,\mathrm{d}x = \eta^2 \int_{\mathit{\Gamma}_0} f\, v \,\mathrm{d}s \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \mu \,\mathrm{d}x - \int_\Omega \nabla \lambda_\varepsilon \cdot \nabla \mu \,\mathrm{d}x = \int_{\mathit{\Gamma}_0} g\, \mu \,\mathrm{d}s. \end{array} \right. \tag{3.5}$$

Note that in formulation (3.5), both data f and g are weakly imposed.

Remark 3.4. The idea of considering a boundary condition weakly is of course not new (see for example [19]). It is also possible to consider the PDE itself weakly, that is as an augmented Lagrangian term. It is for example done in [12] on the discrete level and in [16] on the continuous level.

From Theorem 2.7 and since the uniqueness condition 2.3 is satisfied, the problem (3.5) is well-posed for any Cauchy data $(f,g) \in L^2(\mathit{\Gamma}_0) \times L^2(\mathit{\Gamma}_0)$ and for (f,g) such that problem (3.1) has a (unique) solution u, we have $(u_\varepsilon, \lambda_\varepsilon) \to (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$. Now assume that we measure some noisy data $(f^\delta, g^\delta) \in L^2(\mathit{\Gamma}_0) \times L^2(\mathit{\Gamma}_0)$ so that

$$||f^\delta - f||_{L^2(\mathit{\Gamma}_0)} \leq \delta_f, \quad ||g^\delta - g||_{L^2(\mathit{\Gamma}_0)} \leq \delta_g.$$

Let us apply the weak formulation (2.7). For any $g \in L^2(\mathit{\Gamma}_0)$, there exists a unique $L \in H^1(\Omega)$ with $L|_{\mathit{\Gamma}_1} = 0$ such that for all $\mu \in H^1(\Omega)$ with $\mu|_{\mathit{\Gamma}_1} = 0$,

$$\int_\Omega \nabla L \cdot \nabla \mu \,\mathrm{d}x = \int_{\mathit{\Gamma}_0} g\, \mu \,\mathrm{d}s. \tag{3.6}$$

Assume that L^δ is associated with data $g^\delta \in L^2(\mathit{\Gamma}_0)$ *via* (3.6), so that

$$||f^\delta - f||_{L^2(\mathit{\Gamma}_0)} \leq \delta_f, \quad ||L^\delta - L||_{H^1(\Omega)} \leq \rho\, \delta_g.$$

We assume in addition that

$$\eta^2 ||f^\delta||^2_{L^2(\mathit{\Gamma}_0)} + ||L^\delta||^2_{H^1(\Omega)} \geq \eta^2 \delta_f^2 + \rho^2 \delta_g^2,$$

so that our noisy data (f^δ, L^δ) satisfy (2.10). Theorem 2.10 is hence directly applicable and we obtain that there exists a unique $\varepsilon > 0$ such that the solution $(u^\delta_\varepsilon, \lambda^\delta_\varepsilon)$ to problem (3.5) associated with the corresponding data (f^δ, g^δ) satisfies

$$\sqrt{\eta^2 ||u^\delta_\varepsilon - f^\delta||^2_{L^2(\mathit{\Gamma}_0)} + ||\lambda^\delta_\varepsilon||^2_{H^1(\Omega)}} = \sqrt{\eta^2 \delta_f^2 + \rho^2 \delta_g^2}.$$

In appendix, we present the slightly more complicated example of the backward heat equation, which is also a particular instance of the abstract problem (2.1).

4. An inverse obstacle problem in an acoustic waveguide

4.1. Introduction

We consider a d dimensional waveguide $W = \Sigma \times \mathbb{R}$ for $d \geq 2$, where Σ is a $(d-1)$ dimensional Lipschitz domain. The boundary of W is denoted ∂W. A generic point $x \in W$ has coordinates (x_Σ, x_d) with $x_\Sigma \in \Sigma$ and $x_d \in \mathbb{R}$. Let us consider a smooth Lipschitz domain D such that $\overline{D} \subset W$, referred to as the obstacle. For some wave number $k > 0$ and data $(f, g) \in H^{1/2}(\partial W) \times H^{-1/2}(\partial W)$ that are compactly supported with $(f, g) \neq (0, 0)$, the inverse obstacle problem consists in finding a domain D and a function $u \in H^1_{\mathrm{loc}}(W \setminus \overline{D})$ such that

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad W \setminus \overline{D} \\ (u, \partial_\nu u) = (f, g) & \text{on} \quad \partial W \\ u = 0 & \text{on} \quad \partial D \\ (RC), & \end{cases} \tag{4.1}$$

where

$$H^1_{\mathrm{loc}}(W \setminus \overline{D}) = \{v \in \mathcal{D}'(W \setminus \overline{D}),\ \varphi(x_d)v(x) \in H^1(W \setminus \overline{D}),\ \forall \varphi \in \mathcal{D}(\mathbb{R})\},$$

ν is the outward unit normal to W and (RC) is a radiation condition which forces the field u to be outgoing. Physically, D can be seen as a sound-soft obstacle, u is the pressure field outside D, g is the prescribed normal component of the velocity while f is the resulting measured pressure on the boundary. The first question related to this inverse obstacle problem is identifiability: is the obstacle D uniquely defined from a single pair of Cauchy data (f, g)? Uniqueness for this problem is unknown in general, only a local uniqueness result is known. For example, the following result is proved in [30].

Theorem 4.1. *Let $D_- \subset D_+$ be two obstacles such that $\mathrm{Vol}(D_+ \setminus D_-) < \Omega_d\, k^{-d}$, where Ω_d is the volume of the unit ball in $\mathbb{R}^d$. Let D_j, $j = 1, 2$, be two other obstacles such that $D_- \subset D_j \subset D_+$, and corresponding solutions u_j which satisfy problem (4.1). If we assume in addition that the functions u_j are continuous in $\overline{W} \setminus D_j$, $j = 1, 2$, then $D_1 = D_2$.*

In what follows we will assume that global uniqueness holds in the inverse obstacle problem. In order to solve it, we propose an "exterior approach" coupling a mixed Tikhonov formulation such as (2.2) and the level set method introduced in [5] in the case of the Laplacian. In such iterative approach, for a given estimated defect $\tilde{D}$ we update the solution $\tilde{u}$ with the help of a mixed formulation of quasi-reversibility while for a given estimated solution $\tilde{u}$ we update the defect $\tilde{D}$ with the help of a level set method based on a Poisson problem.

4.2. Application of mixed Tikhonov formulations

First of all, we wish to transform the problem (4.1) into an equivalent one set in a bounded domain. To this aim, we assume that the obstacle D lies within the bounded subdomain W_R of W delimited by the two sections $\Sigma_\pm = \Sigma \times \{x_d = \pm R\}$ and let us introduce $\Omega = W_R \setminus \overline{D}$. The portion of ∂W contained between Σ_- and Σ_+ is denoted by Γ. It is well-known that for $k > 0$ and $(f, g) \in H^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$, the problem (4.1) is equivalent to find a domain D and a function $u \in H^1(\Omega)$ such that

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad \Omega \\ (u, \partial_\nu u) = (f, g) & \text{on} \quad \Gamma \\ u = 0 & \text{on} \quad \partial D \\ \pm\partial_{x_d} u = T_\pm u & \text{on} \quad \Sigma_\pm, \end{cases} \tag{4.2}$$

where the operators $T_\pm\colon H^{1/2}(\Sigma_\pm) \to \tilde{H}^{-1/2}(\Sigma_\pm)$ are the so-called Dirichlet-to-Neumann operators. A convenient way to define such operators consists in introducing the eigenvalues λ_n and the eigenfunctions θ_n of the following eigenvalue problem set in the $(d-1)$ dimensional domain Σ:

$$\begin{cases} (\Delta_\perp + \lambda)v = 0 & \text{in} \quad \Sigma \\ \partial_{\nu_\perp} v = 0 & \text{on} \quad \partial\Sigma, \end{cases} \tag{4.3}$$

where $\Delta_\perp$ is the Laplacian in Σ while $\nu_\perp$ is the outward unit normal vector to domain Σ. The (λ_n) form a non negative and increasing sequence of reals that tends to $+\infty$ while the (θ_n) can be chosen to form a complete orthonormal basis of $L^2(\Sigma)$.

Remark 4.2. In particular, for $d = 2$, $\Sigma = (0, h)$ for some $h > 0$, the solutions to problem (4.3) are given for $n \in \mathbb{N}$ by $\lambda_n = n^2\pi^2/h^2$ and by

$$\begin{cases} \theta_0(x_1) = \sqrt{\frac{1}{h}} \\ \theta_n(x_1) = \sqrt{\frac{2}{h}} \cos(\frac{n\pi}{h} x_1) \quad (n \geq 1). \end{cases} \tag{4.4}$$

Setting for all $n \in \mathbb{N}$

$$\beta_n = \sqrt{k^2 - \lambda_n} \quad \text{with} \quad \mathrm{Re}(\beta_n) \geq 0, \quad \mathrm{Im}(\beta_n) \geq 0, \tag{4.5}$$

the operators $T_\pm$ can be defined, for $\varphi \in H^{1/2}(\Sigma_\pm)$, by

$$T_\pm \varphi = \sum_{n \in \mathbb{N}} i\beta_n (\varphi, \theta_n)_{L^2(\Sigma_\pm)} \theta_n. \tag{4.6}$$

Assume that the obstacle D is known. Given some Cauchy data $(f, g) \in H^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$, let us consider the linear ill-posed problem of finding $u \in H^1(\Omega)$ such that the following problem be satisfied:

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad \Omega \\ (u, \partial_\nu u) = (f, g) & \text{on} \quad \Gamma \\ \pm\partial_{x_d} u = T_\pm u & \text{on} \quad \Sigma_\pm. \end{cases} \tag{4.7}$$

We can easily check that the problem (4.7) is equivalent to a weak formulation of type (2.1).

Lemma 4.3. *For $(f, g) \in H^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$, the function $u \in H^1(\Omega)$ is a solution to problem (4.7) if and only if $u|_\Gamma = f$ and for all $\mu \in H^1(\Omega)$ with $\mu|_{\partial D} = 0$,*

$$\int_\Omega \nabla u \cdot \nabla \overline{\mu} \, \mathrm{d}x - k^2 \int_\Omega u \, \overline{\mu} \, \mathrm{d}x - \left\langle T_\pm u|_{\Sigma_\pm}, \overline{\mu}|_{\Sigma_\pm} \right\rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)} = \langle g, \overline{\mu}|_\Gamma \rangle_{\tilde{H}^{-1/2}(\Gamma), H^{1/2}(\Gamma)},$$

where $\pm$ means the summation of the bracket on Σ_- and the bracket on Σ_+.

The weak formulation of Lemma 4.3 is a particular instance of abstract problem (2.1) with $V = H^1(\Omega)$, $H = H^{1/2}(\Gamma)$, $M = \{\mu \in H^1(\Omega),\ \mu|_{\partial D} = 0\}$, $A\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ is the trace operator on Γ, $V_f = \{u \in H^1(\Omega),\ u|_\Gamma = f\}$ while for $(u, \mu) \in V \times M$,

$$b(u, \mu) = \int_\Omega \nabla u \cdot \nabla \overline{\mu} \, \mathrm{d}x - k^2 \int_\Omega u \, \overline{\mu} \, \mathrm{d}x - \left\langle T_\pm u|_{\Sigma_\pm}, \overline{\mu}|_{\Sigma_\pm} \right\rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)}, \tag{4.8}$$

$$\ell(\mu) = \langle g, \overline{\mu}|_\Gamma \rangle_{\tilde{H}^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \tag{4.9}$$

We have to note that, contrary to the abstract problem presented in Section 2, the functions have complex values, which means in particular that the forms b and ℓ are sesquilinear and antilinear instead of bilinear and linear, respectively. However, it is readily seen that all the results of Section 2 remain valid in this context of complex Hilbert spaces provided a complex modulus be applied to b in the inf–sup condition 2.1. Once again, we have the following proposition, the proof of which is very similar to that of Proposition 3.2.

Proposition 4.4. *For the sesquilinear form b given by (4.8), the conditions 2.2 and 2.3 are satisfied while the condition 2.1 is not.*

The mixed formulation (2.2) of the Tikhonov regularization applies as follows: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_f \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \varepsilon \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \overline{v} \,\mathrm{d}x + \int_\Omega \nabla v \cdot \nabla \overline{\lambda_\varepsilon} \,\mathrm{d}x - k^2 \int_\Omega v \, \overline{\lambda_\varepsilon} \,\mathrm{d}x - \left\langle T_\pm v, \overline{\lambda_\varepsilon} \right\rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)} = 0 \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \overline{\mu} \,\mathrm{d}x - k^2 \int_\Omega u_\varepsilon \, \overline{\mu} \,\mathrm{d}x - \langle T_\pm u_\varepsilon, \overline{\mu} \rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)} \\ - \displaystyle\int_\Omega \nabla \lambda_\varepsilon \cdot \nabla \overline{\mu} \,\mathrm{d}x = \langle g, \overline{\mu} \rangle_{\tilde{H}^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \end{cases} \tag{4.10}$$

Again, from Theorem 2.4 and since condition 2.3 is satisfied, the problem (4.10) is well-posed for any data $(f, g) \in H^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$ and for (f, g) such that problem (4.7) has a (unique) solution u, we have $(u_\varepsilon, \lambda_\varepsilon) \to (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$. Let us now apply the relaxed formulation (2.9) to problem (4.7). To this aim we consider some Cauchy data $(f, g) \in L^2(\Gamma) \times L^2(\Gamma)$ instead of $H^{1/2}(\Gamma) \times \tilde{H}^{-1/2}(\Gamma)$. We now consider $V = H^1(\Omega)$, $H = L^2(\Gamma)$, $M = \{\mu \in H^1(\Omega),\ \mu|_{\partial D} = 0\}$, A: $H^1(\Omega) \to L^2(\Gamma)$ is the trace operator on Γ (which has a dense range), while the sesquilinear form b and the antilinear form ℓ are again given by (4.8) and (4.9) (again, the duality bracket defining ℓ is now an integral since $g \in L^2(\Gamma)$). The relaxed formulation (2.9) is in this case: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in H^1(\Omega) \times \{\mu \in H^1(\Omega),\ \mu|_{\partial D} = 0\}$ such that for all $(v, \mu) \in H^1(\Omega) \times \{\mu \in H^1(\Omega),\ \mu|_{\partial D} = 0\}$,

$$\begin{cases} \varepsilon \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \overline{v} \,\mathrm{d}x + \eta^2 \int_\Gamma u_\varepsilon \, \overline{v} \,\mathrm{d}s + \int_\Omega \nabla v \cdot \nabla \overline{\lambda_\varepsilon} \,\mathrm{d}x - k^2 \int_\Omega v \, \overline{\lambda_\varepsilon} \,\mathrm{d}x \\ - \left\langle T_\pm v, \overline{\lambda_\varepsilon} \right\rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)} = \eta^2 \displaystyle\int_\Gamma f \, \overline{v} \,\mathrm{d}s \\ \displaystyle\int_\Omega \nabla u_\varepsilon \cdot \nabla \overline{\mu} \,\mathrm{d}x - k^2 \int_\Omega u_\varepsilon \, \overline{\mu} \,\mathrm{d}x - \langle T_\pm u_\varepsilon, \overline{\mu} \rangle_{\tilde{H}^{-1/2}(\Sigma_\pm), H^{1/2}(\Sigma_\pm)} - \int_\Omega \nabla \lambda_\varepsilon \cdot \nabla \overline{\mu} \,\mathrm{d}x = \int_\Gamma g \, \overline{\mu} \,\mathrm{d}s. \end{cases} \tag{4.11}$$

From Theorem 2.7 and uniqueness condition 2.3, the problem (4.11) is well-posed for any Cauchy data $(f, g) \in L^2(\Gamma) \times L^2(\Gamma)$ and for (f, g) such that problem (4.7) has a (unique) solution u, we have $(u_\varepsilon, \lambda_\varepsilon) \to (u, 0)$ in $H^1(\Omega) \times H^1(\Omega)$. In case of noisy data $(f^\delta, g^\delta) \in L^2(\Gamma) \times L^2(\Gamma)$ with

$$||f^\delta - f||_{L^2(\Gamma)} \le \delta_f, \quad ||g^\delta - g||_{L^2(\Gamma)} \le \delta_g,$$

the Morozov's principle can be applied exactly as in the case of the Cauchy problem for the Laplace equation.

4.3. The "exterior approach"

In this paragraph, we simply adapt to the Helmholtz equation the approach introduced in [5] in the case of the Laplace equation. We here briefly give a sketch of this approach: the reader will refer to [5] for a more detailed description. For a defect D and a solution u satisfying the inverse obstacle problem (4.2) for Cauchy data (f, g), let us consider a function $c \in H^1(W_R)$ such that $c = |u|$ in Ω and $c \le 0$ in D (this is always possible, take $c = 0$ in D in view of the Dirichlet boundary condition on ∂D) and a distribution $F \in H^{-1}(W_R)$ such that

$F - \Delta c \geq 0$. For some open domain $\omega \subset W_R$ and $G \in H^{-1}(W_R)$, let us denote by $v_{G,\omega}$ the solution $v \in H^1_0(\omega)$ of the Poisson problem $\Delta v = G$ in ω. We now define a sequence of open domains D_n by following induction. We first consider an open domain D_0 such that $D \subset D_0 \Subset W_R$. The domain D_n being given, we define

$$D_{n+1} = D_n \setminus \mathrm{supp}(\sup(\varphi_n, 0)),$$

where (supp denotes the support of a function)

$$\varphi_n = c + v_{G,D_n}, \quad G = F - \Delta c. \tag{4.12}$$

From [21], since the open domains D_n form a decreasing sequence, it converges in the sense of Hausdorff distance to some open domain D_∞, with $D \subset D_\infty$. Lastly, the following convergence theorem justifies the method.

Theorem 4.5. *If we assume that the sequence of functions v_{G,D_n} converges in $H^1_0(W_R)$ to the function v_{G,D_∞} and if k^2 is not a Dirichlet eigenvalue of operator $-\Delta$ in $D_\infty \setminus \overline{D}$, then $D_\infty = D$.*

Since the proof of Theorem 4.5 is very close to that given in [5] for the case of the Laplace equation, it is omitted. However, while 0 is not a Dirichlet eigenvalue of the operator $-\Delta$ in $D_\infty \setminus \overline{D}$, the positive number k^2 for $k > 0$ might be one of them, which explains why there is an assumption on k in the statement of Theorem 4.5.

Remark 4.6. The convergence of the sequence of the v_{G,D_n} functions with respect to the domain D_n is a classical question which is for example extensively studied in [21] (see also [5]). The reader will find in [5, 21] some sufficient assumptions, depending on the dimension d, that guarantee such convergence.

The statement of Theorem 4.5 means that the sequence of open domains D_n defined with the help of the solution u converges under some assumptions to the true obstacle D, in the sense of Hausdorff distance. In practice, the exact solution u is unknown, but it can be approached by the regularized solution u_ε to problem (4.10) or problem (4.11). This is why it is natural to consider the following algorithm.

Algorithm:

(1) Choose an initial guess D_0 such that $D \subset D_0 \Subset W_R$.
(2) Step 1: for a given D_n, compute the quasi-reversibility solution u_n of system (4.10) or (4.11) in Ω_n for sufficiently small ε, where $\Omega_n := W_R \setminus \overline{D_n}$.
(3) Step 2: for a given u_n in Ω_n, compute $c_n(x) = |u_n|$ in Ω_n and the solution φ_n to problem (4.12) for sufficiently large F, which simply reads for smooth D_n as:

$$\begin{cases} \Delta \varphi_n = F & \text{in} \quad D_n \\ \varphi_n = c_n & \text{on} \quad \partial D_n. \end{cases}$$

Compute $D_{n+1} = \{x \in D_n,\ \varphi_n(x) < 0\}$.
(4) Go back to step 1 until some stopping criterion is satisfied.

The convergence rate of the algorithm strongly depends on the function F, the admissible ones depending on the unknown obstacle D. The way F and ε are chosen is explained in the numerical section.

5. Some numerical experiments

In this paragraph we apply the exterior approach in a 2d waveguide of height $h = 1$ (see Rem. 4.2) and by choosing $R = 1$. We consider two different obstacles D:

(1) the disk of center $(-0.2, 0.6)$ and radius 0.1;
(2) the union of the previous disk and the disk of center $(0.3, 0.4)$ and radius 0.15.

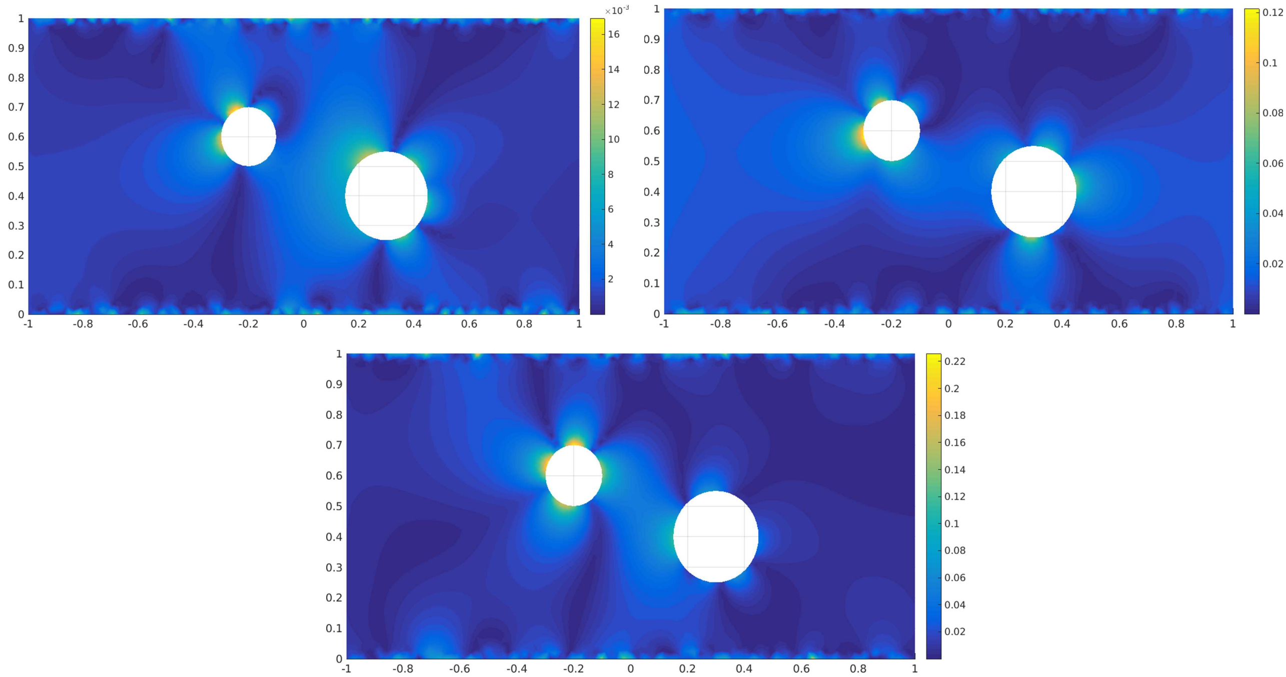

FIGURE 1. Values of $|u_\varepsilon^\delta - u|/(\max|u|)$ in Ω for $\varepsilon = 10^{-3}$. *Top left*: $\sigma = 0.01$. *Top right*: $\sigma = 0.05$. *Bottom*: $\sigma = 0.1$

The artificial data on Γ are obtained by solving the following forward problem in Ω: for $g \in \tilde{H}^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} (\Delta + k^2)u = 0 & \text{in} \quad \Omega \\ \partial_\nu u = g & \text{on} \quad \Gamma \\ u = 0 & \text{on} \quad \partial D \\ \pm\partial_{x_d} u = T_\pm u & \text{on} \quad \Sigma_\pm, \end{cases} \tag{5.1}$$

and then by setting $f = u|_\Gamma$, which provides some cauchy data (f, g) on Γ. In what follows we refer to u as the exact solution and (f, g) as the exact Cauchy data. Here we have chosen $g(x_2) = 20\max(1 - 4|x_2|, 0)$ both on $\Gamma \cap \{x_1 = h\}$ and on $\Gamma \cap \{x_1 = 0\}$. The forward problem (5.1) is discretized with the help of a standard finite element method. Before solving the inverse obstacle problem, let us first apply the mixed formulation (4.10) in Ω to solve problem (4.7), assuming the obstacle D is *a priori* known to be the obstacle (2).

In order to test the robustness of formulation (4.10) with respect to noise on the data, we perturb data (f, g) as follows. Considering f and g as vectors the components of which are the degrees of freedom induced by the finite element space, each component is contaminated pointwise by some Gaussian noise, namely

$$f^\delta = f + \sigma\frac{||f||}{||b_f||}b_f, \quad g^\delta = g + \sigma\frac{||g||}{||b_g||}b_g, \tag{5.2}$$

where b_f, b_g are given by a standard normal distribution, $\sigma > 0$ is a scaling factor and $||.||$ denotes a discretized L^2 norm. With such definition, both f and g are perturbed by a relative error of amplitude σ in L^2 norm. In Figure 1 we have plotted the discrepancy $|u_\varepsilon^\delta - u|/(\max|u|)$ in Ω for $\varepsilon = 10^{-3}$, where u is the solution to problem (5.1) and u_ε^δ is the solution to problem (4.10) with noisy data (f^δ, g^δ), for three different values of σ, that is $\sigma = 0.01$, $\sigma = 0.05$ and $\sigma = 0.1$. It can be seen that the error between the regularized solution u_ε^δ

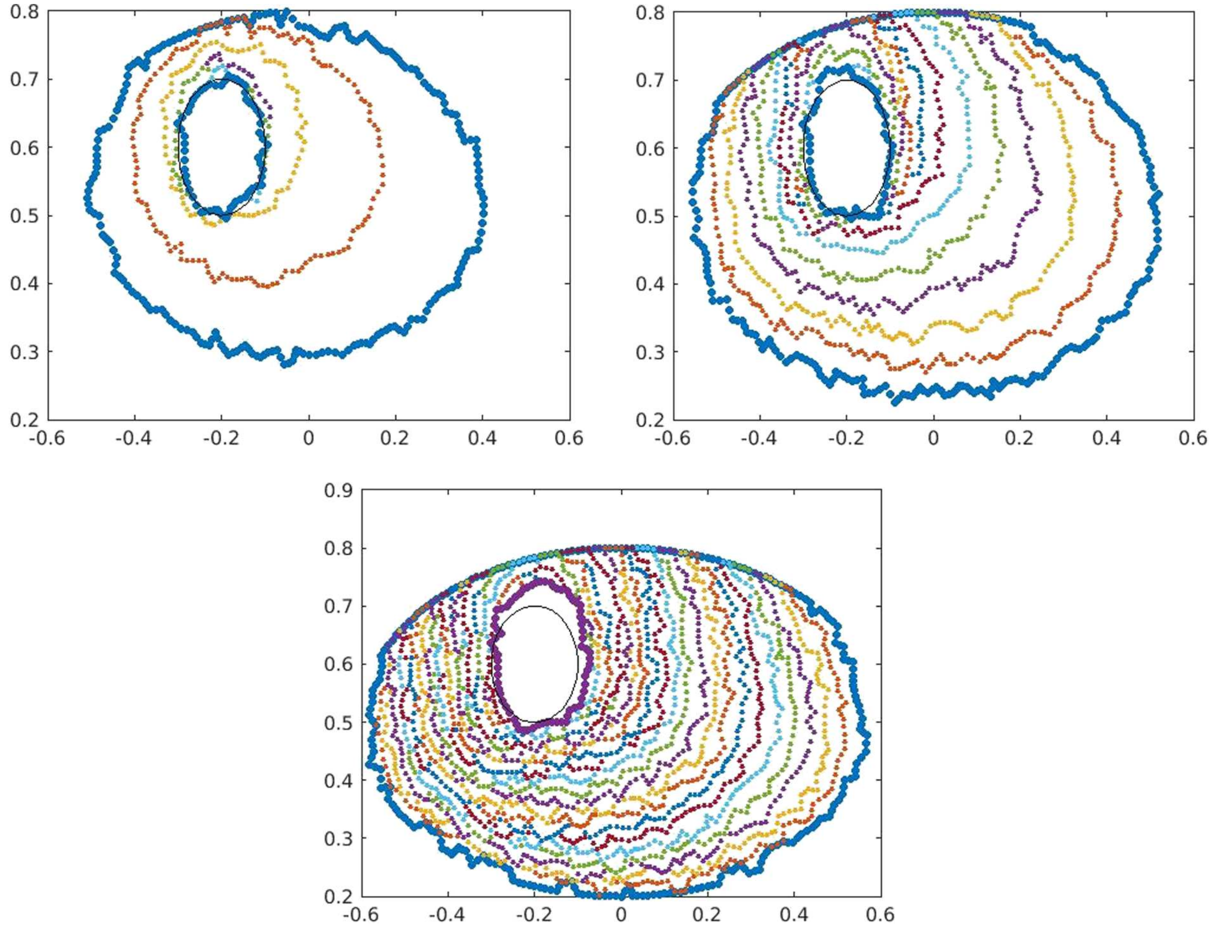

FIGURE 2. Identification of obstacle (1) for $k = 3$. *Top left*: $\sigma = 0.01$. *Top right*: $\sigma = 0.05$. *Bottom*: $\sigma = 0.1$.

and the exact solution u is small everywhere in Ω except near the obstacle D, since no boundary condition is given on ∂D. This localized error obviously explodes with respect to the noise, due to the fact that the problem (4.7) is exponentially ill-posed. Now let us come back to the inverse obstacle problem and show some numerical experiments using the exterior approach algorithm. We emphasize the fact that a single finite element triangular mesh of W_R is used both for the quasi-reversibility problem (4.10) in Ω_n and the Poisson problem (4.12) in D_n. Such mesh is the same for all $n \in \mathbb{N}$ and is different from the one used to obtain the artificial data. Both problems are discretized with standard $P2$ elements. The size of the mesh is such that it corresponds to 80 triangle edges on each part of Γ while the infinite sum (4.6) defining the operators $T_\pm$ is truncated to 100 terms. Again, we arbitrarily set $\varepsilon = 10^{-3}$ in the quasi-reversibility problem. It should be noted that, due to the fact that the common boundary of Ω_n and D_n is supported by a polygonal line based on the finite element mesh, the sequence of domains D_n is stationary for sufficiently large n, which provides a simple stopping criterion for our algorithm. The function F is fixed to a constant in the Poisson problem. As shown in [5], the value of F strongly influences the convergence of the algorithm: if F is too large, the algorithm needs more iterations to converge and can even stop prematurely, overestimating D. If F is too small, which means that the condition $F - \Delta c \geq 0$ is not satisfied, there is no convergence to D and the sequence of D_n collapses to the null-set. To cope with this issue, F is initialized to the arbitrary value 0.05 and is then updated *via* a dichotomy procedure depending on the convergence of the D_n: as the sequence (D_n) becomes stationary, F is taken bigger if the

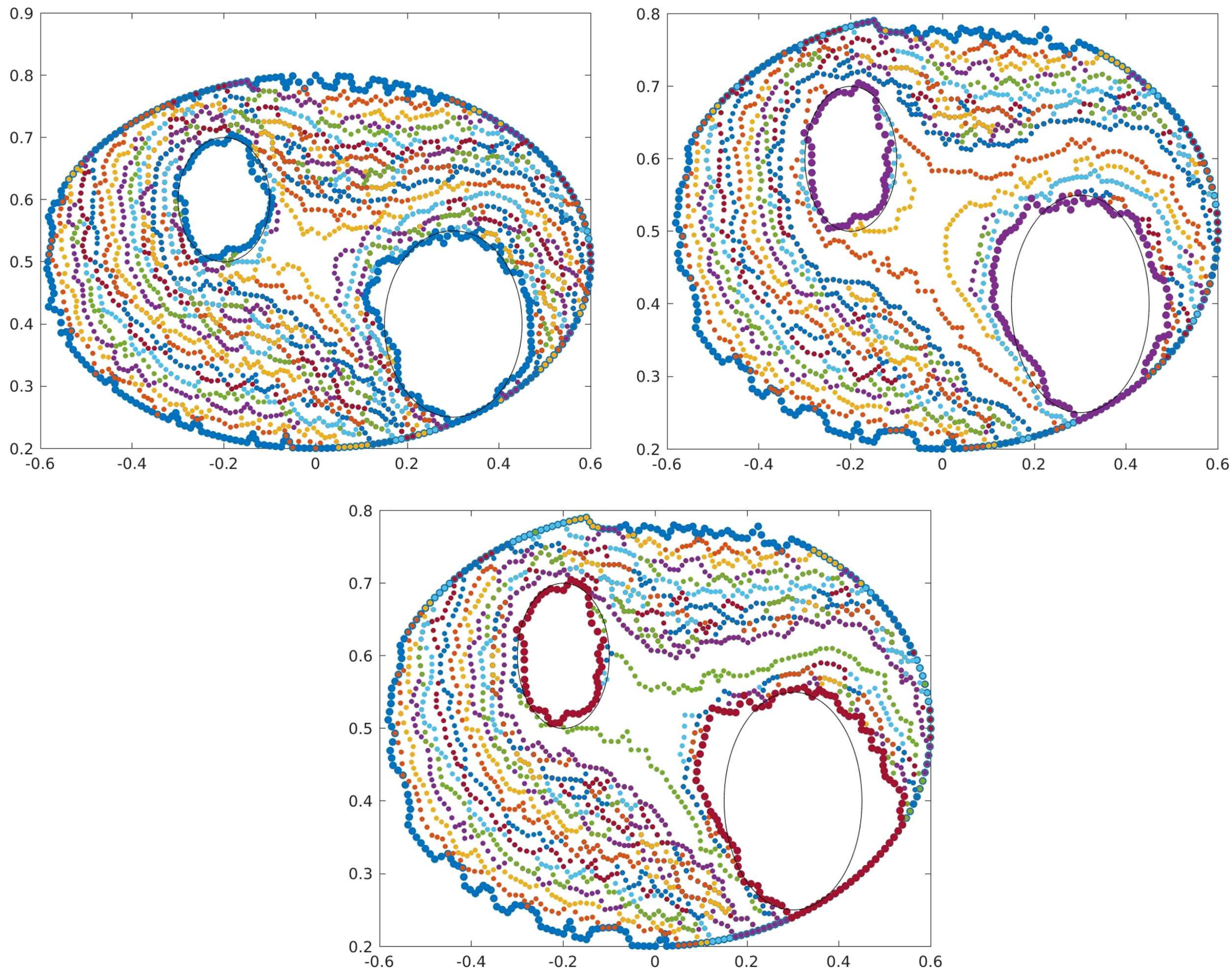

FIGURE 3. Identification of obstacle (2) for $k = 5$. *Top left*: $\sigma = 0.01$. *Top right*: $\sigma = 0.05$. *Bottom*: $\sigma = 0.1$.

limit set is the null-set and smaller otherwise. The initial guess D_0 is an ellipse. In Figure 2, one can see, for obstacle (1), the sequence of computed obstacles D_n until complete stationarity, for wave number $k = 3$ and with our three different amplitudes of noise σ. The retrieved obstacle has to be compared to the true one. The same results are shown in Figure 3 for obstacle (2) and $k = 5$. We mention that those values of k are rather small: $k = 3$ corresponds to a single purely real value of β_n defined by (4.5) while $k = 5$ corresponds to two purely real values of β_n, which means that the number of propagating guided modes (see [8]) is 1 for $k = 3$ and 2 for $k = 5$. We can see on Figure 2 that when the amplitude of noise increases, not only the obstacle is not as well retrieved but the number of iterations in the level set method becomes larger. In the case of two obstacles, we can check on Figure 3 that our level set method manages to separate the two obstacles starting from the connected initial guess D_0, which is well-known for level set methods in general.

Here, in the quasi-reversibility problem (4.10), we have set ε to an arbitrary value of 10^{-3}, which was *a posteriori* a good choice in our numerical experiments. But such value of ε is difficult to guess in general and in order to cope with the drawbacks listed in Remark 3.3 we now use the relaxed mixed formulation of Tikhonov regularization (4.11). Let us focus on the numerical results of formulation (4.11) when the obstacle D is known. In practice,

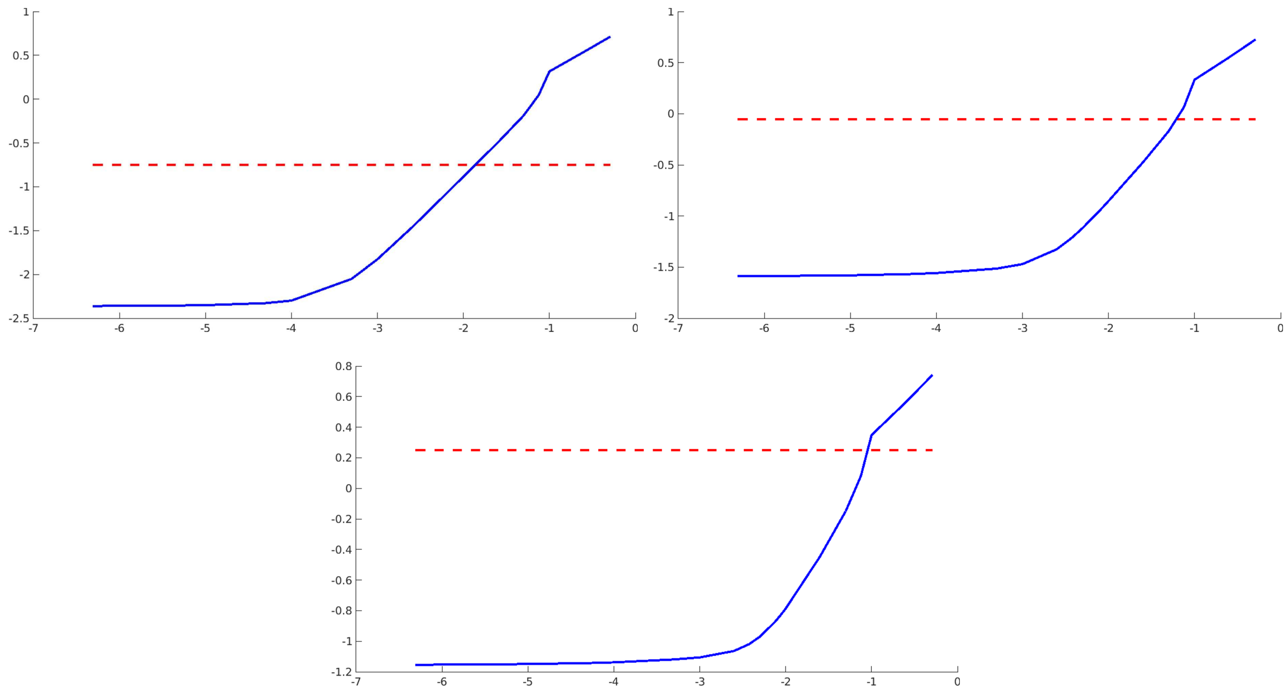

FIGURE 4. Application of Morozov's discrepancy principle. Continuous line: $E^\delta(\varepsilon)$, dashed line: constant Δ ($\log - \log$ scale). *Top left*: $\sigma = 0.01$. *Top right*: $\sigma = 0.05$. *Bottom*: $\sigma = 0.1$.

f^δ and g^δ are artificially computed with the help of (5.2), so that we take

$$\delta_f = \sigma ||f^\delta||_{L^2(\Gamma)}, \quad \delta_g = \sigma ||g^\delta||_{L^2(\Gamma)}, \quad \Delta = \sqrt{\eta^2 \delta_f^2 + \rho^2 \delta_g^2}.$$

We now describe how to choose the constants η and ρ. In view of Proposition 2.8, a natural choice is

$$\eta = \frac{||L^\delta||_{H^1(\Omega)}}{||f^\delta||_{L^2(\Gamma)}}. \tag{5.3}$$

The constant ρ is heuristically defined as follows. We consider many functions g such that

$$g = \frac{\delta_g}{||b_g||} b_g,$$

where b_g is defined as in (5.2), which means that g is 0 up to some random noise of fixed $L^2(\Gamma)$ norm δ_g, and we solve the problems (2.7): for $g \in L^2(\Gamma)$, find the unique $L \in H^1(\Omega)$ with $L|_{\partial D} = 0$ such that for all $\mu \in H^1(\Omega)$ with $\mu|_{\partial D} = 0$,

$$\int_\Omega \nabla L \cdot \nabla \mu \, \mathrm{d}x = \int_\Gamma g \, \mu \, \mathrm{d}s.$$

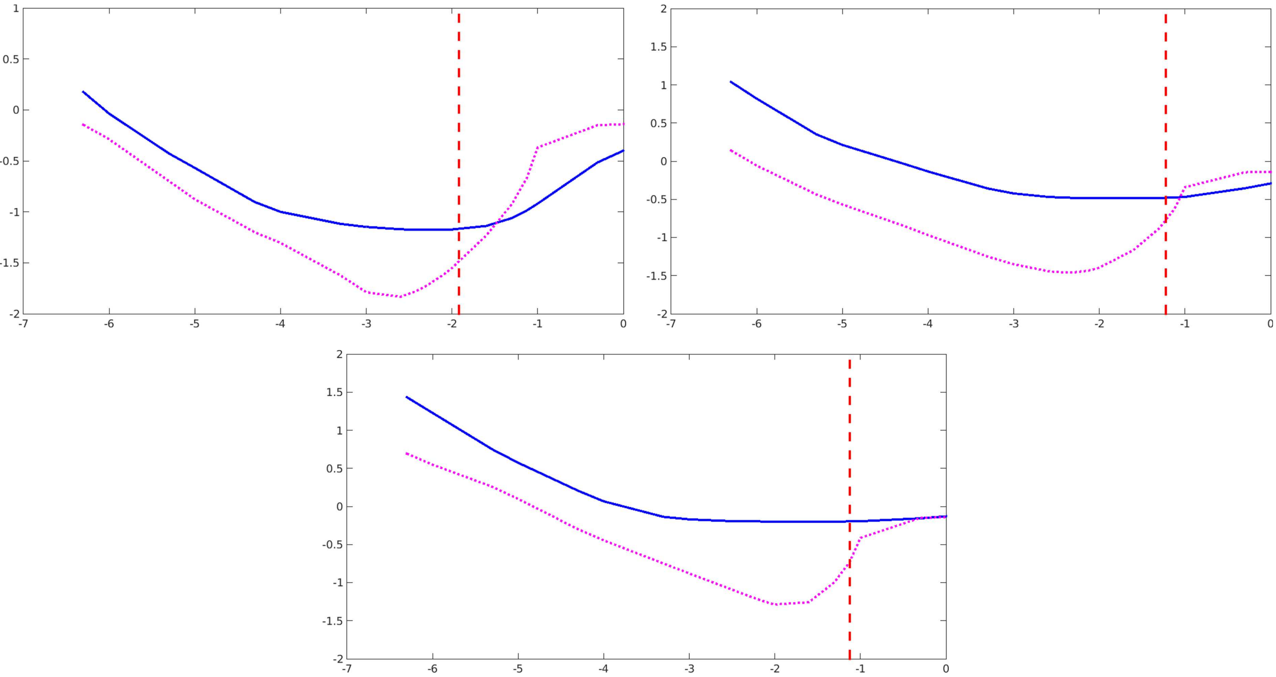

FIGURE 5. Error $||u_\varepsilon^\delta - u||_{H^1(\Omega)}$ between the regularized solution for noisy data (f^δ, g^δ) and the exact solution, as a function of ε ($\log - \log$ scale). Continuous line: u_ε^δ is the solution of (4.10), dashed line: u_ε^δ is the solution of (4.11) (the vertical line represents the Morozov's value of ε). *Top left*: $\sigma = 0.01$. *Top right*: $\sigma = 0.05$. *Bottom*: $\sigma = 0.1$.

Computing the mean value δ_L of the $H^1(\Omega)$ norms of all the obtained L, the constant ρ is given by the ratio $\rho = \delta_L/\delta_g$. The constants η and ρ being determined, the value of ε is now obtained by solving the equation

$$E^\delta(\varepsilon) := \sqrt{\eta^2 ||u_\varepsilon^\delta - f^\delta||^2_{L^2(\Gamma)} + ||\lambda_\varepsilon^\delta||^2_{H^1(\Omega)}} = \sqrt{\eta^2\delta_f^2 + \rho^2\delta_g^2}. \tag{5.4}$$

by a simple dichotomy method. We present some numerical experiments related to the relaxed formulation in the case of obstacle (1). In order to illustrate the fact that equation (5.4) indeed uniquely determines ε in the discretized case, we have plotted on Figure 4 on the one hand the function E^δ of ε and on the other hand the constant Δ, for $\sigma = 0.01$, $\sigma = 0.05$ and $\sigma = 0.1$. In Figure 5, we compare the error between the exact solution u and the regularized solution u_ε^δ in the presence of noisy data (f^δ, g^δ) obtained with the initial formulation (4.10) and with the relaxed formulation (4.11), as a function of ε. We notice that on the whole the relaxed formulation provides a better solution in Ω than the initial formulation. The Figure 5 also illustrates the fact that the Morozov's value of ε is a rather good choice in the relaxed formulation.

We conclude this numerical section by presenting the application of the exterior approach when the regularized solution is obtained with the relaxed formulation (4.11) instead of the initial one (4.10), the data (f^δ, g^δ) being the same as in Figure 2 and ε being determined with the Morozov's procedure. Let us remark that ε, η and ρ depend on the current domain Ω_n and should be determined for each n. This is actually the case for η, but since the computations of ρ and ε are quite heavy, we computed them once and for all at iteration $n = 0$. The identification results are presented in Figure 6 for $\sigma = 0.05$ and $\sigma = 0.1$.

Remark 5.1. In this numerical section, we did not theoretically analyze the discretization of our mixed formulations (4.10) and (4.11). The reader will refer to [2, 4, 7] to find such analysis in slightly different cases.

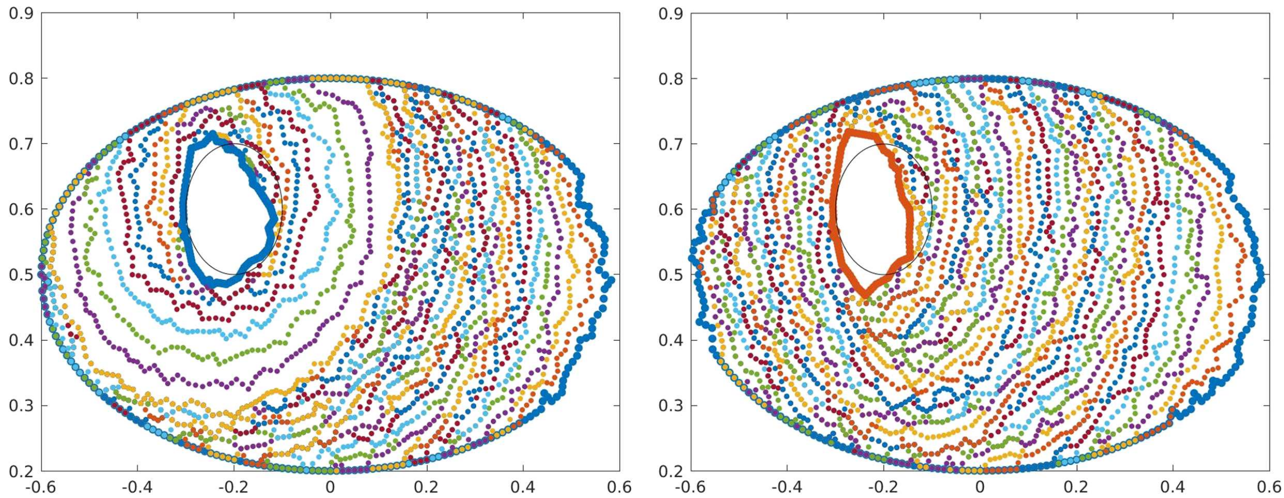

FIGURE 6. Identification of obstacle (1) for $k = 3$ and using the relaxed mixed formulation with Morozov's procedure. *Left*: $\sigma = 0.05$. *Right*: $\sigma = 0.1$.

Implicitly, our mesh size was sufficiently small with respect to ε for the regularization to be compatible with our discretization.

Remark 5.2. The reader could be frustrated by the fact that the identification results of Figure 6 seem not better than the results of Figure 2. This is maybe due to the high sensitivity of the results with respect to η: the choice given by (5.3) is maybe not optimal. Furthermore, as proposed in [13], the mesh size should probably be taken into account in the norms at the discrete level. And lastly, the calibration of ρ should probably benefit from a deeper analysis, in terms of probability, of how some noise on the surface data g propagates into some noise on the volume data L. Some further work on that choices should be continued.

Remark 5.3. We have shown that our method works for small values of the wavenumber k but acknowledge it fails for large values, which is expected in view of the uniqueness Theorem 4.1.

APPENDIX A. THE BACKWARD HEAT EQUATION

We consider Ω as in Section 3 and we consider the domain $Q = \Omega \times (0,T)$ of $\mathbb{R}^{d+1}$, with $T > 0$, $\Sigma = \partial\Omega \times (0,T)$, $S_0 = \Omega \times \{0\}$ and $S_T = \Omega \times \{T\}$. The backward heat equation consists, for some data $h \in H^{-1/2}(S_T)$, to find $u \in L^2(0,T;H^1(\Omega))$ such that

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u = h & \text{on} \quad S_T. \end{cases} \tag{A.1}$$

We now prove that the problem (A.1) is equivalent to a weak formulation of type (2.1).

Lemma A.1. *For $h \in H^{-1/2}(S_T)$, the function $u \in L^2(0,T;H^1(\Omega))$ is the solution to problem (A.1) if and only if $u \in L^2(0,T;H^1_0(\Omega))$ and for all $\mu \in H^1(Q)$ with $\mu|_{\Sigma\cup S_0} = 0$,*

$$-\int_Q u\,\partial_t\mu\,\mathrm{d}x\mathrm{d}t + \int_Q \nabla u\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t = -\langle h, \mu|_{S_T}\rangle_{H^{-1/2}(S_T),\tilde{H}^{1/2}(S_T)}. \tag{A.2}$$

Proof. To begin with, let us assume that $u \in L^2(0,T;H_0^1(\Omega))$ and satisfies the weak formulation (A.2). We have $u=0$ on Σ and by first choosing $\mu=\varphi \in \mathcal{D}(Q)$, we obtain $\partial_t u - \Delta u = 0$ in Q in the distributional sense. Let us now introduce the vector field $\boldsymbol{u} \in \mathbb{R}^{d+1}$ defined in Q by

$$\boldsymbol{u} = (\nabla u, -u) = (\partial_{x_i} u, -u), \quad i = 1, \dots, d. \tag{A.3}$$

We clearly have $\mathrm{div}_{d+1}\boldsymbol{u} = \Delta u - \partial_t u = 0$ in Q, which implies that $\boldsymbol{u} \in H_{\mathrm{div},Q} := \{\boldsymbol{u} \in (L^2(Q))^{d+1},\ \mathrm{div}_{d+1}\boldsymbol{u} \in L^2(Q)\}$. As a consequence we have $\boldsymbol{u}\cdot\nu_{d+1} \in H^{-1/2}(\partial Q)$, where ν_{d+1} is the unit outward normal on ∂Q. In addition, from a classical integration by parts formula, we have for all $\mu \in H^1(Q)$,

$$\int_Q \boldsymbol{u}\cdot\nabla_{d+1}\mu \,\mathrm{d}x = -\int_Q (\mathrm{div}_{d+1}\boldsymbol{u})\,\mu\,\mathrm{d}x + \langle \boldsymbol{u}\cdot\nu_{d+1}, \mu\rangle_{H^{-1/2}(\partial Q), H^{1/2}(\partial Q)},$$

where $\nabla_{d+1} = (\nabla, \partial_t)$ and X is the Lebesgue measure on Q. Now, for $\mu|_{\Sigma\cup S_0} = 0$ and given that $\mathrm{div}_{d+1}\boldsymbol{u} = 0$ in Q, we obtain

$$\int_Q \boldsymbol{u}\cdot\nabla_{d+1}\mu\,\mathrm{d}x = \langle \boldsymbol{u}\cdot\nu_{d+1}, \mu\rangle_{H^{-1/2}(S_T), \tilde{H}^{1/2}(S_T)}. \tag{A.4}$$

Besides, the weak formulation (A.2) is equivalent to

$$\int_Q \boldsymbol{u}\cdot\nabla_{d+1}\mu\,\mathrm{d}x = -\langle h, \mu\rangle_{H^{-1/2}(S_T), \tilde{H}^{1/2}(S_T)}, \quad \forall \mu \in H^1(Q),\ \mu|_{\Sigma\cup S_0} = 0. \tag{A.5}$$

Comparing equations (A.4) and (A.5) we end up with $-\boldsymbol{u}\cdot\nu_{d+1} = h$ in the sense of $H^{-1/2}(S_T)$, which implies that $u = h$ on S_T, that is u solves problem (A.1). We prove similarly that if $u \in L^2(0,T;H^1(\Omega))$ solves the problem (A.1) then it satisfies $u \in L^2(0,T;H_0^1(\Omega))$ and the weak formulation (A.2). □

The weak formulation (A.2) is again an instance of abstract problem (2.1) in the particular case when $V = L^2(0,T;H^1(\Omega))$, $H = L^2(0,T;H^{1/2}(\partial\Omega))$, $M = \{\mu \in H^1(Q),\ \mu|_{\Sigma\cup S_0} = 0\}$, $A : L^2(0,T;H^1(\Omega)) \to L^2(0,T;H^{1/2}(\partial\Omega))$ is the trace operator on Σ, $f = 0$ so that $V_f = V_0 = L^2(0,T;H_0^1(\Omega))$ while for $(u,\mu) \in V \times M$,

$$b(u,\mu) = -\int_Q u\,\partial_t\mu\,\mathrm{d}x\mathrm{d}t + \int_Q \nabla u\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t, \tag{A.6}$$

$$\ell(\mu) = -\langle h, \mu\rangle_{H^{-1/2}(S_T), \tilde{H}^{1/2}(S_T)}. \tag{A.7}$$

Proposition A.2. *For the bilinear form b given by (A.6), the conditions 2.2 and 2.3 are satisfied while the condition 2.1 is not.*

Proof. Let us begin with condition 2.2. For $\mu \in H^1(Q)$ with $\mu|_{\Sigma\cup S_0} = 0$, let us assume that for all $u \in L^2(0,T;H_0^1(\Omega))$,

$$-\int_Q u\,\partial_t\mu\,\mathrm{d}x\mathrm{d}t + \int_Q \nabla u\cdot\nabla\mu\,\mathrm{d}x\mathrm{d}t = 0.$$

We obtain that $\mu \in L^2(0,T;H^1(\Omega))$ satisfies

$$\begin{cases} \partial_t \mu + \Delta \mu = 0 & \text{in} \quad Q \\ \mu = 0 & \text{on} \quad \Sigma \\ \mu = 0 & \text{on} \quad S_0, \end{cases}$$

which from uniqueness in the backward heat equation (see for example [22]) implies that $\mu = 0$ in Q. Similarly, condition 2.3 amounts to prove that if u solves the backward heat equation (A.1) with $h = 0$, then $u = 0$. Lastly, since the backward heat equation (A.1) is ill-posed (see for example [24]), the Brezzi-Nečas-Babuška theorem implies that the inf–sup condition 2.1 is not satisfied. □

The mixed formulation (2.2) of the Tikhonov regularization applies as follows: for $\varepsilon > 0$, find $(u_\varepsilon, \lambda_\varepsilon) \in V_0 \times M$ such that for all $(v, \mu) \in V_0 \times M$,

$$\begin{cases} \varepsilon \displaystyle\int_Q \nabla u_\varepsilon \cdot \nabla v \, \mathrm{d}x\mathrm{d}t - \int_Q v \, \partial_t \lambda_\varepsilon \, \mathrm{d}x\mathrm{d}t + \int_Q \nabla v \cdot \nabla \lambda_\varepsilon \, \mathrm{d}x\mathrm{d}t = 0 \\ -\displaystyle\int_Q u_\varepsilon \, \partial_t \mu \, \mathrm{d}x\mathrm{d}t + \int_Q \nabla u_\varepsilon \cdot \nabla \mu \, \mathrm{d}x\mathrm{d}t - \int_Q \nabla \lambda_\varepsilon \cdot \nabla \mu \, \mathrm{d}x\mathrm{d}t = -\langle h, \mu \rangle_{H^{-1/2}(S_T), \tilde{H}^{1/2}(S_T)}. \end{cases} \tag{A.8}$$

Again, from Theorem 2.4 and since condition 2.3 is satisfied, the problem (A.8) is well-posed for any data $h \in H^{-1/2}(S_T)$ and for h such that problem (A.1) has a (unique) solution u, we have $(u_\varepsilon, \lambda_\varepsilon) \to (u, 0)$ in $L^2(0,T;H^1(\Omega)) \times H^1(Q)$. In [2], a similar mixed formulation as (A.8) is used to regularize the backward heat equation in 1D ($d = 1$). A discretization with a Finite Element Method is analyzed and illustrated with the help of numerical examples.

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