ESAIM: M2AN 52 (2018) 827–867
https://doi.org/10.1051/m2an/2018018

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ERROR ANALYSIS OF A FINITE ELEMENT APPROXIMATION OF A DEGENERATE CAHN-HILLIARD EQUATION

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Abstract. This work considers a Cahn-Hilliard type equation with degenerate mobility and single-well potential of Lennard-Jones type, motivated by increasing interest in diffuse interface modelling of solid tumors. The degeneracy set of the mobility and the singularity set of the potential do not coincide, and the zero of the potential is an unstable equilibrium configuration. This feature introduces a nontrivial difference with respect to the Cahn-Hilliard equation analyzed in the literature. In particular, the singularities of the potential do not compensate the degeneracy of the mobility by constraining the solution to be strictly separated from the degeneracy values. The error analysis of a well posed continuous finite element approximation of the problem, where the positivity of the solution is enforced through a discrete variational inequality, is developed. Whilst in previous works the error analysis of suitable finite element approximations has been studied for second order degenerate and fourth order non degenerate parabolic equations, in this work the *a priori* estimates of the error between the discrete solution and the weak solution to which it converges are obtained for a degenerate fourth order parabolic equation. The theoretical error estimates obtained in the present case state that the norms of the approximation errors, calculated on the support of the solution in the proper functional spaces, are bounded by power laws of the discretization parameters with exponent 1/2, while in the case of the classical Cahn-Hilliard equation with constant mobility the exponent is 1. The estimates are finally succesfully validated by simulation results in one and two space dimensions.

Mathematics Subject Classification. 35Q80, 35K35, 35K65, 65M60, 65K10, 65G99

Received March 2, 2017. Accepted March 9, 2018.

1. Introduction

The Cahn-Hilliard (CH) equation in its original formulation, proposed in [4, 11, 12], describes the dynamics of phase separation in binary alloys. It has been used also as a phenomenological model in several different areas, from the description of multicomponent polymeric systems in [24], and lithium-ion batteries in [32], to the modeling of nanoporosity during dealloying in [16], or inpainting of binary images in [9], and even to the formation of Saturn rings in [29]. Recently, CH type equations have also been employed to describe pattern formation in biological systems (see, for instance, [22, 23]) and diffuse interface tumor growth models, [19, 25]. In particular, a CH equation with degenerate mobility, obtained from the application of mixture theory to solid tumors, is described in [30].

Keywords and phrases: Degenerate Cahn Hilliard equation, single well potential, continuous Galerkin finite element approximation, error analysis.

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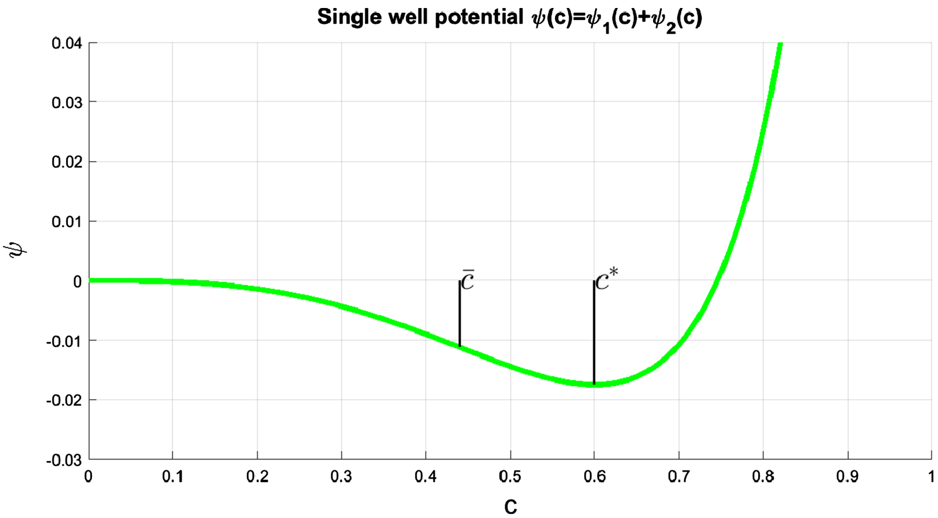

FIGURE 1. Plot of the single well potential (1.4) corresponding to the value $c^* = 0.6$.

In these papers the free energy functional for tumor adhesion is always characterized by a double-well potential, smooth or with singularities of logarithmic type. However, for the description of the evolution of biological cells such a choice seems unphysical (see, *e.g.*, [10]). Indeed, it has been observed that cell-cell interactions are attractive at a moderate cell volume fraction, and repulsive at a high volume fraction, with a zero in the absence of cells and an infinite cell-cell repulsion as the cell concentration tends to 1.

In [14] a Cahn-Hilliard type equation with degenerate mobility and single-well potential of Lennard-Jones type was considered (*cf.* also [13, 15]), as a result of the application of mixture model to solid tumors, which has the form of the following initial and boundary value problem

Problem P: Find $c(\mathbf{x}, t)$ such that

$$\frac{\partial c}{\partial t} = \nabla \cdot (b(c)\nabla(-\gamma\Delta c + \psi'(c))) \quad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.1}$$

$$c(\mathbf{x}, 0) = c_0(\mathbf{x}) \qquad \forall \mathbf{x} \in \Omega, \tag{1.2}$$

$$\nabla c \cdot \boldsymbol{\nu} = b(c)\nabla(-\gamma\Delta c + \psi'(c)) \cdot \boldsymbol{\nu} = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{1.3}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$ is a given bounded domain with a Lipschitz boundary $\partial\Omega$, $\boldsymbol{\nu}$ is the unit normal vector pointing outward to $\partial\Omega$, c is the volume fraction of cancerous cells, c_0 is a given initial concentration and

$$\psi(c) = \psi_1(c) + \psi_2(c), \tag{1.4}$$

where

$$\psi_1(c) = -(1 - c^*)\log(1 - c), \tag{1.5}$$
$$\psi_2(c) = -\frac{c^3}{3} - (1 - c^*)\frac{c^2}{2} - (1 - c^*)c + k.$$

Here c^* is the volume fraction at which the cells would naturally be at mutual equilibrium and $k > 0$. A spinodal decomposition can be triggered if $c < \bar{c}$, where $\psi''(\bar{c}) = 0$. Figure 1 shows a plot of the single well potential (1.4), corresponding to the value $c^* = 0.6$. The derivative of the potential is

$$\psi'(c) = \frac{c^2(c-c^*)}{1-c}. \tag{1.6}$$

Correspondingly, the mobility is given by

$$b(c) = c(1-c)^2. \tag{1.7}$$

Note that ψ_1 is a convex function defined on $[0,1)$ while ψ_2 is concave. Also, the product $b\psi''$ is a continuous function in $[0,1]$.

In [3] the existence of different classes of weak solutions of Problem P and their positivity properties, for the cases of spatial dimension $d=1$ and $d=2,3$ separately, was studied, and a continuous finite element approximation of the problem was formulated, studying its convergencence to the weak solutions. As a consequence of the fact that (1.1) degenerates on the set $\{c=0; c=1\}$, and the singularity is concentrated on the set $\{c=1\}$ only, one cannot exploit the relationship between b and ψ at 0 in order to ensure that $c \geq 0$ at a discrete level. Moreover, the Entropy inequalities obtained in [3], which guarantee the positivity property of the continuous solutions, are not straightforwardly available at the discrete level (see also [8, 18] for details). Therefore, following [6], this condition is imposed as a constraint and a discrete variational inequality of the following form is formulated. Let $\mathcal{T}_h$ be a quasi-uniform conforming decomposition of Ω into $d-$simplices K, $d=1,2,3$, and introduce the following finite element spaces:

$$S^h := \{\chi \in C(\bar{\Omega}) : \chi|_K \in P^1(K)\ \forall K \in \mathcal{T}_h\} \subset H^1(\Omega),$$
$$K^h := \{\chi \in S^h : \chi \geq 0 \text{ in } \Omega\}$$

where $\mathbb{P}_1(K)$ indicates the space of polynomials of total order one on K. Let set $\Delta t = T/N$ for a $N \in \mathbb{N}$ and $t_n = n\Delta t$, $n =, \ldots, N$. For $d=1,2,3$, starting from a datum $c_0 \in H^1(\Omega)$ and $c_h^0 = \pi^h c_0$ (if $d=1$) or $c_h^0 = \hat{P}^h c_0$ (if $d=2,3$), with $0 \leq c_h^0 < 1$, the fully discretized problem reads as

Problem P^h: For $n=1,\ldots,N$, given $c_h^{n-1} \in K^h$, find $(c_h^n, w_h^n) \in K^h \times S^h$ such that for all $(\chi,\phi) \in S^h \times K^h$,

$$\begin{cases} \left(\dfrac{c_h^n - c_h^{n-1}}{\Delta t}, \chi\right)^h + (b(c_h^{n-1})\nabla w_h^n, \nabla\chi) = 0, \\ \gamma(\nabla c_h^n, \nabla(\phi - c_h^n)) + (\psi_1'(c_h^n), \phi - c_h^n)^h \geq (w_h^n - \psi_2'(c_h^{n-1}), \phi - c_h^n)^h, \end{cases} \tag{1.8}$$

where $(\cdot,\cdot)^h$ is the lumped scalar product, defined as

$$(\eta_1,\eta_2)^h = \int_\Omega \pi^h(\eta_1(x)\eta_2(x))\mathrm{d}x \equiv \sum_{j\in J}(1,\chi_j)\eta_1(x_j)\eta_2(x_j), \tag{1.9}$$

for all $\eta_1,\eta_2 \in C(\bar{\Omega})$, where J is the set of nodes of $\mathcal{T}_h$. Note that, due to the use of the lumped product, the boundary of the support of the discrete solution c_h^n cannot propagate more than a distance h at each time step Δt (see *e.g.* [6] for details). The discrete solution is thus able to track compactly supported solutions of (1.1) with a free boundary which moves with a finite speed of velocity $v_{\text{supp}}(t)$ (whose existence is discussed in, *e.g.*, [6]) if one of these two conditions is satisfied:

$$\Delta t = O(h^{1+\epsilon}), \quad \epsilon > 0, \quad \text{or} \tag{1.10}$$

$$\Delta t = Ch, \quad C < \frac{1}{\max_t v_{\text{supp}}(t)}. \tag{1.11}$$

In [3] the existence and uniqueness of the solution of (1.8) was proven, together with the convergence, in the $d = 1$ spatial dimension, for $(h, \Delta t) \to 0$, to a limit point (c, w), with $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\bar{\Omega}_T)$ and $w \in L^2_{\text{loc}}(0 < c < 1)$, $\frac{\partial w}{\partial x} \in L^2_{\text{loc}}(0 < c < 1)$, which satisfies the weak problem

$$\begin{cases} \displaystyle\int_0^T \left\langle \frac{\partial c}{\partial t}, \eta \right\rangle \mathrm{d}t + \int_{0<c<1} \left(b(c) \frac{\partial w}{\partial x}, \frac{\partial \eta}{\partial x} \right) \mathrm{d}t = 0, \quad \forall \eta \in L^2(0,T;H^1(\Omega)), \\ \displaystyle\int_{0<c<1} \gamma \left(\frac{\partial c}{\partial x}, \frac{\partial \theta}{\partial x} \right) \mathrm{d}t + \int_{0<c<1} (\psi'(c), \theta) \mathrm{d}t - \int_{0<c<1} (w, \theta) \mathrm{d}t = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \theta \in L^2(0,T;H^1(\Omega)), \end{cases} \tag{1.12}$$

with $c(\cdot, 0) = c_0(\cdot)$, and with $\text{supp}(\theta) \subset \{0 < c < 1\}$. The following notation has been used in (1.12) for the domain of integration: $\{0 < c < 1\} := \{(x,t) \in \Omega_T : 0 < c(x,t) < 1\}$, with $\Omega_T := \Omega \times (0,T)$. In particular, for ease of notation we defined $\int_{0<c<1} (\cdot,\cdot)\mathrm{d}t := \int_0^T (\cdot,\cdot)_{0<c(\cdot,t)<1} \mathrm{d}t$, with $\{0 < c(\cdot,t) < 1\} := \{x \in \Omega : 0 < c(x,t) < 1\}$ and $(f,g)_\omega = \int_\omega fg\mathrm{d}x$ for $\omega \subset \Omega$.

To the sequence of discrete solutions (c_h^n, w_h^n) of (1.8) let us associate the following piecewise constant-in-time functions

$$\begin{aligned} C_h^+(t) &:= c_h^n, \quad C_h^-(t) := c_h^{n-1}, \\ W_h^+(t) &:= w_h^n, \quad W_h^-(t) := w_h^{n-1}, \end{aligned} \tag{1.13}$$

for $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$. Let us recall that the following convergence properties, for $(h, \Delta t) \to 0$ and for a subsequence of $C_h^\pm$ and W_h^+, are satisfied (see [3]),

$$C_h^\pm \rightharpoonup c \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{1.14}$$

$$C_h^\pm \to c \text{ uniformly on } \bar{\Omega}_T, \tag{1.15}$$

$$W_h^+ \rightharpoonup w, \quad \frac{\partial W_h^+}{\partial x} \rightharpoonup \frac{\partial w}{\partial x} \quad \text{weakly in } L^2_{\text{loc}}(0 < c < 1). \tag{1.16}$$

Moreover, we recall that the discrete solution of (1.8) is unique, whereas the continuous solution of (1.12) is non unique (see [3, 6] for details).

In this paper *a priori* estimates in the appropriate functional spaces are studied for the error between the discrete solution of (1.8) and the solutions of the weak problem (1.12). We remark the fact that, even if the convergence properties (1.14)–(1.16) are valid only up to a subsequence, the *a priori* estimates are valid for the whole sequence of discrete solutions, since the latter is bounded in the proper functional spaces in which the *a priori* estimates are valid. The error analysis is studied in the one dimensional case, since, for the time being, the convergence of the discrete approximation (1.8) to the weak formulation (1.12) is available only in one dimension (see [3]). However, the error estimates obtained in this work could be in principle extended to the general $d = 2, 3$ dimensional cases, provided that the discrete and weak solutions have additional regularity properties, which will be introduce later (see in particular assumptions (3.12) and (3.13)).

The error analysis for degenerate second order parabolic problems is well studied in literature, see *e.g.* [5, 31]. In these cases, the degenerate flux $b(s)\nabla s$, where b is a degenerate mobility and s is the independent variable of the problem, can be rewritten at the continuous level through the Kirchhoff transformation $D(s) = \int_0^s b(\zeta)d\zeta$ as $b(s)\nabla s = \nabla D(s)$. The error associated to the elliptic terms can be than estimated using standard properties of convexity and monotonicity of $D(s)$.

The error analysis for non-degenerate second and fourth order parabolic problems can be studied by standard methods of error analysis, see *e.g.* [21, 26–28]. In particular, in [21] the $L^2(0,T;H^{-1}(\Omega))$ norm of the time increment of the error of the concentration c is estimated in terms of the discretization parameters, and this estimate is sufficient to obtain all other estimates in the error analysis. In this case and for P^1 elements, the norms of the approximation errors are bounded by power laws of the discretization parameters with exponent 1.

In the case of the fourth-order degenerate parabolic problem (1.8) and (1.12), the elliptic terms cannot be handled by a Kirchhoff-like transformation, and in particular the elliptic term in the second equation of (1.12) does not contain a degenerate factor and must be calculated on the set $\{0 < c < 1\}$, *i.e.* on the support of the function c. The fact that the relationship between b and ψ at 0 does not ensure that $c \geq 0$ at a discrete level induces to consider a discrete variational inequality, where $c \geq 0$ is imposed as a constraint, and at the continuous level it imposes to consider test functions θ whith $\text{supp}(\theta) \subset \{0 < c < 1\}$. All these complications make non standard the error analysis for the present problem. In particular, using test functions θ whith $\text{supp}(\theta) \subset \{0 < c < 1\}$ introduces the necessity to calculate the $L^2(0 < c < 1)$ norm of the time increment of the error of c in terms of the discretization parameters.

The main result of this paper states that the norms of the approximation errors for the degenerate CH equation, calculated on the support of the solution in the proper functional spaces and in the hypothesis that $h/\Delta t = O(1)$, which satisfies condition (1.11), are bounded by power laws of the discretization parameters with exponent 1/2. Since this result does not depend on the particular form of ψ, but it is based on the fact that the singularity set of the potential and the degeneracy set of the mobility do not coincide, it can be applied also to the degenerate CH equation with smooth potential.

The paper is organized as follows. In Section 2 some properties of the functional setting used in the error analysis are introduced, together with the main result of this paper regarding the *a priori* error estimates between the discrete solution of (1.8) and the solution of the weak problem (1.12). Section 3 introduces some useful notations and some preliminary lemmas, comprehensive of two crucial lemmas used in the main calculation steps of the proof of the main result. In Section 4 the proof of the main result is shown. In Section 5 the proofs of the main Lemmas introduced in Section 3 are reported. Finally, in Section 6 some numerical experiments in spatial dimensions one and two are presented in order to validate the *a priori* error estimates introduced in Section 2.

2. Functional setting and main result

Before exposing the main result of this paper, we introduce some definitions and fundamental properties of the functional setting introduced in the first part of the paper which will be used in the following calculations.

Let us introduce for any bounded domain ω the "inverse Laplacian" operator $\mathcal{G}_\omega : (H^1(\omega))' \to H^1_0(\omega)$ such that

$$(\nabla \mathcal{G}_\omega v, \nabla \eta)_\omega =< v, \eta >_\omega \quad \forall \eta \in H^1_0(\omega), \tag{2.1}$$

The existence and uniqueness of an element $\mathcal{G}_\omega v \in H^1_0(\omega)$, for any $v \in (H^1(\omega))'$, follows from the Lax-Milgram theorem and the Poincaré inequality.
A norm on $(H^1(\omega))'$ can be defined by

$$||v||_{(H^1(\omega))'} := |\mathcal{G}_\omega v|_{1,\omega} \equiv < v, \mathcal{G} v >^{1/2}_\omega \quad \forall v \in (H^1(\omega))'. \tag{2.2}$$

The following Sobolev interpolation result will be used (see, *e.g.*, [1]). Let $p \in [1, \infty]$, $m \geq 1$, and set

$$r \in \begin{cases} [p, \infty] & \text{if } m - \frac{d}{p} > 0, \\ [p, \infty) & \text{if } m - \frac{d}{p} = 0, \\ [p, -\frac{d}{m-(d/p)}] & \text{if } m - \frac{d}{p} < 0, \end{cases}$$

and $\mu = \frac{d}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$. Then, there is a constant C such that

$$||v||_{0,r} \leq C||v||_{0,p}^{1-\mu}||v||_{m,p}^{\mu} \quad \forall v \in W^{m,p}(\Omega). \tag{2.3}$$

Similarly to (2.1), let us define the operator $\mathcal{G}_\omega^h : (H^1(\omega))' \to S^h \cap H_0^1(\omega)$ such that

$$(\nabla \mathcal{G}_\omega^h v, \nabla \chi)_\omega =< v, \chi >_\omega \quad \forall \chi \in S^h, \text{ with } \mathring{\text{supp}}(\chi) \subset \omega, \tag{2.4}$$

where $\mathring{\text{supp}}(\chi)$ denotes the interior part of $\text{supp}(\chi)$. It is useful to introduce the L^2 projection operator $P^h : L^2(\Omega) \to S^h$ and the operator $\hat{\mathcal{G}}^h : \mathcal{F}^h \to V^h$ as follows

$$(P^h \eta, \chi) = (\eta, \chi) \quad \forall \chi \in S^h, \tag{2.5}$$

$$(\nabla \hat{\mathcal{G}}^h v, \nabla \chi) = (v, \chi)^h \quad \forall \chi \in S^h, \tag{2.6}$$

where $\mathcal{F}^h = \{v \in \bar{C}(\Omega) : (v, 1) = 0\}$ and $V^h = \{v^h \in S^h : (v^h, 1) = 0\}$. The following well-known results will be used (see, *e.g.*, [6]),

$$|\chi|_{1,p} \leq Ch^{-1}||\chi||_{0,p} \quad \forall \chi \in S^h, \ 1 \leq p \leq \infty; \tag{2.7}$$

$$||(I - P^h)\eta||_0 + h|(I - P^h)\eta|_1 \leq Ch^m|\eta|_m \quad \forall \eta \in H^m(\Omega), \ m = 1, 2; \tag{2.8}$$

$$||\chi||_0^2 \leq (\chi, \chi)^h \leq (d + 2)||\chi||_0^2 \quad \forall \chi \in S^h \tag{2.9}$$

$$|(v^h, \chi)^h - (v^h, \chi)| \leq Ch^{1+m}||v^h||_m||\chi||_1 \quad \forall v^h, \chi \in S^h, \ m = 0, 1; \tag{2.10}$$

where d is the spatial dimension. The following bounds will also be used in the following calculations.

Lemma 2.1. *Given $v^h \in S^h$ with $\mathring{\text{supp}}(\chi) \subset \omega$, the following inequalities hold:*

$$||\nabla(\mathcal{G}_\omega^h v^h)||_\omega \leq C||v^h||_\omega; \tag{2.11}$$

$$||v^h||_\omega \leq Ch^{-1}||\nabla(\mathcal{G}_\omega^h v^h)||_\omega. \tag{2.12}$$

Proof. The inequality (2.11) follows directly from the definition (2.4) by choosing $\chi \equiv \mathcal{G}_\omega^h v^h$, using the Cauchy-Schwarz and the Poincaré inequalities. Choosing $\chi \equiv v^h$ in (2.4), using the Cauchy-Schwarz inequality and the inverse inequality (2.7), we get

$$||v^h||_\omega^2 \leq ||\nabla \mathcal{G}_\omega^h v^h||_\omega ||\nabla v^h||_\omega \leq Ch^{-1}||\nabla \mathcal{G}_\omega^h v^h||_\omega ||v^h||_\omega,$$

from which we obtain (2.12). □

Let us moreover introduce the following *discrete Laplacian* operator: given $w \in S^h$, find $\Delta_{h,\omega} w \in S^h \cap H_0^1(\omega)$ such that

$$-(\Delta_{h,\omega} w, \phi)_\omega = (\nabla w, \nabla \phi)_\omega, \quad \forall \phi \in S^h, \text{ with } \mathring{\text{supp}}(\chi) \subset \omega. \tag{2.13}$$

The existence and uniqueness of the element $\Delta_{h,\omega} w$ in (2.13) follows from the Riesz representation theorem and the Poincaré inequality.

Furthermore, C denotes throughout a generic positive constant independent of the unknown variables, the discretization and the regularization parameters, the value of which might change from line to line; $C_1, C_2, \ldots$ indicate generic positive constants whose particular value must be tracked through the calculations; $C(a)$ denotes a constant depending on the non negative parameter a, such that, for $C_1 > 0$, if $a \leq C_1$, there exists a $C_2 > 0$ such that $C(a_1) \leq C_2$.

We also recall the definitions of the sets D_δ^+, $D_\delta^+(t)$ and of the cut-off function $\theta_{4\delta}$, for a $\delta > 0$, introduced in [3]:

$$\begin{aligned}
&D_\delta^+ := \{(x,t) \in \bar{\Omega}_T : \delta < c(x,t) < 1\},\\
&D_\delta^+(t) := \{x \in \bar{\Omega} : \delta < c(x,t) < 1\},\\
&\theta_{4\delta} \in C_0^\infty(D_{2\delta}^+), \quad \theta_{4\delta} \equiv 1 \text{ on } D_{4\delta}^+(t), \quad 0 \leq \theta_{4\delta}(\cdot,t) \leq 1, \quad |\nabla \theta_{4\delta}| \leq C\delta^{-2}.
\end{aligned}$$

On account of the uniform convergence (1.15), for a fixed $\delta > 0$, it follows that there exists a $h(\delta) \in \mathbb{R}^+$ such that, for all $h \leq h(\delta)$,

$$\begin{aligned}
&0 \leq C_h^\pm(x,t) < \min\{2\delta, 1\} \quad \forall (x,t) \notin D_\delta^+,\\
&\frac{1}{2}\delta \leq C_h^\pm(x,t) < 1 \quad \forall (x,t) \in D_\delta^+.
\end{aligned} \tag{2.14}$$

Choosing

$$\phi(\cdot,t) \equiv C_h^+(\cdot,t) \pm \frac{1}{2}\delta \frac{\eta^h(\cdot,t)}{||\eta^h(\cdot,t)||_{L^\infty(\Omega)}} \in K^h, \quad \forall \eta^h \in L^2(0,T;S^h) \text{ with } \operatorname{supp}(\eta^h) \subset D_\delta^+$$

in the second equation of system (1.8) yields, $\forall h < h(\delta)$, that

$$\int_0^T \left[\gamma\left(\frac{\partial C_h^+}{\partial x}, \frac{\partial \eta^h}{\partial x}\right) + (\psi_1'(C_h^+) + \psi_2'(C_h^-), \eta^h)^h\right]\mathrm{d}t = \int_0^T (W_h^+, \eta^h)^h \mathrm{d}t. \tag{2.15}$$

This equality formulation of the inequality in (1.8) will be used in the following calculations.

The main theorem of this paper is now introduced. It will be proved in Section 4.

Theorem 2.2. *Let $d = 1$ and $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^2(D_\delta^+(t)))$ and $w \in L^2(0,T;H^2(D_\delta^+(t)))$, for a given $\delta > 0$ small enough, solutions of the limit system (1.12). Assume moreover that $c_0(\cdot) \in H^1(\Omega) \cap H^4(D_\delta^+(0))$. Under the hypothesis that $\Delta t \sim h$, the following error estimates hold:*

$$||c - C_h^+||_{L^\infty(0,T;H^1(D_\delta^+(t)))} \leq C(h^{1/2} + \Delta t^{1/2}); \tag{2.16}$$

$$||w - W_h^+||_{L^2(0,T;H^1(D_\delta^+(t)))} \leq C(h^{1/2} + \Delta t^{1/2}). \tag{2.17}$$

The required smallness of the parameter δ will be specified in Remarks 4.1 and 4.2.

Before deriving error bounds for the discrete solution of problem P^h, we need to introduce some notation and preliminary Lemmas.

3. Notation and preliminary Lemmas

The analysis will be restricted to the one − dimensional case (see [3], Rem. 3 and the Introduction). We proceed by following the same steps as those introduced in [21], here generalized in a non-standard way and for the first time to the degenerate fourth-order parabolic case.

Let us introduce the projection operator $P_{h,1} : H^1(\Omega) \to S^h$ defined, for $v \in H^1(\Omega)$, by

$$\left(\frac{\partial}{\partial x} P_{h,1} v, \frac{\partial}{\partial x}\chi\right) = \left(\frac{\partial}{\partial x} v, \frac{\partial}{\partial x}\chi\right) \quad \forall \chi \in S^h, \tag{3.1}$$

with $(P_{h,1}v, 1) = (v, 1)$, and the anisotropic projection operator $P^b_{h,1} : H^1(D^+_0(t)) \to S^h$ defined, for $w \in H^1(D^+_0(t))$, $\delta > 0$ and for a.e. $t \in [0, T]$, by

$$\left(b(c)\frac{\partial}{\partial x} P^b_{h,1} w, \frac{\partial}{\partial x}\chi\right)_{D^+_\delta(t)} = \left(b(c)\frac{\partial}{\partial x} w, \frac{\partial}{\partial x}\chi\right)_{D^+_\delta(t)} \quad \forall \chi \in S^h,\ \text{supp}(\chi) \subset\subset D^+_\delta(t), \tag{3.2}$$

with $\text{supp}(P^b_{h,1}w) \subset\subset D^+_\delta(t)$.

The following bounds hold:

$$||v - P_{h,1}(v)|| \le Ch||v - P_{h,1}(v)||_1, \quad \forall v \in H^1(\Omega), \tag{3.3}$$

$$||v - P_{h,1}(v)||_1 \le Ch^l||v||_{l+1} \qquad \forall v \in H^{s+1}(\Omega), \quad l = \min(1, s), \tag{3.4}$$

with analogous bounds, calculated over the set $D^+_\delta(t)$, for the operator P^b_h. For ease of exposition, we introduce the following notation:

$$e^c(\cdot, t) := c(\cdot, t) - C^+_h(\cdot, t),\ e^c_h(\cdot, t) := P_{h,1}c(\cdot, t) - C^+_h(\cdot, t),\ e^c_p(\cdot, t) := c(\cdot, t) - P_{h,1}c(\cdot, t),$$

with $c^0_h = P_h c(\cdot, 0)$,

$$e^w(\cdot, t) := w(\cdot, t) - W^+_h(\cdot, t),\ e^w_h(\cdot, t) := P^b_{h,1}w(\cdot, t) - W^+_h(\cdot, t),\ e^w_p(\cdot, t) := w(\cdot, t) - P^b_{h,1}w(\cdot, t),$$

and

$$\delta_{\Delta t}C^+_h(t) := \frac{c^n_h - c^{n-1}_h}{\Delta t}, \qquad t \in (t_{n-1}, t_n],\ n \ge 1;$$

$$\delta_{\Delta t}c(t) := \frac{c(t) - c(t - \Delta t)}{\Delta t}, \quad t \in (t_{n-1}, t_n],\ n \ge 1;$$

where $C^+_h(t) := c^n_h$, $W^+_h(t) := w^n_h$. We recall here some useful energy estimates for the discrete solution, derived in [2] (see in particular Lem. 2.4 therein).

$$\min_{\frac{\delta}{2} \le C^-_h < 1} b(C^-_h) \int_{D^+_\delta} \left|\frac{\partial W^+_h}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t \le \int_{D^+_\delta} b(C^-_h)\left|\frac{\partial W^+_h}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t \le C, \tag{3.5}$$

$$\int_0^T |\delta_{\Delta t}C^+_h(t)|^2_1 \mathrm{d}t \le C(\Delta t)^{-1}, \tag{3.6}$$

$$\int_0^T \left|\hat{\mathcal{G}}^h(\delta_{\Delta t}C^+_h(t))\right|^2_1 \mathrm{d}t \le C. \tag{3.7}$$

Moreover, we recall that (see Thm. 2 and Lem. 7 in [3])

$$0 \le c, C^\pm_h < 1 \quad \text{in} \quad \bar{\Omega}_T. \tag{3.8}$$

Remark 3.1. By means of the Energy and Entropy inequalities, in Theorems 2 and 3 in [3] it was shown that $c<1$ a.e. in Ω_T. In the sequel (see in particular Lem. 3.2 and bound (4.8)), we need a stronger result, *i.e.* we need that c can not become arbitrarely close to 1 in a critical point in $\bar{\Omega}_T$. Due to the fact that $c\in C^{\frac{1}{2},\frac{1}{8}}_{x,t}(\bar{\Omega}_T)$, we can choose a $\bar{T}$ such that the following *strict separation property* is valid:

$$\text{if } c_0\in[0,1-k], k\in(0,1), \text{ than } \exists\lambda(k)>0|c(t)\leq 1-\lambda(k), \tag{3.9}$$

uniformly for $t\in[0,\bar{T}]$. In the sequel we assume that $T=\bar{T}$. We note that this choice is not restrictive, since it is possible to show that the solution of (1.12) satisfies the strict separation property for a.e. $t\geq 0$. Indeed, since the factor $(1-c)$ in (1.7) has an exponent equal to 2, the weighted Entropy inequalities obtained in [7] for degenerate fourth-order non linear parabolic equations give the strict separation property for a.e. $t\geq 0$ in the $d=1$ case (see in particular Thm. 4.1 and the result (4.2) in [7]). The same result for the $d=2$ case can be obtained studying the regularity properties of $\psi_1(c)$ by similar techniques as those used [17]. Since this result is out of the scope of the present work, its details will be discussed in a forthcoming work.

Next define

$$S^c(\cdot,t):=\delta_{\Delta t}P_{h,1}c(\cdot,t)-\partial_t P_{h,1}c(\cdot,t),\quad t\in(t_{n-1},t_n],\ n\geq 1.$$

We introduce the following lemma.

Lemma 3.2. *For a given $\delta>0$ and if $c_0\in H^4(D^+_\delta(0))$, the solution (c,w) of (1.12) satisfies the properties*

$$c\in L^2(0,T;H^4(D^+_\delta(t))), \tag{3.10}$$

$$w\in L^2(0,T;H^2(D^+_\delta(t))). \tag{3.11}$$

Proof. From the second equation in system (1.12) and elliptic regularity on the set D^+_0 we immediately get $c\in L^2(0,T;H^2(D^+_\delta(t)))$. We choose now $\eta\in H^1(0,T;C^\infty_0(\Omega))$ with $\operatorname{supp}(\eta)\subset\subset D^+_{\delta/2}(t)$, in the first equation in system (1.12), and integrate by parts in time. We get

$$(c(\cdot,T),\eta(\cdot,T))_{D^+_{\delta/2}(T)}-(c(\cdot,0),\eta(\cdot,0))_{D^+_{\delta/2}(0)}-\int_0^T\left(c,\frac{\partial\eta}{\partial t}\right)_{D^+_{\delta/2}(t)}\mathrm{d}t+\int_0^T\left(b(c)\frac{\partial w}{\partial x},\frac{\partial\eta}{\partial x}\right)_{D^+_{\delta/2}(t)}\mathrm{d}t=0,$$

for all $\eta\in H^1(0,T;C^\infty_0(D^+_{\delta/2}(t)))$. Using (1.15), (1.16) and elliptic regularity on the interior of the set $D^+_{\delta/2}(t)$, for each $\delta>0$, we get that $w(t)\in H^2_{\text{loc}}(D^+_{\delta/2}(t))$ for a.e. $t\in(0,T]$, and hence we get (3.11) and (3.10). ☐

Note that, when $\delta>0$ is a finite parameter, if $\delta<\fint c$, then for each $t\in[0,T]$, $D^+_\delta(t)\neq 0$.

We introduce the following assumptions, which are consequences of (1.14)–(1.16), Lem. 3.2, Schauder theory for parabolic equations (see *e.g.* [20]), the definition (2.13) and the additional assumption that $c_0\in H^4(D^+_\delta(0))$:

1.

$$c\in C^0([0,T];H^4(D^+_\delta(t))),\quad \partial_t c\in C^0(D^+_\delta)\cap H^1(0,T;H^1(D^+_\delta(t))); \tag{3.12}$$

2.

$$e^c_h\in C^0([0,T];H^1(D^+_\delta(t))),\quad \Delta_{h,D^+_\delta(t)}e^c_h\in C^0([0,T];L^2(D^+_\delta(t))). \tag{3.13}$$

The following result is valid.

Lemma 3.3. *Let $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))')$ be the solution of the limit system (1.12). The following bounds hold:*

$$||e_p^c(\cdot,t)|| + h||e_p^c(\cdot,t)||_1 \leq Ch \qquad \textit{for a.e. } t \in [0,T], \tag{3.14}$$
$$||e^c(\cdot,t)|| \leq C||e^c(\cdot,t)||_1 \quad \textit{for a.e. } t \in [0,T]. \tag{3.15}$$

Proof. The bound (3.14) is a simple consequence of (3.3) and (3.4). The bound (3.15) is given by the Poincaré inequality. □

By application of (3.3) and (3.4) with $s = 1$, considering the regularity of the solutions of the limit system (1.12), and considering the fact that $b(c) > 0$ on $D_\delta^+(t)$, we get the following result.

Lemma 3.4 (Estimates on $D_\delta^+(t)$)**.** *For each $\delta > 0$,*

$$||e_p^w(\cdot,t)||_{D_\delta^+(t)} + h||e_p^w(\cdot,t)||_{1,D_\delta^+(t)} \leq Ch^2; \quad \textit{for a.e. } t \in [0,T]; \tag{3.16}$$

$$||e_p^c(\cdot,t)||_{D_\delta^+(t)} + h||e_p^c(\cdot,t)||_{1,D_\delta^+(t)} \leq Ch^2; \quad \textit{for a.e. } t \in [0,T]. \tag{3.17}$$

Remark 3.5. In order for (3.16) and (3.17) to be valid as a consequence of the bounds (3.3) and (3.4) restricted on the set $D_\delta^+(t)$, we need that each connected subsection of $D_\delta^+(t)$ is a subset of Ω with a sufficiently regular boundary. This is true in the $d = 1$ case, where each subsection is a segment. In higher dimensions, a result on the regularity of the boundary of $D_\delta^+(t)$ should be obtained.

Moreover, we get the following result.

Lemma 3.6. *Let c be the solution of the system (1.12). The following bounds hold*

$$||S^c||_{L^2(0,T;(H^1(D_\delta^+(t)))')} \leq C\Delta t; \tag{3.18}$$

$$||S^c - \partial_t e_p^c||_{L^2(0,T;(H^1(D_\delta^+(t)))')} \leq C\Delta t + Ch. \tag{3.19}$$

Proof. Let us multiply S^c by $\mathcal{G}_{D_\delta^+(t)} S^c$ and integrate in Ω_T. By expanding in a Taylor series the term $P_{h,1}c(\cdot, t - \Delta t)$ around t and using (2.1), (2.2), the Cauchy-Schwarz and the Poincaré inequalities we get

$$\begin{aligned}
||S^c||^2_{L^2(0,T;(H^1(D_\delta^+(t)))')} &= \int_0^T < S^c, \mathcal{G}_{D_\delta^+(t)} S^c >_{D_\delta^+(t)} \mathrm{d}t \\
&= \int_0^T (\delta_{\Delta t} P_{h,1} c, \mathcal{G}_{D_\delta^+(t)} S^c)_{D_\delta^+(t)} \mathrm{d}t - \int_0^T (\partial_t P_{h,1} c, \mathcal{G}_{D_\delta^+(t)} S^c)_{D_\delta^+(t)} \mathrm{d}t \\
&= -\Delta t \int_0^T (\partial_{tt} P_{h,1} c, \mathcal{G}_{D_\delta^+(t)} S^c)_{D_\delta^+(t)} \mathrm{d}t + O[(\Delta t)^2] \\
&\leq \Delta t ||\partial_{tt} c||_{L^2(D_\delta^+)} ||\nabla \mathcal{G}_{D_\delta^+(t)} S^c||_{L^2(D_\delta^+)} + \Delta t ||\partial_{tt} P_{h,1} c \quad - \partial_{tt} c||_{L^2(D_\delta^+)} ||\nabla \mathcal{G}_{D_\delta^+(t)} S^c||_{L^2(D_\delta^+)} \\
&\quad + O[(\Delta t)^2].
\end{aligned} \tag{3.20}$$

Hence, using (3.3) and (3.12) we get (3.18). Summing to both sides of equation (3.20) the term $-\int_0^T (\partial_t e_p^c, \mathcal{G}_{D_\delta^+(t)} \partial_t e_p^c)_{D_\delta^+(t)}$, and using (3.3) and (3.12), (3.19) follows. □

We introduce now the two main Lemmas of this section, which will be used as fundamental steps in the proof of Theorem 2.2 and which will be proved in Section 5. The first Lemma concerns estimates of the $L^2(0,T;H^{-1}(D_\delta^+(t)))$ norm of the time increment of the error of the concentration c and the $L^2(D_\delta^+)$ norm of the error of the chemical potential w.

Lemma 3.7. *Let $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^2(D_\delta^+(t)))$ and $w \in L^2(0,T;H^2(D_\delta^+(t)))$, for a given $\delta > 0$ small enough, solutions of the limit system (1.12). Under the hypothesis that $\Delta t \sim h$, the following bounds hold:*

$$\int_0^T \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \leq (C + C(\delta^{-4}))h^3 + (C + C(\delta^{-4}))(\Delta t)^{3/2}$$
$$+ (C + C(\delta^{-5})) \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ (C + C(\delta^{-4})) \int_0^T ||e_h^c \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t, \tag{3.21}$$

$$\int_0^T ||e_h^w \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t \leq Ch^2 + C\Delta t + C \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ C \int_0^T ||e_h^c \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t + (C + C(\delta^{-4})) \int_0^T \left|\left| \frac{\partial e_h^c}{\partial x} \theta_{4\delta} \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t. \tag{3.22}$$

The second Lemma concerns estimates of the $L^\infty(0,T;L^2(D_\delta^+(t)))$ norm of the error of the concentration c and the $L^2(D_\delta^+)$ norm of the time increment of the error of the concentration c.

Lemma 3.8. *Let $c \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;(H^1(\Omega))') \cap L^2(0,T;H^2(D_\delta^+(t)))$ and $w \in L^2(0,T;H^2(D_\delta^+(t)))$, for a given $\delta > 0$ small enough, solutions of the limit system (1.12). Under the hypothesis that $\Delta t \sim h$, the following bounds hold:*

$$||e_h^c(\cdot,t)(\theta_{4\delta})||^2_{D_\delta^+(t)} \leq (C + C(\delta)^{-4})h^3 + C(\delta^{-6})\Delta t + C \int_0^T \left(b(c) \frac{\partial e_h^w(\cdot,t)}{\partial x}, \frac{\partial e_h^w(\cdot,t)}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ (C + C(\delta^{-4})) \int_0^T \left|\left| \frac{\partial e_h^c(\cdot,t)}{\partial x} \theta_{4\delta} \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t, \tag{3.23}$$

$$\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t \leq (C + C(\delta^{-5}))h + (C + C(\delta^{-14}))\Delta t$$
$$+ C(\delta^{-8}) \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-14}) \int_0^T \left|\left| \frac{\partial e_h^c}{\partial x} \theta_{4\delta} \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t$$
$$+ \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,\bar{t})}{\partial x} \theta_{4\delta}(\cdot,\bar{t}) \right|\right|^2_{D_\delta^+(\bar{t})} + \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,0)}{\partial x} \theta_{4\delta}(\cdot,0) \right|\right|^2_{D_\delta^+(0)}$$
$$+ \frac{9}{10} \frac{\Delta t}{2} \int_0^T \left|\left| \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta}) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t, \tag{3.24}$$

with a constant $C_4 < 1$ and for a given $\bar{t} \in (T - \Delta t, T]$.

4. Proof of Theorem 2.2

We proceed now with the proof of the main result.

Proof. Rewrite (1.8), using (2.15), in the following way:

$$\begin{cases} \int_0^T \left[(\delta_{\Delta t} C_h^+, \chi) + \left(b(c)\frac{\partial W_h^+}{\partial x}, \frac{\partial \chi}{\partial x}\right)\right]\mathrm{d}t = \int_0^T [(\delta_{\Delta t} C_h^+, \chi) - (\delta_{\Delta t} C_h^+, \chi)^h]\mathrm{d}t \\ \qquad\qquad + \int_0^T \left([b(c) - b(C_h^-)]\frac{\partial W_h^+}{\partial x}, \frac{\partial \chi}{\partial x}\right)\mathrm{d}t, \\ \int_0^T \left[\gamma\left(\frac{\partial C_h^+}{\partial x}, \frac{\partial \phi}{\partial x}\right) + (\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, \phi)\right]\mathrm{d}t = \int_0^T [(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, \phi) \\ \qquad\qquad -(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, \phi)^h]\mathrm{d}t, \end{cases} \tag{4.1}$$

for all $(\chi, \phi) \in L^2(0,T;S^h) \times L^2(0,T;S^h)$, with $\mathrm{supp}(\phi) \subset\subset D_\delta^+$, for each $\delta > 0$, and with $C_h(0) = c_h^0 = \pi_h c(\cdot, 0)$. Moreover, from (1.12) the following system is derived:

$$\begin{cases} \int_0^T (\delta_{\Delta t} P_{h,1} c, \eta)\mathrm{d}t + \int_0^T \left(b(c)\frac{\partial P_{h,1}^b w}{\partial x}, \frac{\partial \eta}{\partial x}\right)_{D_\delta^+(t)}\mathrm{d}t = \int_0^T < S^c - \partial_t e_p^c, \eta > \mathrm{d}t, \\ \int_0^T \left[\gamma\left(\frac{\partial P_{h,1} c}{\partial x}, \frac{\partial \xi}{\partial x}\right)_{D_\delta^+(t)} + (\psi'(c) - P_{h,1}^b w, \xi)_{D_\delta^+(t)}\right]\mathrm{d}t = \int_0^T (e_p^w, \xi)_{D_\delta^+(t)}\mathrm{d}t, \end{cases} \tag{4.2}$$

for all $\eta \in L^2(0,T;S^h)$ and $\xi \in L^2(0,T;S^h)$ with $\mathrm{supp}(\eta) \subset\subset D_\delta^+$, $\mathrm{supp}(\xi) \subset\subset D_\delta^+$, and with $c(\cdot, 0) = c_0(\cdot)$.

Take $\phi = \xi \equiv P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c]$ in the second equation of (4.1) and in the second equation of (4.2), on noting that, for $h \le \bar{h}(\delta)$, $\mathrm{supp}(\phi) \subset\subset D_\delta^+$ (this happens when at least two mesh points are in the set $\{x \in \bar{\Omega} : \delta < c(x,t) < 2\delta\}$), and subtract the former from the latter. Using the equality

$$\frac{\partial}{\partial x}P^h[f] = \frac{\partial}{\partial x}f + \frac{\partial}{\partial x}(P^h - I)f, \text{ for } f \in H^1(D_\delta^+(t)), \tag{4.3}$$

with $f \equiv (\theta_{4\delta})^2 \delta_{\Delta t} e_h^c(t)$, the following identity is obtained

$$\begin{aligned} &\int_0^T (e_h^w, (\theta_{4\delta})^2 \delta_{\Delta t} e_h^c)_{D_\delta^+(t)}\mathrm{d}t \\ &= \int_0^T \gamma\left(\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2\right)_{D_\delta^+(t)}\mathrm{d}t \\ &\quad + \int_0^T \left[\gamma\left(\frac{\partial e_h^c}{\partial x}, \delta_{\Delta t} e_h^c \frac{\partial (\theta_{4\delta})^2}{\partial x}\right)_{D_\delta^+(t)} + (\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-), P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])_{D_\delta^+(t)}\right]\mathrm{d}t \\ &\quad - \int_0^T (e_p^w, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])_{D_\delta^+(t)}\mathrm{d}t + \int_0^T \Big[(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c]) \\ &\quad - (\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])^h\Big]\mathrm{d}t - \int_0^T \gamma\left(\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 \delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)}\mathrm{d}t \end{aligned} \tag{4.4}$$

Furthermore, choose $\chi = \eta \equiv P^h[(\theta_{4\delta})^2 e_h^w]$ in the first equation of (4.1) and in the first equation of (4.2), and subtract the former from the latter. We obtain, using again (4.3) with $f \equiv (\theta_{4\delta})^2 e_h^w(t)$,

$$
\begin{aligned}
&\int_0^T (\delta_{\Delta t} e_h^c, (\theta_{4\delta})^2 e_h^w)_{D_\delta^+(t)} \mathrm{d}t \\
&= -\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t - \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, e_h^w \frac{\partial (\theta_{4\delta})^2}{\partial x} \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T < S^c - \partial_t e_p^c, P^h[(\theta_{4\delta}))^2 e_h^w] >_{D_\delta^+(t)} \mathrm{d}t - \int_0^T \left[(\delta_{\Delta t} C_h^+, P^h[(\theta_{4\delta})^2 e_h^w]) - (\delta_{\Delta t} C_h^+, P^h[(\theta_{4\delta})^2 e_h^w])^h \right] \mathrm{d}t \\
&\quad - \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial e_h^w}{\partial x}(\theta_{4\delta})^2 \right) \mathrm{d}t - \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial (\theta_{4\delta})^2}{\partial x} e_h^w \right) \mathrm{d}t \\
&\quad + \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 e_h^w) \right)_{D_\delta^+(t)} \mathrm{d}t + \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 e_h^w) \right) \mathrm{d}t.
\end{aligned} \tag{4.5}
$$

Combining (4.4) and (4.5) we finally get

$$
\begin{aligned}
&\int_0^T \gamma \left(\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t + \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&= -\int_0^T \gamma \left(\frac{\partial e_h^c}{\partial x}, \delta_{\Delta t} e_h^c \frac{\partial (\theta_{4\delta})^2}{\partial x} \right)_{D_\delta^+(t)} \mathrm{d}t - \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, e_h^w \frac{\partial (\theta_{4\delta})^2}{\partial x} \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T < S^c - \partial_t e_p^c, P^h[(\theta_{4\delta})^2 e_h^w] >_{D_\delta^+(t)} \mathrm{d}t - \int_0^T (\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-), P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T (e_p^w, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])_{D_\delta^+(t)} \mathrm{d}t - \int_0^T \left[(\delta_{\Delta t} C_h^+, P^h[(\theta_{4\delta})^2 e_h^w]) - (\delta_{\Delta t} C_h^+, P^h[(\theta_{4\delta}))^2 e_h^w])^h \right] \mathrm{d}t \\
&\quad - \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial e_h^w}{\partial x}(\theta_{4\delta})^2 \right) \mathrm{d}t - \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial (\theta_{4\delta})^2}{\partial x} e_h^w \right) \mathrm{d}t \\
&\quad - \int_0^T \left[(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c]) - (\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c])^h \right] \mathrm{d}t \\
&\quad + \int_0^T \left[\gamma \left(\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 \delta_{\Delta t} e_h^c) \right)_{D_\delta^+(t)} + \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 e_h^w) \right)_{D_\delta^+(t)} \mathrm{d}t \right. \\
&\quad + \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(I - P^h)((\theta_{4\delta})^2 e_h^w) \right) \mathrm{d}t \\
&= E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9 + E_{10} + E_{11} + E_{12},
\end{aligned} \tag{4.6}
$$

where we indicate the terms on the right hand side of (4.6) with the notation $E_1, \ldots, E_{12}$. In order to proceed, we introduce here two inequalities, which will be frequently used in the sequel. The first one is

$$
||f||_{D_\delta^+(t)}^2 \le C||f\theta_{4\delta}||_{D_\delta^+(t)}^2, \qquad \forall f \in L^2(D_\delta^+(t)), \tag{4.7}
$$

which is given by the fact that

$$||f||^2_{D^+_\delta(t)} = ||f\theta_{4\delta}||^2_{D^+_\delta(t)} + ||f(1-\theta^2_{4\delta})^{1/2}||^2_{D^+_\delta(t)} = ||f\theta_{4\delta}||^2_{D^+_\delta(t)} + ||f(1-\theta^2_{4\delta})^{1/2}||^2_{D^+_\delta(t)\setminus D^+_{2\delta}(t)} + ||f(1-\theta^2_{4\delta})^{1/2}||^2_{D^+_{2\delta}(t)\setminus D^+_{4\delta}(t)} \le ||f\theta_{4\delta}||^2_{D^+_\delta(t)} + C_1||f||^2_{D^+_\delta(t)}$$

and the fact that there exists a $\bar{\delta}$ such that $C_1 < 1$ for each $\delta < \bar{\delta}$.

Remark 4.1. Since the function $1-\theta_{4\delta}$ is equal to one on the set $D^+_\delta(t) \setminus D^+_{2\delta}(t)$ and is less than one on $D^+_{2\delta}(t) \setminus D^+_{4\delta}(t)$, the constant C_1 is less than one if $D^+_\delta(t) \setminus D^+_{2\delta}(t) \subset \mathrm{supp} f$, since than the sum of the factors $||f(1-\theta^2_{4\delta})^{1/2}||^2_{D^+_\delta(t)\setminus D^+_{2\delta}(t)}$ and $||f(1-\theta^2_{4\delta})^{1/2}||^2_{D^+_{2\delta}(t)\setminus D^+_{4\delta}(t)}$ is a fraction of the factor $||f||^2_{D^+_\delta(t)}$. Thus, $\bar{\delta}$ must be chosen in such a way that $D^+_{\bar{\delta}}(t) \setminus D^+_{2\bar{\delta}}(t) \subset \mathrm{supp} f$. Note that, since $f \in L^2(D^+_\delta(t))$ and $f \not\equiv 0$, the value of $\bar{\delta}$ is finite.

The second one is

$$\left|\left|\frac{\partial e^w_h}{\partial x}\right|\right|_{D^+_\delta(t)} \le \frac{1}{(\min_{\delta\le c<1-\lambda(k)} b(c))^{1/2}}\left|\left|(b(c))^{1/2}\frac{\partial e^w_h}{\partial x}\right|\right|_{D^+_\delta(t)} \le C(\delta^{-1/2})\left|\left|(b(c))^{1/2}\frac{\partial e^w_h}{\partial x}\right|\right|_{D^+_\delta(t)}, \tag{4.8}$$

where the constant $\lambda(k) > 0$ has been introduced in (3.9).

We now find upper bounds for the terms $E_1,\dots,E_{12}$ on the right hand side of (4.6), using (3.21)–(3.24), the Cauchy-Schwarz and the Young inequalities and Sobolev inequalities. In particular, the global strategy is to use the Young inequality in order to sum up the bounds on the right hand side of (4.6) isolating the factors $\frac{1}{2}\int_0^T\left(b(c)\frac{\partial e^w_h(\cdot,t)}{\partial x}, \frac{\partial e^w_h(\cdot,t)}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t$, $\gamma\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t$ and $C\int_0^T\left|\left|\frac{\partial e^c_h(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t$ plus terms depending on the discretization parameters (see (4.31)). Note that the first two factors must be multiplied by the coefficient $1/2$, whereas the remaining factors are multiplied by a generic positive constant C. The factors multiplied by the $1/2$ coefficient will be combined with the corresponding factors on the left hand side of (4.6), in order to obtain, after some technicalities, an integral inequality to which Gronwall arguments can be applied (see (4.37)). Finally, we will obtain a bound on $\left|\left|\frac{\partial e^c_h(\cdot,t)}{\partial x}\theta_{4\delta}(\cdot,t)\right|\right|^2_{D^+_\delta(t)}$, from which the bounds in Theorem 2.2 will be derived.

Using the Cauchy-Schwarz and the Young inequalities, the fact that $|\nabla\theta_{4\delta}| \le C\delta^{-2}$ and (4.7), we get

$$|E_1| \le C(\delta^{-4})\int_0^T\left|\left|\frac{\partial e^c_h}{\partial x}\theta_{4\delta}\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + C_1\gamma\int_0^T||\delta_{\Delta t}e^c_h(\theta_{4\delta})||^2_{D^+_\delta(t)}\mathrm{d}t. \tag{4.9}$$

Using (3.24) in (4.9), choosing a constant C_1 in (4.9) such that $C_1\gamma C(\delta^{-8}) = \frac{1}{24}$, where $C(\delta^{-8})$ is the coefficient of the third term on the right hand side of (3.24), and noting that there exists a value $\hat{\delta}$ such that $C_1 < 1$ for each $\delta < \hat{\delta}$, we obtain, at the lowest order,

$$|E_1| \le (C + C(\delta^{-5}))h + (C + C(\delta^{-14}))\Delta t + \frac{1}{24}\int_0^T\left(b(c)\frac{\partial e^w_h}{\partial x}, \frac{\partial e^w_h}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t + C(\delta^{-14})\int_0^T\left|\left|\frac{\partial e^c_h}{\partial x}\theta_{4\delta}\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D^+_\delta(\bar{t})} + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D^+_\delta(0)} + \gamma\frac{9}{10}\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t. \tag{4.10}$$

Remark 4.2. Since $C(\delta^{-8})$ depends polinomially on its argument, and since δ is a small finite parameter (see Rem. 4.1), we have that $C(\delta^{-8})$ is possibly a finite large parameter. Hence, it is sufficient to take $\hat{\delta} \le \bar{\delta}$, where

$\bar{\delta}$ has been introduced in Remark 4.1, so that the factor $C(\hat{\delta}^{-8})$ is large enough to guarantee that C_1 is finite and less than one.

Using the Cauchy-Schwarz and Young inequalities and the fact that $|\nabla\theta_{4\delta}| \le C\delta^{-2}$ we get

$$|E_2| \le \frac{1}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-4})\int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t.$$

Using (3.22) and (3.23), choosing the constant C in the third term on the right hand side of (3.22) and the constant C in the first term in the right hand side of (3.23) such that $C(\delta^{-4})C = \frac{1}{48}$, we get, at the lowest order,

$$|E_2| \le \frac{2}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + (C+C(\delta^{-10}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-4})h^2 + (C+C(\delta^{-10}))\Delta t. \tag{4.11}$$

Using (2.1) and (4.3) with $f \equiv \theta_{4\delta}^2 e_h^w$, we rewrite the term E_3 as

$$\begin{aligned} E_3 &= \int_0^T < S^c - \partial_t e_p^c, P^h[\theta_{4\delta}^2 e_h^w] >_{D_\delta^+(t)} \mathrm{d}t \\ &= \int_0^T \left(\frac{\partial}{\partial x}\mathcal{G}_{D_\delta^+(t)}(S^c - \partial_t e_p^c), \frac{\partial}{\partial x}\theta_{4\delta}^2 e_h^w\right)\mathrm{d}t - \int_0^T \left(\frac{\partial}{\partial x}\mathcal{G}_{D_\delta^+(t)}(S^c - \partial_t e_p^c), \frac{\partial}{\partial x}(I-P^h)[\theta_{4\delta}^2 e_h^w]\right)\mathrm{d}t. \end{aligned}$$

Using the Cauchy-Schwarz inequality, (2.2), the fact that $|\nabla\theta_{4\delta}| \le C\delta^{-2}$, the stability of the P^h projector under the H^1 seminorm, *i.e.* $|v - P^h(v)|_{1,D_\delta^+(t)} \le C|v - P_{h,1}v|_{1,D_\delta^+(t)}$ for $v \in H^1(D_\delta^+(t))$ and (3.4) we obtain

$$E_3 \le C||S^c - \partial_t e_p^c||_{L^2(0,T;(H^1(D_\delta^+(t)))')}\left[\left(\int_0^T \left|\left|\frac{\partial e_h^w}{\partial x}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} + C(\delta^{-2})\left(\int_0^T ||e_h^w\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\right].$$

Using (3.19) and (4.8) we get

$$E_3 \le C(h^2+\Delta t)\left[C(\delta^{-1/2})\left(\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} + C(\delta^{-2})\left(\int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\right].$$

Using the Young inequality and (3.22) and (3.23), choosing the constant C in the third term on the right hand side of (3.22) and the constant C in the first term in the right hand side of (3.23) such that $C(\delta^{-4})C = \frac{1}{48}$, and keeping only the lowest order terms, we get

$$|E_3| \le \frac{2}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + (C+C(\delta^{-10}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-4})h^2 + (C+C(\delta^{-10}))\Delta t. \tag{4.12}$$

Using the following equality

$$P^h[f] = f + (P^h - I)f, \quad \text{for } f \in L^2(D_\delta^+(t)), \tag{4.13}$$

we can rewrite the term E_4 as

$$E_4 = \int_0^T (\theta_{4\delta}(\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-)), (\theta_{4\delta})\delta_{\Delta t}e_h^c)_{D_\delta^+(t)}\mathrm{d}t$$

$$
\begin{aligned}
&-\int_0^T (\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-),(I-P^h)[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])_{D_\delta^+(t)}\mathrm{d}t\\
&=\int_0^T (P^h[\theta_{4\delta}(\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-))],P^h[(\theta_{4\delta})\delta_{\Delta t}e_h^c])_{D_\delta^+(t)}\mathrm{d}t\\
&+\int_0^T ((I-P^h)[\theta_{4\delta}(\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-))],(\theta_{4\delta})\delta_{\Delta t}e_h^c)_{D_\delta^+(t)}\mathrm{d}t\\
&-\int_0^T (\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-),(I-P^h)[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])_{D_\delta^+(t)}\mathrm{d}t.
\end{aligned}
$$

Using (2.4), the Cauchy-Schwarz and Young inequalities, (2.8) on the set D_δ^+ with $m=1$ and (2.12), we get, keeping only the lowest order terms,

$$
\begin{aligned}
|E_4|\le&\left(\int_0^T\left|\left|\frac{\partial}{\partial x}(P^h[\theta_{4\delta}(\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)])\right|\right|_{D_\delta^+(t)}^2\mathrm{d}t\right)^{1/2}\left(\int_0^T\left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c)\right|\right|_{D_\delta^+(t)}^2\mathrm{d}t\right)^{1/2}\\
&+(C+C(\delta^{-4}))\int_0^T||\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)||^2_{D_\delta^+(t)}\mathrm{d}t+C\int_0^T\left|\left|\frac{\partial}{\partial x}(\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\\
&+Ch^2\int_0^T||\theta_{4\delta}\delta_{\Delta t}e_h^c||^2_{D_\delta^+(t)}\mathrm{d}t+Ch^2\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\\
\le&(C+C(\delta^{-4}))\int_0^T||\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)||^2_{D_\delta^+(t)}\mathrm{d}t+C\int_0^T\left|\left|\frac{\partial}{\partial x}(\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\\
&+C\int_0^T\left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t. \qquad (4.14)
\end{aligned}
$$

In order to bound the first and the second terms in the last line of (4.14), we write

$$
\psi'(c)-\psi_1'(C_h^+)-\psi_2'(C_h^-)=\psi_1'(c)-\psi_1'(C_h^+)+\psi_2'(c)-\psi_2'(C_h^+)+\psi_2'(C_h^+)-\psi_2'(C_h^-),
$$

and use the inequality $||a+b||^2\le 2||a||^2+2||b||^2$. Noting that $\psi_1'(\cdot)\in C^1([0,1))$, $\psi_2'(\cdot)\in C^1([0,1])$, using (3.8) and (4.7), we obtain

$$
\begin{aligned}
|E_4|\le&(C+C(\delta^{-4}))\int_0^T\left(||e^c\theta_{4\delta}||^2_{D_\delta^+(t)}+(\Delta t)^2||\delta_{\Delta t}C_h^+||^2_{D_\delta^+(t)}\right)\mathrm{d}t\\
&+C\int_0^T\left(\left|\left|\frac{\partial e^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}+(\Delta t)^2\left|\left|\frac{\partial}{\partial x}\delta_{\Delta t}C_h^+\right|\right|^2_{D_\delta^+(t)}\right)\mathrm{d}t+C\int_0^T\left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t. \qquad (4.15)
\end{aligned}
$$

Now, from (3.6) and from (2.6), (2.10) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_0^T ||\delta_{\Delta t}C_h^+||^2\mathrm{d}t &\le \int_0^T (\delta_{\Delta t}C_h^+,\delta_{\Delta t}C_h^+)^h\mathrm{d}t + \int_0^T |(\delta_{\Delta t}C_h^+,\delta_{\Delta t}C_h^+)^h - (\delta_{\Delta t}C_h^+,\delta_{\Delta t}C_h^+)|\mathrm{d}t\\ &\le \int_0^T \left(\frac{\partial}{\partial x}\hat{\mathcal{G}}^h(\delta_{\Delta t}C_h^+), \frac{\partial}{\partial x}\delta_{\Delta t}C_h^+\right)\mathrm{d}t + Ch^2\int_0^T ||\delta_{\Delta t}C_h^+||_1^2\mathrm{d}t\\ &\le \left(\int_0^T \left|\hat{\mathcal{G}}^h(\delta_{\Delta t}C_h^+)\right|_1^2\mathrm{d}t\right)^{1/2}\left(\int_0^T |\delta_{\Delta t}C_h^+|_1^2\mathrm{d}t\right)^{1/2}\\ &\quad + Ch^2\int_0^T ||\delta_{\Delta t}C_h^+||^2\mathrm{d}t + Ch^2(\Delta t)^{-1}. \end{aligned}\tag{4.16}$$

Using the hypothesis that $\Delta t \sim h$ and from (3.6) and (3.7) we get, at the lowest order,

$$\int_0^T ||\delta_{\Delta t}C_h^+||^2\mathrm{d}t \le C(\Delta t)^{-1/2}.\tag{4.17}$$

Remark 4.3. Note that (4.17) is more generally valid if we make in (4.16) the assumption that $\Delta t = Ch^n$, with $n \le 4$, which, together with (1.10) and (1.11), becomes $\Delta t = Ch^m$, with $1 \le m \le 4$. Since the condition $\Delta t \sim h$ is the necessary assumption for the validity of Theorem 2.2, we use it in the calculations.

Using the fact that $e^c = e_p^c + e_h^c$ and (3.17), using moreover (3.6) and (4.17) in (4.15) we get, at the lowest order,

$$\begin{aligned}|E_4| \le Ch^2 + C\Delta t + (C + C(\delta^{-4}))\int_0^T ||e_h^c\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t + C\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\\ + C\int_0^T \left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t.\end{aligned}\tag{4.18}$$

Using finally (3.21) and (3.23) in (4.18), choosing a constant C in the last term in (4.18) such that $(C + C(\delta^{-5}))C = \frac{1}{48}$, where $C + C(\delta^{-5})$ is the coefficient of the third term on the right hand side of (3.21), choosing moreover a constant C in the first term on the right hand side of (3.23) such that $(C + C(\delta^{-4}))C = \frac{1}{48}$, we get, at the lowest order,

$$|E_4| \le Ch^2 + (C + C(\delta^{-10}))\Delta t + \frac{1}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t + (C + C(\delta^{-8}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t.\tag{4.19}$$

Using (3.16), (4.13), (2.8) with $m = 1$, the Cauchy-Schwarz and Young inequalities and (2.12) we get, at the lowest order,

$$|E_5| \le C\left(\int_0^T ||e_p^w||^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\left(\int_0^T ||\theta_{4\delta}\delta_{\Delta t}e_h^c||^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} \le Ch^2 + C\int_0^T \left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t.\tag{4.20}$$

Using now (3.21) and (3.23), choosing a constant C in the last term in (4.20) such that $(C + C(\delta^{-5}))C = \frac{1}{48}$, choosing moreover a constant C in the first term on the right hand side of (3.23) such that $(C + C(\delta^{-4}))C = \frac{1}{48}$, we get, at the lowest order,

$$|E_5| \le Ch^2 + (C + C(\delta^{-10}))\Delta t + \frac{1}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + (C + C(\delta^{-8}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t. \tag{4.21}$$

Using (2.10) with $m = 1$, (3.6), (4.17), (4.3), (4.13), (2.8) with $m = 1$, (4.8), the Cauchy-Schwarz and Young inequalities and the hypothesis that $\Delta t \sim h$ we get, at the lowest order,

$$\begin{aligned} |E_6| &\le Ch^2\left(\int_0^T ||\delta_{\Delta t} C_h^+||^2_{1,D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\left(\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||^2_{1,D_\delta^+(t)}\mathrm{d}t\right)^{1/2} \\ &\le Ch^2(\Delta t)^{-1/2}\left[C(\delta^{-1/2})\left(\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} + (C + C(\delta^{-2}))\left(\int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\right] \\ &\le (C + C(\delta^{-4}))h^3 + \frac{1}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t + C\int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t. \end{aligned} \tag{4.22}$$

Using (3.22) and (3.23), choosing an appropriate value of the constant C in the last term in (4.22) and an appropriate value of the constant C in the first term on the right hand side of (3.23), and keeping only the lowest order terms, we get

$$|E_6| \le Ch^2 + (C + C(\delta^{-6}))\Delta t + \frac{2}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + (C + C(\delta^{-6}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t. \tag{4.23}$$

Using the Cauchy-Schwarz inequality, (3.5), (4.8), (4.7), (2.3) with $r = \infty$, $d = 1$, $m = 1$, $p = 2$, the Lipschtitz continuity of $b(\cdot)$, (3.6), (4.17), the hypothesis that $\Delta t \sim h$ and the Young inequality, we get

$$\begin{aligned} |E_7| &\le C(\delta^{-1})\left(\int_0^T ||b(c) - b(C_h^+)||^2_{\infty,D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} \\ &\quad + C(\delta^{-1})\left(\int_0^T ||b(C_h^+) - b(C_h^-)||^2_{\infty,D_\delta^+(t)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2} \\ &\le C(\delta^{-2})\int_0^T ||e_h^c\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t + C(\delta^{-2})\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t \\ &\quad + C(\delta^{-2})\Delta t + \frac{1}{48}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t. \end{aligned} \tag{4.24}$$

Using (3.23) in (4.24), choosing a constant C in the first term on the right hand side of (3.23) such that $C(\delta^{-2})C = \frac{1}{48}$, we get, at the lowest order,

$$|E_7| \le (C + C(\delta^{-6}))h^3 + C(\delta^{-2})\Delta t + (C + C(\delta^{-6}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + \frac{1}{24}\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t. \tag{4.25}$$

Similarly to (4.25), we get

$$\begin{aligned}|E_8| \leq C(\delta^{-5/2})\Big(\int_0^T ||b(c)-b(C_h^+)||^2_{\infty,D_\delta^+(t)}\mathrm{d}t\Big)^{1/2}\Big(\int_0^T ||e_h^w\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t\Big)^{1/2}\\ + C(\delta^{-5/2})\Big(\int_0^T ||b(C_h^+)-b(C_h^-)||^2_{\infty,D_\delta^+(t)}\mathrm{d}t\Big)^{1/2}\Big(\int_0^T ||e_h^w\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t\Big)^{1/2}\\ \leq C(\delta^{-5})\int_0^T ||e_h^c\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t + C(\delta^{-5})\int_0^T \Big|\Big|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t + C(\delta^{-5})\Delta t + C\int_0^T ||e_h^w\theta_{4\delta}||^2_{D_\delta^+(t)}\mathrm{d}t.\end{aligned} \tag{4.26}$$

Using (3.22) and (3.23), choosing an appropriate value of the constant C in the last term in (4.26) and an appropriate value of the constant C in the first term on the right hand side of (3.23), and keeping only the lowest order terms, we get

$$|E_8| \leq Ch^2 + (C + C(\delta^{-11}))\Delta t + (C + C(\delta^{-9}))\int_0^T \Big|\Big|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t + \frac{1}{24}\int_0^T \Big(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\Big)_{D_\delta^+(t)}\mathrm{d}t. \tag{4.27}$$

Noting that $\psi_1'(\cdot) \in C^1([0,1))$, $\psi_2'(\cdot) \in C^1([0,1])$, that $C_h^\pm \in L^\infty(0,T;H^1(\Omega))$, that $W_h^+ \in L^2(0,T;H^1(D_\delta^+(t)))$, using (2.10) with $m = 1$, (2.12), (4.3), the Cauchy-Schwarz and Young inequalities and the hypothesis that $\Delta t \sim h$ we get, at the lowest order,

$$\begin{aligned}|E_9| \leq Ch^2\Big(\int_0^T\Big|\Big|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t\Big)^{1/2} + C(\delta^{-2})h\Big(\int_0^T\Big|\Big|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t\Big)^{1/2}\\ \leq C(\delta^{-4})h^2 + \gamma\frac{1}{20}\frac{\Delta t}{2}\int_0^T\Big|\Big|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t + C\int_0^T\Big|\Big|\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t.\end{aligned} \tag{4.28}$$

Remark 4.4. Note that (4.28), together with (1.10) and (1.11), are more generally valid if we make the assumption $\Delta t = Ch^n$, with $1 \leq m \leq 2$.

Using (3.21), (3.23) in (4.28), choosing a constant C in the last term on the right hand side of (4.28) such that $(C + C(\delta^{-5}))C = \frac{1}{48}$ and choosing a constant C in the first term on the right hand side of (3.23) such that $(C + C(\delta^{-4}))C = \frac{1}{48}$, we get, at the lowest order,

$$\begin{aligned}|E_9| \leq C(\delta^{-4})h^2 + C(\delta^{-10})\Delta t + \frac{1}{24}\int_0^T\Big(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\Big)_{D_\delta^+(t)}\mathrm{d}t\\ + (C + C(\delta^{-8}))\int_0^T\Big|\Big|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t + \gamma\frac{1}{20}\frac{\Delta t}{2}\int_0^T\Big|\Big|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\Big|\Big|^2_{D_\delta^+(t)}\mathrm{d}t.\end{aligned} \tag{4.29}$$

The term E_{10} can be bounded using a generalized version of identity (2.13) with a $\phi \equiv (I - P^h)[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c] \in H_0^1(D_\delta^+(t))$, using moreover the Cauchy-Schwarz and Young inequalities and (2.8) with $m = 1$, obtaining, keeping only the lowest order terms,

$$
\begin{aligned}
|E_{10}| \le Ch\int_0^T ||\Delta_{D^+_\delta(t)} e^c_h||^2_{D^+_\delta(t)}\mathrm{d}t + C(\delta^{-4})h\int_0^T ||\delta_{\Delta t} e^c_h(\theta_{4\delta})||^2_{D^+_\delta(t)}\mathrm{d}t \\
+\gamma\frac{1}{20}\frac{h}{2}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h)(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t \le Ch + \gamma\frac{1}{20}\frac{h}{2}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h)(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t. \qquad (4.30)
\end{aligned}
$$

The terms E_{11} and E_{12} can be bounded easily using (2.8) with $m = 1$ and the hypothesis that $\Delta t \sim h$.

Finally, using (4.10)–(4.30) in (4.6) and the identity $(a,(a-b)c^2) = \frac{1}{2}a^2c^2 - \frac{1}{2}b^2c^2 + \frac{1}{2}(a-b)^2c^2$ we get, writing only the lowest order terms and omitting to write explicitly the dependence of some constants on the finite parameter δ,

$$
\begin{aligned}
&\gamma\frac{1}{2\Delta t}\int_0^T\left(\frac{\partial e^c_h(\cdot,t)}{\partial x},\frac{\partial e^c_h(\cdot,t)}{\partial x}(\theta_{4\delta})^2\right)_{D^+_\delta(t)}\mathrm{d}t - \gamma\frac{1}{2\Delta t}\int_0^T\left(\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x},\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta})^2\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad+\gamma\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + \int_0^T\left(b(c)\frac{\partial e^w_h(\cdot,t)}{\partial x},\frac{\partial e^w_h(\cdot,t)}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\le Ch + C\Delta t + C\int_0^T\left|\left|\frac{\partial e^c_h(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + \frac{1}{2}\int_0^T\left(b(c)\frac{\partial e^w_h(\cdot,t)}{\partial x},\frac{\partial e^w_h(\cdot,t)}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad+\gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,\bar t)}{\partial x}\theta_{4\delta}(\cdot,\bar t)\right|\right|^2_{D^+_\delta(\bar t)} + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D^+_\delta(0)} + \gamma\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t. \qquad (4.31)
\end{aligned}
$$

In order to treat the first two terms on the left hand side of (4.31), we write $\theta_{4\delta}(\cdot,t) = \theta_{4\delta}(\cdot,t-\Delta t) - \Delta t\frac{\partial}{\partial t}\theta_{4\delta}(\cdot,t)|_{t=\bar t}$, where $\bar t \in (t-\Delta t, t)$, in the second term in the left hand side of (4.31), and obtain, at the lowest order,

$$
\begin{aligned}
&\gamma\frac{1}{2\Delta t}\int_0^T\left(\frac{\partial e^c_h(\cdot,t)}{\partial x},\frac{\partial e^c_h(\cdot,t)}{\partial x}(\theta_{4\delta})^2\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad-\gamma\frac{1}{2\Delta t}\int_0^T\left(\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x},\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta}(\cdot,t-\Delta t))^2\right)_{D^+_\delta(t)-D^+_\delta(t-\Delta t)}\mathrm{d}t \\
&\quad-\gamma\frac{1}{2\Delta t}\int_0^T\left(\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x},\frac{\partial e^c_h(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta}(\cdot,t-\Delta t))^2\right)_{D^+_\delta(t-\Delta t)}\mathrm{d}t \\
&\quad+\gamma\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + \int_0^T\left(b(c)\frac{\partial e^w_h(\cdot,t)}{\partial x},\frac{\partial e^w_h(\cdot,t)}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\le Ch + C\Delta t + C\int_0^T\left|\left|\frac{\partial e^c_h(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t + \frac{1}{2}\int_0^T\left(b(c)\frac{\partial e^w_h(\cdot,t)}{\partial x},\frac{\partial e^w_h(\cdot,t)}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad+\gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,\bar t)}{\partial x}\theta_{4\delta}(\cdot,\bar t)\right|\right|^2_{D^+_\delta(\bar t)} + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e^c_h(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D^+_\delta(0)} + \gamma\frac{\Delta t}{2}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e^c_h(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t. \qquad (4.32)
\end{aligned}
$$

Note that the integral on $D^+_\delta(t) - D^+_\delta(t-\Delta t)$ in (4.32), due to the continuity of the integrands in time (see (3.13)) and to the fact that the support of c changes in time by a finite value [6], is proportional to Δt multiplied by finite terms on $\partial D_\delta^+(t)$, and can be bounded by a term like the third on the right hand side of (4.32). Changing variables as $(t-\Delta t) \to t$ in the third term in the left hand side of (4.32) we rewrite the sum of the first and the third terms in the left hand side of (4.32) as

$$\gamma \frac{1}{2\Delta t}\int_{T-\Delta t}^{T}\left(\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta})^2\right)_{D_\delta^+(t)} \mathrm{d}t - \gamma\frac{1}{2\Delta t}\int_{-\Delta t}^{0}\left(\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta}(\cdot,t))^2\right)_{D_\delta^+(t)} \mathrm{d}t. \quad (4.33)$$

Noting (3.13), we can use in (4.33) the mean value theorem for integrals and obtain that there exists a $\hat{t} \in (T-\Delta t, T]$ such that (4.33) can be rewritten as

$$\gamma\frac{1}{2}\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D_\delta^+(\bar{t})} - \gamma\frac{1}{2}\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)}. \quad (4.34)$$

Using (4.34) in (4.32), we obtain that there exists a $\hat{t} \in (T-\Delta t, T]$ such that

$$\begin{aligned}
&\gamma\frac{1}{2}\left|\left|\frac{\partial e_h^c(\cdot,\hat{t})}{\partial x}\theta_{4\delta}(\cdot,\hat{t})\right|\right|^2_{D_\delta^+(\hat{t})} + \frac{1}{2}\int_0^T\left(b(c)\frac{\partial e_h^w(\cdot,t)}{\partial x}, \frac{\partial e_h^w(\cdot,t)}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t \\
&\le \gamma\frac{1}{2}\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)} + Ch + C\Delta t + C\int_0^{\hat{t}}\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t \\
&\quad + C\int_{\hat{t}}^T\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} dt + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D_\delta^+(\bar{t})} + \gamma\frac{C_4}{2}\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)}. \quad (4.35)
\end{aligned}$$

Rewrite the sixth term on the right hand side of (4.35), using (4.3) and (3.12), as

$$\begin{aligned}
\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|_{D_\delta^+(\bar{t})} &\le \left|\left|\frac{\partial}{\partial x}(P_h(c(\cdot,\bar{t}) - c(\cdot,\hat{t})))\theta_{4\delta}(\cdot,\bar{t})\right|\right|_{D_\delta^+(\bar{t})} + \left|\left|\frac{\partial}{\partial x}(P_h(c(\cdot,\hat{t})) - C_h^+(\cdot,\hat{t}))\theta_{4\delta}(\cdot,\bar{t})\right|\right|_{D_\delta^+(\bar{t})} \\
&\le C\Delta t\left|\left|\frac{\partial c(\cdot,t)}{\partial t}\right|\right|_{L^\infty(0,T;H^1(D_\delta^+(t)))} + \left|\left|\frac{\partial e_h^c(\cdot,\hat{t})}{\partial x}\theta_{4\delta}(\cdot,\hat{t})\right|\right|_{D_\delta^+(\hat{t})} + C\Delta t. \quad (4.36)
\end{aligned}$$

Using (4.36) in (4.35), noting that

$$\int_{\hat{t}}^T\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} dt \le C\Delta t,$$

noting moreover that $e_h^c(\cdot,0) \equiv 0$ and that $C_4 < 1$, we get

$$\left|\left|\frac{\partial e_h^c(\cdot,\hat{t})}{\partial x}\theta_{4\delta}(\cdot,\hat{t})\right|\right|^2_{D_\delta^+(\hat{t})} \le Ch + C\Delta t + C\int_0^{\hat{t}}\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t. \quad (4.37)$$

Choosing $T = t_n$, $n = 1,\ldots,N$, in (4.37) and using a Gronwall inequality argument we derive that there exists a set $\hat{t}_n \in ((n-1)\Delta t, n\Delta t]$, $n = 1,\ldots,N$, such that

$$\left|\left|\frac{\partial e_h^c(\cdot,\hat{t}_n)}{\partial x}\right|\right|^2_{D_\delta^+(\hat{t}_n)} \le C\left|\left|\frac{\partial e_h^c(\cdot,\hat{t}_n)}{\partial x}\theta_{4\delta}(\cdot,\hat{t}_n)\right|\right|^2_{D_\delta^+(\hat{t}_n)} \le Ch + C\Delta t. \quad (4.38)$$

For $t \in (t_{n-1}, t_n]$, $n = 1, \dots, N$, using (4.38), we have that

$$\begin{aligned}\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\right|\right|_{D_\delta^+(t)} &= \left|\left|\frac{\partial}{\partial x}(P_h c(\cdot,t) - C_h^+(\cdot,t))\right|\right|_{D_\delta^+(t)} \\ &\le \left|\left|\frac{\partial}{\partial x}(P_h(c(\cdot,t) - c(\cdot,\hat{t}_n)))\right|\right|_{D_\delta^+(t)} + \left|\left|\frac{\partial}{\partial x}(P_h(c(\cdot,\hat{t}_n)) - C_h^+(\cdot,\hat{t}_n))\right|\right|_{D_\delta^+(t)} \\ &\le C\Delta t\left|\left|\frac{\partial c(\cdot,t)}{\partial t}\right|\right|_{L^\infty(0,T;H^1(D_\delta^+(t)))} + \left|\left|\frac{\partial e_h^c(\cdot,\hat{t}_n)}{\partial x}\right|\right|_{D_\delta^+(\hat{t}_n)} + C\Delta t \le Ch^{1/2} + C\Delta t^{1/2}. \end{aligned} \tag{4.39}$$

We can finally derive the bounds in Theorem 2.2.

From the bounds in (4.38), (4.39), (3.17) and (3.23) we have the bound (2.16). From (4.35), (3.22), (4.38), (4.39), (3.16) and (3.17) we get (2.17). ☐

Remark 4.5. The error estimates on the whole domain Ω_T are not theoretically studied here, but will be numerically investigated by the test cases in Section 6. Here we only note that, as a consequence of the fact that $c, C_h^+ \in L^\infty(0,T;H^1(\Omega))$, the following bound is valid

$$||c - C_h^+||_{L^\infty(0,T;H^1(\Omega))} \le C. \tag{4.40}$$

Indeed, using the fact that $|e^c|_1 \le |e_h^c|_1 + |e_p^c|_1$, the bounds (3.14) and (3.15), the fact that $c, C_h^+ \in L^\infty(0,T;H^1(\Omega))$ and (4.39), we get

$$\begin{aligned}||e^c||_{L^\infty(0,T;L^2(\Omega))} &\le Ch + C\left(\left|\left|\frac{\partial e_h^c}{\partial x}\right|\right|^2_{L^\infty(0,T;L^2(\Omega\setminus D_\delta^+(t)))} + \left|\left|\frac{\partial e_h^c}{\partial x}\right|\right|^2_{L^\infty(0,T;L^2(D_\delta^+(t)))}\right)^{1/2} \\ &\le C, \end{aligned} \tag{4.41}$$

and

$$\begin{aligned}\left|\left|\frac{\partial e^c}{\partial x}\right|\right|_{L^\infty(0,T;L^2(\Omega))} &\le C + \left(\left|\left|\frac{\partial e_h^c}{\partial x}\right|\right|^2_{L^\infty(0,T;L^2(\Omega\setminus D_\delta^+(t)))} + \left|\left|\frac{\partial e_h^c}{\partial x}\right|\right|^2_{L^\infty(0,T;L^2(D_\delta^+(t)))}\right)^{1/2} \\ &\le C, \end{aligned} \tag{4.42}$$

which give the bound (4.40). In the case in which the continuous solution c has a fixed in time support, we can use the convergence property (1.15) and the fact that $c(t), C_h^+(t) \in C^{1/2}(\Omega)$ in order to improve the estimate (4.40). Let us choose a point $\bar{x}$ such that $c(\bar{x},0) = 0$ and let us define the set $S := \text{supp}(C_h^+(x,t)) \setminus \text{supp}(c(x,t))$. There exists a value $\bar{h}$ such that $|c(\bar{x},0) - c_h^0(\bar{x})| \le Ch$ and $|\text{supp}(c_h^0(x)) \setminus \text{supp}(c(x,0))| \le Ch$ for all $h \le \bar{h}$. Since the set $\text{supp}(C_h^+(t))$ moves at each time step by a distance proportional to h (see [3] for details), we should have that $|c(\bar{x},t) - C_h^+(\bar{x},t)| \le Ch$ and $|S| \le Ch$ for all $h \le \bar{h}$. Hence,

$$|c(x,t) - C_h^+(x,t)| \le |c(x,t) - c(\bar{x},t)| + |c(\bar{x},t) - C_h^+(\bar{x},t)| + |C_h^+(x,t) - C_h^+(\bar{x},t)| \le Ch^{1/2}, \tag{4.43}$$

$\forall x \,|\, |x - \bar{x}| \le h$. Taking

$$\sum_{K\in\mathcal{T}_h} \int_K |c(x,t) - C_h^+(x,t)|^2 \mathrm{d}x,$$

using (4.43) and (2.16), we obtain

$$||e^c||_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{1/2}. \tag{4.44}$$

It can also be observed that the $L^\infty(0,T;L^2(\Omega \setminus D_0^+(t)))$ norm of $\partial e^c/\partial x$ is different from zero only in the set S if $\text{supp}(c(x,t)) \subset \text{supp}(C_h^+(x,t))$, for which we have that $|S| \leq Ch$. Moreover, from (1.14) and the inverse of the Vitali convergence theorem we deduce that $\left|\frac{\partial e^c}{\partial x}\right|$ is a uniformly integrable function. As a consequence we get that

$$\left|\left|\frac{\partial e^c}{\partial x}\right|\right|^2_{L^\infty(0,T;L^2(\Omega\setminus D_0^+(t)))} \leq Ch. \tag{4.45}$$

From (4.45) and (2.16), it follows that

$$\left|\left|\frac{\partial e^c}{\partial x}\right|\right|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{1/2}. \tag{4.46}$$

5. Proofs of Lemmas 3.5 and 3.6

The proofs of Lemmas 3.7 and 3.8 are given below.

5.1. Proof of Lemma 3.5

Proof. Choose $\chi = \eta \equiv P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))]$ in the first equation of (4.1) and in the first equation of (4.2), and subtract the former from the latter, on noting (2.4) and the fact that $(a_h, P^h(fb_h)) = (b_h, P^h(fa_h))$, for each $a_h, b_h \in S^h$, $f \in L^2(\Omega)$. We obtain

$$\begin{aligned}
&\int_0^T \left(\frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c)), \frac{\partial}{\partial x}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))\right)_{D_\delta^+(t)} \mathrm{d}t \\
&= \int_0^T (P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c), \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c)))\mathrm{d}t \\
&= -\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))]\right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T < S^c - \partial_t e_p^c, P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))] >_{D_\delta^+(t)} \mathrm{d}t \\
&\quad - \int_0^T \left[(\delta_{\Delta t}C_h^+, P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))]) - (\delta_{\Delta t}C_h^+, P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))])^h\right]\mathrm{d}t \\
&\quad - \int_0^T \left([b(c) - b(C_h^-)]\frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e_h^c))]\right)_{D_\delta^+(t)} \mathrm{d}t.
\end{aligned} \tag{5.1}$$

The second term on the right hand side of (5.1) can be rewritten using (2.1) and (4.3) with

$f \equiv \theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))$,

$$\begin{aligned}
&\int_0^T < S^c - \partial_t e^c_p, P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))] >_{D^+_\delta(t)} \mathrm{d}t \\
&= \int_0^T \left(\frac{\partial}{\partial x}\mathcal{G}_{D^+_\delta(t)}(S^c - \partial_t e^c_p), \frac{\partial}{\partial x}\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right)\mathrm{d}t \\
&\quad - \int_0^T \left(\frac{\partial}{\partial x}\mathcal{G}_{D^+_\delta(t)}(S^c - \partial_t e^c_p), \frac{\partial}{\partial x}(I - P^h)[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))]\right)\mathrm{d}t.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, (2.2), the fact that $|\nabla\theta_{4\delta}| \le C\delta^{-2}$, the stability of the P^h projector under the H^1 seminorm, *i.e.* $|v - P^h(v)|_{1,D^+_\delta(t)} \le C|v - P_{h,1}v|_{1,D^+_\delta(t)}$ for $v \in H^1(D^+_\delta(t))$, (3.3) and (3.4) we obtain, at the lowest order,

$$\begin{aligned}
\int_0^T < S^c - \partial_t e^c_p, P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))] >_{D^+_\delta(t)} \mathrm{d}t &\le C||S^c - \partial_t e^c_p||_{L^2(0,T;(H^1(D^+_\delta(t)))')} \\
&\times \left[\left(\int_0^T \left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t\right)^{1/2}\right. \\
&\left. + C(\delta^{-2})\left(\int_0^T ||\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))||^2_{D^+_\delta(t)}\mathrm{d}t\right)^{1/2}\right].
\end{aligned}$$

Using finally (3.19) and the Poincaré inequality applied to the function $\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))$, we get

$$\begin{aligned}
&\int_0^T < S^c - \partial_t e^c_p, P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))] >_{D^+_\delta(t)} \mathrm{d}t \\
&\le (C + C(\delta^{-2}))(h^2 + \Delta t)\left(\int_0^T \left|\left|\frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)}\mathrm{d}t\right)^{1/2}
\end{aligned} \tag{5.2}$$

The last term on the right hand side of (5.1) can be rewritten using the fact that $-W^+_h = e^w_h + e^w_p - w$,

$$\begin{aligned}
&- \int_0^T \left([b(c) - b(C^-_h)]\frac{\partial W^+_h}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))]\right)_{D^+_\delta(t)}\mathrm{d}t \\
&= \int_0^T \left([b(c) - b(C^-_h)]\frac{\partial(e^w_h + e^w_p)}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))]\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad - \int_0^T \left([b(c) - b(C^+_h)]\frac{\partial w}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))]\right)_{D^+_\delta(t)}\mathrm{d}t \\
&\quad - \int_0^T \left([b(C^+_h) - b(C^-_h)]\frac{\partial w}{\partial x}, \frac{\partial}{\partial x}P^h[\theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))]\right)_{D^+_\delta(t)}\mathrm{d}t.
\end{aligned} \tag{5.3}$$

In order to bound the first term on the right hand side of (5.3), we use the Lipschitz continuity property of $b(\cdot)$, (3.16), (4.3) with $f \equiv \theta_{4\delta}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))$, the fact that $|\nabla\theta_{4\delta}| \le C\delta^{-2}$, the Poincaré inequality for the function $\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))$, the stability of the P^h projector under the H^1 seminorm, (3.3), (3.4), the

Cauchy-Schwarz inequality and the inequality (4.8), obtaining, at the lowest order,

$$\int_0^T \left([b(c) - b(C_h^-)] \frac{\partial (e_h^w + e_p^w)}{\partial x}, \frac{\partial}{\partial x} P^h[\theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))] \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$\leq C(\delta^{-5/2}) ||b(c) - b(C_h^-)||_{\infty, D_\delta^+} \int_0^T \left|\left| (b(c))^{1/2} \frac{\partial e_h^w}{\partial x} \right|\right|_{D_\delta^+(t)} \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|_{D_\delta^+(t)} \mathrm{d}t. \tag{5.4}$$

For what concerns the second term on the right hand side of (5.3), using the Lipschitz continuity property of $b(\cdot)$, the Cauchy-Schwarz inequality, (4.7), (4.3) with $f \equiv \theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))$, the fact that $|\nabla \theta_{4\delta}| \leq C\delta^{-2}$, the stability of the P^h projector under the H^1 seminorm, the Poincaré inequality for the function $\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))$, (3.3), (3.4), at the lowest order, we get

$$\int_0^T \left([b(c) - b(C_h^+)] \frac{\partial w}{\partial x}, \frac{\partial}{\partial x} P^h[\theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))] \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$\leq (C + C(\delta^{-2})) \left|\left| \frac{\partial w}{\partial x} \right|\right|_{\infty, D_\delta^+} \int_0^T ||e^c \theta_{4\delta}||_{D_\delta^+(t)} \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|_{D_\delta^+(t)} \mathrm{d}t. \tag{5.5}$$

The third term on the right hand side of (5.3) can be bounded similarly to the second term, considering the fact that $C_h^+ - C_h^- = \Delta t(\delta_{\Delta t} C_h^+(t))$, obtaining

$$\int_0^T \left([b(C_h^+) - b(C_h^-)] \frac{\partial w}{\partial x}, \frac{\partial}{\partial x} P^h[\theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))] \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$\leq (C + C(\delta^{-2}))\Delta t \left|\left| \frac{\partial w}{\partial x} \right|\right|_{\infty, D_\delta^+} \int_0^T ||\delta_{\Delta t} C_h^+(t)||_{D_\delta^+(t)} \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|_{D_\delta^+(t)} \mathrm{d}t. \tag{5.6}$$

Using (5.4), (5.5) and (5.6) in (5.3), using moreover the Cauchy-Schwarz inequality and (4.7), the property (1.15), (3.12) and (4.17), we get

$$- \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x} P^h[\theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))] \right)_{D_\delta^+(t)} \mathrm{d}t$$
$$\leq (C + C(\delta^{-2})) \left(\int_0^T ||e^c \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \left(\int_0^T \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2}$$
$$+ (C + C(\delta^{-2}))(\Delta t)^{3/4} \left(\int_0^T \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2}$$
$$+ C(\delta^{-5/2}) \left(\int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \left(\int_0^T \left|\left| \frac{\partial}{\partial x} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c)) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2}. \tag{5.7}$$

We can bound the first term on the right hand side of (5.1) using the Cauchy-Schwarz and Young inequalities, (4.7) and (4.3) with $f \equiv \theta_{4\delta} \mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))$, the fact that $|\nabla \theta_{4\delta}| \leq C\delta^{-2}$, the stability of the P^h projector under the H^1 seminorm, the Poincaré inequality for the function $\mathcal{G}^h_{D_\delta^+(t)}(P^h(\theta_{4\delta}\delta_{\Delta t} e_h^c))$, (3.3) and (3.4). Moreover, applying (2.10) with $m = 1$, (3.6) and (4.17) we can bound the third term on the right hand side of (5.1). In the hypothesis that $\Delta t \sim h$ (as in Rem. 4.4, we could choose $\Delta t = Ch^n$, with $1 \leq m \leq 2$), at the lowest order, we obtain

$$
\begin{aligned}
&\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t \\
&\le (C + C(\delta^{-2}))||b(c)||^{1/2}_{L^\infty(D^+_\delta)}\left(\int_0^T \left(b(c)\frac{\partial e^w_h}{\partial x}, \frac{\partial e^w_h}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2}\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + (C + C(\delta^{-2}))(h^2+\Delta t)\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + (C + C(\delta^{-2}))h^2(\Delta t)^{-1/2}\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + (C + C(\delta^{-2}))\left(\int_0^T ||e^c\theta_{4\delta}||^2_{D^+_\delta(t)}\mathrm{d}t\right)^{1/2}\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + (C + C(\delta^{-2}))(\Delta t)^{3/4}\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + C(\delta^{-5/2})\left(\int_0^T \left(b(c)\frac{\partial e^w_h}{\partial x}, \frac{\partial e^w_h}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2}\left(\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t\right)^{1/2} \\
&\le (C + C(\delta^{-4}))h^3 + (C + C(\delta^{-4}))(\Delta t)^{3/2} + (C + C(\delta^{-4}))\int_0^T ||e^c\theta_{4\delta}||^2_{D^+_\delta(t)}\mathrm{d}t \\
&\quad + (C + C(\delta^{-5}))\int_0^T \left(b(c)\frac{\partial e^w_h}{\partial x}, \frac{\partial e^w_h}{\partial x}\theta^2_{4\delta}\right)_{D^+_\delta(t)} \mathrm{d}t + \frac{1}{2}\int_0^T \left|\left| \frac{\partial}{\partial x}\mathcal{G}^h_{D^+_\delta(t)}(P^h(\theta_{4\delta}\delta_{\Delta t}e^c_h))\right|\right|^2_{D^+_\delta(t)} \mathrm{d}t,
\end{aligned}
$$

from which, using the fact that $e^c = e^c_p + e^c_h$ and the bound (3.17), writing only the lowest order terms we obtain (3.21).

Choose $\phi = \xi \equiv P^h[(\theta_{4\delta})^2 e^w_h]$ in the second equation of (4.1) and in the second equation of (4.2), and subtract the former from the latter, using (4.3) with $f \equiv (\theta_{4\delta})^2 e^w_h$. We obtain

$$
\begin{aligned}
\int_0^T (e^w_h, (\theta_{4\delta})^2 e^w_h)_{D^+_\delta(t)}\mathrm{d}t &= \int_0^T \gamma\left(\frac{\partial e^c_h}{\partial x}, \frac{\partial e^w_h}{\partial x}(\theta_{4\delta})^2\right)_{D^+_\delta(t)} \mathrm{d}t + \int_0^T \left[\gamma\left(\frac{\partial e^c_h}{\partial x}, e^w_h\frac{\partial(\theta_{4\delta})^2}{\partial x}\right)_{D^+_\delta(t)}\right. \\
&\quad \left. + (\psi'(c) - \psi_1'(C^+_h) - \psi_2'(C^-_h), P^h[(\theta_{4\delta})^2 e^w_h])_{D^+_\delta(t)}\right]\mathrm{d}t - \int_0^T (e^w_p, P^h[(\theta_{4\delta})^2 e^w_h])_{D^+_\delta(t)}\mathrm{d}t \\
&\quad + \int_0^T \left[(\psi_1'(C^+_h) + \psi_2'(C^-_h) - W^+_h, P^h[(\theta_{4\delta})^2 e^w_h]) - (\psi_1'(C^+_h) + \psi_2'(C^-_h) - W^+_h, P^h[(\theta_{4\delta})^2 e^w_h])^h\right]\mathrm{d}t \\
&\quad - \int_0^T \gamma\left(\frac{\partial e^c_h}{\partial x}, \frac{\partial}{\partial x}(I - P^h)(\theta^2_{4\delta}e^w_h)\right)_{D^+_\delta(t)} \mathrm{d}t \qquad (5.8)
\end{aligned}
$$

Noting that $\psi_1'(\cdot) \in C^1([0,1))$, $\psi_2'(\cdot) \in C^1([0,1])$, using (3.8), the Cauchy-Schwarz inequality, the Sobolev embedding result (2.3) with $d = 1$, $r = \infty$, $p = 2$ and $m = 1$, the Young inequality, the embedding of $L^2(D^+_\delta(t))$ in

$L^1(D_\delta^+(t))$ and (4.7), we bound the third term in (5.8) as

$$\begin{aligned}
&\int_0^T (\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-), P^h[(\theta_{4\delta})^2 e_h^w])_{D_\delta^+(t)} \mathrm{d}t \\
&\leq \left(\int_0^T ||\psi_1'(c) - \psi_1'(C_h^+)||^2_{L^\infty(D_\delta^+(t))} \mathrm{d}t\right)^{1/2} \left(\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||^2_{L^1(D_\delta^+(t))} \mathrm{d}t\right)^{1/2} \\
&\quad + \left(\int_0^T ||\psi_2'(c) - \psi_2'(C_h^-)||^2_{L^\infty(D_\delta^+(t))} \mathrm{d}t\right)^{1/2} \left(\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||^2_{L^1(D_\delta^+(t))} \mathrm{d}t\right)^{1/2} \\
&\leq C\left[\int_0^T \left(||e^c\theta_{4\delta}||^2_{D_\delta^+(t)} + \left|\left|\frac{\partial e^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\right)\mathrm{d}t\right]^{1/2} \left(\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||^2_{D_\delta^+(t)} \mathrm{d}t\right)^{1/2} \\
&\quad + C\Delta t\left[\int_0^T \left(||\delta_{\Delta t} C_h^+||^2_{D_\delta^+(t)} + \left|\left|\frac{\partial}{\partial x}\delta_{\Delta t} C_h^+\right|\right|^2_{D_\delta^+(t)}\right)\mathrm{d}t\right]^{1/2} \left(\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||^2_{D_\delta^+(t)} \mathrm{d}t\right)^{1/2}. \qquad (5.9)
\end{aligned}$$

Using (4.13) with $f \equiv (\theta_{4\delta})^2 e_h^w$ and the bound (2.8), obtained on the set $D_\delta^+(t)$, with $m = 1$, as well as the Young inequality, (3.6) and (4.17) in (5.9), we get, at the lowest order,

$$\begin{aligned}
&\int_0^T (\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-), P^h[(\theta_{4\delta})^2 e_h^w])_{D_\delta^+(t)} \mathrm{d}t \\
&\leq C\int_0^T ||e^c\theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t + C\int_0^T \left|\left|\frac{\partial e^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + C\Delta t + \frac{1}{10}\int_0^T ||e_h^w \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.10)
\end{aligned}$$

Using the inequality (2.10) with $m = 0$, the equality (4.13) with $f \equiv (\theta_{4\delta})^2 e_h^w$, the bound (2.8) on the set $D_\delta^+(t)$ with $m = 1$ and the Young inequality, we can bound the fifth term on the right hand side of (5.8), writing only the lowest order terms, as

$$\begin{aligned}
&\int_0^T \left[(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 e_h^w]) - (\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+, P^h[(\theta_{4\delta})^2 e_h^w])^h\right] \mathrm{d}t \\
&\leq Ch\int_0^T ||P^h[(\theta_{4\delta})^2 e_h^w]||_{D_\delta^+(t)} ||\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+||_{1,D_\delta^+(t)} \mathrm{d}t \leq Ch^2 + \frac{1}{10}\int_0^T ||e_h^w\theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t. \quad (5.11)
\end{aligned}$$

Using (5.10) and (5.11) in (5.8), the fact that $e^c = e_p^c + e_h^c$, the Cauchy-Schwarz and the Young inequalities, the bounds (3.17), (4.8), (4.7), (4.13) with $f \equiv (\theta_{4\delta})^2 e_h^w$ and the bound (2.8) on the set $D_\delta^+(t)$ with $m = 1$, we get, at the lowest order,

$$\begin{aligned}
\int_0^T (e_h^w, (\theta_{4\delta})^2 e_h^w)_{D_\delta^+(t)} \mathrm{d}t &\leq C\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t + C\int_0^T ||e_h^c\theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + (C + C(\delta^{-4}))\int_0^T \left|\left|\frac{\partial e_h^c}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + Ch^2 \\
&\quad + C\Delta t + \frac{5}{10}\int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t, \qquad (5.12)
\end{aligned}$$

from which we obtain (3.22). □

5.2. Proof of Lemma 3.8

Proof. Choose $\chi = \eta \equiv P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c]$ in the first equation of (4.1) and in the first equation of (4.2), and subtract the former from the latter, using (4.3). We obtain

$$
\begin{aligned}
&\int_0^T ||\delta_{\Delta t}e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t + \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})^2\right)_{D_\delta^+(t)}\mathrm{d}t \\
&= -\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}((\theta_{4\delta})^2)\delta_{\Delta t}e_h^c\right)_{D_\delta^+(t)}\mathrm{d}t + \int_0^T < S^c - \partial_t e_p^c, P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c] >_{D_\delta^+(t)} \mathrm{d}t \\
&\quad - \int_0^T [(\delta_{\Delta t}C_h^+, P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c]) - (\delta_{\Delta t}C_h^+, P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])^h]_{D_\delta^+(t)}\mathrm{d}t \\
&\quad - \int_0^T \left([b(c) - b(C_h^-)]\frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t \\
&\quad - \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(I - P^h)(\delta_{\Delta t}e_h^c(\theta_{4\delta})^2)\right)_{D_\delta^+(t)}\mathrm{d}t. \qquad (5.13)
\end{aligned}
$$

Taking $\phi(\cdot,t) = \xi(\cdot,t) \equiv \zeta^h(\cdot,t)$, with $\zeta^h(\cdot,t) \in S^h$ and $\mathrm{supp}(\zeta^h) \subset\subset D_\delta^+(t)$, in the second equation of (4.1) and in the second equation of (4.2), subtracting the former from the latter, and using (2.13) and the definition of the lumped scalar product, we get

$$
\begin{aligned}
e_h^w|_{D_\delta^+(t)} &= -\gamma\Delta_{h,D_\delta^+(t)}e_h^c + P^h(\psi'(c) - \psi_1'(C_h^+) - \psi_2'(C_h^-))|_{D_\delta^+(t)} - P^h(e_p^w)|_{D_\delta^+(t)} \\
&\quad - \left(P^h(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+) - \sum_{i\in\bar{J}_\delta(t), j\in\bar{I}_\delta(t)} \bar{M}_{ij}^{-1}(\psi_1'(C_h^+(x_i)) + \psi_2'(C_h^-(x_i)) - W_h^+(x_i))(1,\chi_i)\chi_j\right)|_{D_\delta^+(t)} \\
&\quad + \sum_{k\in\bar{I}_\delta(t)\setminus\bar{J}_\delta(t)} C_k\chi_k, \qquad (5.14)
\end{aligned}
$$

for a.e. $t \in (0,T]$, where $\bar{M}_{ij}$ is the mass matrix (χ_i,χ_j), $\bar{I}_\delta(t)$ is the set of nodes inside $D_\delta^+(t)$, $\bar{J}_\delta(t)$ is the set of nodes inside $D_\delta^+(t)$ except the nearest node to $\partial D_\delta^+(t)$, $\bar{M}_{ij}^{-1}$ is the right inverse of $\bar{M}_{ij}$ and C_k are finite constants. Note that the last term in (5.14) is due to the fact that $\dim\mathrm{Ker}(\bar{M}) = 1$. For ease of clarity, we indicate the term in the parenthesis in (5.14) formally as

$$
P^h(I - (\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+),
$$

even if the projector $\hat{P}^h$ is not invertible, since we will be only interested in obtaining a bound for a bilinear form containing this term using (2.10).

Let us introduce the following equality

$$
b(c)\frac{\partial e_h^w}{\partial x}(\theta_{4\delta})^2 = \frac{\partial}{\partial x}(P^h[b(c)e_h^w(\theta_{4\delta})^2]) - \frac{\partial b(c)}{\partial x}e_h^w(\theta_{4\delta})^2 - b(c)e_h^w\frac{\partial(\theta_{4\delta})^2}{\partial x} + \frac{\partial}{\partial x}((I - P^h)[b(c)e_h^w(\theta_{4\delta})^2]). \quad (5.15)
$$

Using (5.14) and (5.15) in (5.13), the Cauchy-Schwarz and Young inequalities, a generalized version of identity (2.13) with a $\phi \equiv (I - P^h)[b(c)e_h^w(\theta_{4\delta})^2] \in H_0^1(D_\delta^+(t))$ and (2.8) on $D_\delta^+(t)$ with $m = 1$, using moreover (4.7), similar calculations to those used in (5.2), (2.10) with $m = 0$ and (4.13), we get, at the lowest order,

$$\begin{aligned}
&\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t - \gamma \int_0^T \left(\frac{\partial}{\partial x} \left(P^h [b(c) \Delta_{h, D_\delta^+(t)} e_h^c (\theta_{4\delta})^2] \right), \frac{\partial}{\partial x} \delta_{\Delta t} e_h^c \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\le C \int_0^T \left|\left| \frac{\partial}{\partial x} (b(c) e_h^w (\theta_{4\delta})^2) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + Ch^2 (\Delta t)^{-1} \int_0^T \left(\Delta_{D_\delta^+(t)} (\delta_{\Delta t} e_h^c), \Delta_{D_\delta^+(t)} (e_h^c(t) - e_h^c(t - \Delta t)) \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T \left(\frac{\partial}{\partial x} (b(c)) e_h^w (\theta_{4\delta})^2, \frac{\partial}{\partial x} \delta_{\Delta t} e_h^c \right)_{D_\delta^+(t)} \mathrm{d}t + 2 \int_0^T \left(b(c) e_h^w \theta_{4\delta} \frac{\partial}{\partial x} \theta_{4\delta}, \frac{\partial}{\partial x} \delta_{\Delta t} e_h^c \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad - \int_0^T \left(b(c) \frac{\partial}{\partial x} (\psi_1'(c) - \psi_1'(C_h^+)), \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad - \int_0^T \left(b(c) \frac{\partial}{\partial x} (\psi_2'(c) - \psi_2'(C_h^-)), \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t + \int_0^T \left(b(c) \frac{\partial}{\partial x} e_p^w, \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \int_0^T \left(b(c) \frac{\partial}{\partial x} \left((I - (\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+) \right), \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \sum_{k \in \bar{I}_\delta(t) \setminus \bar{J}_\delta(t)} \int_0^T \left(b(c) C_k \frac{\partial \chi_k}{\partial x}, \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-4}) \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&\quad + \frac{1}{10} \int_0^T ||\delta_{\Delta t} e_h^c (\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t + C(h^2 + \Delta t) \left[\left(\int_0^T \left|\left| \frac{\partial}{\partial x} (\delta_{\Delta t} e_h^c)(\theta_{4\delta}) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \right. \\
&\quad \left. + C(\delta^{-2}) \left(\int_0^T ||\delta_{\Delta t} e_h^c (\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \right] + Ch \left(\int_0^T ||\delta_{\Delta t} C_h^+||^2_{1, D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \left(\int_0^T ||\delta_{\Delta t} e_h^c (\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t \right)^{1/2} \\
&\quad - \int_0^T \left([b(c) - b(C_h^-)] \frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x} (P^h [(\theta_{4\delta})^2 \delta_{\Delta t} e_h^c]) \right)_{D_\delta^+(t)} \mathrm{d}t - \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x} (I - P^h)(\delta_{\Delta t} e_h^c (\theta_{4\delta})^2) \right)_{D_\delta^+(t)} \mathrm{d}t \\
&= E_{1,1} + E_{1,2} + E_{1,3} + E_{1,4} + E_{1,5} + E_{1,6} + E_{1,7} + E_{1,8} + E_{1,9} + E_{1,10} + E_{1,11} + E_{1,12} + E_{1,13} + E_{1,14} + E_{1,15}, \quad (5.16)
\end{aligned}$$

where we indicate the terms on the right hand side of (5.16) as $E_{1,1}, \dots, E_{1,15}$. Using (3.10), (3.11) and (1.16), we get that

$$\frac{\partial}{\partial x}(b(c)) e_h^w (\theta_{4\delta})^2 \in L^2(0, T; H^1(D_\delta^+(t))); \quad b(c) e_h^w \theta_{4\delta} \frac{\partial}{\partial x} \theta_{4\delta} \in L^2(0, T; H^1(D_\delta^+(t))).$$

Hence, we can integrate by parts in the terms $E_{1,3}$ and $E_{1,4}$. Using the Cauchy-Schwarz and Young inequalities, the facts that $\partial/\partial x(b(c))$ and $\partial^2/\partial x^2(b(c))$ are in $L^\infty(D_\delta^+)$ as a consequence of (3.12), using (4.7), the fact that $|\nabla \theta_{4\delta}| \le C\delta^{-2}$ and (4.8) we get, at the lowest order,

$$\begin{aligned}
|E_{1,3}| \le (C + C(\delta^{-4})) \int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-1}) \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
+ \frac{1}{10} \int_0^T ||\delta_{\Delta t} e_h^c (\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t, \qquad (5.17)
\end{aligned}$$

$$|E_{1,4}| \le C(\delta^{-8}) \int_0^T ||e_h^w(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-5}) \int_0^T \left(b(c) \frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x} \theta_{4\delta}^2 \right)_{D_\delta^+(t)} \mathrm{d}t + \frac{1}{10} \int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.18)$$

In order to isolate terms in e_p^c, which can be bounded using (3.17), and terms in e_h^c, we rewrite the term $E_{1,5}$ as

$$
\begin{aligned}
E_{1,5} =& \int_0^T \left(b(c)\psi_1''(c)\frac{\partial}{\partial x}(c - P_{h,1}(c)), \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \int_0^T \left(b(c)\psi_1''(c)\frac{\partial}{\partial x}(P_{h,1}(c) - C_h^+), \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \int_0^T \left(b(c)(\psi_1''(c) - \psi_1''(P_{h,1}(c)) + \psi_1''(P_{h,1}(c)) - \psi_1''(C_h^+))\frac{\partial C_h^+}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.19)
\end{aligned}
$$

Using (3.17), (1.5) and (1.7), the identity $(a, (a-b)c) = \frac{1}{2}a^2c - \frac{1}{2}b^2c + \frac{1}{2}(a-b)^2c$, the fact that $\psi_1''(x)$ is Lipschitz continuous for $x < 1$, the Cauchy-Schwarz and Young inequalities, we get

$$
\begin{aligned}
|E_{1,5}| \le& Ch + \frac{h}{400}\int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta}) \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t + \frac{(1-c^*)}{2\Delta t}\int_0^T \left(c\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \frac{(1-c^*)}{2\Delta t}\int_0^T \left(c\frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}, \frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ (1-c^*)\max_{D_\delta^+(t)}[c]\frac{\Delta t}{2}\int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta}) \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t Ch^2 + Ch^2 \int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta}) \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t \\
&- \int_0^T \left(b(c)(\psi_1''(P_h(c)) - \psi_1''(C_h^+))\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \int_0^T \left(b(c)(\psi_1''(P_h(c)) - \psi_1''(C_h^+))\frac{\partial c}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.20)
\end{aligned}
$$

Note that in the last two terms we have written $C_h^+ = C_h^+ - P_{h,1}c + P_{h,1}c - c + c$ and we have used (3.17), in order to isolate a term containing $\partial e_h^c/\partial x$ and in order to be able to integrate by parts in the term containing $\partial c/\partial x$. Let us now write $\theta_{4\delta}(\cdot,t) = \theta_{4\delta}(\cdot,t-\Delta t) - \Delta t\frac{\partial}{\partial t}\theta_{4\delta}(\cdot,t)|_{t=\bar{t}}$ and $c(\cdot,t) = c(\cdot,t-\Delta t)) - \Delta t\frac{\partial}{\partial t}c(\cdot,t))|_{t=\bar{t}}$, where $\bar{t} \in (t-\Delta t, t)$, in the fourth term in the right hand side of equation (5.20), and obtain, using (3.12), integrating by parts the last term in the right hand side of equation (5.20), and writing only the lowest order terms,

$$
\begin{aligned}
|E_{1,5}| \le& Ch + \frac{h}{400}\int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta}) \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t + \frac{(1-c^*)}{2\Delta t}\int_0^T \left(c\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \frac{(1-c^*)}{2\Delta t}\int_0^T \left(c(\cdot,t-\Delta t)\frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}, \frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta}(\cdot,t-\Delta t))^2 \right)_{D_\delta^+(t)-D_\delta^+(t-\Delta t)} \mathrm{d}t \\
&- \frac{(1-c^*)}{2\Delta t}\int_0^T \left(c(\cdot,t-\Delta t)\frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}, \frac{\partial e_h^c(\cdot,t-\Delta t)}{\partial x}(\theta_{4\delta}(\cdot,t-\Delta t))^2 \right)_{D_\delta^+(t-\Delta t)} \mathrm{d}t \\
&+ C\int_0^T \left|\left| \frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta} \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t + C(\Delta t)^2 + (1-c^*)\max_{D_\delta^+(t)}[c]\frac{\Delta t}{2}\int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta}) \right|\right|_{D_\delta^+(t)}^2 \mathrm{d}t \\
&+ \int_0^T \left(b(c)(\psi_1''(P_h(c)) - \psi_1''(C_h^+))\frac{\partial e_h^c}{\partial x}, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2 \right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \frac{1}{10}\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||_{D_\delta^+(t)}^2 \mathrm{d}t + (C + C(\delta)^{-4})\int_0^T ||e_h^c(\theta_{4\delta})||_{D_\delta^+(t)}^2 \mathrm{d}t. \qquad (5.21)
\end{aligned}
$$

Note that the integral on $D_\delta^+(t) - D_\delta^+(t-\Delta t)$ in (5.21), due to the continuity of the integrands in time and to the fact that the support of c changes in time by a finite value [6], is proportional to Δt times finite terms on $\partial D_\delta^+(t)$, and can be bounded by a term like the sixth on the right hand side of (5.21) multiplied by Δt. Changing variables as $(t-\Delta t) \to t$ in the fifth term in the right hand side of (5.21), let us rewrite the sum of the third and the fifth terms in the right hand side of (5.21) as

$$\frac{(1-c^*)}{2\Delta t}\int_{T-\Delta t}^{T}\left(c\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta})^2\right)_{D_\delta^+(t)}\mathrm{d}t - \frac{(1-c^*)}{2\Delta t}\int_{-\Delta t}^{0}\left(c(\cdot,t)\frac{\partial e_h^c(\cdot,t)}{\partial x}, \frac{\partial e_h^c(\cdot,t)}{\partial x}(\theta_{4\delta}(\cdot,t))^2\right)_{D_\delta^+(t)}\mathrm{d}t. \tag{5.22}$$

Recalling (3.13), we can use in (5.22) the mean value theorem for integrals and obtain that there exists a $\bar{t} \in (T-\Delta t, T]$ such that (5.22) can be controlled by

$$\frac{(1-c^*)}{2}\max_{D_\delta^+}[c]\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D_\delta^+(\bar{t})} + \frac{(1-c^*)}{2}\max_{D_\delta^+}[c]\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)}. \tag{5.23}$$

In order to bound the last term in (5.21), choose $\chi = \eta \equiv P^h[(\theta_{4\delta})^2 e_h^c]$ in the first equation of (4.1) and in the first equation of (4.2), and subtract the former from the latter. We obtain

$$\begin{aligned}\int_0^T (\delta_{\Delta t}e_h^c, e_h^c(\theta_{4\delta})^2)_{D_\delta^+(t)}\mathrm{d}t = &-\int_0^T\left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2 e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t + \int_0^T < S^c - \partial_t e_p^c, P^h[(\theta_{4\delta})^2 e_h^c] >_{D_\delta^+(t)}\mathrm{d}t\\ &-\int_0^T[(\delta_{\Delta t}C_h^+, P^h[(\theta_{4\delta})^2 e_h^c]) - (\delta_{\Delta t}C_h^+, P^h[(\theta_{4\delta})^2 e_h^c])^h]_{D_\delta^+(t)}\mathrm{d}t\\ &-\int_0^T\left([b(c)-b(C_h^-)]\frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2 e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t.\end{aligned} \tag{5.24}$$

The term on the left hand side of (5.24) can be treated as the second term on the right hand side of (5.19) and can be rewritten using the same calculations which led to (5.23). The first term on the right hand side of (5.24) can be bounded using the Cauchy-Schwarz and Young inequalities, (4.3), (4.7) and (2.8) with $m = 1$. The second term on the right hand side of (5.24) can be bounded using similar calculations to those used in (5.2). The third term on the right hand side of (5.24) can be bounded using bound (2.10), (3.6), (3.7) and the hypothesis that $\Delta t \sim h$. Finally, the last term on the right hand side of (5.24) can be bounded using similar calculations to those used in (5.3)−(5.7). Writing only the lowest order terms, we obtain that there exists a $\bar{t} \in (T-\Delta t, T]$ such that

$$\begin{aligned}||e_h^c(\cdot,\bar{t})(\theta_{4\delta})||^2_{D_\delta^+(\bar{t})} \leq\ & C\int_0^T\left(b(c)\frac{\partial e_h^w(\cdot,t)}{\partial x}, \frac{\partial e_h^w(\cdot,t)}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t + (C + C(\delta^{-4}))\int_0^T\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\\ &+ C(\delta^{-5})\int_0^{\bar{t}}||e_h^c(\cdot,t)(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t + C(\delta^{-5})\int_{\bar{t}}^T||e_h^c(\cdot,t)(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t\\ &+ (C + C(\delta)^{-4})h^3 + (C + C(\delta)^{-4})(\Delta t)^{3/2}.\end{aligned} \tag{5.25}$$

Choosing $T = t_n$, $n = 1, \ldots, N$ in (5.25), noting that $\int_{\bar{t}}^T ||e_h^c(\cdot,t)(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t \leq C\Delta t$ and using a Gronwall inequality, we derive that there exists a set $\bar{t}_n \in ((n-1)\Delta t, n\Delta t]$, $n = 1, \ldots, N$, such that, at the lowest order,

$$\begin{aligned}||e_h^c(\cdot,\bar{t}_n)(\theta_{4\delta})||^2_{D_\delta^+(\bar{t}_n)} \leq\ & C\int_0^T\left(b(c)\frac{\partial e_h^w(\cdot,t)}{\partial x}, \frac{\partial e_h^w(\cdot,t)}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)}\mathrm{d}t + (C + C(\delta^{-4}))\int_0^T\left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t\\ &+ (C + C(\delta)^{-4})h^3 + C(\delta^{-6})\Delta t.\end{aligned} \tag{5.26}$$

For $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$, using (3.12) and (5.26), we have that

$$\begin{aligned}||e_h^c(\cdot,t)\theta_{4\delta}||_{D_\delta^+(t)} &= ||(P_h c(\cdot,t) - C_h^+(\cdot,t))\theta_{4\delta}||_{D_\delta^+(t)} \\ &\leq ||(P_h(c(\cdot,t) - c(\cdot,\bar{t}_n)))\theta_{4\delta}||_{D_\delta^+(t)} + ||(P_h(c(\cdot,\bar{t}_n)) - C_h^+(\cdot,\bar{t}_n))\theta_{4\delta}||_{D_\delta^+(t)} \\ &\leq C\Delta t\left|\left|\frac{\partial c(\cdot,t)}{\partial t}\right|\right|_{L^\infty(D_\delta^+)} + ||e_h^c(\cdot,\bar{t}_n)\theta_{4\delta}(\cdot,\bar{t}_n)||_{D_\delta^+(\bar{t}_n)} + C\Delta t. \qquad (5.27)\end{aligned}$$

Combining (5.26) and (5.27) we get (3.23).

Note that there exists an $\bar{h}$ such that $b(c)\psi_1''(c) - b(c)\psi_1''(P_h(c)) + b(c)\psi_1''(C_h^+) < 1$ for each $h < \bar{h}$. Hence we can include the ninth term on the right hand side of (5.21) in the second term on the right hand side of (5.19) and introduce a constant C_2, with $C_2 < 1$, such that, using (3.23) in (5.21), and writing only the lowest order terms, we get

$$\begin{aligned}|E_{1,5}| \leq Ch + C\Delta t &+ \frac{h}{400}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + \frac{C_2}{2}\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D_\delta^+(\bar{t})} \\ &+ \frac{C_2}{2}\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)} + (C + C(\delta^{-4}))\int_0^T \left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \\ &+ (1 - c^*)\frac{\Delta t}{2}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \\ &+ \frac{1}{10}\int_0^T ||\delta_{\Delta t}e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)}\mathrm{d}t + C(\delta^{-6})\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.28)\end{aligned}$$

Let us rewrite the term $E_{1,6}$ as

$$E_{1,6} = \int_0^T \left(b(c)\frac{\partial}{\partial x}(\psi_2'(c) - \psi_2'(c(t-\Delta t)) + \psi_2'(c(t-\Delta t)) - \psi_2'(C_h^-)), \frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})^2\right)_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.29)$$

Expanding in a Taylor series to first order the term $\psi_2'(c(t-\Delta t))$ around t, using the Lipschitz continuity of $\psi_2''(\cdot)$, (3.12), the Cauchy-Schwarz and Young inequalities we get that

$$\begin{aligned}E_{1,6} \leq C\Delta t &+ \frac{\Delta t}{160}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \\ &+ \int_0^T \left(b(c)\frac{\partial}{\partial x}(\psi_2'(c(t-\Delta t)) - \psi_2'(C_h^-)), \frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})^2\right)_{D_\delta^+(t)} \mathrm{d}t. \qquad (5.30)\end{aligned}$$

The third term on the right hand side of (5.30) is similar to $E_{1,5}$ and can be treated in a similar way, repeating the calculations which led to (5.28), on noting the Lipschitz continuity of $\psi_2''(\cdot)$ and the fact that $|b(c)\psi_2''(c)| < 1$.

We get

$$
\begin{aligned}
|E_{1,6}| \leq{}& Ch + C\Delta t + \left(\frac{h}{400} + \frac{\Delta t}{160}\right)\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + \frac{C_3}{2}\left|\left|\frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t})\right|\right|^2_{D_\delta^+(\bar{t})} \\
&+ \frac{C_3}{2}\left|\left|\frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0)\right|\right|^2_{D_\delta^+(0)} + (C + C(\delta^{-4}))\int_0^T \left|\left|\frac{\partial e_h^c(\cdot,t)}{\partial x}\theta_{4\delta}\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \\
&+ (1-c^*)\frac{\Delta t}{2}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t \\
&+ \frac{1}{10}\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t + C(\delta^{-6})\int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t,
\end{aligned}
\tag{5.31}
$$

with $C_3 < 1$. Using (3.16), the Cauchy-Schwarz and Young inequalities, we get

$$
|E_{1,7}| \leq Ch + \frac{h}{400}\int_0^T \left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)} \mathrm{d}t. \tag{5.32}
$$

Let us rewrite the term $E_{1,8}$, using (2.13), as

$$
\begin{aligned}
E_{1,8} ={}& \int_0^T \left(\frac{\partial}{\partial x}P^h[b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2], \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \int_0^T \left(\frac{\partial b(c)}{\partial x}\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \int_0^T \left(b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)\frac{\partial}{\partial x}(\theta_{4\delta})^2, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \int_0^T \left(\frac{\partial}{\partial x}(I-P^h)[b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2], \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
={}& -\int_0^T \left(b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2, \Delta_{h,D_\delta^+(t)}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \int_0^T \left(\frac{\partial b(c)}{\partial x}\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&- \int_0^T \left(b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)\frac{\partial}{\partial x}(\theta_{4\delta})^2, \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t \\
&+ \int_0^T \left(\frac{\partial}{\partial x}(I-P^h)[b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\big)(\theta_{4\delta})^2], \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)\right)_{D_\delta^+(t)} \mathrm{d}t
\end{aligned}
\tag{5.33}
$$

Choosing $v^h \equiv P^h[(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)(\theta_{4\delta})]$, $\chi \equiv P^h(I-(\hat{P}^h)^{-1})[(\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)(\theta_{4\delta})]$ and $m = 1$ in (2.10), using (2.8) and (2.9), we get that

$$
\begin{aligned}
||(I-(\hat{P}^h)^{-1})((\psi_1'(C_h^+) + \psi_2'(C_h^-) - W_h^+)\theta_{4\delta})||_{D_\delta^+(t)} \leq{}& ||((P^h + (I-P^h))(I-(\hat{P}^h)^{-1})[(\psi_1'(C_h^+) + \psi_2'(C_h^-) \\
&- W_h^+)\theta_{4\delta}])||_{D_\delta^+(t)} \leq Ch^2.
\end{aligned}
\tag{5.34}
$$

The last term in (5.33) can be bounded using a generalized version of identity (2.13) with a $\phi \equiv (I-P^h)[b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+)+\psi_2'(C_h^-)-W_h^+)\big)(\theta_{4\delta})^2]$, the Cauchy-Schwarz inequality and (5.34), obtaining

$$\int_0^T\left(\frac{\partial}{\partial x}(I-P^h)[b(c)\big((I-(\hat{P}^h)^{-1})(\psi_1'(C_h^+)+\psi_2'(C_h^-)-W_h^+)\big)(\theta_{4\delta})^2], \frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)\right)_{D_\delta^+(t)}\mathrm{d}t$$
$$\leq Ch^2(\Delta t)^{-1}\left(\int_0^T ||\Delta_{D_\delta^+(t)}(e_h^c(t)-e_h^c(t-\Delta t))||_{D_\delta^+(t)}\mathrm{d}t\right)^{1/2}. \tag{5.35}$$

Using (3.12), (3.13), (5.34), (5.35), the Cauchy-Schwarz and Young inequalities in (5.33) and the hypothesis that $\Delta t \sim h$ we get, at the lowest order,

$$|E_{1,8}| \leq Ch + \frac{h}{400}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t + C\int_0^T ||\Delta_{h,D_\delta^+(t)}e_h^c||^2_{D_\delta^+(t)}dt. \tag{5.36}$$

The term $E_{1,9}$ can be bounded using the Cauchy-Schwarz and Young inequalities, on noting that

$$\left|\left|b(c)C_k\frac{\partial \chi_k}{\partial x}\right|\right|_{D_\delta^+(t)} \leq Ch,$$

obtaining

$$|E_{1,9}| \leq Ch + \frac{h}{400}\int_0^T\left|\left|\frac{\partial}{\partial x}(\delta_{\Delta t}e_h^c)(\theta_{4\delta})\right|\right|^2_{D_\delta^+(t)}\mathrm{d}t. \tag{5.37}$$

Let us rewrite the term $E_{1,14}$ as

$$\begin{aligned}E_{1,14} = &\int_0^T\left([b(c)-b(c(t-\Delta t))]\frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t\\
&+\int_0^T\left([b(c(t-\Delta t))-b(P_h(c(t-\Delta t)))]\frac{\partial W_h^+}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t\\
&-\int_0^T\left([b(P_h(c(t-\Delta t)))-b(C_h^-)]\frac{\partial e_h^w}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t\\
&-\int_0^T\left([b(P_h(c(t-\Delta t)))-b(C_h^-)]\frac{\partial e_p^w}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t\\
&+\int_0^T\left([b(P_h(c(t-\Delta t)))-b(C_h^-)]\frac{\partial w}{\partial x}, \frac{\partial}{\partial x}(P^h[(\theta_{4\delta})^2\delta_{\Delta t}e_h^c])\right)_{D_\delta^+(t)}\mathrm{d}t.\end{aligned} \tag{5.38}$$

Noting that, for a given $\hat{h}(\delta)$, $b(C_h^-)-b(P_h(c(t-\Delta t))) < b(c)$ for each $h < \hat{h}(\delta)$, we can absorb the third term on the right hand side of (5.38) into the second term on the left hand side and the first term on the right hand side of (5.13). Using (3.12), (3.17), (3.16), (4.7), (3.5) and (2.3) with $r=\infty, m=1, p=2$, integrating by parts the last term on the right hand side of (5.38) and using the Cauchy-Schwarz and Young inequalities, we get, at the lowest order,

$$|E_{1,14}| \leq (C + C(\delta^{-5}))h + C(\delta^{-5})\Delta t + \left(\frac{h}{400} + \frac{\Delta t}{160}\right) \int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c)(\theta_{4\delta}) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t$$
$$+ C(\delta)^{-4} \int_0^T ||e_h^c \theta_{4\delta}||^2_{D_\delta^+(t)} \mathrm{d}t + \frac{1}{10} \int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t. \tag{5.39}$$

Note that, since $b(c)\psi''(c) < 1$ for $c \in [0,1)$, there exists a constant C_4 such that $C_2 + C_3 = C_4 < 1$. Finally, in order to bound the term $E_{1,15}$, we use a similar trick to (5.15), integration by parts and a generalized version of identity (2.13) with a $\phi \equiv (I - P^h)(\delta_{\Delta t} e_h^c(\theta_{4\delta})^2)$, (2.8) with $m = 1$ and the Cauchy-Schwarz and Young inequalities. Using (2.13), (3.6), (4.17), (3.22), (5.25), (3.23), the Cauchy-Schwarz and Young inequalities in (5.16), we get, with the hypothesis that $\Delta t \sim h$, the fact that $c^* < 1$ and writing only the lowest order terms,

$$\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t + \frac{\gamma}{2\Delta t} \int_0^T \big(b(c)\Delta_{h,D_\delta^+(t)} e_h^c(\cdot,t)(\theta_{4\delta})^2, \Delta_{h,D_\delta^+(t)} e_h^c(\cdot,t)\big)_{D_\delta^+(t)} \mathrm{d}t$$
$$- \frac{\gamma}{2\Delta t} \int_0^T \big(b(c)\Delta_{h,D_\delta^+(t)} e_h^c(\cdot,t-\Delta t)(\theta_{4\delta})^2, \Delta_{h,D_\delta^+(t)} e_h^c(\cdot,t-\Delta t)\big)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ \frac{\gamma}{2}\Delta t \int_0^T \big(b(c)\Delta_{h,D_\delta^+(t)}(\delta_{\Delta t} e_h^c)(\theta_{4\delta})^2, \Delta_{h,D_\delta^+(t)}(\delta_{\Delta t} e_h^c)\big)_{D_\delta^+(t)} \mathrm{d}t$$
$$\leq C \int_0^T ||\Delta_{h,D_\delta^+(t)} e_h^c||^2_{D_\delta^+(t)} \mathrm{d}t + (C + C(\delta^{-5}))h + (C + C(\delta^{-14}))\Delta t + C(\delta^{-8}) \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ C(\delta^{-14}) \int_0^T \left|\left| \frac{\partial e_h^c}{\partial x}\theta_{4\delta} \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t}) \right|\right|^2_{D_\delta^+(\bar{t})} + \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0) \right|\right|^2_{D_\delta^+(0)}$$
$$+ \frac{9}{10}\frac{\Delta t}{2} \int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta}) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t. \tag{5.40}$$

Using similar calculations to those used in (5.21)-(5.23) in order to treat the second and third terms on the left hand side of (5.40), using moreover (3.13) and similar Gronwall arguments to those used in (5.25), we get

$$\int_0^T ||\delta_{\Delta t} e_h^c(\theta_{4\delta})||^2_{D_\delta^+(t)} \mathrm{d}t \leq (C + C(\delta^{-5}))h + (C + C(\delta^{-14}))\Delta t + C(\delta^{-8}) \int_0^T \left(b(c)\frac{\partial e_h^w}{\partial x}, \frac{\partial e_h^w}{\partial x}\theta_{4\delta}^2\right)_{D_\delta^+(t)} \mathrm{d}t$$
$$+ C(\delta^{-14}) \int_0^T \left|\left| \frac{\partial e_h^c}{\partial x}\theta_{4\delta} \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t + \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,\bar{t})}{\partial x}\theta_{4\delta}(\cdot,\bar{t}) \right|\right|^2_{D_\delta^+(\bar{t})}$$
$$+ \frac{C_4}{2} \left|\left| \frac{\partial e_h^c(\cdot,0)}{\partial x}\theta_{4\delta}(\cdot,0) \right|\right|^2_{D_\delta^+(0)} + \frac{9}{10}\frac{\Delta t}{2} \int_0^T \left|\left| \frac{\partial}{\partial x}(\delta_{\Delta t} e_h^c(\cdot,t))(\theta_{4\delta}) \right|\right|^2_{D_\delta^+(t)} \mathrm{d}t, \tag{5.41}$$

with $C_4 < 1$ and for a $\bar{t} \in (T - \Delta t, T]$, which is (3.24). □

6. Numerical results.

In this section different test cases are simulated in order to validate the error analysis introduced in Theorem 2.2. In order to simplify the notation, let us indicate in the sequel

$$e_{0,\delta} := ||c - C_h^+||_{L^\infty(0,T;L^2(D_\delta^+(t)))},$$

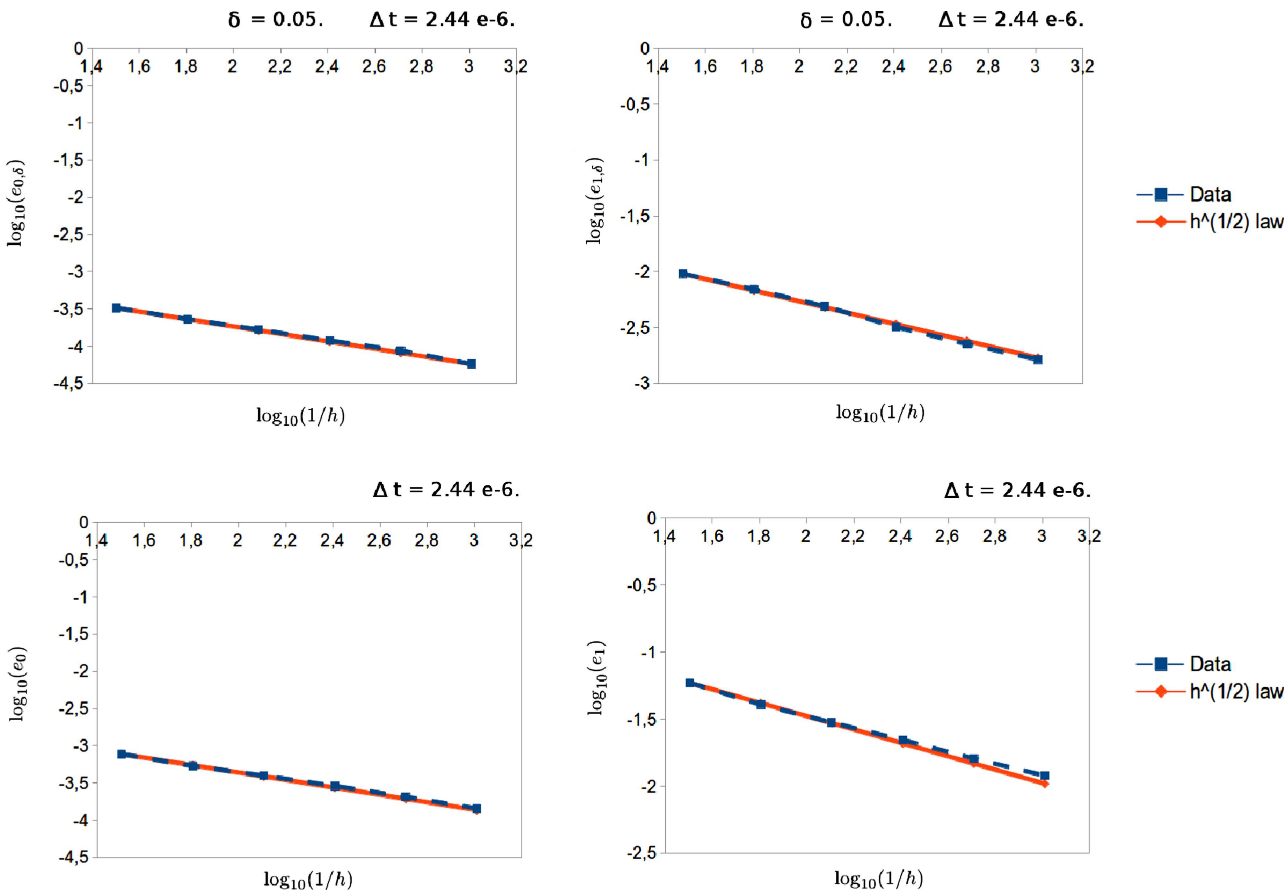

FIGURE 2. Test case 1. Values of $\log_{10}(e_{0,\delta})$, $\log_{10}(e_{1,\delta})$, $\log_{10}(e_0)$, $\log_{10}(e_1)$ *vs.* $\log_{10}(1/h)$, for $\delta = 0.05$, $\Delta t = 2.44 \cdot 10^{-6}$, and an exact decay law proportional to $h^{1/2}$.

$$
\begin{aligned}
e_{1,\delta} &:= \left|\left|\frac{\partial}{\partial x}(c - C_h^+)\right|\right|_{L^\infty(0,T;L^2(D_\delta^+(t)))},\\
e_0 &:= ||c - C_h^+||_{L^\infty(0,T;L^2(\Omega))},\\
e_1 &:= \left|\left|\frac{\partial}{\partial x}(c - C_h^+)\right|\right|_{L^\infty(0,T;L^2(\Omega))}.
\end{aligned}
$$

Three test cases will be studied in one and two space dimensions in which proper right hand sides are added to equation (1.1) in such a way that exact solutions are known. In the first test case a one dimensional stationary $H^1(\Omega)$ exact solution will be considered. In the second test case a one dimensional time dependent $H^2(\Omega)$ exact solution is considered; note that a more regular solution could result in a higher rate of convergence for the error estimates. In the third test case a two dimensional $H^1(\Omega)$ exact solution is considered.

In order to solve the variational inequality at each time step in (1.8) we use the splitting algorithm presented in [3] (in Sect. 5), in which a scalar variational inequality is only solved, by means of a projected gradient method, on the nodes of the triangulation whose associated basis functions span the set on which the degenerate operator can be inverted.

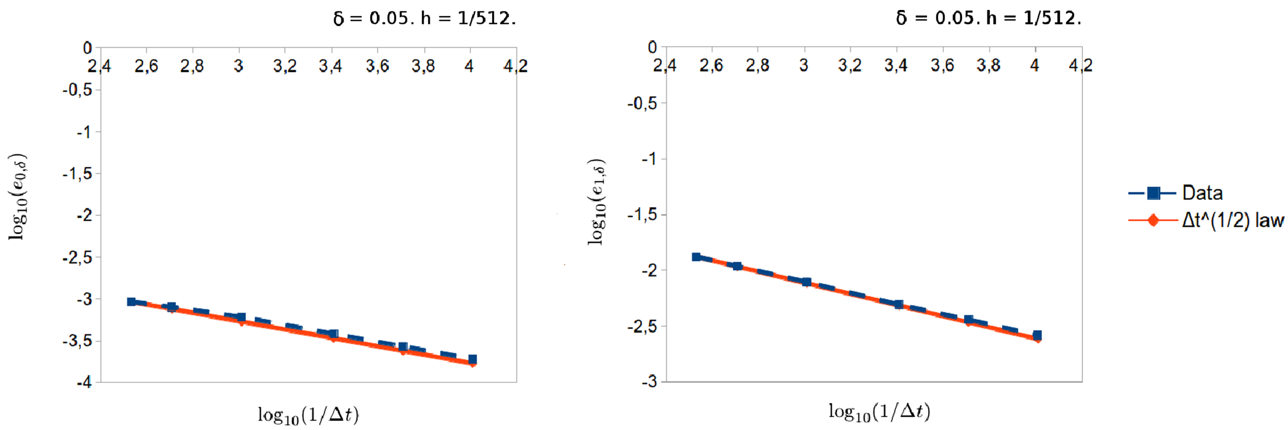

FIGURE 3. Test case 1. Values of $\log_{10}(e_{0,\delta})$, $\log_{10}(e_{1,\delta})$, *vs.* $\log_{10}(1/\Delta t)$, for $\delta = 0.05$, $h = 1/512$, and an exact decay law proportional to $\Delta t^{1/2}$.

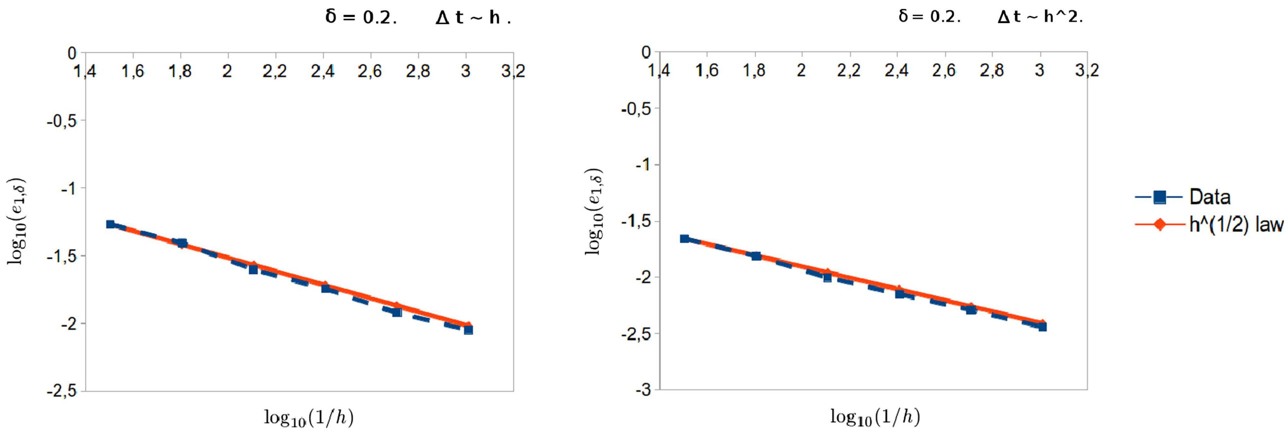

FIGURE 4. Test case 1. Values of $\log_{10}(e_{1,\delta})$, *vs.* $\log_{10}(1/h)$, for $\delta = 0.2$, with $\Delta t \sim h$ and $\Delta t = h^2$, and an exact decay law proportional to $h^{1/2}$.

6.1. Test case 1 - One dimensional stationary $H^1(\Omega)$ solution

Let us study a first test case in which a proper right hand side is added to equation (1.1) in such a way that the function

$$\begin{cases} c(x,t) = \frac{1}{2}\cos\left(\frac{x}{\sqrt{\gamma}} + \pi\right) & \text{if } \frac{\pi\sqrt{\gamma}}{2} \le x \le \frac{3\pi\sqrt{\gamma}}{2}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{6.1}$$

is a stationary exact solution. This solution has $H^1(\Omega)$ regularity, since its space derivative has a jump discontinuity. As data the values $\gamma = 0.0196$, $c^* = 0.6$ are taken. The domain is $\Omega = [0,1]$, and $T = 100\Delta t$.

In Figures 2–4 the convergence behaviours of the errors $e_{0,\delta}$, $e_{1,\delta}$, calculated inside the support of the solution (6.1) for different values of δ, and of the errors e_0, e_1 on the whole domain, are shown, by plotting the $\log_{10}$ of the error norm and seminorm in function of $\log_{10}(1/h)$ and $\log_{10}(1/\Delta t)$.

In Figure 2 the convergence behaviour is studied by varying the parameter h and keeping a small fixed Δt.

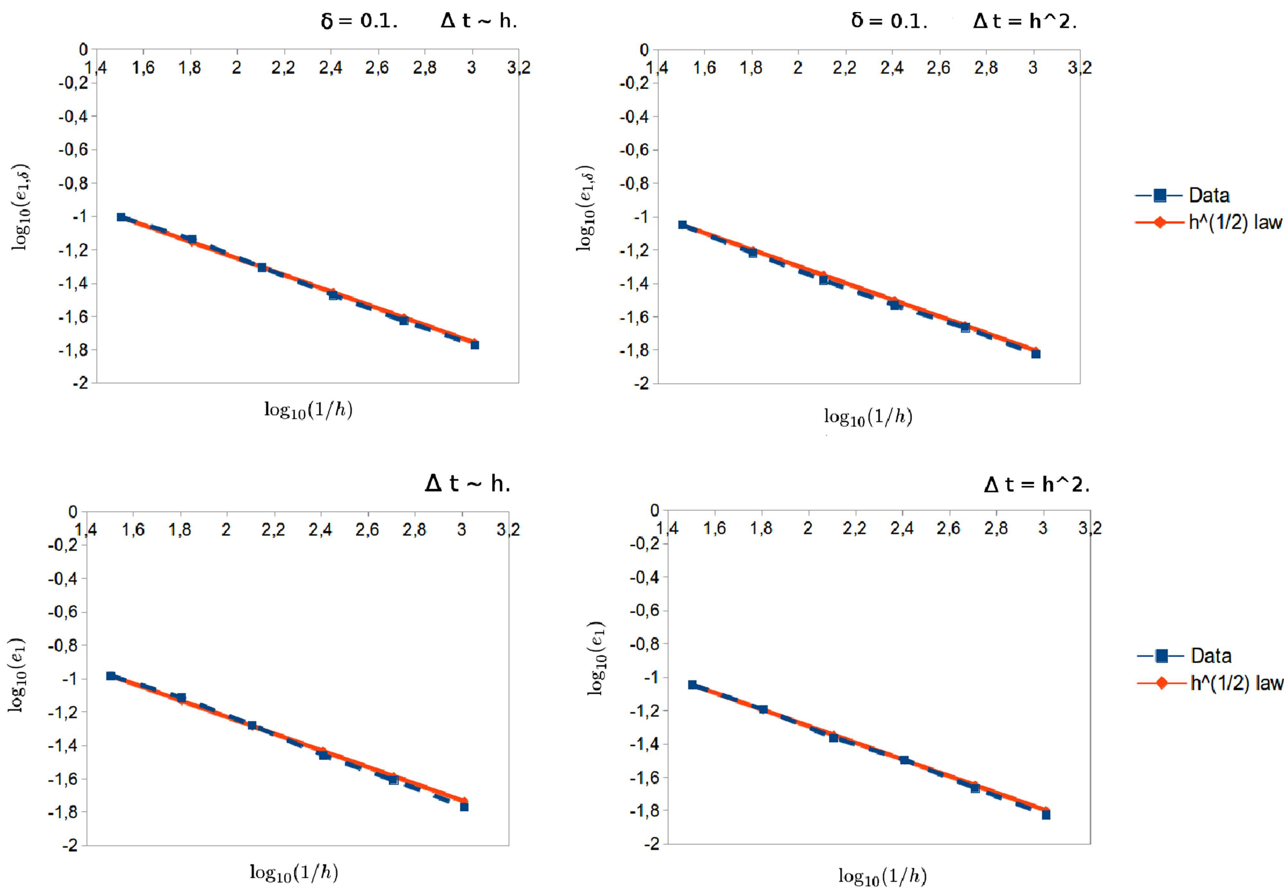

FIGURE 5. Test case 2. Values of $\log_{10}(e_{1,\delta})$ and $\log_{10}(e_1)$ *vs.* $\log_{10}(1/h)$, for $\delta = 0.1$, for the cases $\Delta t \sim h$ and $\Delta t = h^2$, and an exact decay law proportional to $h^{1/2}$.

From Figure 2 it can be observed that the $h^{1/2}$ behaviour of the error estimate (2.16) is recovered. Moreover, note that (4.40) is a rough estimate, and that effectively

$$||c - C_h^+||_{L^\infty(0,T;H^1(\Omega))} \leq Ch^{1/2}, \tag{6.2}$$

as predicted in (4.44) and (4.46) (note that the support of (6.1) is fixed in time).

In Figure 3 the convergence behaviour is studied by varying the parameter Δt and keeping a small fixed h.

From Figure 3 it can be noted that the $\Delta t^{1/2}$ behaviour of the error estimate (2.16) is recovered.

In Figure 4 the convergence behaviour is studied both for the cases $\Delta t \sim h$ and $\Delta t = h^2$, in order to observe if the rate of convergence changes if we change the assumption $\Delta t \sim h$ to $\Delta t = h^2$, which satisfies the constraint introduced in Remark 4.4.

From Figure 4 it can be noted that the $h^{1/2}$ behaviour of the error estimate (2.16) is recovered both for the cases $\Delta t \sim h$ and $\Delta t = h^2$.

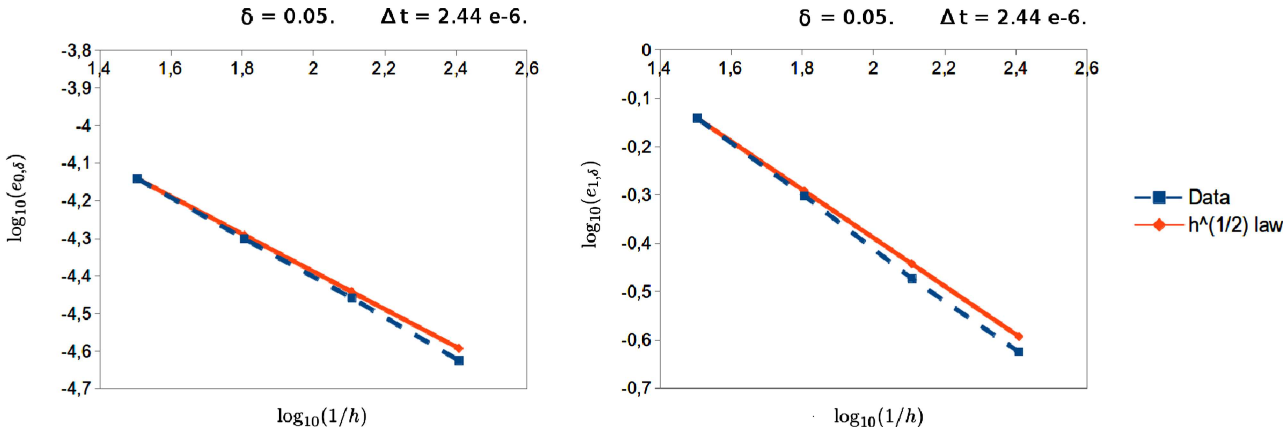

FIGURE 6. Test case 3. Values of $\log_{10}(e_{0,\delta})$ and $\log_{10}(e_{1,\delta})$ *vs.* $\log_{10}(1/h)$, for $\delta = 0.05$, for $\Delta t = 2.44 \cdot 10^{-6}$, and an exact decay law proportional to $h^{1/2}$.

6.2. Test case 2 - One dimensional time dependent $H^2(\Omega)$ solution

A second test case is considered in which a proper right hand side is added to equation (1.1) in such a way that the function

$$\begin{cases} c(x,t) = \frac{\exp(-t)}{2}\sin^2\left(\frac{x}{\sqrt{\gamma}} - \frac{\pi}{2}\right) & \text{if } \frac{\pi\sqrt{\gamma}}{2} \le x \le \frac{3\pi\sqrt{\gamma}}{2}, \\ 0 \quad \text{otherwise}, \end{cases} \tag{6.3}$$

is an exact solution. Differently from (6.1), this solution is time dependent and has $H^2(\Omega)$ regularity, since its space derivative is continuous. As data the values $\gamma = 0.0196$, $c^* = 0.6$ are taken. The domain is $\Omega = [0,1]$, and $T = 100\Delta t$. The convergence behaviour is studied for the cases $\Delta t \sim h$ and $\Delta t = h^2$.

In Figure 5 the convergence behaviours of the error $e_{1,\delta}$, calculated inside the support of the solution (6.3), and of the error e_1 on the whole domain, are shown.

Note from Figure 5 that the $h^{1/2}$ behaviour of the error estimate (2.16) is recovered both for the cases $\Delta t \sim h$ and $\Delta t = h^2$. Moreover, observe that (6.2) is also valid in this test case, both for the cases $\Delta t \sim h$ and $\Delta t = h^2$.

6.3. Test case 3 - Two dimensional $H^1(\Omega)$ solution

Let us study a test case in $2-d$ dimensions in order to test the convergence properties of the discrete scheme for the $d = 2$ case. A proper right hand side is added to equation (1.1) in such a way that the function

$$\begin{cases} c(x,y,t) = \frac{1}{2}\cos\left(\frac{x^2}{\sqrt{\gamma}} + \frac{y^2}{\sqrt{\gamma}}\right) & \text{if } x^2 + y^2 \le \frac{\pi\sqrt{\gamma}}{2}, \\ 0 \quad \text{otherwise}. \end{cases} \tag{6.4}$$

is a stationary exact $H^1(\Omega)$ solution. As data the values $\gamma = 0.0196$, $c^* = 0.6$ are taken. The domain is $\Omega = [-1,1] \times [-1,1]$, and $T = 100\Delta t$. The convergence behaviour is studied by varying the parameter h and keeping a small fixed Δt. In Figure 6 the convergence behaviours of the errors $e_{0,\delta}$ and $e_{1,\delta}$, calculated inside the support of the solution (6.1), are shown, by plotting the $\log_{10}$ of the error norm and seminorm in function of $\log_{10}(1/h)$.

Note from Figure 6 that the $h^{1/2}$ behaviour of the error estimate (2.16) is recovered also in the $2-d$ case.

7. Conclusions

This work investigated the error analysis of a discrete finite element approximation of the degenerate Cahn-Hilliard equation, with degenerate mobility and single-well potential introduced in [3]. In contrast to the CH equations studied in the literature, where the degeneracy and the singularity sets coincide, here the degeneracy set $\{c = 0, c = 1\}$ and the singularity set $\{c = 1\}$ do not coincide. This constitutive choice introduces further complications, as explained in the Introduction, which causes the error analysis to be non-standard with respect to the standard CH case with constant mobility. Starting from some preliminary Lemmas, introduced in Section 3 and shown in Section 5, which in particular give the estimates of the $L^2(0, T; H^{-1}(\Omega))$ and $L^2(0 < c < 1)$ norms of the time increment of the error of the concentration c in terms of the discretization parameters, the main *a priori* error estimates were derived, which describe the fact that the norms of the approximation errors for the concentration variable c and for the chemical potential variable w, calculated on the support of the solution c in the spaces $L^\infty(0, T; H^1(D^+_\delta(t)))$ and $L^2(0, T; H^1(D^+_\delta(t)))$ respectively, are bounded by power laws of the discretization parameters with exponent $1/2$. These estimates are obtained for discretization parameters h and Δt which satisfy the condition $\Delta t \sim h$, which guarantees that the discrete solution is able to track compactly supported solutions of (1.1) with a free boundary which moves with a finite speed of velocity (see in particular condition (1.11)). This property is peculiar to a degenerate fourth order parabolic equation. The obtained *a priori* estimates are indeed different from that obtained in the case of the classical CH equation with constant mobility (see *e.g.* [21] for details), where the exponent in the power laws of the discretization parameters is 1 and no relation between the discretization parameters has to be satisfied. The main result of this paper is introduced in Theorem 2.2, and shown in Section 4 in the $d = 1$ dimensional case. Let us however note that the error estimates could be in principle extended to the general $d = 2, 3$ dimensional cases, as described in the Introduction.

Finally, in Section 6 some numerical results for different test cases with known exact solutions both in one and two space dimensions were reported, which validated the *a priori* error estimates introduced in Section 2. The numerical results validated the theoretical error estimates in the case of a one dimensional stationary $H^1(\Omega)$ exact solution (test case 1), in the case of a more regular exact solution, *i.e.* a one dimensional time dependent $H^2(\Omega)$ exact solution (test case 2), and in the case of a two dimensional stationary $H^1(\Omega)$ exact solution (test case 3).

Future work will concern the study of the convergence of the finite element approximation (1.8) to the weak formulation (1.12) in the $d = 2, 3$ dimensional cases, together with the natural extension of the error estimates in Theorem 2.2 to the multidimensional case. Moreover, the error analysis of the discrete solution obtained using a finite element approximations with discontinuous elements will be investigated. In this case, no lumping of the scalar product has to be introduced in order to have discrete solutions which track compactly supported solutions with moving support, and hence better convergence properties than the ones obtained within the present discretization could be expected.

Acknowledgements. The author acknowledges P. Ciarletta, M. Grasselli, M. Verani and P. F. Antonietti for some useful discussions.

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