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ERROR ESTIMATES OF THE THIRD ORDER RUNGE-KUTTA ALTERNATING EVOLUTION DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS

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Abstract. In this paper, we present the stability analysis and error estimates for the alternating evolution discontinuous Galerkin (AEDG) method with third order explicit Runge-Kutta temporal discretization for linear convection-diffusion equations. The scheme is shown stable under a CFL-like stability condition $c_0\tau \le \epsilon \le c_1 h^2$. Here ϵ is the method parameter, and h is the maximum spatial grid size. We further obtain the optimal L^2 error of order $O(\tau^3 + h^{k+1})$. Key tools include two approximation finite element spaces to distinguish overlapping polynomials, coupled global projections, and energy estimates of errors. For completeness, the stability analysis and error estimates for second order explicit Runge-Kutta temporal discretization is included in the appendix.

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1. Introduction

In this paper, we present the stability analysis and *a priori* error estimates of Runge-Kutta alternating evolution discontinuous Galerkin (RKAEDG) method to smooth solutions of linear convection-diffusion equation

$$\partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi, \quad (x,t) \in [a,b] \times (0,T], \tag{1.1a}$$
$$\phi(x,0) = \phi_0(x), \quad x \in [a,b], \tag{1.1b}$$

here $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$ are given constants. We do not pay attention to boundary conditions in this paper, hence the solution is considered to be periodic; though other boundary conditions can also be studied along the same lines.

The AEDG method is a grid-based discontinuous Galerkin (DG) method, which was introduced by Liu and Pollack first in [7] for Hamilton-Jacobi equations, and further developed in [8] for nonlinear convection-diffusion

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equation

$$\partial_t \phi + \nabla_x \cdot f(\phi) = \Delta_x B(\phi), \tag{1.2}$$

in one and multi-dimensional setting, where $f(\phi)$ is a given flux function, and $B(\phi)$ a non-decreasing function. These and earlier works [13, 15] are all based on the alternating evolution (AE) framework introduced in [6]. The scheme construction is carried out by allowing the neighboring polynomials to overlap. In particular, the AEDG method involves only one approximating polynomial near each grid point, independent of the spatial dimension, hence providing a unique high order approximation locally around each grid point. The AEDG method is similar to the central DG method in the sense of avoiding numerical fluxes. The central DG methods developed in [4, 10, 11, 12] use overlapping cells [5] and hence duplicative information. Developed in [12] are two versions of the central local discontinuous Galerkin (LDG) methods for solving the diffusion equation, $u_t = u_{xx}$, based on discretizing an equivalent first order system, $u_t - r_x = 0,\ r - u_x = 0$ on overlapping cells. For one dimensional setting of same meshes, the central LDG method involves four times the computational cost and storage requirement than the AEDG method presented in this paper. This distinction is more pronounced in multi-dimensional setting. We refer to [7, 13] for comments on differences between the AEDG method and the central DG methods [4, 10, 11] for hyperbolic problems.

The one-dimensional semi-discrete AEDG scheme introduced in [8] has the following form:

$$\begin{aligned}\int_{I_j} (\partial_t \Phi_j + \partial_x f(\Phi_j^{SN}) - \partial_x^2 B(\Phi_j^{SN}))\eta \mathrm{d}x = \left(-[f(\Phi_j^{SN})]\eta + [\partial_x B(\Phi_j^{SN})]\eta - [B(\Phi_j^{SN})]\partial_x \eta\right)\Big|_{x=x_j} \\ + \frac{1}{\epsilon}\left(\int_{I_j} \Phi_j^{SN} \eta \mathrm{d}x - \int_{I_j} \Phi_j \eta \mathrm{d}x\right),\end{aligned}$$

where x_j is the grid point in cell I_j, in which the numerical solution is denoted by Φ_j; Φ_j^{SN} are sampled from neighboring polynomials $\Phi_{j\pm1}$, with $[g(\Phi_j^{SN})]|_{x_j}$ standing for the difference of two neighboring functions at x_j in the sense that $[g(\Phi_j^{SN})]|_{x_j} = g(\Phi_{j+1}(x_j^+)) - g(\Phi_{j-1}(x_j^-))$.

In contrast to other DG methods, the stability analysis of the AEDG method is more subtle since stability property is less obvious from the scheme formulation. For linear convection–diffusion equation (1.1a), the L^2 stability of the semi-discrete AEDG method has been proven if $\epsilon \le Qh^2$, for some Q and mesh size h in [8], in which the technical difficulty was resolved by a special regrouping of mixed terms combined with the use of some inverse inequalities.

Further in [9] the authors obtained the first optimal L^2 error estimates based on the stability result established in [8] for the semi-discrete AEDG scheme (2.2). For a class of fully discrete $\theta-$schemes, the stability condition relating ϵ to the time step τ of the form $c_0\tau \le \epsilon < Qh^2$ for some $c_0 > 1$ is shown sufficient for obtaining the following optimal error estimate in [9] as

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}^n(x) - \phi(x,t^n)|^2 + |\Phi_j^n(x) - \phi(x,t^n)|^2}{2} \mathrm{d}x \le C(|1-2\theta|\tau + \tau^2 + h^{k+1})^2,$$

for $\theta = 0$ the Euler forward method, $\theta = 1$ the Euler backward, and $\theta = 1/2$ the Crank-Nicolson. Here of course Φ_j^n is the numerical solution at time level n near grid x_j, τ is the time step, and the positive constant C is independent of τ, h and the numerical solution. This estimate differs from the usual L^2 error since the AEDG method uses overlapping polynomials. These features require new techniques in the error estimates.

High order fully discrete schemes are usually obtained by applying certain Runge-Kutta time discretization; we refer to Cockburn and Shu [2] for a review of the development of the RKDG methods for nonlinear convection–dominated problems. With the third order explicit SSP Runge–Kutta time discretization [3], Liu and Pollack

[8] presented excellent numerical results for applying the AEDG method to both one and two dimensional convection–diffusion problems. In this paper, we are to carry out the priori error estimates for such RK3AEDG method approximating the smooth solutions of (1.1). The error analysis for RK2AEDG is simpler, and included in the appendix. A general discussion of the AEDG method and background references on the error estimates for the DG methods for convection-diffusion problems are given in the introduction to [9]. In this paper, we have two objectives:
(i) to present the stability analysis of the RK3AEDG method;
(ii) to estimate the difference in L^2 norm between the exact solution and the approximate ones.
The stability analysis for (i) is based on the AE formulation and carried out by identifying a sufficient condition on the time step restriction, relating to the method parameter ϵ.

The error estimates for (ii) are based on Taylor's expansion and energy estimates, following the recent works on the RKDG methods for hyperbolic conservation laws [18, 19, 20], in which the error equations involve a nonlinear operator. In [18, 20] the authors obtained error estimates for the second order explicit RKDG method for smooth solutions, under the stronger time step restriction $\tau = o(h)$. In contrast, the optimal error estimates in [19] for the third order RKDG method are obtained under the standard temporal-spatial restriction $\tau \leq Ch$ for convection. With the RKAEDG method for linear convection-diffusion equations, our error analysis is carried out by solving a coupled system involving two bilinear operators, and we essentially use several tools developed in [9], including two approximation spaces $V_h \times U_h$ associating with odd and even grids, respectively, with which the AE scheme can be reformulated using two bi-linear operators; the two global projections on V_h and U_h, coupled through the ϵ-dependent term dictated by the AEDG formulation, the projection errors, as well as the ϵ-dependent energy norm in $V_h \times U_h$, involving a special term of the form $h^{-1}\|u - v\|$ for $(v, u) \in V_h \times U_h$. The error analysis for AEDG methods is more involved because the coupling between overlapping polynomials must be carefully handled. Nevertheless, for both RK2AEDG and RK3AEDG methods, we are able to obtain the optimal error estimates under the standard temporal-spatial restriction $\tau \leq Ch^2$ for diffusion, with polynomials of arbitrary degree $k \geq 1$. The error estimate for other fully discrete DG methods has also been made available recently, see, *e.g.* [16, 17] for the LDG method coupled with a third order Runge–Kutta time discretization to solve linear convection-diffusion equations.

We now mention results related to the central DG methods in the literature. The development of the central DG technique for hyperbolic conservation laws first appeared in [10], then stability analysis and error estimates were obtained in [11] for linear hyperbolic problems. Further development of the central DG methods can be found in [4] for Hamilton-Jacobi equations, and in [12] for diffusion equations. The interesting comparison analysis by Reyna and Li [14] for linear convection problems indicates that for a fixed stable time discretization, the time step allowed for the DG method is typically smaller than that for the central DG method.

The rest of this article is organized as follows: in Section 2 we present both the semi-discrete and fully discrete AEDG schemes with third order Runge-Kutta time discretization for the one-dimensional linear convection-diffusion equation, and the main results of both stability and optimal L^2 error estimates. In Section 3 we reformulate the RK3AEDG scheme as a coupled system using two bi-linear operators, and then review several useful tools and known results from [9]. In Section 4, we figure out a sufficient condition on the time step restriction so that the RK3AEDG can be shown stable. Finally, optimal L^2 error estimates are given in Section 5. The stability analysis and error estimates for second order explicit Runge-Kutta temporal discretization is included in Appendix A.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset [a, b]$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = [a, b]$, we omit the subscript D; and if $p = 2$, we set $W^{m,p}(D) = H^m(D)$, $\|\cdot\|_{m,p,D} = \|\cdot\|_{m,D}$, and $|\cdot|_{m,p,D} = |\cdot|_{m,D}$. We use either $\|\cdot\|_{0,D}$ or $\|\cdot\|$ when $D = [a, b]$ to denote the usual L^2 norm. We also use the notation $A \lesssim B$ to indicate that A can be bounded by B multiplied by a constant independent of the mesh size τ, h. $A \sim B$ stands for $A \lesssim B$ and $B \lesssim A$. We will also use C to denote a positive constant independent of h and τ, which may depend on solutions of (1.1).

2. Alternating evolution DG methods

The AEDG method consists of a semi-discrete formulation based on sampling of the AE system on alternating grids and a fully discrete version by using an appropriate Runge-Kutta solver.

2.1. Setting of semi-discrete AEDG method

Recall the AEDG method for the one-dimensional convection-diffusion equation

$$\partial_t \phi + \partial_x f(\phi) = \partial_x^2 (B(\phi)) \tag{2.1}$$

subject to initial data $\phi_0(x)$ and periodic boundary conditions.

Partition the spatial domain $[a,b]$ into a grid with grid points $\{x_j\}$ such that $x_1 = a, x_N = b$. Set $I_j = (x_{j-1}, x_{j+1})$ for $j = 1, 2, \ldots, N-1$, while $I_1 = (x_0, x_2)$ in which (x_0, x_1) is the periodic shift of (x_{N-1}, x_N) and $h_j = \frac{x_{j+1} - x_{j-1}}{2}$, and we define the quantities

$$h = \max_{1 \le j \le N-1} h_j \text{ and } \rho = \min_{1 \le j \le N-1} h_j.$$

For simplicity of presentation we would like to assume that the ratio of h and ρ is upper bounded by a fixed positive constant ν^{-1} when h goes to zero so that $\nu h \le \rho \le h$. We shall analyze the uniform grid case $\nu = 1$, knowing that the techniques can be easily carried over to the case $\nu \neq 1$.

Centered at each grid $\{x_j\}$, the numerical approximation is a polynomial $\Phi|_{I_j} = \Phi_j(x) \in P^k$, where P^k denotes a linear space of all polynomials of degree at most k:

$$P^k := \{p \mid p(x)|_{I_j} = \sum_{0 \le i \le k} a_i (x - x_j)^i, \quad a_i \in \mathbb{R}\}.$$

We denote $v(x^{\pm}) = \lim_{\epsilon \to 0\pm} v(x+\epsilon)$, and $v_j^{\pm} = v(x_j^{\pm})$. The jump at x_j is $[v]|_{x_j} = v(x_j^+) - v(x_j^-)$. Note that the solution space here differs from the usual finite element space since it allows the overlapping of two neighboring polynomials of Φ_j and Φ_{j+1} over $I_j \cap I_{j+1} = [x_j, x_{j+1}] \neq \emptyset$.

The semi-discrete AEDG scheme introduced in [8] is to find $\Phi|_{I_j} \in P^k$ such that for all $\eta \in P^k(I_j)$,

$$\begin{aligned}\int_{I_j} (\partial_t \Phi_j + \partial_x f(\Phi_j^{SN}) - \partial_x^2 B(\Phi_j^{SN}))\eta \mathrm{d}x &= \left(-[f(\Phi_j^{SN})]\eta + [\partial_x B(\Phi_j^{SN})]\eta - [B(\Phi_j^{SN})]\eta_x\right)\Big|_{x=x_j} \\ &+ \frac{1}{\epsilon}\left(\int_{I_j} \Phi_j^{SN} \eta \mathrm{d}x - \int_{I_j} \Phi_j \eta \mathrm{d}x\right),\end{aligned} \tag{2.2}$$

where Φ_j^{SN} is defined as

$$\Phi_j^{SN} = \begin{cases} \Phi_{j-1}(x), & x_{j-1} < x < x_j, \\ \Phi_{j+1}(x), & x_j < x < x_{j+1} \end{cases}$$

with periodic boundary conditions. $\Phi_N(x)$ is regarded to be identical to $\Phi_1(x)$, which is computed over $I_1 = [x_0, x_2] = [a-h, a+h]$. Numerical solution on $[x_{N-1}, x_N]$ is simply taken from Φ_1 over $[x_0, x_1]$. Note that $\Phi_1(x, 0) = \Phi_N(x, 0)$ for initial data.

The semi-discrete AEDG scheme is also shown to be conservative and stable for linear problems in [8].

Theorem 2.1. *([8], Thms. 3.1 and 3.2) Let Φ be computed from the AEDG scheme (2.2) for the linear convection-diffusion equation*

$$\partial_t \phi + \alpha \partial_x \phi = \beta \partial_x^2 \phi,$$

with periodic boundary conditions. Then
(i) the scheme is conservative in the sense that

$$\frac{d}{dt}\left(\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\Phi_{j+1}+\Phi_j}{2}\, dx\right) = 0; \mathit{and}$$

(ii) the scheme using polynomials of degree $k \geq 1$ is L^2 stable if $\epsilon \leq Qh^2$. Moreover,

$$\frac{d}{dt}\left(\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\Phi_{j+1}^2+\Phi_j^2}{2}\, dx\right) \leq -\beta \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{(\partial_x \Phi_{j+1})^2 + (\partial_x \Phi_j)^2}{2}\, dx + \left(\frac{1}{Qh^2} - \frac{1}{\epsilon}\right) \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} (\Phi_{j+1} - \Phi_j)^2\, dx \tag{2.3}$$

with

$$Q = \frac{1}{\beta (k+1)^2 (17(k+1)^2 - 1)}. \tag{2.4}$$

2.2. Fully discrete AEDG method with third order Runge-Kutta time discretization

We now turn to time discretization of (2.2). Let $\{t^n\}, n = 0, 1, \ldots, K$ be a uniform partition of the time interval $[0, T]$ and denote the time step size as τ. The initial data for $\Phi_j(x, 0)$ is taken as the L^2 projection of ϕ_0 on I_j for $j = 1, \ldots, N-1$:

$$\int_{I_j} \Phi_j(x,0)\eta \mathrm{d}x = \int_{I_j} \phi_0(x)\eta \mathrm{d}x, \quad \forall \eta \in P^k(I_j), \quad j = 1, \ldots, N-1. \tag{2.5}$$

Denote $\Psi = [\psi_1, \ldots, \psi_J]^\top$ the unknown coefficients of the numerical solution against the basis in the DG space, the ODE system (2.2) can be written as

$$\partial_t \Psi = L(\Psi),$$

where $L(\cdot)$ is some spatial differential operator defined by (2.2).

We use the third order explicit SSP Runge-Kutta method [3] for time discretization. In details, let $\Psi^{n,0}$ be the solution at time level n, and $\Psi^{n,i}, i = 1, 2$ be the solution at intermediate step between t^n and t^{n+1}, thus we can write

$$\Psi^{n,1} = \Psi^{n,0} + \tau L(\Psi^{n,0}), \tag{2.6}$$

$$\Psi^{n,2} = \frac{3}{4}\Psi^{n,0} + \frac{1}{4}\left(\Psi^{n,1} + \tau L(\Psi^{n,1})\right), \tag{2.7}$$

$$\Psi^{n+1} = \frac{1}{3}\Psi^{n,0} + \frac{2}{3}\left(\Psi^{n,2} + \tau L(\Psi^{n,2})\right). \tag{2.8}$$

Based on the above setting, we are able to show the scheme is stable under some restriction on the time step τ and ϵ, and further obtain the optimal L^2 error estimates for (2.2) with third order time discretization (2.6)–(2.8). The main results are summarized in the following.

Theorem 2.2. *Let Φ^n be the numerical solution computed from (2.6)–(2.8) with $\epsilon = cQh^2$, $0 < c < 1$, then there exists $c_0 > 0$ such that for h small,*

$$c_0\tau \le \epsilon = cQh^2, \tag{2.9}$$

we have

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\left(\Phi_{j+1}^{n+1}\right)^2 + \left(\Phi_j^{n+1}\right)^2}{2} dx \le \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{\left(\Phi_{j+1}^{n}\right)^2 + \left(\Phi_j^{n}\right)^2}{2} dx.$$

Theorem 2.3. *Let ϕ be the smooth solution to (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, and $\Phi_j^n \in P^k(I_j)(k \ge 1)$ be the numerical solution to (2.6)–(2.8) with $c_0\tau \le \epsilon = cQh^2$ for some $c_0 > 0$ and $0 < c < 1$, then the following error estimate holds:*

$$\sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}^n(x) - \phi(x, t^n)|^2 + |\Phi_j^n(x) - \phi(x, t^n)|^2}{2} dx \le C(\tau^6 + h^{2k+2}), \quad n\tau \le T, \tag{2.10}$$

where C is a constant independent of τ, h and n.

We defer the proof of Theorem 2.2 to Section 4 and Theorem 2.3 to Section 5.

Remark 2.4. The CFL condition given in (2.9) is a sufficient condition rather than necessary to preserve L^2 stability of numerical solutions, which ensure the optimal error estimates. Therefore, in practice, such CFL condition is strictly enforced only in the case the stability property is violated. Technically, a slightly sharper estimate (4.26) of c_0 can be obtained in the proof of Theorem 2.2, also Theorem 4.2.

3. Scheme reformulation and useful tools

3.1. Scheme reformulation

Following [9], we introduce two solution spaces of piecewise polynomials as

$$V_h = \{\eta \in L^2, \eta \in P^k(I_j), \quad j = \text{odd}\}, \quad U_h = \{\eta \in L^2, \eta \in P^k(I_j), \quad j = \text{even}\}. \tag{3.1}$$

Note that for N odd, the set $\{j = even\} = \{2, 4, \dots, N-1\}$, and $\{j = \text{odd}\} = \{1, 3, \dots, N-2\}$; For N even, the set $\{j = \text{even}\} = \{2, 4, \dots, N-2\}$ and $\{j = \text{odd}\} = \{1, 3, \dots, N-1\}$. This way the periodic boundary condition is always satisfied through $\Phi_1 = \Phi_N$, with $\Phi_1 \in V_h$, no matter N is odd or even.

Taking $f(w) = \alpha w$ and $B(w) = \beta w$ in AEDG scheme (2.2), summing over $j = odd$ and $j = even$, respectively, we obtain a coupled system

$$\langle \partial_t v, \xi\rangle + A_{21}(u, \xi) = \frac{1}{\epsilon}\langle u - v, \xi\rangle, \quad \xi \in V_h, \tag{3.2}$$

$$\langle \partial_t u, \eta\rangle + A_{12}(v, \eta) = \frac{1}{\epsilon}\langle v - u, \eta\rangle, \quad \eta \in U_h, \tag{3.3}$$

where inner product is defined as $\langle w,\xi\rangle = \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} w\xi \mathrm{d}x$ and the two bilinear operators are defined by

$$A_{21}(u,\xi) = \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} \partial_x J(u)\xi \mathrm{d}x + \sum_{j=odd}([J(u)]\xi + \beta[u]\partial_x\xi)|_{x_j}, (u,\xi)\in U_h\times V_h,$$
$$A_{12}(v,\eta) = \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} \partial_x J(v)\eta \mathrm{d}x + \sum_{j=even}([J(v)]\eta + \beta[v]\partial_x\eta)|_{x_j}, (v,\eta)\in V_h\times U_h,$$

where $J(w)=\alpha w-\beta\partial_x w$. Note that for N odd, $\int_{x_{N-1}}^{x_N}\partial_x J(v)\eta \mathrm{d}x$ is defined by $\int_{x_0}^{x_1}\partial_x J(v)\eta \mathrm{d}x$; and for N even, $\int_{x_{N-1}}^{x_N}\partial_x J(u)\xi \mathrm{d}x$ is defined by $\int_{x_0}^{x_1}\partial_x J(v)\xi \mathrm{d}x$, using the periodicity of the numerical solution. We remark that the subscripts in the operator A_{12} or A_{21} indicate the odd and even (or even and odd) spaces to which the corresponding arguments belong.

For notational convenience, in the following we use $\|\partial_x v\|^2 := \int_a^b(\partial_x v)^2\mathrm{d}x$ and $\|\partial_x u\|^2 := \int_a^b(\partial_x u)^2\mathrm{d}x$ to denote

$$\sum_{j=odd}\int_{x_{j-1}}^{x_{j+1}}|\partial_x v|^2\mathrm{d}x, \qquad \sum_{j=even}\int_{x_{j-1}}^{x_{j+1}}|\partial_x u|^2\mathrm{d}x \tag{3.4}$$

respectively if $(v,u)\in V_h\times U_h$, unless otherwise stated. Also, we define the L^2 norm and energy norm of $(v,u)\in V_h\times U_h$ as

$$\|(v,u)\|^2 := \|v\|^2+\|u\|^2 \tag{3.5}$$

and

$$\|(v,u)\|_E^2 := \|(v,u)\|^2 + \|(\partial_x v,\partial_x u)\|^2 + \frac{1}{h^2}\|v-u\|^2, \tag{3.6}$$

respectively, where $\|\cdot\|$ is the L^2 norm for functions in V_h and U_h shown in (3.4).

For semi-discrete AEDG scheme (2.2), the stability result obtained in [8] is as follows.

Lemma 3.1. *For any $(v,u)\in V_h\times U_h$, we have*

$$A_{21}(u,v)+A_{12}(v,u) \ge \frac{\beta}{2}\|(\partial_x v,\partial_x u)\|^2 - \frac{1}{Qh^2}\|v-u\|^2,$$

where Q is defined in (2.4).

The AEDG method with the third order SSP Runge-Kutta method for time discretization gives the RK3AEDG scheme (2.6)–(2.8), which can be rewritten as

$$\langle v^{n,1},\xi\rangle = \langle v^n,\xi\rangle - \tau A_{21}(u^n,\xi) + \frac{\tau}{\epsilon}\langle u^n-v^n,\xi\rangle, \qquad (\xi,\eta)\in V_h\times U_h, \tag{3.7a}$$
$$\langle u^{n,1},\eta\rangle = \langle u^n,\eta\rangle - \tau A_{12}(v^n,\eta) + \frac{\tau}{\epsilon}\langle v^n-u^n,\eta\rangle, \qquad (v^n,u^n)\in V_h\times U_h; \tag{3.7b}$$

$$\langle v^{n,2},\xi\rangle = \frac14\langle 3v^n+v^{n,1},\xi\rangle - \frac{\tau}{4}A_{21}(u^{n,1},\xi) + \frac{\tau}{4\epsilon}\langle u^{n,1}-v^{n,1},\xi\rangle, \quad (\xi,\eta)\in V_h\times U_h, \tag{3.8a}$$
$$\langle u^{n,2},\eta\rangle = \frac14\langle 3u^n+u^{n,1},\eta\rangle - \frac{\tau}{4}A_{12}(v^{n,1},\eta) + \frac{\tau}{4\epsilon}\langle v^{n,1}-u^{n,1},\eta\rangle, \quad (v^{n,1},u^{n,1})\in V_h\times U_h; \tag{3.8b}$$

$$\langle v^{n+1}, \xi\rangle = \frac{1}{3}\langle v^n + 2v^{n,2}, \xi\rangle - \frac{2\tau}{3}A_{21}(u^{n,2}, \xi) + \frac{2\tau}{3\epsilon}\langle u^{n,2} - v^{n,2}, \xi\rangle, \quad (\xi, \eta) \in V_h \times U_h, \tag{3.9a}$$

$$\langle u^{n+1}, \eta\rangle = \frac{1}{3}\langle u^n + 2u^{n,2}, \eta\rangle - \frac{2\tau}{3}A_{12}(v^{n,2}, \eta) + \frac{2\tau}{3\epsilon}\langle v^{n,2} - u^{n,2}, \eta\rangle, \quad (v^{n,2}, u^{n,2}) \in V_h \times U_h. \tag{3.9b}$$

3.2. Projection and projection errors

In this subsection, we review the global projections introduced in [9] and the associated properties proved therein.

Suppose w is a smooth periodic function, the two coupled projections $(\Pi_v w, \Pi_u w) \in V_h \times U_h$ introduced in [9] are as follows

$$\langle \Pi_v w - w, \xi\rangle + A_{21}(\Pi_u w - w, \xi) = \frac{1}{\epsilon}(\Pi_u w - \Pi_v w, \xi), \quad \xi \in V_h, \tag{3.10}$$

$$\langle \Pi_u w - w, \eta\rangle + A_{12}(\Pi_v w - w, \eta) = \frac{1}{\epsilon}(\Pi_v w - \Pi_u w, \eta), \quad \eta \in U_h. \tag{3.11}$$

Here, we again construct $\Pi_v w, \Pi_u w$ over the extended cell $I_1 = [x_0, x_2]$, and set

$$\Pi_v w|_{[x_{N-1}, x_N]} = \Pi_v w|_{[x_0, x_1]}, \quad N = \text{odd},$$
$$\Pi_u w|_{[x_{N-1}, x_N]} = \Pi_u w|_{[x_0, x_1]}, \quad N = \text{even}.$$

Such extension for both $\Pi_v w$ and $\Pi_u w$ is made so that they become periodic.

Theorem 3.2. *([9], Lem. 3.2, Thms. 3.3 and 4.4) Let w be a smooth periodic function that belongs to H^m, if $\epsilon = cQh^2$ for $0 < c < 1$, then the two projection operators Π_v, Π_u defined in (3.10) and (3.11) have the following projection error*

$$\|(\Pi_v w - w, \Pi_u w - w)\|_E \le Ch^{\min\{k,m\}}|w|_m, \tag{3.12}$$

$$\|\Pi_v w - w\| + \|\Pi_u w - w\| \le Ch^{\min\{k+1,m\}}|w|_m, \tag{3.13}$$

where C is a constant independent of h.

We recall some local approximation results, see, *e.g.*,[1], Lemma 4.3.8, which will be used for the energy estimates.

Lemma 3.3. *([9], Lem. 4.1) If $w \in H^m(\Omega)$ is a periodic function, then there exists polynomials $(v_I, u_I) \in V_h \times U_h$ that satisfy optimal approximation properties, i.e., $v_I \in V_h$, $u_I \in U_h$ are polynomials in I_j, for $j =$ odd and $j =$ even, respectively,*

$$|w - v_I|_{s,I_j} \le Ch^{\min\{m,k+1\}-s}|w|_{m,I_j}, j = odd,$$
$$|w - u_I|_{s,I_j} \le C'h^{\min\{m,k+1\}-s}|w|_{m,I_j}, j = even,$$

for $0 \le s \le \min\{m, k+1\}$, where C and C' are two constants independent of mesh size h.

Lemma 3.4. *([9], Lem. 4.2) Let $I = [c, d] \subset [a, b]$ be an interval of length $|I|$, and $v \in P^m(I)$, then*

$$\max\{|v(c)|, |v(d)|\} \le (m+1)|I|^{-1/2}\|v\|_{0,I}, \tag{3.14a}$$

$$\|\partial_x v\|_{0,I} \le (m+1)\sqrt{m(m+2)}|I|^{-1}\|v\|_{0,I}, \tag{3.14b}$$

$$\|v(\cdot)\|_{\infty,I}^2 \le \frac{\sqrt{5}+1}{2}(|I|^{-1}\|v\|_{0,I}^2 + |I|\|\partial_x v\|_{0,I}^2), \quad \text{if} \quad v \in H^1(I). \tag{3.14c}$$

We shall also use the following bound of two bilinear operators A_{21} and A_{12}.

Lemma 3.5. *([9], Lem. 5.1) For any $(\xi,\eta),(v,u) \in V_h \times U_h$, it holds*

$$A_{21}(u,\xi) \le h^{-1}\Gamma(\|u_x\| + h^{-1}\|v-u\|)\|\xi\|, \tag{3.15}$$
$$A_{12}(v,\eta) \le h^{-1}\Gamma(\|v_x\| + h^{-1}\|v-u\|)\|\eta\|, \tag{3.16}$$

where

$$\Gamma := \max\{|\alpha|h + \beta\gamma_k + 2(k+1)(1+\gamma_k^2)^{1/2}, 2(k+1)(1+\gamma_k^2)^{1/2}(|\alpha|h + \beta\gamma_k)\}, \tag{3.17}$$

with $\gamma_k = (k+1)\sqrt{k(k+2)}$.

4. Stability analysis

In this section, we present the L^2 stability analysis while relating time step τ to ϵ for the RK3AEDG method. We start by introducing some notations:

$$\mathcal{L}_v^{n,1} = v^{n,1} - v^n, \mathcal{L}_v^{n,2} = 2v^{n,2} - v^{n,1} - v^n, \mathcal{L}_v^{n,3} = v^{n+1} - 2v^{n,2} + v^n, \tag{4.1a}$$
$$\mathcal{L}_u^{n,1} = u^{n,1} - u^n, \mathcal{L}_u^{n,2} = 2u^{n,2} - u^{n,1} - u^n, \mathcal{L}_u^{n,3} = u^{n+1} - 2u^{n,2} + u^n. \tag{4.1b}$$

Lemma 4.1. *For the fully discrete RK3AEDG method (3.7)–(3.9), we can write*

$$\langle \mathcal{L}_v^{n,1}, \xi\rangle = -\tau A_{21}(u^n,\xi) + \frac{\tau}{\epsilon}\langle u^n - v^n, \xi\rangle, \quad \xi \in V_h, \tag{4.2a}$$
$$\langle \mathcal{L}_v^{n,2}, \xi\rangle = -\frac{\tau}{2} A_{21}(\mathcal{L}_u^{n,1},\xi) + \frac{\tau}{2\epsilon}\langle \mathcal{L}_u^{n,1} - \mathcal{L}_v^{n,1}, \xi\rangle, \quad \xi \in V_h, \tag{4.2b}$$
$$\langle \mathcal{L}_v^{n,3}, \xi\rangle = -\frac{\tau}{3} A_{21}(\mathcal{L}_u^{n,2},\xi) + \frac{\tau}{3\epsilon}\langle \mathcal{L}_u^{n,2} - \mathcal{L}_v^{n,2}, \xi\rangle, \quad \xi \in V_h, \tag{4.2c}$$
$$\langle \mathcal{L}_u^{n,1}, \eta\rangle = -\tau A_{12}(v^n,\eta) + \frac{\tau}{\epsilon}\langle v^n - u^n, \eta\rangle, \quad \eta \in U_h, \tag{4.2d}$$
$$\langle \mathcal{L}_u^{n,2}, \eta\rangle = -\frac{\tau}{2} A_{12}(\mathcal{L}_v^{n,1},\eta) + \frac{\tau}{2\epsilon}\langle \mathcal{L}_v^{n,1} - \mathcal{L}_u^{n,1}, \eta\rangle, \quad \eta \in U_h, \tag{4.2e}$$
$$\langle \mathcal{L}_u^{n,3}, \eta\rangle = -\frac{\tau}{3} A_{12}(\mathcal{L}_v^{n,2},\eta) + \frac{\tau}{3\epsilon}\langle \mathcal{L}_v^{n,2} - \mathcal{L}_u^{n,2}, \eta\rangle, \quad \eta \in U_h. \tag{4.2f}$$

Proof. Using notations in (4.1a), (4.2a) is straightforward from (3.7a). We can obtain (4.2b) by calculating $2 \times$ (3.8a) $- \frac{1}{2} \times$ (3.7a). To prove (4.2c), substituting the left hand side of (3.7a) into (3.8a), we have

$$\langle v^{n,2} - v^n, \xi\rangle = -\frac{\tau}{4} A_{21}(u^{n,1} + u^n, \xi) + \frac{\tau}{4\epsilon}\langle u^{n,1} + u^n - (v^{n,1} + v^n), \xi\rangle, \tag{4.3}$$

then by applying (3.9a) $- \frac{4}{3} \times$ (4.3), we get (4.2c). In entirely same manner, we can obtain the claimed relations (4.2d)–(4.2f) for $\mathcal{L}_u^{n,1}, \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,3}$ too. □

Stability result in Theorem 2.2 can be reformulated in terms of (v^n, u^n) as follows.

Theorem 4.2. *Let v^n and u^n be the numerical solution computed from (3.7)–(3.9) with $\epsilon = cQh^2$, $0 < c < 1$, there exists $c_0 > 0$ such that for h small,*

$$c_0\tau \le \epsilon = cQh^2, \tag{4.4}$$

then

$$\|(v^{n+1}, u^{n+1})\| \le \|(v^n, u^n)\|.$$

Proof. Taking $(\xi, \eta) = (v^n, u^n)$ in (3.7), $(\xi, \eta) = (4v^{n,1}, 4u^{n,1})$ in (3.8), $(\xi, \eta) = (6v^{n,2}, 6u^{n,2})$ in (3.9) and summing up the resulting relations, we obtain

$$\begin{aligned}&-\tau\left(A_{21}(u^n, v^n) + A_{21}(u^{n,1}, v^{n,1}) + 4A_{21}(u^{n,2}, v^{n,2}) + A_{12}(v^n, u^n) + A_{12}(v^{n,1}, u^{n,1})\right)\\&-4\tau A_{12}(v^{n,2}, u^{n,2}) - \frac{\tau}{\epsilon}\|v^n - u^n\|^2 - \frac{\tau}{\epsilon}\|v^{n,1} - u^{n,1}\|^2 - \frac{4\tau}{\epsilon}\|v^{n,2} - u^{n,2}\|^2 = \int_a^b \mathbb{V}\mathrm{d}x + \int_a^b \mathbb{U}\mathrm{d}x,\end{aligned} \tag{4.5}$$

in which $\mathbb{V}$ and $\mathbb{U}$ can be written as

$$\begin{aligned}\mathbb{V} &= -2v^n v^{n,1} - (v^n)^2 + 4v^{n,1}v^{n,2} - (v^{n,1})^2 + 6v^{n,2}v^{n+1} - 2v^n v^{n,2} - 4(v^{n,2})^2\\&= 3\left((v^{n+1})^2 - (v^n)^2\right) - (2v^{n,2} - v^{n,1} - v^n)^2 - 3(v^{n+1} - v^n)(v^{n+1} - 2v^{n,2} + v^n),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathbb{U} &= -2u^n u^{n,1} - (u^n)^2 + 4u^{n,1}u^{n,2} - (u^{n,1})^2 + 6u^{n,2}u^{n+1} - 2u^n u^{n,2} - 4(u^{n,2})^2\\&= 3\left((u^{n+1})^2 - (u^n)^2\right) - (2u^{n,2} - u^{n,1} - u^n)^2 - 3(u^{n+1} - u^n)(u^{n+1} - 2u^{n,2} + u^n).\end{aligned} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5), we have

$$3\|(v^{n+1}, u^{n+1})\|^2 - 3\|(v^n, u^n)\|^2 = \mathit{\Pi}_1 + \mathit{\Pi}_2, \tag{4.8}$$

where $\mathit{\Pi}_i (i = 1, 2)$ are defined by

$$\begin{aligned}\mathit{\Pi}_1 = &-\tau\left(A_{21}(u^n, v^n) + A_{21}(u^{n,1}, v^{n,1}) + 4A_{21}(u^{n,2}, v^{n,2})\right)\\&-\tau\left(A_{12}(v^n, u^n) + A_{12}(v^{n,1}, u^{n,1}) + 4A_{12}(v^{n,2}, u^{n,2})\right)\\&-\frac{\tau}{\epsilon}\|v^n - u^n\|^2 - \frac{\tau}{\epsilon}\|v^{n,1} - u^{n,1}\|^2 - \frac{4\tau}{\epsilon}\|v^{n,2} - u^{n,2}\|^2,\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\mathit{\Pi}_2 &= \int_a^b \left[(2v^{n,2} - v^{n,1} - v^n)^2 + 3(v^{n+1} - v^n)(v^{n+1} - 2v^{n,2} + v^n)\right]\mathrm{d}x\\&\quad + \int_a^b \left[(2u^{n,2} - u^{n,1} - u^n)^2 + 3(u^{n+1} - u^n)(u^{n+1} - 2u^{n,2} + u^n)\right]\mathrm{d}x\\&= \langle \mathcal{L}_v^{n,2}, \mathcal{L}_v^{n,2}\rangle + 3\langle \mathcal{L}_v^{n,1} + \mathcal{L}_v^{n,2} + \mathcal{L}_v^{n,3}, \mathcal{L}_v^{n,3}\rangle + \langle \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,2}\rangle + 3\langle \mathcal{L}_u^{n,1} + \mathcal{L}_u^{n,2} + \mathcal{L}_u^{n,3}, \mathcal{L}_u^{n,3}\rangle.\end{aligned} \tag{4.10}$$

Here we have used notations in (4.1) and the fact that $v^{n+1} - v^n = \mathcal{L}_v^{n,1} + \mathcal{L}_v^{n,2} + \mathcal{L}_v^{n,3}$ (similarly for $u^{n+1} - u^n$) in the last equality of (4.10).

From Lemma 3.1 and the fact $\frac{1}{Qh^2} = \frac{c}{\epsilon}$, it follows that

$$\begin{aligned}\mathit{\Pi}_1 \leq & -\frac{\beta\tau}{2}\|(\partial_x v^n, \partial_x u^n)\|^2 - \frac{\tau}{\epsilon}(1-c)\|v^n - u^n\|^2 \\ & -\frac{\beta\tau}{2}\|(\partial_x v^{n,1}, \partial_x u^{n,1})\|^2 - \frac{\tau}{\epsilon}(1-c)\|v^{n,1} - u^{n,1}\|^2 \\ & -2\beta\tau\|(\partial_x v^{n,2}, \partial_x u^{n,2})\|^2 - \frac{4\tau}{\epsilon}(1-c)\|v^{n,2} - u^{n,2}\|^2 \leq 0.\end{aligned} \tag{4.11}$$

Next we show that under some restriction on τ and ϵ, $\mathit{\Pi}_2$ can be bounded by $|\mathit{\Pi}_1|$ for $0 < c < 1$. To this aim, we denote

$$\mathit{\Pi}_2 = \mathit{\Pi}_2^1 + \mathit{\Pi}_2^2 + \mathit{\Pi}_2^3, \tag{4.12}$$

where

$$\begin{aligned}\mathit{\Pi}_2^1 =& \langle \mathcal{L}_v^{n,2}, \mathcal{L}_v^{n,2}\rangle + 3\langle \mathcal{L}_v^{n,3}, \mathcal{L}_v^{n,1}\rangle + \langle \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,2}\rangle + 3\langle \mathcal{L}_u^{n,3}, \mathcal{L}_u^{n,1}\rangle, \\ \mathit{\Pi}_2^2 =& 3\langle \mathcal{L}_v^{n,3}, \mathcal{L}_v^{n,2}\rangle + 3\langle \mathcal{L}_u^{n,3}, \mathcal{L}_u^{n,2}\rangle, \\ \mathit{\Pi}_2^3 =& 3\langle \mathcal{L}_v^{n,3}, \mathcal{L}_v^{n,3}\rangle + 3\langle \mathcal{L}_u^{n,3}, \mathcal{L}_u^{n,3}\rangle.\end{aligned}$$

Regrouping terms in $\mathit{\Pi}_2^1$ we have

$$\begin{aligned}\mathit{\Pi}_2^1 =& -\langle \mathcal{L}_v^{n,2}, \mathcal{L}_v^{n,2}\rangle + 2\langle \mathcal{L}_v^{n,2}, \mathcal{L}_v^{n,2}\rangle + 3\langle \mathcal{L}_v^{n,3}, \mathcal{L}_v^{n,1}\rangle - \langle \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,2}\rangle + 2\langle \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,2}\rangle + 3\langle \mathcal{L}_u^{n,3}, \mathcal{L}_u^{n,1}\rangle \\ =& -\|\mathcal{L}_v^{n,2}\|^2 - \|\mathcal{L}_u^{n,2}\|^2 - \tau A_{21}(\mathcal{L}_u^{n,1}, \mathcal{L}_v^{n,2}) + \frac{\tau}{\epsilon}\langle \mathcal{L}_u^{n,1} - \mathcal{L}_v^{n,1}, \mathcal{L}_v^{n,2}\rangle \\ & -\tau A_{21}(\mathcal{L}_u^{n,2}, \mathcal{L}_v^{n,1}) + \frac{\tau}{\epsilon}\langle \mathcal{L}_u^{n,2} - \mathcal{L}_v^{n,2}, \mathcal{L}_v^{n,1}\rangle - \tau A_{12}(\mathcal{L}_v^{n,1}, \mathcal{L}_u^{n,2}) + \frac{\tau}{\epsilon}\langle \mathcal{L}_v^{n,1} - \mathcal{L}_u^{n,1}, \mathcal{L}_u^{n,2}\rangle \\ & -\tau A_{12}(\mathcal{L}_v^{n,2}, \mathcal{L}_u^{n,1}) + \frac{\tau}{\epsilon}\langle \mathcal{L}_v^{n,2} - \mathcal{L}_u^{n,2}, \mathcal{L}_u^{n,1}\rangle,\end{aligned} \tag{4.13}$$

where we have used (4.2b), (4.2c) and (4.2e), (4.2f) in the last equality.

For those bilinear forms in (4.13), from Lemma 3.1, we can derive

$$-A_{21}(\mathcal{L}_u^{n,1}, \mathcal{L}_v^{n,2}) - A_{12}(\mathcal{L}_v^{n,2}, \mathcal{L}_u^{n,1}) \leq -\frac{\beta}{2}\|(\partial_x \mathcal{L}_v^{n,2}, \partial_x \mathcal{L}_u^{n,1})\|^2 + \frac{1}{Qh^2}\|\mathcal{L}_u^{n,1} - \mathcal{L}_v^{n,2}\|^2 \tag{4.14}$$

and

$$-A_{21}(\mathcal{L}_u^{n,2}, \mathcal{L}_v^{n,1}) - A_{12}(\mathcal{L}_v^{n,1}, \mathcal{L}_u^{n,2}) \leq -\frac{\beta}{2}\|(\partial_x \mathcal{L}_v^{n,1}, \partial_x \mathcal{L}_u^{n,2})\|^2 + \frac{1}{Qh^2}\|\mathcal{L}_u^{n,2} - \mathcal{L}_v^{n,1}\|^2. \tag{4.15}$$

Choosing $\xi = \mathcal{L}_v^{n,1}$ in (4.2a), by Lemma 3.5 and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}\|\mathcal{L}_v^{n,1}\|^2 =& -\tau A_{21}(u^n, \mathcal{L}_v^{n,1}) + \frac{\tau}{\epsilon}\langle u^n - v^n, \mathcal{L}_v^{n,1}\rangle \\ \leq & \frac{\tau\mathit{\Gamma}}{h}(\|\partial_x u^n\| + h^{-1}\|v^n - u^n\|)\|\mathcal{L}_v^{n,1}\| + \frac{\tau}{\epsilon}\|v^n - u^n\|\|\mathcal{L}_v^{n,1}\|,\end{aligned}$$

then, canceling the common term $\|\mathcal{L}_v^{n,1}\|$ and noting $\frac{1}{h^2}=\frac{cQ}{\epsilon}$, we write

$$\|\mathcal{L}_v^{n,1}\| \le \frac{\tau\Gamma}{h}\|\partial_x u^n\| + \frac{\tau}{\epsilon}(1+cQ\Gamma)\|v^n-u^n\|. \tag{4.16}$$

Since we can obtain a similar estimate as (4.16) for $\|\mathcal{L}_u^{n,1}\|$, we have

$$\|(\mathcal{L}_v^{n,1},\mathcal{L}_u^{n,1})\|^2 \le \frac{2\tau^2\Gamma^2}{h^2}\|(\partial_x v^n,\partial_x u^n)\|^2 + \frac{4\tau^2}{\epsilon^2}(1+cQ\Gamma)^2\|v^n-u^n\|^2. \tag{4.17}$$

Thus substituting (4.14) and (4.15) into (4.13), noticing $\epsilon = cQh^2$, we estimate Π_2^1 as

$$\begin{aligned}
\Pi_2^1 \le &-\|(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2})\|^2 - \frac{\beta\tau}{2}(\|(\partial_x\mathcal{L}_v^{n,1},\partial_x\mathcal{L}_u^{n,1})\|^2 + \|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2)\\
&+\frac{c\tau}{\epsilon}\|\mathcal{L}_u^{n,1}-\mathcal{L}_v^{n,2}\|^2 + \frac{c\tau}{\epsilon}\|\mathcal{L}_u^{n,2}-\mathcal{L}_v^{n,1}\|^2 + \frac{2\tau}{\epsilon}\langle\mathcal{L}_u^{n,1}-\mathcal{L}_v^{n,1},\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\rangle\\
\le &-\|(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2})\|^2 - \frac{\beta\tau}{2}\left(\|(\partial_x\mathcal{L}_v^{n,1},\partial_x\mathcal{L}_u^{n,1})\|^2 + \|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2\right)\\
&+\frac{2c\tau}{\epsilon}\left(\|\mathcal{L}_u^{n,1}\|^2+\|\mathcal{L}_v^{n,1}\|^2+\|\mathcal{L}_v^{n,2}\|^2+\|\mathcal{L}_u^{n,2}\|^2\right) + \frac{\tau}{\epsilon}\left(\delta\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2 + \frac{1}{\delta}\|\mathcal{L}_u^{n,1}-\mathcal{L}_v^{n,1}\|^2\right)\\
\le &-(1-\frac{2c\tau}{\epsilon})\|(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2})\|^2 - \frac{\beta\tau}{2}(\|(\partial_x\mathcal{L}_v^{n,1},\partial_x\mathcal{L}_u^{n,1})\|^2 + \|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2)\\
&+\frac{4\tau}{\epsilon}(c+\frac{1}{\delta})\left(\frac{\tau^2\Gamma^2}{h^2}\|(\partial_x v^n,\partial_x u^n)\|^2 + \frac{2\tau^2}{\epsilon^2}(1+cQ\Gamma)^2\|v^n-u^n\|^2\right) + \frac{\delta\tau}{\epsilon}\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2,
\end{aligned} \tag{4.18}$$

where $\delta\in(0,1)$ in the second inequality is a constant to be specified, and we have used (4.17) in the last inequality (4.18).

For Π_2^2, choosing $\xi=\mathcal{L}_v^{n,2}$ in (4.2c) and $\eta=\mathcal{L}_u^{n,2}$ in (4.2f), summing the results up, we have

$$\begin{aligned}
\Pi_2^2 &= 3\langle\mathcal{L}_v^{n,3},\mathcal{L}_v^{n,2}\rangle + 3\langle\mathcal{L}_u^{n,3},\mathcal{L}_u^{n,2}\rangle\\
&= -\tau A_{21}(\mathcal{L}_u^{n,2},\mathcal{L}_v^{n,2}) + \frac{\tau}{\epsilon}\langle\mathcal{L}_u^{n,2}-\mathcal{L}_v^{n,2},\mathcal{L}_v^{n,2}\rangle - \tau A_{12}(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2}) + \frac{\tau}{\epsilon}\langle\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2},\mathcal{L}_u^{n,2}\rangle\\
&\le -\frac{\beta\tau}{2}\|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2 - \frac{\tau}{\epsilon}(1-c)\|\mathcal{L}_u^{n,2}-\mathcal{L}_v^{n,2}\|^2,
\end{aligned} \tag{4.19}$$

where we have used Lemma 3.1 and $\epsilon=cQh^2$ in the above inequality (4.19).

For Π_2^3, choosing $\xi=\mathcal{L}_v^{n,3}$ in (4.2c), in a similar way to get (4.16) and (4.17), we obtain

$$\|\mathcal{L}_v^{n,3}\| \le \frac{\tau\Gamma}{3h}\|\partial_x\mathcal{L}_u^{n,2}\| + \frac{\tau}{3\epsilon}(1+cQ\Gamma)\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\| \tag{4.20}$$

and

$$\|(\mathcal{L}_v^{n,3},\mathcal{L}_u^{n,3})\|^2 \le \frac{2\tau^2\Gamma^2}{9h^2}\|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2 + \frac{4\tau^2}{9\epsilon^2}(1+cQ\Gamma)^2\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2. \tag{4.21}$$

Thus, Π_2^3 can be bounded by

$$\Pi_2^3 = 3\|(\mathcal{L}_v^{n,3},\mathcal{L}_u^{n,3})\|^2$$

$$\leq \frac{2\tau^2\mathit{\Gamma}^2}{3h^2}\|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2+\frac{4\tau^2}{3\epsilon^2}(1+cQ\mathit{\Gamma})^2\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2. \tag{4.22}$$

Thus substituting (4.18), (4.19) and (4.22) into (4.12), we have

$$\begin{aligned}
\mathit{\Pi}_2 \leq &-\left(1-\frac{2c\tau}{\epsilon}\right)\|(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2})\|^2-\frac{\beta\tau}{2}\|(\partial_x\mathcal{L}_v^{n,1},\partial_x\mathcal{L}_u^{n,1})\|^2-\left(\beta\tau-\frac{2\tau^2\mathit{\Gamma}^2}{3h^2}\right)\|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2\\
&+\frac{4\tau}{\epsilon}(c+\frac{1}{\delta})\left(\frac{\tau^2\mathit{\Gamma}^2}{h^2}\|(\partial_x v^n,\partial_x u^n)\|^2+\frac{2\tau^2}{\epsilon^2}(1+cQ\mathit{\Gamma})^2\|v^n-u^n\|^2\right)\\
&-\frac{\tau}{\epsilon}\left(1-c-\delta-\frac{4\tau}{3\epsilon}(1+cQ\mathit{\Gamma})^2\right)\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2.
\end{aligned} \tag{4.23}$$

Combining those estimates in (4.11) and (4.23) and collecting the common terms, we obtain

$$\begin{aligned}
\mathit{\Pi}_1+\mathit{\Pi}_2 \leq &-\tau\left(\frac{\beta}{2}-(c+\frac{1}{\delta})\frac{4\tau^2\mathit{\Gamma}^2}{\epsilon h^2}\right)\|(\partial_x v^n,\partial_x u^n)\|^2\\
&-\frac{\tau}{\epsilon}\left(1-c-(c+\frac{1}{\delta})(1+cQ\mathit{\Gamma})^2\frac{8\tau^2}{\epsilon^2}\right)\|v^n-u^n\|^2\\
&-\frac{\beta\tau}{2}\left(\|(\partial_x v^{n,1},\partial_x u^{n,1})\|^2+4\|(\partial_x v^{n,2},\partial_x u^{n,2})\|^2+\|(\partial_x\mathcal{L}_v^{n,1},\partial_x\mathcal{L}_u^{n,1})\|^2\right)\\
&-\frac{\tau}{\epsilon}(1-c)\left(\|v^{n,1}-u^{n,1}\|^2+4\|v^{n,2}-u^{n,2}\|^2\right)-\left(1-\frac{2c\tau}{\epsilon}\right)\|(\mathcal{L}_v^{n,2},\mathcal{L}_u^{n,2})\|^2\\
&-\left(\beta\tau-\frac{2\tau^2\mathit{\Gamma}^2}{3h^2}\right)\|(\partial_x\mathcal{L}_v^{n,2},\partial_x\mathcal{L}_u^{n,2})\|^2-\frac{\tau}{\epsilon}\left(1-c-\delta-\frac{4\tau}{3\epsilon}(1+cQ\mathit{\Gamma})^2\right)\|\mathcal{L}_v^{n,2}-\mathcal{L}_u^{n,2}\|^2.
\end{aligned} \tag{4.24}$$

Recall (4.8) we see that the desired stability will follow if each term on the right side of the above inequality is nonpositive, this is indeed so if

$$\begin{cases}
\frac{\beta}{2}-(c+\frac{1}{\delta})\frac{4\tau^2\mathit{\Gamma}^2}{\epsilon h^2}\geq 0, \quad 1-c-(c+\frac{1}{\delta})(1+cQ\mathit{\Gamma})^2\frac{8\tau^2}{\epsilon^2}\geq 0,\\
1-\frac{2c\tau}{\epsilon}\geq 0, \quad \beta-\frac{2\tau\mathit{\Gamma}^2}{3h^2}\geq 0, \quad 1-c-\delta-\frac{4\tau}{3\epsilon}(1+cQ\mathit{\Gamma})^2\geq 0
\end{cases} \tag{4.25}$$

for any $\delta\in(0,1-c)$. These are implied by (4.4) if we choose

$$c_0=\max\left\{\left(\frac{8cQ\mathit{\Gamma}^2(c+\frac{1}{\delta})}{\beta}\right)^{1/2}, 2\left(1+cQ\mathit{\Gamma}\right)\left(\frac{2(c+\frac{1}{\delta})}{1-c}\right)^{1/2}, 2c, \frac{2cQ\mathit{\Gamma}^2}{3\beta}, \frac{4(1+cQ\mathit{\Gamma})^2}{3(1-c-\delta)}\right\} \tag{4.26}$$

with $\delta=\frac{1-c}{2}$.

□

5. Error estimates

In this section, based on the stability analysis presented in Section 4, we obtain the optimal L^2 error estimates of the fully discrete AEDG method (3.7)–(3.9). We first prepare the error function and two lemmas for later use.

5.1. Reference solution and error representation

Set $\phi^{n,0} = \phi(x, t^n)$, according to the 3-stage RK3AEDG method (3.7)–(3.9), we define the reference solutions of (1.1) as

$$\phi^{n,1} = \phi^{n,0} - \tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0}, \tag{5.1a}$$

$$\phi^{n,2} = \frac{3}{4}\phi^{n,0} + \frac{1}{4}\phi^{n,1} - \frac{\tau}{4}\left(\alpha\partial_x - \beta\partial_x^2\right)\phi^{n,1}. \tag{5.1b}$$

Lemma 5.1. *If $\|\partial_t^4\phi\|$ is bounded uniformly for any $t \in [0, T]$, we have*

$$\phi(x, t^n + \tau) = \frac{1}{3}\phi^{n,0} + \frac{2}{3}\phi^{n,2} - \frac{2\tau}{3}\left(\alpha\partial_x - \beta\partial_x^2\right)\phi^{n,2} + F(n; x), \tag{5.2}$$

where $F(n; x)$ is the local truncation error in time and $\|F(n; x)\| = O(\tau^4)$ uniformly for any time $t \in [0, T]$.

Proof. By Taylor's expansion in variable t,

$$\phi(x, t^n + \tau) = \phi(x, t^n) + \tau\partial_t\phi(x, t^n) + \frac{\tau^2}{2}\partial_t^2\phi(x, t^n) + \frac{\tau^3}{6}\partial_t^3\phi(x, t^n) + \frac{\tau^4}{24}\partial_t^4\phi(x, t') \tag{5.3}$$

where $t' \in (t^n, t^{n+1})$. The right hand side of (5.2) (RHS) with notations in (5.1) reduces to

$$RHS = \phi^{n,0} - \tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0} + \frac{\tau^2}{2}(\alpha\partial_x - \beta\partial_x^2)^2\phi^{n,0} - \frac{\tau^3}{6}(\alpha\partial_x - \beta\partial_x^2)^3\phi^{n,0} + F(n; x). \tag{5.4}$$

Using the fact that $\phi(x, t^n) = \phi^{n,0}$ and $\phi^{n,0}$ is a solution of (1.1a), we have

$$F(n; x) = \frac{\tau^4}{24}\partial_t^4\phi(x, t').$$

This completes the proof. $\square$

Since all reference solutions $\phi^{n,i}, i = 1, 2$ are smooth in $[a, b]$, the consistency of the AEDG scheme (see [8]) yields

$$\begin{cases} \langle\phi^{n,1}, \xi\rangle & = \langle\phi^{n,0}, \xi\rangle - \tau A_{21}(\phi^{n,0}, \xi) + \dfrac{\tau}{\epsilon}\langle\phi^{n,0} - \phi^{n,0}, \xi\rangle, \\ \langle\phi^{n,1}, \eta\rangle & = \langle\phi^{n,0}, \eta\rangle - \tau A_{12}(\phi^{n,0}, \eta) + \dfrac{\tau}{\epsilon}\langle\phi^{n,0} - \phi^{n,0}, \eta\rangle, \end{cases} \tag{5.5}$$

$$\begin{cases} \langle\phi^{n,2}, \xi\rangle & = \dfrac{1}{4}\langle 3\phi^{n,0} + \phi^{n,1}, \xi\rangle - \dfrac{\tau}{4}A_{21}(\phi^{n,1}, \xi) + \dfrac{\tau}{4\epsilon}\langle\phi^{n,1} - \phi^{n,1}, \xi\rangle, \\ \langle\phi^{n,2}, \eta\rangle & = \dfrac{1}{4}\langle 3\phi^{n,0} + \phi^{n,1}, \eta\rangle - \dfrac{\tau}{4}A_{12}(\phi^{n,1}, \eta) + \dfrac{\tau}{4\epsilon}\langle\phi^{n,1} - \phi^{n,1}, \eta\rangle, \end{cases} \tag{5.6}$$

$$\begin{cases} \langle\phi^{n+1}, \xi\rangle & = \dfrac{1}{3}\langle\phi^{n,0} + 2\phi^{n,2}, \xi\rangle - \dfrac{2\tau}{3}A_{21}(\phi^{n,2}, \xi) + \dfrac{2\tau}{3\epsilon}\langle\phi^{n,2} - \phi^{n,2}, \xi\rangle + \langle F(n; x), \xi\rangle, \\ \langle\phi^{n+1}, \eta\rangle & = \dfrac{1}{3}\langle\phi^{n,0} + 2\phi^{n,2}, \eta\rangle - \dfrac{2\tau}{3}A_{12}(\phi^{n,2}, \eta) + \dfrac{2\tau}{3\epsilon}\langle\phi^{n,2} - \phi^{n,2}, \eta\rangle + \langle F(n; x), \eta\rangle, \end{cases} \tag{5.7}$$

where $(\xi, \eta) \in V_h \times U_h$ at each stage.

Noting that $v^{n,0} = v^n, u^{n,0} = u^n$, we split the solution errors as follows.

$$\phi^{n,i} - v^{n,i} = e_1^{n,i} - \epsilon_1^{n,i}, \qquad \phi^{n,i} - u^{n,i} = e_2^{n,i} - \epsilon_2^{n,i} \tag{5.8}$$

for $i = 0,\ 1,\ 2$, where

$$\begin{aligned} e_1^{n,i} &= \Pi_v \phi^{n,i} - v^{n,i},\ \epsilon_1^{n,i} = \Pi_v \phi^{n,i} - \phi^{n,i}, \\ e_2^{n,i} &= \Pi_u \phi^{n,i} - u^{n,i},\ \epsilon_2^{n,i} = \Pi_u \phi^{n,i} - \phi^{n,i}. \end{aligned}$$

Each equation in scheme (3.7)–(3.9) when subtracted from the corresponding relation in (5.5)–(5.7), leads to

$$\begin{aligned} \langle e_1^{n,1} - e_1^{n,0}, \xi\rangle + \tau A_{21}(e_2^{n,0}, \xi) =& \langle \epsilon_1^{n,1} - \epsilon_1^{n,0}, \xi\rangle + \tau A_{21}(\epsilon_2^{n,0}, \xi) \\ & - \frac{\tau}{\epsilon}\langle u^{n,0} - v^{n,0}, \xi\rangle, \end{aligned} \tag{5.9a}$$

$$\begin{aligned} \langle e_2^{n,1} - e_2^{n,0}, \eta\rangle + \tau A_{12}(e_1^{n,0}, \eta) =& \langle \epsilon_2^{n,1} - \epsilon_2^{n,0}, \eta\rangle + \tau A_{12}(\epsilon_1^{n,0}, \eta) \\ & - \frac{\tau}{\epsilon}\langle v^{n,0} - u^{n,0}, \eta\rangle, \end{aligned} \tag{5.9b}$$

$$\langle 4e_1^{n,2} - 3e_1^{n,0} - e_1^{n,1}, \xi\rangle + \tau A_{21}(e_2^{n,1}, \xi) = \langle 4\epsilon_1^{n,2} - 3\epsilon_1^{n,0} - \epsilon_1^{n,1}, \xi\rangle + \tau A_{21}(\epsilon_2^{n,1}, \xi) - \frac{\tau}{\epsilon}\langle u^{n,1} - v^{n,1}, \xi\rangle, \tag{5.10a}$$

$$\langle 4e_2^{n,2} - 3e_2^{n,0} - e_2^{n,1}, \eta\rangle + \tau A_{12}(e_1^{n,1}, \eta) = \langle 4\epsilon_2^{n,2} - 3\epsilon_2^{n,0} - \epsilon_2^{n,1}, \eta\rangle + \tau A_{12}(\epsilon_1^{n,1}, \eta) - \frac{\tau}{\epsilon}\langle v^{n,1} - u^{n,1}, \eta\rangle, \tag{5.10b}$$

$$\begin{aligned} \langle 6e_1^{n+1} - 2e_1^{n,0} - 4e_1^{n,2}, \xi\rangle + 4\tau A_{21}(e_2^{n,2}, \xi) =& \langle 6\epsilon_1^{n+1} - 2\epsilon_1^{n,0} - 4\epsilon_1^{n,2}, \xi\rangle + 4\tau A_{21}(\epsilon_2^{n,2}, \xi) \\ & - \frac{4\tau}{\epsilon}\langle u^{n,2} - v^{n,2}, \xi\rangle + \langle 6F(n;x), \xi\rangle, \end{aligned} \tag{5.11a}$$

$$\begin{aligned} \langle 6e_2^{n+1} - 2e_2^{n,0} - 4e_2^{n,2}, \eta\rangle + 4\tau A_{12}(e_1^{n,2}, \eta) =& \langle 6\epsilon_2^{n+1} - 2\epsilon_2^{n,0} - 4\epsilon_2^{n,2}, \eta\rangle + 4\tau A_{12}(\epsilon_1^{n,2}, \eta) \\ & - \frac{4\tau}{\epsilon}\langle v^{n,2} - u^{n,2}, \eta\rangle + \langle 6F(n;x), \eta\rangle \end{aligned} \tag{5.11b}$$

for $(\xi, \eta) \in V_h \times U_h$ at each stage, respectively.

Lemma 5.2. *If the time step satisfies $c_0\tau \le \epsilon = cQh^2$, then the following inequalities hold true*

$$\|(e_1^{n,1}, e_2^{n,1})\|^2 \le C\|(e_1^{n,0}, e_2^{n,0})\|^2 + Ch^{2k+2}, \tag{5.12a}$$

$$\|(e_1^{n,2}, e_2^{n,2})\|^2 \le C\|(e_1^{n,0}, e_2^{n,0})\|^2 + C\|(e_1^{n,1}, e_2^{n,1})\|^2 + Ch^{2k+2}, \tag{5.12b}$$

where C is a constant independent τ, h, n.

Proof. Taking $\xi = e_1^{n,1}$ in (5.9a), we obtain

$$\begin{aligned} \|e_1^{n,1}\|^2 - \langle e_1^{n,0}, e_1^{n,1}\rangle + \tau A_{21}(e_2^{n,0}, e_1^{n,1}) =& \langle \epsilon_1^{n,1} - \epsilon_1^{n,0}, e_1^{n,1}\rangle + \tau A_{21}(\epsilon_2^{n,0}, e_1^{n,1}) \\ & - \frac{\tau}{\epsilon}\langle u^{n,0} - v^{n,0}, e_1^{n,1}\rangle. \end{aligned} \tag{5.13}$$

Using the Young's inequality, $ab \le \delta a^2 + \frac{1}{4\delta} b^2$ with $\delta = 3/2$, we have

$$\langle e_1^{n,0}, e_1^{n,1} \rangle \le \frac{3}{2} \|e_1^{n,0}\|^2 + \frac{1}{6} \|e_1^{n,1}\|^2. \tag{5.14}$$

From Lemma 3.5 it follows

$$\begin{aligned} \tau A_{21}(e_2^{n,0}, e_1^{n,1}) &\le \tau \left(\frac{\Gamma}{h} \|\partial_x e_2^{n,0}\| + \frac{\Gamma}{h^2} (\|e_1^{n,0}\| + \|e_2^{n,0}\|) \right) \|e_1^{n,1}\| \\ &\le \frac{C\tau}{h^2} (\|e_1^{n,0}\| + \|e_2^{n,0}\|) \|e_1^{n,1}\| \\ &\le \frac{1}{6} \|e_1^{n,1}\|^2 + C \|(e_1^{n,0}, e_2^{n,0})\|^2, \end{aligned} \tag{5.15}$$

where we have used the inverse inequality (3.14b) in the second inequality and $\tau \le cQh^2/c_0$ in the last inequality.

Again from Lemma 3.5 and the projection error in Theorem 3.2, we obtain

$$\begin{aligned} \tau A_{21}(\epsilon_2^{n,0}, e_1^{n,1}) &\le \tau \left(\frac{\Gamma}{h} \|\partial_x \epsilon_2^{n,0}\| + \frac{\Gamma}{h^2} (\|\epsilon_1^{n,0}\| + \|\epsilon_2^{n,0}\|) \right) \|e_1^{n,1}\| \\ &\le \frac{C\tau}{h^2} \left(\|\epsilon_1^{n,0}\| + \|\epsilon_2^{n,0}\| + h \|\partial_x \epsilon_2^{n,0}\| \right) \|e_1^{n,1}\| \\ &\le \frac{1}{6} \|e_1^{n,1}\|^2 + Ch^{2k+2}. \end{aligned} \tag{5.16}$$

Using the L^2 projection error in (3.13), we obtain

$$\langle \epsilon_1^{n,1} - \epsilon_1^{n,0}, e_1^{n,1} \rangle \le (\|\epsilon_1^{n,1}\| + \|\epsilon_1^{n,0}\|) \|e_1^{n,1}\| \le \frac{1}{6} \|e_1^{n,1}\|^2 + Ch^{2k+2}, \tag{5.17}$$

similarly,

$$\begin{aligned} \langle u^{n,0} - v^{n,0}, e_1^{n,1} \rangle &= \langle e_1^{n,0} - \epsilon_1^{n,0} - (e_2^{n,0} - \epsilon_2^{n,0}), e_1^{n,1} \rangle \\ &\le (Ch^{k+1} + \|e_1^{n,0}\| + \|e_2^{n,0}\|) \|e_1^{n,1}\| \\ &\le \frac{1}{6} \|e_1^{n,1}\|^2 + C \|(e_1^{n,0}, e_2^{n,0})\|^2 + Ch^{2k+2}. \end{aligned} \tag{5.18}$$

Plugging (5.14)–(5.18) into (5.13) we arrive at

$$\|e_1^{n,1}\|^2 \le C \|(e_1^{n,0}, e_2^{n,0})\|^2 + Ch^{2k+2}. \tag{5.19}$$

Likewise, we have

$$\|e_2^{n,1}\|^2 \le C \|(e_1^{n,0}, e_2^{n,0})\|^2 + Ch^{2k+2}. \tag{5.20}$$

Taking summation of (5.19) and (5.20) leads to (5.12a). In a similar manner, we can also prove (5.12b). □

5.2. Error estimates of RK3AEDG in L^2 norm

Based on the stability analysis and the error representation, we set out to derive the error estimates of scheme (3.7)–(3.9). The result stated in Theorem 2.3 can be reformulated in terms of (v^n, u^n) as follows.

Theorem 5.3. *Let ϕ be the smooth solution of (2.1) subject to initial data $\phi_0(x)$ and periodic boundary conditions, $(v^n, u^n) \in V_h \times U_h$ be the numerical solution computed through the fully discrete scheme (3.7)–(3.9), then we have the following error estimate:*

$$\|\phi(\cdot, t^n) - v^n(\cdot)\| + \|\phi(\cdot, t^n) - u^n(\cdot)\| \le C(\tau^3 + h^{k+1}), \quad n\tau \le T, \tag{5.21}$$

where C is a constant independent of τ, h and n.

Proof. Taking $w = \phi^{n,i}$ with $\xi = e_1^{n,i}$ in (3.10), and $\eta = e_2^{n,i}$ in (3.11) for $i = 0, 1, 2$, respectively, we have

$$\begin{aligned}\langle \epsilon_1^{n,i}, e_1^{n,i}\rangle + A_{21}(\epsilon_2^{n,i}, e_1^{n,i}) &= \frac{1}{\epsilon}\langle \epsilon_2^{n,i} - \epsilon_1^{n,i}, e_1^{n,i}\rangle,\\ \langle \epsilon_2^{n,i}, e_2^{n,i}\rangle + A_{12}(\epsilon_1^{n,i}, e_2^{n,i}) &= \frac{1}{\epsilon}\langle \epsilon_1^{n,i} - \epsilon_2^{n,i}, e_2^{n,i}\rangle.\end{aligned}$$

This together with $v^{n,i} - u^{n,i} = e_2^{n,i} - e_1^{n,i} - (\epsilon_2^{n,i} - \epsilon_1^{n,i})$ gives

$$\begin{aligned}A_{21}(\epsilon_2^{n,i}, e_1^{n,i}) + A_{12}(\epsilon_1^{n,i}, e_2^{n,i}) &= \frac{1}{\epsilon}\langle \epsilon_2^{n,i} - \epsilon_1^{n,i}, e_1^{n,i} - e_2^{n,i}\rangle - \langle \epsilon_1^{n,i}, e_1^{n,i}\rangle - \langle \epsilon_2^{n,i}, e_2^{n,i}\rangle\\ &= -\frac{1}{\epsilon}\|e_1^{n,i} - e_2^{n,i}\|^2 - \frac{1}{\epsilon}\langle v^{n,i} - u^{n,i}, e_1^{n,i} - e_2^{n,i}\rangle - \langle \epsilon_1^{n,i}, e_1^{n,i}\rangle - \langle \epsilon_2^{n,i}, e_2^{n,i}\rangle,\end{aligned}$$

which is equivalent to

$$A_{21}(\epsilon_2^{n,i}, e_1^{n,i}) + A_{12}(\epsilon_1^{n,i}, e_2^{n,i}) + \frac{1}{\epsilon}\langle v^{n,i} - u^{n,i}, e_1^{n,i} - e_2^{n,i}\rangle = -\frac{1}{\epsilon}\|e_1^{n,i} - e_2^{n,i}\|^2 - \langle \epsilon_1^{n,i}, e_1^{n,i}\rangle - \langle \epsilon_2^{n,i}, e_2^{n,i}\rangle. \tag{5.22}$$

Taking $(\xi, \eta) = (e_1^{n,0}, e_2^{n,0})$, $(\xi, \eta) = (e_1^{n,1}, e_2^{n,1})$ and $(\xi, \eta) = (e_1^{n,2}, e_2^{n,2})$ in each stage of (5.9)–(5.11), respectively, summing the result up, noticing (5.22), according to the stability analysis (4.8), we obtain

$$3\|(e_1^{n+1}, e_2^{n+1})\|^2 = 3\|(e_1^n, e_2^n)\|^2 + \Pi_1' + \Pi_2' + G_1 + G_2, \tag{5.23}$$

where

$$\begin{aligned}\Pi_1' = &-\tau\left(A_{21}(e_2^{n,0}, e_1^{n,0}) + A_{21}(e_2^{n,1}, e_1^{n,1}) + 4A_{21}(e_2^{n,2}, e_1^{n,2})\right)\\ &-\tau\left(A_{12}(e_1^{n,0}, e_2^{n,0}) + A_{12}(e_1^{n,1}, e_2^{n,1}) + 4A_{12}(e_1^{n,2}, e_2^{n,2})\right)\\ &-\frac{\tau}{\epsilon}\|e_1^{n,0} - e_2^{n,0}\|^2 - \frac{\tau}{\epsilon}\|e_1^{n,1} - e_2^{n,1}\|^2 - \frac{4\tau}{\epsilon}\|e_1^{n,2} - e_2^{n,2}\|^2,\end{aligned} \tag{5.24}$$

$$\begin{aligned}\Pi_2' = &\int_a^b \left[(2e_1^{n,2} - e_1^{n,1} - e_1^{n,0})^2 + 3(e_1^{n+1} - e_1^{n,0})(e_1^{n+1} - 2e_1^{n,2} + e_1^{n,0})\right] \mathrm{d}x\\ &+\int_a^b \left[(2e_2^{n,2} - e_2^{n,1} - e_2^{n,0})^2 + 3(e_2^{n+1} - e_2^{n,0})(e_2^{n+1} - 2e_2^{n,2} + e_2^{n,0})\right] \mathrm{d}x,\end{aligned} \tag{5.25}$$

$$\begin{aligned}G_1 = &\langle \epsilon_1^{n,1} - \epsilon_1^{n,0}, e_1^{n,0}\rangle + \langle \epsilon_2^{n,1} - \epsilon_2^{n,0}, e_2^{n,0}\rangle + \langle 4\epsilon_1^{n,2} - 3\epsilon_1^{n,0} - \epsilon_1^{n,1}, e_1^{n,1}\rangle + \langle 4\epsilon_2^{n,2} - 3\epsilon_2^{n,0} - \epsilon_2^{n,1}, e_2^{n,1}\rangle\\ &+\langle 6\epsilon_1^{n+1} - 2\epsilon_1^{n,0} - 4\epsilon_1^{n,2}, e_1^{n,2}\rangle + \langle 6\epsilon_2^{n+1} - 2\epsilon_2^{n,0} - 4\epsilon_2^{n,2}, e_2^{n,2}\rangle,\end{aligned} \tag{5.26}$$

and

$$G_2 = -\tau \sum_{i=0}^{1} \left(\langle \epsilon_1^{n,i}, e_1^{n,i} \rangle + \langle \epsilon_2^{n,i}, e_2^{n,i} \rangle \right) - 4\tau \langle \epsilon_1^{n,2}, e_1^{n,2} \rangle - 4\tau \langle \epsilon_2^{n,2}, e_2^{n,2} \rangle + \langle 6F(n,x), e_1^{n,2} + e_2^{n,2} \rangle. \tag{5.27}$$

From the stability analysis in leading to (4.8) and (4.24), under the stability condition (4.4), we have

$$\Pi_1' + \Pi_2' \leq 0. \tag{5.28}$$

Noticing the fact that $\phi^{n,0}$ is a solution of (1.1), using (5.1) and the Taylor expansion, we obtain

$$\phi^{n,1} - \phi^{n,0} = -\tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0} = \tau\partial_t\phi^{n,0}, \tag{5.29a}$$

$$\begin{aligned} 4\phi^{n,2} - 3\phi^{n,0} - \phi^{n,1} &= -\tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,1} \\ &= -\tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0} + \tau^2(\alpha\partial_x - \beta\partial_x^2)^2\phi^{n,0}, \end{aligned} \tag{5.29b}$$

$$\begin{aligned} 3\phi^{n+1} - \phi^{n,0} - 2\phi^{n,2} &= 3\phi^{n+1} - 3\phi^{n,0} + \tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0} - \frac{\tau^2}{2}(\alpha\partial_x - \beta\partial_x^2)^2\phi^{n,0} \\ &= 2\tau(\alpha\partial_x - \beta\partial_x^2)\phi^{n,0} - \tau^2(\alpha\partial_x - \beta\partial_x^2)^2\phi^{n,0} - \frac{\tau^3}{2}(\alpha\partial_x - \beta\partial_x^2)^3\phi^{n,0}. \end{aligned} \tag{5.29c}$$

Applying the projection error (3.13) and (5.29), we have

$$\begin{aligned} \|\epsilon_1^{n,1} - \epsilon_1^{n,0}\| =& \|\Pi_v\left(\phi^{n,1} - \phi^{n,0}\right) - (\phi^{n,1} - \phi^{n,0})\| \\ &\leq C\tau h^{\min\{k+1,m\}} \left|\partial_t\phi^{n,0}\right|_m \leq C\tau h^{\min\{k+1,m\}}\|\phi^{n,0}\|_{m+2}. \end{aligned} \tag{5.30a}$$

$$\begin{aligned} \left\|4\epsilon_1^{n,2} - 3\epsilon_1^{n,0} - \epsilon_1^{n,1}\right\| &= \left\|\Pi_v\left(4\phi^{n,2} - 3\phi^{n,0} - \phi^{n,1}\right) - (4\phi^{n,2} - 3\phi^{n,0} - \phi^{n,1})\right\| \\ &\leq C\tau h^{\min\{k+1,m\}}\|\phi^{n,0}\|_{m+4}. \end{aligned} \tag{5.30b}$$

$$\begin{aligned} \left\|6\epsilon_1^{n+1} - 2\epsilon_1^{n,0} - 4\epsilon_1^{n,2}\right\| =& \|\Pi_v(6\phi^{n+1} - 2\phi^{n,0} - 4\phi^{n,2}) - (6\phi^{n+1} - 2\phi^{n,0} - 4\phi^{n,2})\| \\ &\leq C\tau h^{\min\{k+1,m\}}\|\phi^{n,0}\|_{m+6}. \end{aligned} \tag{5.30c}$$

Same estimates as (5.30a)–(5.30c) hold true for $\epsilon_2^{n,i}$.

From the Young's inequality, for $w \in L^2$, we have

$$\langle w, e_j^{n,i} \rangle \leq \|e_j^{n,i}\|^2 + \frac{1}{4}\|w\|^2, \tag{5.31}$$

where $i = 0, 1, 2$ and $j = 1, 2$.

This when applied to each term in G_1 and taking $m = k + 1$, together with (5.30a)–(5.30c) gives

$$|G_1| \leq \sum_{i=0}^{2} \tau\|(e_1^{n,i}, e_2^{n,i})\|^2 + C\tau h^{2k+2}. \tag{5.32}$$

Now we turn to the estimate of G_2, again from (5.31),

$$|G_2| \leq \tau \sum_{i=0}^{2} \|(e_1^{n,i}, e_2^{n,i})\|^2 + \frac{\tau}{4}\left(\sum_{i=0}^{1} \|(\epsilon_1^{n,i}, \epsilon_2^{n,i})\|^2 + 16\|(\epsilon_1^{n,2}, \epsilon_2^{n,2})\|^2 + \frac{1}{\tau^2}\|F(n;\cdot)\|^2 \right) \tag{5.33}$$

$$\leq \tau \sum_{i=0}^{2} \|(e_1^{n,i}, e_2^{n,i})\|^2 + C\tau(h^{2k+2} + \tau^6)$$

thus, collecting the estimate in (5.32) and (5.33), it is easy to see that

$$\begin{aligned} |G_1 + G_2| &\leq 2\tau \sum_{i=0}^{2} \|(e_1^{n,i}, e_2^{n,i})\|^2 + C\tau(h^{2k+2} + \tau^6) \\ &\leq C\tau\|(e_1^{n,0}, e_2^{n,0})\|^2 + C\tau(h^{2k+2} + \tau^6), \end{aligned} \tag{5.34}$$

where we have used (5.12) in the last inequality.

Plugging (5.28) and (5.34) into (5.23) leads to

$$\|(e_1^{n+1}, e_2^{n+1})\|^2 \leq (1 + C\tau)\|(e_1^n, e_2^n)\|^2 + C\tau(h^{2k+2} + \tau^6),$$

this gives

$$\|(e_1^n, e_2^n)\| \leq (1 + C\tau)^{n/2}\|(e_1^0, e_2^0)\| + (1 + C\tau)^{n/2}(h^{k+1} + \tau^3).$$

From the choice of the initial data in (2.5), projection error in Theorem 3.2 and local approximation property in Lemma 3.3, we have

$$\|e_1^0\| = \|e_1(\cdot, 0)\| = \|\Pi_v \phi_0 - v(\cdot, 0)\| \leq \|\Pi_v \phi_0 - \phi_0\| + \|\phi_0 - v(\cdot, 0)\| \leq Ch^{k+1}, \tag{5.35}$$

similarly $\|e_2^0\| \leq Ch^{k+1}$.

Further using the initial error as given in (5.35), so that

$$\|(e_1^n, e_2^n)\| \leq C(h^{k+1} + \tau^3).$$

This together with the projection error for ϵ_i^n when inserted into (5.8) yields the desired estimate (5.21). □

Appendix A

If one considers the second order explicit SSP Runge-Kutta method of the form

$$\Psi^{n,1} = \Psi^{n,0} + \tau L(\Psi^{n,0}), \tag{A.1a}$$

$$\Psi^{n+1} = \frac{1}{2}\Psi^{n,0} + \frac{1}{2}\Psi^{n,1} + \frac{\tau}{2}L(\Psi^{n,1}) \tag{A.1b}$$

for the time discretization, then the corresponding RK2AEDG can be reformulated as

$$\langle v^{n,1}, \xi\rangle = \langle v^n, \xi\rangle - \tau A_{21}(u^n, \xi) + \frac{\tau}{\epsilon}\langle u^n - v^n, \xi\rangle, \qquad (\xi, \eta) \in V_h \times U_h, \tag{A.2a}$$

$$\langle u^{n,1}, \eta\rangle = \langle u^n, \eta\rangle - \tau A_{12}(v^n, \eta) + \frac{\tau}{\epsilon}\langle v^n - u^n, \eta\rangle, \qquad (v^n, u^n) \in V_h \times U_h; \tag{A.2b}$$

$$\langle v^{n+1}, \xi\rangle = \frac{1}{2}\langle v^n + v^{n,1}, \xi\rangle - \frac{\tau}{2}A_{21}(u^{n,1}, \xi) + \frac{\tau}{2\epsilon}\langle u^{n,1} - v^{n,1}, \xi\rangle, \quad (\xi, \eta) \in V_h \times U_h, \tag{A.3a}$$

$$\langle u^{n+1}, \eta\rangle = \frac{1}{2}\langle u^n + u^{n,1}, \eta\rangle - \frac{\tau}{2}A_{12}(v^{n,1}, \eta) + \frac{\tau}{2\epsilon}\langle v^{n,1} - u^{n,1}, \eta\rangle, \quad (v^{n,1}, u^{n,1}) \in V_h \times U_h. \tag{A.3b}$$

The stability analysis for the RK2AEDG is similar, we state the result in the following

Theorem A.1. *Let Φ^n be the numerical solution computed from (A.2)–(A.3) with $\epsilon = cQh^2$, $0 < c < 1$, then for τ satisfying*

$$c^*\tau \le \epsilon = cQh^2 \quad \textit{with} \quad c^* := \max\left\{\frac{20cQ\Gamma^2}{\beta}, \frac{12(1+cQ\Gamma)^2}{1-c}\right\}, \tag{A.4}$$

we have

$$\sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} \frac{(\Phi_{j+1}^{n+1})^2 + (\Phi_j^{n+1})^2}{2}\,dx \le \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} \frac{(\Phi_{j+1}^{n})^2 + (\Phi_j^{n})^2}{2}\,dx.$$

Moreover, the following error estimate holds:

$$\sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} \frac{|\Phi_{j+1}^n(x) - \phi(x,t^n)|^2 + |\Phi_j^n(x) - \phi(x,t^n)|^2}{2}\,dx \le C(\tau^4 + h^{2k+2}), \quad n\tau \le T, \tag{A.5}$$

where C is a constant independent of τ, h and n.

We now outline the main steps of the stability proof.

Step 1 (Regrouping). Taking $(\xi,\eta) = (v^n, u^n)$ in (A.2) and $(\xi,\eta) = (2v^{n,1}, 2u^{n,1})$ in (A.3), respectively, and summing up the resulting relations, we obtain

$$\begin{aligned}\mathbb{V}^1 + \mathbb{U}^1 = &-\tau\left(A_{21}(u^n, v^n) + A_{12}(v^n, u^n) + A_{21}(u^{n,1}, v^{n,1}) + A_{12}(v^{n,1}, u^{n,1})\right)\\ &- \frac{\tau}{\epsilon}\|v^n - u^n\|^2 - \frac{\tau}{\epsilon}\|v^{n,1} - u^{n,1}\|^2\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}\mathbb{V}^1 &= 2\langle v^{n+1}, v^{n,1}\rangle - \|v^{n,1}\|^2 - \|v^n\|^2 = \|v^{n+1}\|^2 - \|v^n\|^2 - \|v^{n+1} - v^{n,1}\|^2,\\ \mathbb{U}^1 &= 2\langle u^{n+1}, u^{n,1}\rangle - \|u^{n,1}\|^2 - \|u^n\|^2 = \|u^{n+1}\|^2 - \|u^n\|^2 - \|u^{n+1} - u^{n,1}\|^2.\end{aligned}$$

Thus, we can rewrite (A.6) as

$$\|(v^{n+1}, u^{n+1})\|^2 - \|(v^n, u^n)\|^2 = \Pi^1 + \Pi^2 \tag{A.7}$$

in which

$$\begin{aligned}\Pi^1 = &-\tau\left(A_{21}(u^n, v^n) + A_{12}(v^n, u^n) + A_{21}(u^{n,1}, v^{n,1}) + A_{12}(v^{n,1}, u^{n,1})\right)\\ &- \frac{\tau}{\epsilon}\|v^n - u^n\|^2 - \frac{\tau}{\epsilon}\|v^{n,1} - u^{n,1}\|^2\end{aligned}$$

and

$$\Pi^2 = \|v^{n+1} - v^{n,1}\|^2 + \|u^{n+1} - u^{n,1}\|^2.$$

Step 2 (Estimate of Π^1). From Lemma 3.1 and the fact $\frac{1}{Qh^2} = \frac{c}{\epsilon}$, it follows that

$$\begin{aligned}\Pi^1 \leq & -\frac{\beta\tau}{2}\|(\partial_x v^n, \partial_x u^n)\|^2 - \frac{\tau}{\epsilon}(1-c)\|v^n - u^n\|^2 \\ & -\frac{\beta\tau}{2}\|(\partial_x v^{n,1}, \partial_x u^{n,1})\|^2 - \frac{\tau}{\epsilon}(1-c)\|v^{n,1} - u^{n,1}\|^2 \leq 0. \end{aligned} \tag{A.8}$$

Step 3 (Estimate of Π^2). Subtracting (A.2a) from (A.3a), then taking $\xi = v^{n+1} - v^{n,1}$ in the resulting equation, we obtain

$$\begin{aligned}\|v^{n+1} - v^{n,1}\|^2 =& \frac{1}{2}\langle v^{n,1} - v^n, v^{n+1} - v^{n,1}\rangle - \tau A_{21}\left(\frac{u^{n,1}}{2} - u^n, v^{n+1} - v^{n,1}\right) \\ & + \frac{\tau}{\epsilon}\left(\frac{1}{2}(u^{n,1} - v^{n,1}) - (u^n - v^n), v^{n+1} - v^{n,1}\right) \\ \leq & \frac{1}{2}\|v^{n,1} - v^n\|\|v^{n+1} - v^{n,1}\| + \frac{\tau}{\epsilon}\left(\frac{1}{2}\|u^{n,1} - v^{n,1}\| + \|u^n - v^n\|\right)\|v^{n+1} - v^{n,1}\| \\ & + \left(\frac{\tau\Gamma}{h}\|\partial_x(\frac{1}{2}u^{n,1} - u^n)\| + \frac{\tau\Gamma}{h^2}\left(\frac{1}{2}\|u^{n,1} - v^{n,1}\| + \|u^n - v^n\|\right)\right)\|v^{n+1} - v^{n,1}\|\end{aligned}$$

where we have used the Cauchy-Schwartz inequality, the triangle inequality, and Lemma 3.5, respectively. By canceling the factor $\|v^{n+1} - v^{n,1}\|$ on both sides, collecting the common terms, and using the fact that $\frac{1}{h^2} = \frac{cQ}{\epsilon}$, we obtain

$$\|v^{n+1} - v^{n,1}\| \leq \frac{1}{2}\|v^{n,1} - v^n\| + \frac{\tau}{\epsilon}(1 + cQ\Gamma)\left(\frac{1}{2}\|u^{n,1} - v^{n,1}\| + \|u^n - v^n\|\right) + \frac{\tau\Gamma}{h}\|\partial_x(\frac{u^{n,1}}{2} - u^n)\|,$$

which upon squaring yields

$$\begin{aligned}\|v^{n+1} - v^{n,1}\|^2 \leq & \|v^{n,1} - v^n\|^2 + \frac{\tau^2}{\epsilon^2}(1 + cQ\Gamma)^2\left(\|u^{n,1} - v^{n,1}\|^2 + 4\|u^n - v^n\|^2\right) \\ & + \frac{2\tau^2\Gamma^2}{h^2}\|\partial_x u^{n,1}\|^2 + \frac{8\tau^2\Gamma^2}{h^2}\|\partial_x u^n\|^2. \end{aligned} \tag{A.9}$$

In an entirely same manner, we derive

$$\begin{aligned}\|u^{n+1} - u^{n,1}\|^2 \leq & \|u^{n,1} - u^n\|^2 + \frac{\tau^2}{\epsilon^2}(1 + cQ\Gamma)^2\left(\|u^{n,1} - v^{n,1}\|^2 + 4\|u^n - v^n\|^2\right) \\ & + \frac{2\tau^2\Gamma^2}{h^2}\|\partial_x v^{n,1}\|^2 + \frac{8\tau^2\Gamma^2}{h^2}\|\partial_x v^n\|^2. \end{aligned} \tag{A.10}$$

Note that the bounds for $\mathcal{L}_v^{n,1} = v^{n,1} - v^n$, $\mathcal{L}_u^{n,1} = u^{n,1} - u^n$ given in (4.17) can still be used. Thus (A.9) and (A.10) together with (4.17) allow us to bound Π^2 as follows.

$$\begin{aligned}\Pi^2 \leq & \frac{2\tau^2\Gamma^2}{h^2}\|(\partial_x v^n, \partial_x u^n)\|^2 + \frac{4\tau^2}{\epsilon^2}(1 + cQ\Gamma)^2\|v^n - u^n\|^2 \\ & + \frac{2\tau^2}{\epsilon^2}(1 + cQ\Gamma)^2\left(\|u^{n,1} - v^{n,1}\|^2 + 4\|u^n - v^n\|^2\right) \\ & + \frac{2\tau^2\Gamma^2}{h^2}\|(\partial_x v^{n,1}, \partial_x u^{n,1})\|^2 + \frac{8\tau^2\Gamma^2}{h^2}\|(\partial_x v^n, \partial_x u^n)\|^2\end{aligned}$$

$$
\begin{aligned}
=&\frac{10\tau^2\mathit{\Gamma}^2}{h^2}\|(\partial_x v^n,\partial_x u^n)\|^2+\frac{12\tau^2}{\epsilon^2}(1+cQ\mathit{\Gamma})^2\|v^n-u^n\|^2\\
&+\frac{2\tau^2\mathit{\Gamma}^2}{h^2}\|(\partial_x v^{n,1},\partial_x u^{n,1})\|^2+\frac{2\tau^2}{\epsilon^2}(1+cQ\mathit{\Gamma})^2\|v^{n,1}-u^{n,1}\|^2. \qquad (A.11)
\end{aligned}
$$

Step 4 (Finding stability condition) Plugging the estimate of $\mathit{\Pi}^1$ in (A.8) and estimate of $\mathit{\Pi}^2$ in (A.11) into (A.7), we obtain

$$
\begin{aligned}
\mathit{\Pi}^1+\mathit{\Pi}^2\le&-\frac{\tau}{2}\left(\beta-\frac{20\tau\mathit{\Gamma}^2}{h^2}\right)\|(\partial_x v^n,\partial_x u^n)\|^2-\frac{\tau}{\epsilon}\left(1-c-\frac{12\tau}{\epsilon}(1+cQ\mathit{\Gamma})^2\right)\|v^n-u^n\|^2\\
&-\frac{\tau}{2}\left(\beta-\frac{4\tau\mathit{\Gamma}^2}{h^2}\right)\|(\partial_x v^{n,1},\partial_x u^{n,1})\|^2-\frac{\tau}{\epsilon}\left(1-c-\frac{2\tau}{\epsilon}(1+cQ\mathit{\Gamma})^2\right)\|v^{n,1}-u^{n,1}\|^2.
\end{aligned}
$$

It suffices for the RK2AEDG to be stable if each term on the right hand side of the above inequality is non-positive, this is indeed so if

$$
\tau\le\frac{\beta h^2}{20\mathit{\Gamma}^2},\quad \tau\le\frac{(1-c)\epsilon}{12(1+cQ\mathit{\Gamma})^2},
$$

which is implied by the assumption (A.4).

The error estimate for the 2-stage RK2AEDG method (A.2) and (A.3) can be carried out similarly to that for the RK3AEDG method. We outline the key ingredients in the following.

Define the reference solutions of (1.1) as

$$
\phi^{n,0}=\phi(x,t^n), \qquad (A.12a)
$$
$$
\phi^{n,1}=\phi^{n,0}-\tau(\alpha\partial_x-\beta\partial_x^2)\phi^{n,0}, \qquad (A.12b)
$$

then the use of Taylor's expansion as in the proof of Lemma 5.1 gives the following result.

Lemma A.2. *If $\|\partial_t^3\phi\|$ is bounded uniformly for any $t\in[0,T]$, we have*

$$
\phi(x,t^n+\tau)=\frac{\phi^{n,0}}{2}+\frac{\phi^{n,1}}{2}-\frac{\tau}{2}\left(\alpha\partial_x-\beta\partial_x^2\right)\phi^{n,1}+F^1(n;x),
$$

where $F^1(n;x)=\frac{\tau^3}{6}\partial_t^3\phi(x,\cdot)$, with $\|F^1(n;\cdot)\|=O(\tau^3)$ uniformly for any time $t\in[0,T]$.

We use the same notation as given in (5.8) for $i=0,1$ only, the first part of Lemma 5.2 remains valid, stated as follows:

Lemma A.3. *If the time step satisfies $c^*\tau\le\epsilon=cQh^2$, then the following inequality holds true*

$$
\|(e_1^{n,1},e_2^{n,1})\|^2\le C\|(e_1^{n,0},e_2^{n,0})\|^2+Ch^{2k+2},
$$

where C is a constant independent τ,h,n.

Equipped with Lemmas A.2 and A.3, the rest of the proof of (A.5) follows the same pattern as the proof of Theorem 5.3, now under the stability condition (A.4). For completeness, we include the main steps.

Step 1 (Error equations). In a similar manner to that leading to (5.9)–(5.11), we write the error equations as

$$
\langle e_1^{n,1}-e_1^{n,0},\xi\rangle+\tau A_{21}(e_2^{n,0},\xi)=\langle\epsilon_1^{n,1}-\epsilon_1^{n,0},\xi\rangle+\tau A_{21}(\epsilon_2^{n,0},\xi)-\frac{\tau}{\epsilon}\langle u^{n,0}-v^{n,0},\xi\rangle, \qquad (A.13a)
$$

$$\langle e_2^{n,1} - e_2^{n,0}, \eta\rangle + \tau A_{12}(e_1^{n,0}, \eta) = \langle \epsilon_2^{n,1} - \epsilon_2^{n,0}, \eta\rangle + \tau A_{12}(\epsilon_1^{n,0}, \eta) - \frac{\tau}{\epsilon}\langle v^{n,0} - u^{n,0}, \eta\rangle, \tag{A.13b}$$

$$\begin{aligned}\langle 2e_1^{n+1} - e_1^{n,0} - e_1^{n,1}, \xi\rangle + \tau A_{21}(e_2^{n,1}, \xi) =&\langle 2\epsilon_1^{n+1} - \epsilon_1^{n,0} - \epsilon_1^{n,1}, \xi\rangle + \tau A_{21}(\epsilon_2^{n,1}, \xi)\\ &- \frac{\tau}{\epsilon}\langle u^{n,1} - v^{n,1}, \xi\rangle + \langle 2F^1(n;x), \xi\rangle, \end{aligned}\tag{A.14a}$$

$$\begin{aligned}\langle 2e_2^{n+1} - e_2^{n,0} - e_2^{n,1}, \eta\rangle + \tau A_{12}(e_1^{n,1}, \eta) =&\langle 2\epsilon_2^{n+1} - \epsilon_2^{n,0} - \epsilon_2^{n,1}, \eta\rangle + \tau A_{12}(\epsilon_1^{n,1}, \eta)\\ &- \frac{\tau}{\epsilon}\langle v^{n,1} - u^{n,1}, \eta\rangle + \langle 2F^1(n;x), \eta\rangle\end{aligned}\tag{A.14b}$$

for $(\xi, \eta) \in V_h \times U_h$ at each stage, respectively.

Step 2 (Regrouping against proper test functions). Taking $(\xi, \eta) = (e_1^{n,0}, e_2^{n,0})$ and $(\xi, \eta) = (e_1^{n,1}, e_2^{n,1})$ in stage (A.13)–(A.14), respectively, summing the result up, using (5.22) again, according to the stability analysis (A.7), we obtain

$$\|(e_1^{n+1}, e_2^{n+1})\|^2 - \|(e_1^n, e_2^n)\|^2 = \Pi_3 + \Pi_4 + G_3 + G_4, \tag{A.15}$$

where

$$\begin{aligned}\Pi_3 =& -\tau\Big(A_{21}(e_2^{n,0}, e_1^{n,0}) + A_{12}(e_1^{n,0}, e_2^{n,0}) + A_{21}(e_2^{n,1}, e_1^{n,1}) + A_{12}(e_1^{n,1}, e_2^{n,1})\Big)\\ &- \frac{\tau}{\epsilon}\|e_1^{n,0} - e_2^{n,0}\|^2 - \frac{\tau}{\epsilon}\|e_1^{n,1} - e_2^{n,1}\|^2,\\ \Pi_4 =&\|e_1^{n+1} - e_1^{n,1}\|^2 + \|e_2^{n+1} - e_2^{n,1}\|^2\end{aligned}$$

and

$$\begin{aligned}G_3 &= \langle \epsilon_1^{n,1} - \epsilon_1^{n,0}, e_1^{n,0}\rangle + \langle \epsilon_2^{n,1} - \epsilon_2^{n,0}, e_2^{n,0}\rangle + \langle 2\epsilon_1^{n+1} - \epsilon_1^{n,0} - \epsilon_1^{n,1}, e_1^{n,1}\rangle + \langle 2\epsilon_2^{n+1} - \epsilon_2^{n,0} - \epsilon_2^{n,1}, e_2^{n,1}\rangle,\\ G_4 &= -\tau\sum_{i=0}^{1}\Big(\langle \epsilon_1^{n,i}, e_1^{n,i}\rangle + \langle \epsilon_2^{n,i}, e_2^{n,i}\rangle\Big) + \langle 2F^1(n,x), e_1^{n,1} + e_2^{n,1}\rangle.\end{aligned}$$

Step 3 (Term by term estimates). Under stability condition (A.4), we know that

$$\Pi_3 + \Pi_4 \le 0. \tag{A.16}$$

Next, $|G_3 + G_4|$ can be bounded as for bounding $|G_1 + G_2|$ in (5.34). Here, instead, we use Lemmas A.2 and A.3 so that

$$\begin{aligned}|G_3 + G_4| &\le 2\tau\sum_{i=0}^{1}\|(e_1^{n,i}, e_2^{n,i})\|^2 + C\tau(h^{2k+2} + \tau^4)\\ &\le C\tau\|(e_1^{n,0}, e_2^{n,0})\|^2 + C\tau(h^{2k+2} + \tau^4).\end{aligned}\tag{A.17}$$

Step 4 (Final substitution). Inserting (A.16) and (A.17) into (A.15) gives

$$\|(e_1^{n+1}, e_2^{n+1})\|^2 \le (1 + C\tau)\|(e_1^n, e_2^n)\|^2 + C\tau(h^{2k+2} + \tau^4),$$

this together with the initial error (5.35) leads to

$$\|(e_1^n, e_2^n)\| \le C(h^{k+1} + \tau^2).$$

The above inequality and the projection error for ϵ_i^n when inserted into (5.8) yields the desired estimate (A.5).

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