ESAIM: M2AN 52 (2018) 1023–1049
https://doi.org/10.1051/m2an/2018021

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

DENSITY AND TRACE RESULTS IN GENERALIZED FRACTAL NETWORKS

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Abstract. The first aim of this paper is to give different necessary and sufficient conditions that guarantee the density of the set of compactly supported functions into the Sobolev space of order one in infinite p-adic weighted trees. The second goal is to define properly a trace operator in this Sobolev space if it makes sense.

Mathematics Subject Classification. 35B53, 35B40, 35B65, 35J05

Received 12 September, 2017. Accepted 16 March, 2018.

1. Introduction

Recent applications, such as electrical circuits, arterial networks, networks of open channels [7], traffic flows on networks [9, 12, 14], the respiratory system model [8, 19], involve partial differential equations set on finite or infinite one-dimensional networks (coupled via some transmission conditions at the nodes). Existence results for such problems need usually the introduction of Sobolev spaces on such structures, while some related inverse problems require Dirichlet to Neumann (or the converse) maps. For this last question, the first step is to define properly the trace spaces of these Sobolev spaces. When the network is finite, *i.e.*, it has a finite number of edges, the characterization of the different Sobolev spaces and its trace ones is relatively easy as the boundary of the network is made of a finite number of points. On the contrary, when the network is an infinite one, these questions become more delicate, because its boundary is no more a finite set. Up to our best knowledge, the only case that is fully understood is the case of dyadic trees that admit some similarities [8, 19].

Hence our goal is to attack such questions in the case of p-adic trees with arbitrary weights. More precisely, after having given the exact setting, we define the Sobolev space H^1_μ of order 1 as well as its subspace, defined as the closure of compactly supported functions. Then we give different necessary and sufficient conditions that guarantee that both spaces coincide. One of these conditions is the well known property from harmony analysis, the so-called Liouville property, that says that any bounded harmonic function is constant. Another fully explicit one says that the resistance of the truncated tree at the generation n goes to infinity, as n goes to infinity. In a second step, if both spaces differ, we show that elements of H^1_μ admit a trace at infinity. As underlined before, such a trace result has potential applications to inverse problem.

Keywords and phrases: Laplace equation, fractal, graph domain, Liouville property, boundary operator.

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Similar questions at a discrete level or on higher-dimensional domains are also considered, let us quote discrete laplacian on infinite networks [11, 13, 15, 18, 20–23], two-dimensional domains with a fractal boundary [1–4], pre-fractal domains approximating the Koch snowflake [10]. Let us finally mention some related problems such as the Hamilton-Jacobi equation [5] and the Gauss-Bonnet operator on infinite graphs [6].

The paper is organized as follows. In Section 2, we give the definition of the Sobolev spaces in an infinite weighted tree. In Section 3, we give four necessary and sufficient conditions on the density of compactly supported functions, *i.e.*, functions that do not vanish on a finite number of edges, into the previously defined Sobolev spaces. Finally, in Section 4, we build a trace operator at the end of the tree, when it makes sense to define such an operator.

Finally in the whole paper, the notation $A \lesssim B$ is used for the estimate $A \leqslant C\ B$, where C is a generic constant that does not depend on A and B.

2. p-adic trees and Sobolev spaces

2.1. General p-adic trees

In this section, we introduce some definitions and notations about general p-adic trees.

Definition 2.1. Given p in $\mathbb{N}^*$, we introduce the following set of indexes in $\mathbb{N}^2$:

- $\mathbb{E}_p$ defined as

$$\mathbb{E}_p = \left\{(\ell, j) \in \mathbb{N}^2 \text{ such that } 0 \leqslant j \leqslant p^\ell - 1\right\},$$

- $\mathbb{V}_p$ defined as

$$\mathbb{V}_p = (0,0) \cup \left\{(\ell, j) \in \mathbb{N}^2 \text{ such that } \ell \geqslant 1 \text{ and } 0 \leqslant j \leqslant p^{\ell-1} - 1\right\}.$$

Definition 2.2 (p-adic tree)**.** One says that $\mathcal{T}$ is a p-adic tree of $\mathbb{R}^d$ if there exists two families $\mathcal{E} = (e_{\ell,j})_{(\ell,j)\in\mathbb{E}_p}$ and $\mathcal{V} = (v_{\ell,j})_{(\ell,j)\in\mathbb{V}_p}$ such that

- each $v_{\ell,j}$ is a point of $\mathbb{R}^d$,
- each $e_{\ell,j}$ is a straight segment in $\mathbb{R}^d$, whose extremities are $v_{\ell,\lfloor j/p\rfloor}$ and $v_{\ell+1,j}$,
- for $((\ell, j), (\ell', j')) \in \mathbb{V}_p^2$, one has

$$(\ell, j) \neq (\ell', j') \Rightarrow v_{\ell,j} \neq v_{\ell',j'},$$

- for $((\ell, j), (\ell', j')) \in \mathbb{E}_p^2$, one has

$$(\ell, j) \neq (\ell', j') \Rightarrow e_{\ell,j} \cap e_{\ell',j'} = \emptyset.$$

One says that $\mathcal{V}$ is the set of nodes of the p-adic tree $\mathcal{T}$ and $\mathcal{E}$ is the set of edges of $\mathcal{T}$. We shall denote $\mathcal{T} = (\mathcal{V}, \mathcal{E})$. An example is given by Figure 1, where we can see the utility of notations introduced in Definition 2.1.

Definition 2.3 (Subtree of a p-adic tree)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a p-adic tree and let us fix $(\ell, j) \in \mathbb{E}_p$. The subtree $\mathcal{T}_{e_{\ell,j}}$ of main edge $e_{\ell,j}$ is the tree $(\mathcal{V}', \mathcal{E}')$ given by

$$\begin{aligned} e'_{m,k} &= e_{\ell+m,p^m j+k}, \text{ for } (m, k) \in \mathbb{E}_p, \\ v'_{m,k} &= v_{\ell+m,p^{m-1} j+k}, \text{ for } (m, k) \in \mathbb{V}_p. \end{aligned}$$

One example of subtree is illustrated in Figure 2.

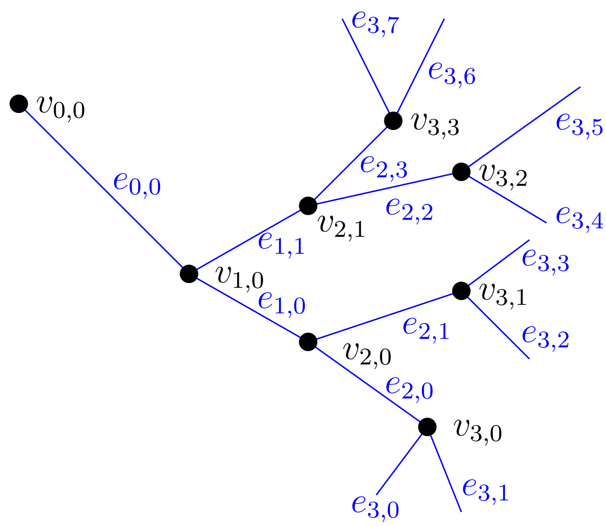

FIGURE 1. An example of dyadic tree. We circle nodes and we color in blue edges.

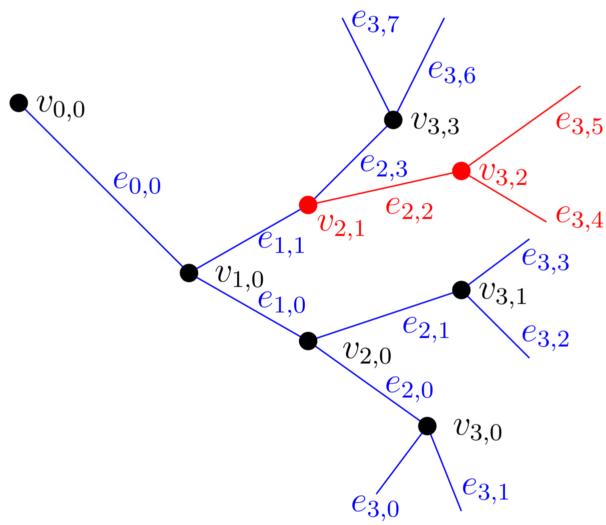

FIGURE 2. An example of dyadic tree. We plot in red the subtree $\mathcal{T}_{e_{2,2}}$

Remark 2.4. In the relation linking v and v', for $m = 0$, we take $\lfloor j/p \rfloor$ instead of j/p. Moreover, taking $n = m - 1$ in this relation ensures that, for $(m, k) \in \mathbb{V}_p$ with $m \geqslant 1$, one has $(n, k) \in \mathbb{E}_p$, and one can see that $v_{\ell+n+1,p^n j+k}$ is one of the node of the edge $e_{\ell+n,p^n j+k}$.

Definition 2.5 (Generations and partial p-adic tree)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a p-adic tree.

- Given $\ell \in \mathbb{N}$, one defines the ℓth generation $\mathcal{G}^\ell(\mathcal{T})$ as the closure (in $\mathbb{R}^d$) of the set

$$\bigcup_{j/(\ell,j)\in\mathbb{E}_p} e_{\ell,j}.$$

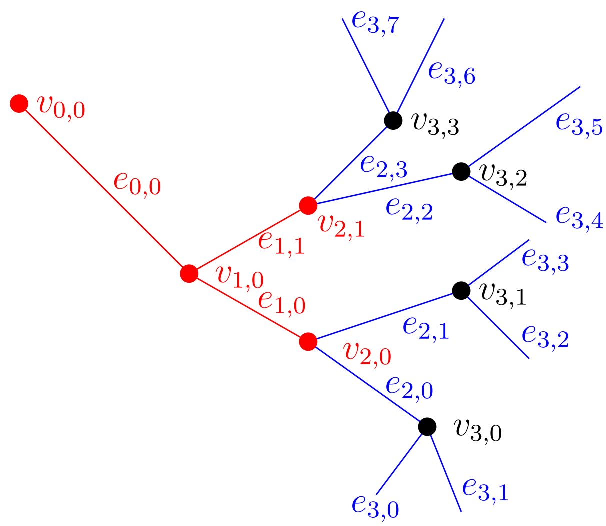

FIGURE 3. An example of dyadic tree. We plotted in red $\mathcal{T}^1$.

- Given $\ell \in \mathbb{N}$, one defines the partial p-adic tree $\mathcal{T}^\ell$ as

$$\mathcal{T}^\ell = \bigcup_{\ell'=0}^{\ell} \mathcal{G}^{\ell'}(\mathcal{T}) = \bigcup_{\ell'=0}^{\ell} \bigcup_{j/(\ell,j)\in\mathbb{E}_p} e_{\ell',j}.$$

An example is given by Figure 3.

Definition 2.6 (Projection)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a tree. For any $(\ell, j) \in \mathbb{E}_p$, one defines the canonical projection (we omit the indexes associated with $\mathcal{T}$ for convenience):

$$\begin{aligned} \varphi_{\ell,j} : (0,1) \to e_{\ell,j} : \\ x \mapsto \varphi_{\ell,j} = v_{\ell,\lfloor j/p \rfloor} + (v_{\ell+1,j} - v_{\ell,\lfloor j/p \rfloor})x. \end{aligned}$$

2.2. Sobolev spaces

Definition 2.7 (Compact supportness of a function)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ be a tree and $u : \mathcal{E} \to \mathbb{C}$ be a function. One says that u is compactly supported if and only if there exists $\ell \in \mathbb{N}$ such that, for any $(m, j) \in \mathbb{E}_p$ with $m > \ell$, the restriction of u on $e_{m,j}$ is identically equal to 0. In this case, one says that the support of u is included in $\mathcal{T}^l$.

Definition 2.8 (Weight on a p-adic tree)**.** Let us consider a p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ and a function $\mu : \mathbb{E}_p \to \mathbb{R}$. One says that μ is a weight on $\mathcal{T}$ if and only if

$$0 < \mu_{\ell,j} := \mu(\ell, j) < \infty, \quad \forall (\ell, j) \in \mathbb{E}_p.$$

In this case, we denote the weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$. By abuse of notation, we also denote by μ the function from $\mathcal{E}$ to $\mathbb{R}$ defined by

$$\mu(\mathbf{x}) = \mu_{\ell,j}, \quad \forall \mathbf{x} \in e_{\ell,j}.$$

Definition 2.9 (L^q_μ-norm)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ be a weighted tree and $q \in [1, \infty)$.

1. Given $(\ell, j) \in \mathbb{E}_p$, a function $u : e_{\ell,j} \to \mathbb{C}$ will be in the set $\mathrm{L}^q_\mu(e_{\ell,j})$ if and only if the function

$$\widehat{u} = u \circ \varphi_{\ell,j}$$

is in $\mathrm{L}^q_\mu(0,1) := \mathrm{L}^q(0,1)$, thanks to the following change of variable $\mathbf{x} = \varphi_{\ell,j}(x)$ (one denotes $L_{\ell,j}$ the length of $e_{\ell,j}$):

$$\int_{e_{\ell,j}} \mu(\mathbf{x})|u(\mathbf{x})|^q \mathrm{d}\mathbf{x} = L_{\ell,j} \int_0^1 (\mu \circ \varphi_{\ell,j})(x)|\widehat{u}(x)|^q \mathrm{d}x < +\infty.$$

In this case, one shall denote

$$\|u\|^q_{\mathrm{L}^q_\mu(e_{\ell,j})} = \int_{e_{\ell,j}} \mu(\mathbf{x})|u(\mathbf{x})|^q \mathrm{d}\mathbf{x} = L_{\ell,j} \int_0^1 (\mu \circ \varphi_{\ell,j})(x)|\widehat{u}(x)|^q \mathrm{d}x.$$

2. A function $u : \mathcal{E} \to \mathbb{R}$ will be in the set $\mathrm{L}^q_\mu(\mathcal{T})$ if and only if:
   (a) the restriction of u to each element $e \in \mathcal{E}$ belongs to $\mathrm{L}^q_\mu(e)$,
   (b) the following quantity

$$\sum_{(\ell,j)\in\mathbb{E}_p} \|u\|^q_{\mathrm{L}^q_\mu(e_{\ell,j})}$$

   is finite. By convention, one shall write

$$\|u\|^q_{\mathrm{L}^q_\mu(\mathcal{T})} := \int_{\mathcal{T}} \mu(\mathbf{x})|u(\mathbf{x})|^q \mathrm{d}\mathbf{x} = \sum_{(\ell,j)\in\mathbb{E}_p} \|u\|^q_{\mathrm{L}^q_\mu(e_{\ell,j})}. \tag{2.1}$$

3. We shall denote by $\mathrm{L}^2_{\mu,\mathrm{loc}}(\mathcal{T})$ the set of functions u defined on the weighted tree $\mathcal{T}$ such that $u\Phi \in \mathrm{L}^2_\mu(\mathcal{T})$ for any bounded function Φ with compact support.

In accordance with (2.1), for all n and $u \in L^1(\mathcal{T}) = \mathrm{L}^1_1(\mathcal{T})$, we often write

$$\int_{\mathcal{T}^n} u(\mathbf{x})\mathrm{d}\mathbf{x} = \sum_{\ell \leqslant n} \sum_{j=0}^{p^\ell - 1} \int_{e_{\ell,j}} u(\mathbf{x})\mathrm{d}\mathbf{x}.$$

Remark 2.10. Taking notations of Definition 2.8, if there exist (ℓ, j) in $\mathbb{E}_p$ and $n \in \mathbb{N}$ such that $u \in \mathrm{H}^n_\mu(e_{\ell,j}) := \mathrm{H}^n(e_{\ell,j})$, then $\widehat{u} \in \mathrm{H}^n_\mu(0,1) = \mathrm{H}^n(0,1)$, and we have the following relation, for any $m \leqslant n$:

$$|u|^2_{\mathrm{H}^m_\mu(e_{\ell,j})} := \int_{e_{\ell,j}} \mu(\mathbf{x})|u^{(m)}(\mathbf{x})|^2 \mathrm{d}\mathbf{x} = L^{1-2m}_{\ell,j} \int_0^1 (\mu \circ \varphi_{\ell,j})(x)|\widehat{u}^{(m)}(x)|^2 \mathrm{d}x. \tag{2.2}$$

Definition 2.11 ($\mathcal{H}^1_\mu$ space and $\mathcal{H}^1_\mu$-norm)**.** Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ be a weighted tree. We shall denote by $\mathcal{H}^1_\mu(\mathcal{T})$ the following set

$$\mathcal{H}^1_\mu(\mathcal{T}) = \left\{ u \in \mathrm{L}^2_{\mu,\mathrm{loc}}(\mathcal{T}) \ / \ u' \in \mathrm{L}^2_\mu(\mathcal{T}) \right\}, \tag{2.3}$$

where u' is the derivative of u (in a weak sense, this implies that function u is continuous on the whole graph, even on each point $v \in \mathcal{V}$). This space is a Hilbert space with associated norm

$$\|u\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = |u(v_{0,0})|^2 + \|u'\|^2_{\mathrm{L}^2_\mu(\mathcal{T})} \tag{2.4}$$

and associated semi-norm

$$|u|_{\mathcal{H}^1_\mu(\mathcal{T})} = \|u'\|_{\mathrm{L}^2_\mu(\mathcal{T})}\,. \tag{2.5}$$

Remark 2.12. One particular function in any $\mathcal{H}^1_\mu(\mathcal{T})$ is the function $\mathbb{1}$ defined by $\mathbb{1}(\mathbf{x}) = 1$, for any $e \in \mathcal{E}$ and for any $\mathbf{x} \in e$. This function will play a particular role in Section 3.

Definition 2.13 ($\mathcal{H}^1_{\mu,\mathrm{c}}$ and $\mathcal{H}^1_{\mu,0}$ spaces)**.** Let $\mathcal{T} = (\mathcal{E}, \mathcal{V}, \mu)$ be a weighted tree.

- We denote by $\mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ the subset of functions $u \in \mathcal{H}^1_\mu(\mathcal{T})$ whose support is compact, in the sense of

$$\mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T}) = \left\{u \in \mathcal{H}^1_\mu(\mathcal{T}) \ / \ \exists n \in \mathbb{N}, \quad u = 0 \text{ in } \mathcal{T} \setminus \mathcal{T}^n\right\}. \tag{2.6}$$

- We denote by $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ the closure of $\mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ in $\mathcal{H}^1_\mu(\mathcal{T})$ for the norm (2.4).

Since functions in $\mathcal{H}^1_\mu(\mathcal{T})$ are continuous, we can associate with the space $\mathcal{H}^1_\mu(\mathcal{T})$ a discrete counterpart. More precisely according to [19, 20, 22], we make the following definition.

Definition 2.14 ($\mathcal{H}^1_{d,\nu}$ and $\mathcal{H}^1_{d,\nu,0}$ spaces)**.** Let $\mathcal{T} = (\mathcal{E}, \mathcal{V}, \nu)$ be a weighted tree.

- We denote by $\mathcal{H}^1_{d,\nu}(\mathcal{T})$ the subset of functions $p \in \mathbb{R}^{\mathcal{V}}$ such that

$$\sum_{(\ell,j)\in\mathbb{E}_p} \nu_{\ell,j} |p(v_{\ell+1,j}) - p(v_{\ell,\lfloor j/p \rfloor})|^2 < \infty. \tag{2.7}$$

- We denote by $D(\mathcal{T})$ the set of finitely supported functions in $\mathbb{R}^{\mathcal{V}}$ and define $\mathcal{H}^1_{d,\nu,0}(\mathcal{T})$ the closure in $\mathcal{H}^1_{d,\nu}(\mathcal{T})$ of $D(\mathcal{T})$.

$\mathcal{H}^1_{d,\nu}(\mathcal{T})$ and $\mathcal{H}^1_{d,\nu,0}(\mathcal{T})$ are Hilbert spaces with norm

$$\|u\|^2_{\mathcal{H}^1_{d,\nu}(\mathcal{T})} = |p(v_{0,0})|^2 + \sum_{(\ell,j)\in\mathbb{E}_p} \nu_{\ell,j} |p(v_{\ell+1,j}) - p(v_{\ell,\lfloor j/p \rfloor})|^2. \tag{2.8}$$

As anticipated before, we have the following embedding.

Lemma 2.15. *Let $\mathcal{T} = (\mathcal{E}, \mathcal{V}, \nu)$ be a weighted tree. Then $\mathcal{H}^1_\mu(\mathcal{T})$ is continuously embedded into $\mathcal{H}^1_{d,\nu}(\mathcal{T})$ with the weights $\nu_{\ell,j} = \frac{\mu_{\ell,j}}{L_{\ell,j}}$, namely the mapping*

$$u \to (u(v_{\ell,j}))_{\ell,j\in\mathbb{V}_p}$$

is continuous from $\mathcal{H}^1_\mu(\mathcal{T})$ into $\mathcal{H}^1_{d,\nu}(\mathcal{T})$.

Proof. Fix $u \in \mathcal{H}^1_\mu(\mathcal{T})$, then for every $(\ell, j) \in \mathbb{E}_p$, we may write

$$u(v_{\ell+1,j}) - u(v_{\ell,\lfloor j/p \rfloor}) = \int_{e_{\ell,j}} u'(\mathbf{x})\mathrm{d}\mathbf{x},$$

hence by Cauchy-Schwarz's inequality we deduce that

$$\frac{\mu_{\ell,j}}{L_{\ell,j}}|u(v_{\ell+1,j}) - u(v_{\ell,\lfloor j/p\rfloor})|^2 \leqslant \mu_{\ell,j}\int_{e_{\ell,j}} |u'(\mathbf{x})|^2\mathrm{d}\mathbf{x}.$$

The conclusion follows by taking the sum on $(\ell, j) \in \mathbb{E}_p$. □

Remark 2.16. Note that the converse embedding is not valid since $\mathcal{H}^1_{d,\nu}(\mathcal{T})$ can be identified with the subspace of affine functions from $\mathcal{H}^1_\mu(\mathcal{T})$ with $\mu_{\ell,j} = \nu_{\ell,j}L_{\ell,j}$.

3. Some results on Sobolev spaces

At this point, one natural issue that can be asked is: on which condition over the triplet $(\mathcal{E}, \mathcal{V}, \mu)$ the sets $\mathcal{H}^1_\mu(\mathcal{T})$ and $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ are the same? If not, how can we characterize $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ (in another way than Def. 2.13)? This will be answered partially here. We answer to the first issue in two steps: the first step is to show an equivalent relation to $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$, the second step is to characterize this equivalent relation.

In the following, we consider $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ a weighted tree.

3.1. $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$: a first implicit necessary and sufficient condition

A first necessary and sufficient to have $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$ is given by the following theorem:

Theorem 3.1. *We have the following equivalence:*

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T}). \tag{3.1}$$

Proof. The proof takes some ideas from the proof of Theorem 2.12 in the paper of B. Maury *et al.* [19], adapted in our case. The part $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \Rightarrow \mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ is a trivial case, since $\mathbb{1}$ always belongs to $\mathcal{H}^1_\mu(\mathcal{T})$. Now let us show the converse implication $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T}) \Rightarrow \mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$: given $u \in \mathcal{H}^1_\mu(\mathcal{T})$, let us show that there exists a family of functions $(u_n)_{n\in\mathbb{N}} \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ such that

$$\|u - u_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0 \quad \text{as} \quad n \to \infty.$$

By hypothesis about the function $\mathbb{1}$, there exists a family of functions $(\mathbb{1}_n)_{n\in\mathbb{N}} \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ such that

$$\|\mathbb{1} - \mathbb{1}_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0 \quad \text{as}\, n \to \infty,$$

which implies

$$|\mathbb{1}_n|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0 \quad \text{as}\, n \to \infty. \tag{3.2}$$

Since $u \in \mathcal{H}^1_\mu(\mathcal{T})$, for any $\varepsilon > 0$, there exists $\ell_0 \in \mathbb{N}$ such that

$$\sum_{(\ell,j)\in\mathbb{E}_p, \ell\geqslant\ell_0} \|u'\|_{\mathrm{L}^2_\mu(e_{\ell,j})} < \varepsilon. \tag{3.3}$$

As $\mathcal{H}^1_\mu(\mathcal{T})$ is continuously embedded into $\mathrm{C}^0(\overline{\mathcal{T}})$, the property (3.2) implies that

$$\mathbb{1}_n(v) \to 1 \quad \text{as} \quad n \to \infty$$

for all nodes v. In particular,

$$\mathbb{1}_n(v_{\ell_0,j}) \to 1 \quad \text{as} \quad n\to\infty, \text{ for any } j \text{ s.t. } (\ell_0,j)\in\mathbb{V}_p.$$

Since this set is finite, there exists $n_0\in\mathbb{N}$ such that, for any $n\geqslant n_0$, one has

$$\mathbb{1}_n(v_{\ell_0,j}) \geqslant \frac{1}{2} \quad \text{for any } j \text{ such that } (\ell_0,j)\in\mathbb{V}_p.$$

We now build the sequence $(u_n)_{n\in\mathbb{N}}$ (we omit here that u_n depends on ℓ_0) as follows:

- for any edge $e_{\ell,j}$ with $(\ell,j)\in\mathbb{E}_p$ and $\ell<\ell_0$, we take $u_n=u$,
- on each $\mathcal{T}_{e_{\ell_0,j}}$ (see back Fig. 2 p. 3 for an example of subtrees) with $(\ell_0,j)\in\mathbb{E}_p$, we take u_n as a multiple of $\mathbb{1}_n$, the multiplicity constant is given by the fact that u_n has to be a continuous fonction on the node $v_{\ell_0,k}$ connected to the edge $e_{\ell_0,j}$ (remember that Def. 2.2 says we have $k=\lfloor j/p\rfloor$).

In other words ($u_{n,\ell,j}$ means the restriction of u_n to the edge $e_{\ell,j}$):

$$u_{n,\ell,j} = \begin{cases} u_{\ell,j}, & \ell<\ell_0,\\ \dfrac{u(v_{\ell_0,k})}{\mathbb{1}_n(v_{\ell_0,k})}\mathbb{1}_{n,\ell,j}, & e_{\ell,j}\in\mathcal{T}_{e_{\ell_0,l}}, \lfloor l/p\rfloor = k.\end{cases}$$

By construction, u_n coincides with u up to the generation ℓ_0 and therefore

$$\|u-u_n\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = \sum_{(\ell,j)\in\mathbb{E}_p,\ell\geqslant\ell_0} \|u'-u_n'\|^2_{\mathrm{L}^2_\mu(e_{\ell,j})}.$$

We use the classical inequality $|u'-u_n'|^2\leqslant 2|u'|^2+2|u_n'|^2$ to get

$$\|u-u_n\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} \leqslant 2\sum_{(\ell,j)\in\mathbb{E}_p,\ell\geqslant\ell_0}\|u'\|^2_{\mathrm{L}^2_\mu(e_{\ell,j})} \tag{3.4-i}$$

$$+\ 2\sum_{k,(\ell_0,k)\in\mathbb{V}_p}\left(\frac{u(v_{\ell_0,k})}{\mathbb{1}_n(v_{\ell_0,k})}\right)^2\sum_{l=0}^{p-1}|\mathbb{1}_n|^2_{\mathcal{H}^1_\mu(\mathcal{T}_{e_{\ell_0,pk+l}})}. \tag{3.4-ii}$$

The term (3.4-i) is estimated with the help of (3.3), while for the term (3.4-ii) we simply notice that the assumption $\mathbb{1}_n(v)\geqslant 1/2$ for all nodes $(v_{\ell_0,j})\in\mathcal{V}$ and the fact that u is bounded on a compact set yield

$$\|u-u_n\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} \leqslant 2\sum_{(\ell,j)\in\mathbb{E}_p,\ell\geqslant\ell_0}\|u'\|^2_{\mathrm{L}^2_\mu(e_{\ell,j})}$$

$$+\ 2\sum_{k,(\ell_0,k)\in\mathbb{V}_p}\left(\frac{u(v_{\ell_0,k})}{\mathbb{1}_n(v_{\ell_0,k})}\right)^2\sum_{l=0}^{p-1}|\mathbb{1}_n|^2_{\mathcal{H}^1_\mu(\mathcal{T}_{e_{\ell_0,pk+l}})}.$$

The conclusion follows from (3.2). □

Remark 3.2. Due to Remark 2.16, the discrete counterpart of Theorem 3.1 (see for instance Thm. 3.63 of [20] or Thm. 2.12 of [19]) is not helpful to prove this Theorem.

3.2. Study of auxiliary Laplacian problems and additional implicit necessary and sufficient conditions

In this section, we want to determine if $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ or not. To answer this question, we look for the solution of some auxiliary Laplacian problems on $\mathcal{T}$.

3.2.1. Generalized Laplacian problems on an infinite tree

In this subsection, we will give another way to determine if $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ or not. To do so, we consider the two following problems:

$$\begin{aligned}
(\mathcal{P}_\mathrm{N}) \quad & \text{Find } u_\mathrm{N} \in \mathcal{H}^1_\mu(\mathcal{T}) \text{ such that } u_\mathrm{N}(v_{0,0}) = 1 \text{ and} \\
& \int_\mathcal{T} \mu(\mathbf{x}) u_\mathrm{N}'(\mathbf{x}) \overline{\varphi'(\mathbf{x})} \mathrm{d}\mathbf{x} = 0, \quad \forall \varphi \in \mathcal{H}^1_\mu(\mathcal{T}),\ \varphi(v_{0,0}) = 0, \\
(\mathcal{P}_\mathrm{D}) \quad & \text{Find } u_\mathrm{D} \in \mathcal{H}^1_{\mu,0}(\mathcal{T}) \text{ such that } u_\mathrm{D}(v_{0,0}) = 1 \text{ and} \\
& \int_\mathcal{T} \mu(\mathbf{x}) u_\mathrm{D}'(\mathbf{x}) \overline{\varphi'(\mathbf{x})} \mathrm{d}\mathbf{x} = 0, \quad \forall \varphi \in \mathcal{H}^1_{\mu,0}(\mathcal{T}),\ \varphi(v_{0,0}) = 0.
\end{aligned}$$

We start with two propositions whose proof are direct and left to the reader.

Proposition 3.3. *Problems $(\mathcal{P}_\mathrm{N})$ and $(\mathcal{P}_\mathrm{D})$ are well posed, in other words, they have a unique solution $u_\mathrm{N} \in \mathcal{H}^1_\mu(\mathcal{T})$ and $u_\mathrm{D} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ respectively.*

Proposition 3.4. *$u_\mathrm{N} = \mathbb{1}$ is the solution of $(\mathcal{P}_\mathrm{N})$, and we have the following equivalence:*

$$\mathbb{1} \text{ is solution of } (\mathcal{P}_\mathrm{D}) \text{ (or } u_\mathrm{D} = \mathbb{1}) \quad \Longleftrightarrow \quad \mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T}). \tag{3.5}$$

3.2.2. Study of a bounded auxiliary Laplacian problem

This subsection is a generalization of ([17], Sect. 3.2) to general weights μ. Here, we study the Dirichlet problem on the finite subtree $\mathcal{T}^n$, and we look at the behaviour of its solution when $n \to \infty$. We first give general results on this kind of solution.

Let us introduce the following spaces:

$$\mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T}) = \left\{ u \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T}) \text{ such that } \operatorname{supp} u \subset \mathcal{T}^n \right\}, \tag{3.6}$$

$$\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T}) = \left\{ u \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T}) \text{ such that } u(v_{0,0}) = 0 \text{ and } \operatorname{supp} u \subset \mathcal{T}^n \right\}. \tag{3.7}$$

For all $n \in \mathbb{N}^*$, we consider the following problem: find $u^n \in \mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T})$ such that, for any test function $\varphi \in \mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T})$, one has

$$\int_\mathcal{T} \mu(\mathbf{x}) (u^n)'(\mathbf{x}) \overline{\varphi'(\mathbf{x})} \mathrm{d}\mathbf{x} = 0 \quad \text{and} \quad u^n(v_{0,0}) = 1. \tag{3.8}$$

Proposition 3.5. *Problem (3.8) is well posed in $\mathcal{H}^{1,n}_{\mu,c}(\mathcal{T})$. Moreover, one has*

$$\frac{1}{2} \left| u^n \right|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = \min_{u \in \mathcal{H}^{1,n}_{\mu,c}(\mathcal{T}) : u(v_{0,0}) = 1} \frac{1}{2} \left| u \right|^2_{\mathcal{H}^1_\mu(\mathcal{T})}. \tag{3.9}$$

Proof. We look for u^n under the form $u^n = \chi + \mathbf{u}^n$, where χ is the affine function on each edge $e \in \mathcal{E}$ with $\chi(v_{0,0}) = 1$ and $\chi(v) = 0$ for any other $v \in \mathcal{V}$, and hence $\mathbf{u}^n$ belongs to $\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T})$. We subsitute this splitting in (3.8) and get equivalently

$$\int_{\mathcal{T}} \mu(\mathbf{x})(\mathbf{u}^n)'(\mathbf{x})\overline{\varphi'(\mathbf{x})}\mathrm{d}\mathbf{x} = -\int_{\mathcal{T}} \mu(\mathbf{x})\chi'(\mathbf{x})\overline{\varphi'(\mathbf{x})}\mathrm{d}\mathbf{x} \tag{3.10}$$

Now Lax-Milgram's lemma gives that (3.10) is well-posed in $\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T})$ and has a unique solution $\mathbf{u}^n$ in $\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T})$. Moreover, as the left-hand side of (3.10) is symmetric one has

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{T}} \mu(\mathbf{x})\left|(\mathbf{u}^n)'(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathcal{T}} \mu(\mathbf{x})\chi'(\mathbf{x})\overline{(\mathbf{u}^n)(\mathbf{x})}\mathrm{d}\mathbf{x}\\ &= \operatorname{argmin}_{\mathbf{u}\in\mathcal{H}^{1,n}_{\mu,\mathrm{c},0}(\mathcal{T})}\left(\frac{1}{2}\int_{\mathcal{T}} \mu(\mathbf{x})\left|\mathbf{u}'(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x} + \int_{\mathcal{T}} \mu(\mathbf{x})\chi'(\mathbf{x})\overline{\mathbf{u}'(\mathbf{x})}\mathrm{d}\mathbf{x}\right).\end{aligned}$$

Now, writing $u^n = \chi + \mathbf{u}^n$, we get that u^n is uniquely defined, and adding one half of seminorm of χ in $\mathcal{H}^1_\mu(\mathcal{T})$ gives the relation (3.9). □

Let us now give some useful properties of u^n.

Proposition 3.6. *u^n satisfies*

$$0 \leqslant u^n \leqslant 1, \tag{3.11}$$

as well as

$$(u^n_{\ell,j})' \leqslant 0, \forall \ell, j, \tag{3.12}$$

where $u^n_{\ell,j}$ is the restriction of u^n to the edge $e_{\ell,j}$.

Proof. First for all sub-tree $\mathcal{S}$ of $\mathcal{T}^n$, the maximum principle yields that u^n attains its maximum (and its minimum) at the boundary of $\mathcal{S}$. Taking $\mathcal{S} = \mathcal{T}^n$, we find the first assertion. For the second assertion, we start by the final subtrees $\mathcal{S}_{n-1,j} = e_{n-1,j} \cup \cup_{k=j}^{j+p-1} e_{n,k}$, for any $j = 0, \cdots, p^{n-1} - 1$. As u^n is zero at the vertices $v_{n+1,k}$ for $k = j, \cdots, j+p-1$, we find that u^n attains it maximum at $v_{n-1,j}$, hence u^n decreases from $v_{n-1,j}$ to $v_{n+1,k}$ for $k = j, \cdots, j+p-1$, this yields (3.12) for $\ell = n-1$ and n. We iterate this procedure by taking the subtrees $\mathcal{S}_{n-2,j}$ made of the edge $e_{n-2,j}$ plus its descendants, to find that u^n attains its maximum at $v_{n-2,j}$ and consequently, u^n decreases from $v_{n-2,j}$ to $v_{n-1,k}$, for any $k = j, \cdots, j+p-1$, hence (3.12) holds for $\ell = n-2$. The conclusion follows by iteration with the subtrees made of $e_{m,j}$ plus its descendants, with decreasing values of m from $n-2$ to 0. □

Proposition 3.7. *One has*

$$|u^n|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = -\mu(v_{0,0})(u^n)'(v_{0,0}). \tag{3.13}$$

Proof. We use that the function u^n is solution of (3.8), and we take $\varphi = u^n - \chi$, where χ is the function defined in the proof of Proposition 3.5. Then, using a Green-Riemann formula and using $\chi(v_{0,0}) = 1$ yields (3.13). □

Now, we give a lemma about the behaviour of the norm $|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})}$.

Lemma 3.8. *The sequence $|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})}$ is a non increasing sequence.*

Proof. For a given n, (3.9) gives that: for any $u \in \mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T})$, we get that

$$|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} \leqslant |u|_{\mathcal{H}^1_\mu(\mathcal{T})}. \tag{3.14}$$

In particular, choosing $u = u^{n-1}$ (since $\mathcal{T}^{n-1} \subset \mathcal{T}^n$) yields the lemma. ☐

We have then proved that $|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})}$ is a non increasing positive sequence, so it admits a limit l. Later on we shall discuss whether $l = 0$ or not. But before, let us show that the sequence $(u^n)_n$ converges to u_{D}.

Theorem 3.9. *We have*

$$u^n \to u_{\mathrm{D}} \ \textit{in}\ \mathcal{H}^1_\mu(\mathcal{T}) \quad \textit{as}\ n \to \infty.$$

Proof. Owing to Lemma 3.8 and the fact that $u^n(v_{0,0}) = 1$, we deduce that $\|u^n\|_{\mathcal{H}^1_\mu(\mathcal{T})}$ is bounded so that there exists a subsequence $(u^{n_k})_{k\in\mathbb{N}}$ that converges weakly in $\mathcal{H}^1_\mu(\mathcal{T})$ to u^∞. Since $u^n \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ for any $n \in \mathbb{N}$, and in particular for any n_k, $u^\infty \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$. Given $k \in \mathbb{N}$, we recall the problem (3.8) satisfied by u^{n_k}: for any $\mathbf{u} \in \mathcal{H}^1_\mu(\mathcal{T}^{n_k})$ with $\mathbf{u}(v_{0,0}) = 0$ and $\mathbf{u}(v_{n_k+1,j}) = 0$ for any $0 \leqslant j < p^{n_k}$, one has

$$\int_{\mathcal{T}^{n_k}} \mu(\mathbf{x})(u^{n_k})'(\mathbf{x})\overline{\mathbf{u}'}(\mathbf{x})\mathrm{d}\mathbf{x} = 0. \tag{3.15}$$

Given now $k_0 \in \mathbb{N}$, and for any $k \geqslant k_0$, we can choose test functions in (3.15) whose support is included in $\mathcal{T}^{n_{k_0}}$. We can also choose these functions as test functions for the problem $(\mathcal{P}_{\mathrm{D}})$. Writing the difference between $(\mathcal{P}_{\mathrm{D}})$ and (3.15) for these test functions gives: for any $k \geqslant k_0$, for any $\mathbf{u} \in \mathcal{H}^1_\mu(\mathcal{T}^{n_{k_0}})$ with $\mathbf{u}(v_{0,0}) = 0$ and $\mathbf{u}(v_{n_{k_0}+1,j}) = 0$ for any $0 \leqslant j < p^{n_{k_0}}$, and extended by 0 outside $\mathcal{T}^{n_{k_0}}$, one has

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{n_k})'(\mathbf{x})\overline{\mathbf{u}'}(\mathbf{x})\mathrm{d}\mathbf{x} = 0. \tag{3.16}$$

We now use that $(u^{n_k})_{k\in\mathbb{N}}$ converges weakly in $\mathcal{H}^1_\mu(\mathcal{T})$ to u^∞ to take the limit when $k \to \infty$ in (3.16) to get: for any $\mathbf{u} \in \mathcal{H}^1_\mu(\mathcal{T}^{n_{k_0}})$ with $\mathbf{u}(v_{0,0}) = 0$ and $\mathbf{u}(v_{n_{k_0}+1,j}) = 0$ for any $0 \leqslant j < p^{n_{k_0}}$, and extended by 0 outside $\mathcal{T}^{n_{k_0}}$, one has

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{\infty})'(\mathbf{x})\overline{\mathbf{u}'}(\mathbf{x})\mathrm{d}\mathbf{x} = 0. \tag{3.17}$$

As $\mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ is dense in $\mathcal{H}^1_{\mu,0}(\mathcal{T})$, the identity (3.17) remains valid for any test function $\mathbf{u} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$ with $\mathbf{u}(v_{0,0}) = 0$. Using again the weak convergence gives $u^\infty(v_{0,0}) = 1$, so we can choose $\mathbf{u} = u_{\mathrm{D}} - u^\infty$ in (3.17). That gives

$$\|u_{\mathrm{D}} - u^\infty\|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = 0. \tag{3.18}$$

We have thus shown that there exists a subsequence $(u^{n_k})_{k\in\mathbb{N}}$ that converges weakly in $\mathcal{H}^1_\mu(\mathcal{T})$ to u_{D}. Since

$$\|u\|_{1,e_{0,0}} \lesssim |u(v_{0,0})| + |u|_{1,e_{0,0}} \lesssim \|u\|_{\mathcal{H}^1_\mu(\mathcal{T})},$$

we deduce (again up to a subsequence) that the subsequence converges strongly in $C(\bar{e}_{0,0})$ to u_{D}. In particular this implies that

$$u^{n_k}(v_{1,0}) \to u_{\mathrm{D}}(v_{1,0}) \text{ as } k \to \infty. \tag{3.19}$$

Now in (3.16) we take the test-function $\mathbf{u} = u^{n_k} - \chi$, where χ is the function fixed in the proof of Proposition 3.5 to get

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{n_k})'(\mathbf{x})(u^{n_k} - \chi)'(\mathbf{x})\mathrm{d}\mathbf{x} = 0.$$

This shows that

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{n_k})'(\mathbf{x})(u^{n_k})'(\mathbf{x})\mathrm{d}\mathbf{x} = \int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{n_k})'(\mathbf{x})\chi'(\mathbf{x})\mathrm{d}\mathbf{x},$$

and integrating by parts in this right-hand side we find

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^{n_k})'(\mathbf{x})(u^{n_k})'(\mathbf{x})\mathrm{d}\mathbf{x} = \mu(v_{0,0})(u_{\mathrm{D}} - u^{n_k})(v_{1,0})\chi'_{0,0}.$$

This identity can be equivalently written

$$\int_{\mathcal{T}} \mu(\mathbf{x})|(u^{n_k})'(\mathbf{x})|^2\mathrm{d}\mathbf{x} = \int_{\mathcal{T}} \mu(\mathbf{x})u'_{\mathrm{D}}(\mathbf{x})(u^{n_k})'(\mathbf{x})\mathrm{d}\mathbf{x} - \mu(v_{0,0})(u_{\mathrm{D}} - u^{n_k})(v_{1,0})\chi'_{0,0}.$$

Hence by Cauchy-Schwarz's and Young's inequalities we get

$$\begin{aligned}\int_{\mathcal{T}} \mu(\mathbf{x})|(u^{n_k})'(\mathbf{x})|^2\mathrm{d}\mathbf{x} \leqslant \frac{1}{2}\int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\mathcal{T}} \mu(\mathbf{x})|(u^{n_k})'(\mathbf{x})|^2\mathrm{d}\mathbf{x}\\ +\mu(v_{0,0})|(u_{\mathrm{D}} - u^{n_k})(v_{1,0})||\chi'_{0,0}|,\end{aligned}$$

or equivalently

$$\int_{\mathcal{T}} \mu(\mathbf{x})|(u^{n_k})'(\mathbf{x})|^2\mathrm{d}\mathbf{x} \leqslant \int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2\mathrm{d}\mathbf{x} + 2\mu(v_{0,0})|(u_{\mathrm{D}} - u^{n_k})(v_{1,0})||\chi'_{0,0}|.$$

By (3.19), we deduce that

$$\lim_{k\to\infty}\int_{\mathcal{T}} \mu(\mathbf{x})|(u^{n_k})'(\mathbf{x})|^2\mathrm{d}\mathbf{x} \leqslant \int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2\mathrm{d}\mathbf{x},$$

and consequently

$$\lim_{k\to\infty}\|u^{n_k}\|_{\mathcal{H}^1_\mu(\mathcal{T})} \leqslant \|u_{\mathrm{D}}\|_{\mathcal{H}^1_\mu(\mathcal{T})}.$$

This implies that $(u^{n_k})_{k\in\mathbb{N}}$ converges strongly in $\mathcal{H}^1_\mu(\mathcal{T})$ to u_{D}. Further by the previous estimate and again Lemma 3.8, we have

$$\|u_{\mathrm{D}}\|_{\mathcal{H}^1_\mu(\mathcal{T})} = \lim_{n\to\infty}\|u^n\|_{\mathcal{H}^1_\mu(\mathcal{T})}. \tag{3.20}$$

In particular this yields

$$|u_{\mathrm{D}}|_{\mathcal{H}^1_\mu(\mathcal{T})} = \lim_{n\to\infty}|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})}. \tag{3.21}$$

Now using (3.16) with n instead of n_k, we get

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^n)'(\mathbf{x})\overline{\mathbf{u}'}(\mathbf{x})\mathrm{d}\mathbf{x} = 0,$$

for all $\mathbf{u} \in \mathcal{H}^{1,m}_{\mu,\mathrm{c},0}(\mathcal{T})$, for all $m \leqslant n$, which implies that

$$\lim_{n\to\infty} \int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^n)'(\mathbf{x})\overline{\varphi'}(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \forall \varphi \in \mathcal{H}^1_{\mu,0}(\mathcal{T}),\ \varphi(v_{0,0}) = 0. \tag{3.22}$$

Now for any $\psi \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$, we set

$$\varphi = \psi - \psi(v_{0,0})\chi,$$

and deduce that

$$\begin{aligned}(u^n - u_{\mathrm{D}}, \varphi)_{\mathcal{H}^1_\mu(\mathcal{T})} = &\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^n)'(\mathbf{x})\overline{\psi'}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &- \bar{\psi}(v_{0,0}) \int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^n)'(\mathbf{x})\overline{\chi'}(\mathbf{x})\mathrm{d}\mathbf{x}.\end{aligned}$$

Hence Green's formula in the last term of this right-hand side yields

$$\begin{aligned}(u^n - u_{\mathrm{D}}, \varphi)_{\mathcal{H}^1_\mu(\mathcal{T})} = &\int_{\mathcal{T}} \mu(\mathbf{x})(u_{\mathrm{D}} - u^n)'(\mathbf{x})\overline{\psi'}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &+ \bar{\psi}(v_{0,0})\mu(v_{0,0})(u_{\mathrm{D}} - u^n)'(v_{0,0}).\end{aligned} \tag{3.23}$$

By (3.13) and the property

$$|u_{\mathrm{D}}|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = -\mu(v_{0,0})(u_{\mathrm{D}})'(v_{0,0}), \tag{3.24}$$

proved similarly than (3.13), we get that

$$\lim_{n\to\infty} \mu(v_{0,0})(u_{\mathrm{D}} - u^n)'(v_{0,0}) = \lim_{n\to\infty} |u^n|^2_{\mathcal{H}^1_\mu(\mathcal{T})} - |u_{\mathrm{D}}|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = 0,$$

by (3.21). Using this property and (3.22) into (3.23) leads to

$$\lim_{n\to\infty} (u^n - u_{\mathrm{D}}, \psi)_{\mathcal{H}^1_\mu(\mathcal{T})} = 0.$$

Combined with (3.21), this shows the announced strong convergence. □

Let us give the following consequences.

Corollary 3.10. *For all $\ell \leqslant n$, one has*

$$u^n \to u_{\mathrm{D}} \text{ in } C^1(\bar{e}_{\ell,j}), \quad \forall j = 0, \dots, p^\ell - 1.$$

Proof. Direct consequence of the Sobolev embedding theorem and the fact that u^n and u_{D} are affine functions on each edge. □

Theorem 3.11. *One has*

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \lim_{n\to\infty} |u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} = 0. \tag{3.25}$$

Proof.

$\Leftarrow$: Since by assumption $|u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0$ and since $u^n(v_{0,0}) = 1$, we can write that

$$\lim_{n\to\infty} \|u^n - \mathbb{1}\|_{\mathcal{H}^1_\mu(\mathcal{T})} = 0. \tag{3.26}$$

As $u^n \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$, one gets that $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$, and $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$ by using Theorem 3.1.

$\Rightarrow$: We use a contradiction argument. Assume that

$$\lim_{n\to\infty} |u^n|_{\mathcal{H}^1_\mu(\mathcal{T})} > 0.$$

Then owing to the relation (3.21), one has

$$|u_\mathrm{D}|_{\mathcal{H}^1_\mu(\mathcal{T})} > 0,$$

and therefore the solution u_D of $(\mathcal{P}_\mathrm{D})$ is not $\mathbb{1}$. By relation (3.5) of Proposition 3.4 and Theorem 3.1, we conclude that $\mathcal{H}^1_\mu(\mathcal{T}) \neq \mathcal{H}^1_{\mu,0}(\mathcal{T})$. □

3.2.3. Relation with the Liouville property

We start with the definition of harmonic functions on a weighted p-adic tree and then of the Liouville property (see [24], Def. 3.2) in the case $\mu = 1$).

Definition 3.12. Let $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ be a weighted p-adic tree. A function h on $\mathcal{T}$ is called harmonic if it is continuous on $\mathcal{T}$ and satisfies

$$\begin{cases} (\mu h')' = 0 & \text{in } e_{\ell,j},\ 0 \leqslant j < p^\ell, \quad \ell \in \mathbb{N}, \\ [\mu h'] = 0 & \text{on } v_{\ell,j},\ 0 \leqslant j < p^{\ell-1}, \quad \ell \in \mathbb{N}\setminus\{0\}. \end{cases} \tag{3.27}$$

Definition 3.13. A weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$ is called a Liouville network if and only if every bounded harmonic function on $\mathcal{T}$ is constant.

Let us now show the relation of this property with our first necessary and sufficient condition.

Proposition 3.14. *We have the following equivalence*

$$u_\mathrm{D} = \mathbb{1} \quad \Longleftrightarrow \quad \mathcal{T} \text{ is a Liouville network.} \tag{3.28}$$

Proof. The implication $\Leftarrow$ is direct, since u_D is harmonic and bounded (it even satisfies $0 \leqslant u_\mathrm{D} \leqslant 1$) owing to (3.11) and Corollary 3.10. For the converse implication, let us fix a bounded harmonic function h on $\mathcal{T}$. As the assumption is that $u_\mathrm{D} = \mathbb{1}$, by Proposition 3.4, this is equivalent to $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$. Hence let us fix a sequence of functions $(\mathbb{1}_n)_{n\in\mathbb{N}} \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ such that

$$\|\mathbb{1} - \mathbb{1}_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0 \quad \text{as } n \to \infty.$$

Now for any $n \in \mathbb{N}$, let us show that

$$\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} = -2\int_{\mathcal{T}} \mu(\mathbf{x})h'(\mathbf{x})\mathbb{1}_n(\mathbf{x})\mathbb{1}_n'(\mathbf{x})h(\mathbf{x})\mathrm{d}\mathbf{x} + \mu(v_{0,0})h'(v_{0,0})\mathbb{1}_n^2(v_{0,0})h(v_{0,0}). \tag{3.29}$$

Indeed, we write the left-hand side of (3.29) as

$$\int_{\mathcal{T}} \mu(\mathbf{x})(h)'(\mathbf{x})\mathbb{1}_n^2(\mathbf{x})(h)'(\mathbf{x})\mathrm{d}\mathbf{x},$$

hence a simple consequence of Green's formula on each edge yields

$$\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} = -\int_{\mathcal{T}} (\mu h'\mathbb{1}_n^2)'(\mathbf{x})h(\mathbf{x})\mathrm{d}\mathbf{x} + \mu(v_{0,0})h'(v_{0,0})\mathbb{1}_n^2(v_{0,0})h(v_{0,0}),$$

by noticing that the boundary terms cancel since h satisfies the Kirchoff law at the interior nodes and $\mathbb{1}_n$ has a compact support. Leibniz's rule yields the conclusion recalling that $(\mu h')' = 0$ on each edge.

Using Cauchy-Schwarz's inequality in the first term of the right-hand side of (3.29), we get

$$\begin{aligned}&\textstyle\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} \leqslant \mu(v_{0,0})|h'(v_{0,0})|\mathbb{1}_n^2(v_{0,0})|h(v_{0,0})| \\ &+2\left(\textstyle\int_{\mathcal{T}} \mu(\mathbf{x})(h')^2(\mathbf{x})\mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\left(\textstyle\int_{\mathcal{T}} \mu(\mathbf{x})(\mathbb{1}_n')^2(\mathbf{x})h^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}.\end{aligned}$$

As h is bounded, there exists $R > 0$ such that

$$|h(\mathbf{x})| \leqslant R, \forall \mathbf{x} \in \mathcal{T},$$

and consequently the next estimate becomes

$$\begin{aligned}&\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} \leqslant \mu(v_{0,0})|h'(v_{0,0})|\mathbb{1}_n^2(v_{0,0})|h(v_{0,0})| \\ &\quad +2R\left(\int_{\mathcal{T}} \mu(\mathbf{x})(h')^2(\mathbf{x})\mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\left(\int_{\mathcal{T}} \mu(\mathbf{x})(\mathbb{1}_n')^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}.\end{aligned}$$

By Young's inequality, we deduce that

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} \leqslant \mu(v_{0,0})|h'(v_{0,0})|\mathbb{1}_n^2(v_{0,0})|h(v_{0,0})|. \\ &\quad +2R^2\int_{\mathcal{T}} \mu(\mathbf{x})(\mathbb{1}_n')^2(\mathbf{x})\mathrm{d}\mathbf{x}.\end{aligned}$$

Since the second term of this right-hand side tends to zero and $\mathbb{1}_n^2(v_{0,0}) \to 1$ as n goes to infty, the left-hand side of this estimate remains bounded uniformly in n, in other words

$$\int_{\mathcal{T}} \mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x})\mathrm{d}\mathbf{x} \lesssim 1, \quad \forall n \in \mathbb{N}.$$

As

$$\mu(\mathbf{x})(h'(\mathbf{x}))^2 \mathbb{1}_n^2(\mathbf{x}) \to \mu(\mathbf{x})(h'(\mathbf{x}))^2, \quad \forall \mathbf{x} \in \mathcal{T}, \quad \text{as } n \to \infty,$$

by Fatou's Lemma, we deduce that $\mu(h')^2$ belongs to $L^1(\mathcal{T})$. Combined with the boundedness of h, this shows that h belongs to $\mathcal{H}^1_\mu(\mathcal{T})$, hence to $\mathcal{H}^1_{\mu,0}(\mathcal{T})$. Thanks to Proposition 3.3, this directly implies that $h = h(v_{0,0})u_{\mathrm{D}} = h(v_{0,0})\mathbb{1}$, hence the conclusion. □

3.3. Explicit necessary and sufficient condition

We recall here problem (3.8) given in $\mathcal{H}^{1,n}_{\mu,\mathrm{c}}(\mathcal{T})$, but written under the PDE form: we look for $u^n \in \mathcal{H}^1_\mu(\mathcal{T})$ such that

$$\left\{\begin{array}{rcl} (\mu(u^n)')' &=& 0 \quad \text{in } e_{\ell,j},\ 0 \leqslant \ell \leqslant n,\ 0 \leqslant j < p^\ell, \\ [\mu(u^n)'] &=& 0 \quad \text{on } v_{\ell,j},\ 1 \leqslant \ell \leqslant n, 0 \leqslant j < p^{\ell-1}, \\ u^n(v_{0,0}) &=& 1, \\ u^n &=& 0 \quad \text{in } \mathcal{T} \setminus \mathcal{T}^n. \end{array}\right. \tag{3.30}$$

On each edge $e_{\ell,j}$, we introduce the resistance $R_{\ell,j}$ given by

$$R_{\ell,j} = \int_{e_{\ell,j}} \frac{\mathrm{d}\mathbf{x}}{\mu(\mathbf{x})}, \tag{3.31}$$

and the new unknowns

- $U^n_{\ell,j} = u^n(v_{\ell+1,j}) - u^n(v_{\ell,\lfloor p^{-1}j \rfloor})$,
- $I^n_{\ell,j} = \mu_{\ell,j}(u^n_{\ell,j})'$.

This new set of unknowns $(U^n_{\ell,j}, I^n_{\ell,j})$ allows us to re-write (3.30) in the following equivalent form:

- $I^n_{\ell,j}$ is constant on each $e_{\ell,j}$,
- on each edge $e_{\ell,j}$, we have $U^n_{\ell,j} = R_{\ell,j} I_{\ell,j}$,
- for any $j \in \{0, \ldots, p^n - 1\}$, we have

$$\sum_{\ell=0}^{n} U^n_{\ell,\lfloor p^{\ell-n} j \rfloor} = -1, \tag{3.32}$$

- for any $0 \leqslant \ell \leqslant n-1$, we have

$$I^n_{\ell,j} = \sum_{k=0}^{p-1} I^n_{\ell+1,pj+k}. \tag{3.33}$$

Note that this last identity (3.33) corresponds to the so-called Kirchoff law.

We have actually rewritten problem (3.30) as a general electrical problem. Let us call R^n the equivalent resistance of the finite tree $\mathcal{T}^n$. Namely, the global voltage between the entrance and the boundary of the tree is equal to -1, thanks to relation (3.32). The intensity is equal to the intensity in the main edge $e_{0,0}$ and Ohm law gives

$$-1 = R^n I^n_{0,0}. \tag{3.34}$$

Now, using the definition of $I^n_{0,0}$ and relation (3.13) of Proposition 3.7 gives

$$|u^n|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = (R^n)^{-1}. \tag{3.35}$$

Lemma 3.8 already gives that R^n is a non decreasing sequence. Then, according to Theorem 3.11, we have the following theorem:

Theorem 3.15 (Explicit necessary and sufficient condition). *One has*

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \lim_{n\to\infty} R^n = +\infty. \tag{3.36}$$

Corollary 3.16. *One has*

$$\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \lim_{n\to\infty} R^n = +\infty. \tag{3.37}$$

Proof. Direct consequence of the previous Theorem and of Theorem 3.1. □

Let us finish this subsection by an explicit expression for the equivalent resistance R^n in terms of the local resistances $R_{\ell,j}$.

Proposition 3.17 (Equivalent resistance). *One has*

$$R^n = R_{0,0} + \left(\sum_{j_1=0}^{p-1} \left(R_{1,j_1} + \left(\sum_{j_2=0}^{p-1} \left(R_{2,pj_1+j_2} + \dots \left(\sum_{j_n=0}^{p-1} R^{-1}_{n,\sum_{k=1}^n p^{n-k} j_k} \right)^{-1} \right)^{-1} \right)^{-1} \right)^{-1} \right)^{-1}. \tag{3.38}$$

Proof. The proof is very technical and will be only sketched here, see ([20] p. 27) for a similar argument. It is proved by induction on n. For $n = 0$, the formula (3.38) is easily checked as $u^0(\mathbf{x}) = 1 - \frac{\mathbf{x}}{L_{0,0}}$. Now, given $n \geqslant 1$, let us assume that we have proven formula (3.38) for any resistive tree with $n-1$ generations. Given $j \in \{0, \dots, p^{n-1} - 1\}$, we consider subtree of $\mathcal{T}^n$ starting from $e_{n-1,j}$. Since all resistances $R_{n,pj+k}$, for $0 \leqslant k \leqslant p - 1$ are in parallel, we can use an equivalent resistance equal to their harmonic sum (inverse of sum of inverses), and we add with $R_{n-1,j}$, to get

$$R_{n-1,j} + \left(\sum_{k=0}^{p-1} (R_{n,pj+k})^{-1} \right)^{-1},$$

which is nothing else than the last term in (3.38). □

3.4. A characterization of the necessary and sufficient condition in some particular cases

Relation (3.36) of Theorem 3.15 gives an intrinsic condition to determine if $\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T})$ or not, but computation of R^n with the help of formula (3.38) is rather difficult. However, in some particular cases, we can simplify this formula. The first precise statement is the following one.

Proposition 3.18. *Let us assume that there exists a sequence $(M_n)_{n\in\mathbb{N}}$ of positive real numbers such that, for any $n \in \mathbb{N}$ and for any $0 \leqslant j \leqslant p^{n-1}$, one has*

$$\left(\sum_{k=0}^{p-1} R^{-1}_{n+1,pj+k} \right)^{-1} = M_n R_{n,j}. \tag{3.39}$$

Then

$$R^n = R_{0,0} \left(\sum_{m=0}^{n} \prod_{\ell=0}^{m-1} M_\ell \right), \tag{3.40}$$

with the convention

$$\prod_{\ell=0}^{-1} M_\ell = 1.$$

Proof. The proof is easily done by finite reversal reccurence on n. First, we use (3.39) for $n = n-1$ to write, for any $0 \leqslant j < p^{n-1}$:

$$\begin{aligned} R_{n-1,j} + \left(\sum_{k=0}^{p-1} \left(R_{n,pj+k}\right)^{-1}\right)^{-1} &= R_{n-1,j} + M_{n-1} R_{n-1,j} \\ &= R_{n-1,j}(1 + M_{n-1}). \end{aligned}$$

Then, we use again (3.39) for $n = n-2$ and we use the previous relation to write, for any $0 \leqslant j < p^{n-2}$:

$$\begin{aligned} R_{n-2,j} + \left(\sum_{k=0}^{p-1} \left(R_{n-1,pj+k}(1 + M_{n-1})\right)^{-1}\right)^{-1} &= R_{n-2,j} + (1 + M_{n-1}) \left(\sum_{k=0}^{p-1} R_{n-1,pj+k}^{-1}\right)^{-1} \\ &= R_{n-2,j} + M_{n-2}(1 + M_{n-1}) R_{n-2,j} \\ &= R_{n-2,j}(1 + M_{n-2} + M_{n-2} M_{n-1}). \end{aligned}$$

Using a finite induction gives (3.40). □

Remark 3.19. The assumption (3.39) of Proposition 3.18 says that M_n is independent of j.

Remark 3.20. Consider the case of a self-similar p-adic tree [16, 17], namely assume that there exist p direct similitudes $\sigma_0, \sigma_1, \dots, \sigma_{p-1}$ of respective amplitude $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$ such that

$$\mathcal{T} = e_{0,0} \cup \bigcup_{k=0}^{p-1} \sigma_k(\mathcal{T}),$$

and that there exist p positive constants $\mu_0, \mu_1, \dots, \mu_{p-1}$ such that

$$\mu \circ \sigma_k = \mu_k \mu.$$

In this case, we can easily see that

$$M_n = \left(\sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}\right)^{-1},$$

and then

$$R^n = \sum_{m=0}^{n} \left(\sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i}\right)^{-m}$$

which, combined with Theorem 3.15, gives the result ([17] Prop. 3.11 and Thm. 3.13)

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \sum_{i=0}^{p-1} \frac{\mu_i}{\alpha_i} \leqslant 1.$$

Proposition 3.21. *Let us assume that for all $\ell \in \mathbb{N}$ there exists $\mu_\ell > 0$ such that*

$$\int_{e_{\ell,j}} \frac{\mathrm{d}x}{\mu_{\ell,j}(x)} = \frac{L_{\ell,j}}{\mu_{\ell,j}} = \mu_\ell^{-1}, \forall j = 0, \cdots, p^\ell - 1. \tag{3.41}$$

Then

$$R^n = \sum_{\ell=0}^{n} p^{-\ell} \mu_\ell^{-1}. \tag{3.42}$$

Proof. Direct consequence of (3.38) and of the fact that

$$R_{\ell,j} = \mu_\ell^{-1}, \forall j = 0, \cdots, p^\ell - 1.$$

□

Corollary 3.22. *Under the assumption of Proposition 3.21, one has*

$$\mathcal{H}^1_\mu(\mathcal{T}) = \mathcal{H}^1_{\mu,0}(\mathcal{T}) \quad \Longleftrightarrow \quad \sum_{\ell=0}^{\infty} p^{-\ell} \mu_\ell^{-1} \text{ diverges.}$$

In the case $p = 2$, Corollary 3.22 is nothing else than the continuous version of Theorem 2.12 of [19]. The fact that this necessary and sufficient condition is the same than the one from Theorem 2.12 of [19] simply follows from the fact that u^n is affine on each edge. Hence u^n can also be seen as the solution of the discrete Dirichlet problem.

4. Trace results

In this section, we consider a weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$, and we clarify how to define a trace at infinity

Definition 4.1. We call p-recursive partition of the unit interval a sequence of *real* numbers $\gamma^n_{p,j} \in [0, 1]$ defined for $n \geqslant 1$ and $0 \leqslant j \leqslant p^{n-1}$ such that

- $\gamma^n_{p,0} = 0$ for any $n \geqslant 1$,
- $\gamma^n_{p,p^{n-1}} = 1$ for any $n \geqslant 1$,
- $\gamma^n_{p,j} \leqslant \gamma^n_{p,j+1}$, for all $0 \leqslant j < p^{n-1}$,
- $\gamma^{n+1}_{p,pj} = \gamma^n_{p,j}$ for any $n \geqslant 1$ and for any $0 \leqslant j \leqslant p^{n-1}$.

Remark 4.2. According to the definition of p-recursive partition, we can make the two following remarks:

- the set of intervals $]\gamma^n_{p,j}, \gamma^n_{p,j+1}[$, $0 \leqslant j < p^{n-1}$, gives a subdivision of $]0, 1[$,
- moreover, the set of intervals $]\gamma^{n+1}_{p,pj+k}, \gamma^{n+1}_{p,pj+k+1}[$, $0 \leqslant k < p$ gives a subdivision of $]\gamma^n_{p,j}, \gamma^n_{p,j+1}[$.

Definition 4.3. Given $\gamma = (\gamma^n_{p,j})_{n \geqslant 1, 0 \leqslant j \leqslant p^{n-1}}$ a p-recursive partition of the unit interval, we can define the following trace operators:

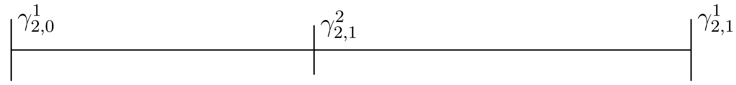

FIGURE 4. Example of 2-recursive partition

- the nth trace operator T^n_γ defined as follows: for $u \in \mathcal{H}^1_\mu(\mathcal{T})$, we define $T^n_\gamma(u) \in \mathrm{L}^2(]0,1[)$ as

$$T^n_\gamma(u)(x) = u(v_{n,j}), \quad \forall x \in \left]\gamma^n_{p,j}, \gamma^n_{p,j+1}\right[, \tag{4.1}$$

- the trace operator $T^\infty_\gamma(u)$ defined as the limit of $T^n_\gamma(u)$ in $\mathrm{L}^2(]0,1[)$, as n goes to infinity (if it exists).

Remark 4.4. For any choice of γ, one always has $T^\infty_\gamma(\mathbb{1}) = 1$.

Proposition 4.5. *If $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$, then there exists no p-recursive partition γ of the unit interval such that T^∞_γ is continuous from $\mathcal{H}^1_\mu(\mathcal{T})$ to $\mathrm{L}^2(]0,1[)$.*

Proof. Assume that $\mathbb{1} \in \mathcal{H}^1_{\mu,0}(\mathcal{T})$, and let γ be a p-recursive partition of the unit interval. By hypothesis, there exists a sequence $(\mathbb{1}_n) \in \mathcal{H}^1_{\mu,\mathrm{c}}(\mathcal{T})$ such that

$$\|\mathbb{1} - \mathbb{1}_n\|_{\mathcal{H}^1_\mu(\mathcal{T})} \to 0, \quad \text{as } n \to \infty.$$

Now, it is easy to see that, for a given n, $T^\infty_\gamma(\mathbb{1} - \mathbb{1}_n) = 1$, then

$$\left\|T^\infty_\gamma(\mathbb{1} - \mathbb{1}_n)\right\|_{\mathrm{L}^2(]0,1[)} = 1 \not\to 0, \quad \text{as } n \to \infty.$$

□

In view of the previous Proposition and Corollary 3.16, a trace result is only available if we assume that

$$R = \lim_{n\to\infty} R^n < \infty. \tag{4.2}$$

Before considering the general case, let us mention that for dyadic trees satisfying the assumptions of Corollary 3.22 with $\mu_\ell = \mu_0 \alpha^{-\ell}$, for some $\alpha \in]0,2[$, and combining Lemma 2.15 with Theorem 4.11 of [19] (based on a multiscale analysis and the above choice of μ_ℓ), one deduces that there exists a 2-recursive partition γ of $]0,1[$ such that T^∞_γ is continuous from $\mathcal{H}^1_\mu(\mathcal{T})$ to $H^{s'}((]0,1[)$ (hence into $\mathrm{L}^2(]0,1[)$) with $s' = s$ if $s = \frac{1-\ln\alpha/\ln 2}{2} < \frac{1}{2}$ and $s' < \frac{1}{2}$ else. Note that in such a situation, for any $u \in \mathcal{H}^1_\mu(\mathcal{T})$, its trace $T^\infty_\gamma u$ is not continuous in general as s' is always strictly smaller than $\frac{1}{2}$. Our goal is to show a similar result under the sole assumption (4.2), since in such a case, we cannot use a multiscale analysis, in a first attempt we will restrict ourselves to the case $s' = 0$.

To prove our trace result, we note that the identities (3.13) and (3.24) and the assumption (4.2) yield the identity

$$|u_\mathrm{D}|^2_{\mathcal{H}^1_\mu(\mathcal{T})} = \frac{1}{R},$$

which implies that

$$-\mu(v_{0,0})(u_\mathrm{D})'(v_{0,0}) = \frac{1}{R}. \tag{4.3}$$

Lemma 4.6. *For any $u \in \mathcal{H}^1_\mu(\mathcal{T})$ real valued and all $n \in \mathbb{N}$, one has*

$$\int_{\mathcal{T}^n} \mu(\mathbf{x}) u'_{\mathrm{D}}(\mathbf{x})(u^2)'(\mathbf{x})\mathrm{d}\mathbf{x} = \sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 - \mu(v_{0,0}) u'_{\mathrm{D}}(v_{0,0}) |u(v_{0,0})|^2, \tag{4.4}$$

where, here and below, $u_{\mathrm{D},n,j}$ means the restrition of u_{D} to the edge $e_{n,j}$.

Proof. Simple consequence of Green's formula recalling that $(\mu u'_{\mathrm{D}})' = 0$ on each edge. ☐

By the identity (4.3), the identity (4.4) can be equivalently written as

$$-\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 = -\int_{\mathcal{T}^n} \mu(\mathbf{x}) u'_{\mathrm{D}}(\mathbf{x})(u^2)'(\mathbf{x})\mathrm{d}\mathbf{x} + \frac{1}{R}|u(v_{0,0})|^2.$$

Hence by Leibniz's rule, one obtains

$$-\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 = -2\int_{\mathcal{T}^n} \mu(\mathbf{x}) u'_{\mathrm{D}}(\mathbf{x}) u(\mathbf{x}) u'(\mathbf{x})\mathrm{d}\mathbf{x} + \frac{1}{R}|u(v_{0,0})|^2$$

and by Cauchy-Schwarz's inequality we get

$$-\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 \leqslant \frac{1}{R}|u(v_{0,0})|^2 + 2\left(\int_{\mathcal{T}^n} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|u(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}} \left(\int_{\mathcal{T}^n} \mu(\mathbf{x})|u'(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}. \tag{4.5}$$

Hence if we are able to show that

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|u(\mathbf{x})|^2\mathrm{d}\mathbf{x} \lesssim \|u\|^2_{\mathcal{H}^1_\mu(\mathcal{T})}, \tag{4.6}$$

the previous estimate will become

$$-\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} |u(v_{n+1,j})|^2 \lesssim \|u\|^2_{\mathcal{H}^1_\mu(\mathcal{T})}, \tag{4.7}$$

and consequently the candidates for the lenghts $\ell^{n+1}_{p,j}$ are

$$\ell^{n+1}_{p,j} = -\beta \mu_{n,j} u'_{\mathrm{D},n,j},$$

with $\beta > 0$ that is chosen so that

$$\sum_{j=0}^{p^n-1} \ell^{n+1}_{p,j} = 1. \tag{4.8}$$

Indeed taking $u = 1$ in (4.4), we find that

$$\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} - \mu(v_{0,0}) u'_{\mathrm{D}}(0) = 0,$$

and again by (4.3) we find that

$$-\sum_{j=0}^{p^n-1} \mu_{n,j} u'_{\mathrm{D},n,j} = \frac{1}{R}.$$

Hence the choice $\beta = R$ leads to (4.8).

It then remains to prove (4.6), for that purpose, we start with the next result.

Lemma 4.7. *For any $\varphi \in \mathcal{H}^1_\mu(\mathcal{T})$ real valued with $\varphi(v_{0,0}) = 0$, one has*

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} = -2 \int_{\mathcal{T}^n} \mu(\mathbf{x})(u^n)'(\mathbf{x})\varphi(\mathbf{x})\varphi'(\mathbf{x}) u^n(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{4.9}$$

Proof. We write the left-hand side of (4.9) as

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})(u^n)'(\mathbf{x})\varphi^2(\mathbf{x})(u^n)'(\mathbf{x}) \mathrm{d}\mathbf{x},$$

hence a simple consequence of Green's formula on each edge yields

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} = -\int_{\mathcal{T}^n} (\mu (u^n)' \varphi^2)'(\mathbf{x}) u^n(\mathbf{x}) \mathrm{d}\mathbf{x},$$

by noticing that the boundary terms cancel since u^n satisfies the Kirchoff law at the interior nodes, u^n is zero at the nodes $v_{n+1,j}$ and $\varphi(v_{0,0}) = 0$. Leibniz's rule yields the conclusion recalling that $(\mu(u^n)')' = 0$ on each edge. □

Corollary 4.8. *For any $\varphi \in \mathcal{H}^1_\mu(\mathcal{T})$ real valued with $\varphi(v_{0,0}) = 0$, one has*

$$\int_{\mathcal{T}} \mu(\mathbf{x})(u'_{\mathrm{D}}(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} \leqslant 4 \int_{\mathcal{T}} \mu(\mathbf{x})(\varphi')^2(\mathbf{x}) \mathrm{d}\mathbf{x} = 4|\varphi|^2_{\mathcal{H}^1_\mu(\mathcal{T})}. \tag{4.10}$$

Proof. Indeed by Cauchy-Schwarz's inequality (4.9) implies

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} \leqslant 2 \left(\int_{\mathcal{T}^n} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}} \left(\int_{\mathcal{T}^n} \mu(\mathbf{x})(\varphi'(\mathbf{x}))^2 (u^n(\mathbf{x}))^2 \mathrm{d}\mathbf{x} \right)^{\frac{1}{2}}.$$

By simplification and using (3.11), one gets

$$\int_{\mathcal{T}^n} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2 \varphi^2(\mathbf{x}) \mathrm{d}\mathbf{x} \leqslant 4 \int_{\mathcal{T}^n} \mu(\mathbf{x})(\varphi')^2(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

In particular, for all $m \leqslant n$, it holds

$$\int_{\mathcal{T}^m} \mu(\mathbf{x})((u^n)'(\mathbf{x}))^2\varphi^2(\mathbf{x})\mathrm{d}\mathbf{x} \leqslant 4\int_{\mathcal{T}} \mu(\mathbf{x})(\varphi')^2(\mathbf{x})\mathrm{d}\mathbf{x}.$$

For a fixed m, passing to the limit in n and using Corollary 3.10, one finds that

$$\int_{\mathcal{T}^m} \mu(\mathbf{x})((u_{\mathrm{D}})'(\mathbf{x}))^2\varphi^2(\mathbf{x})\mathrm{d}\mathbf{x} \leqslant 4\int_{\mathcal{T}} \mu(\mathbf{x})(\varphi')^2(\mathbf{x})\mathrm{d}\mathbf{x}.$$

This estimate and the application of Fatou's Lemma lead to (4.10). □

Corollary 4.9. *For any $\varphi \in \mathcal{H}^1_\mu(\mathcal{T})$ real valued, one has*

$$\int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|\varphi(\mathbf{x})|^2\mathrm{d}\mathbf{x} \lesssim \|\varphi\|^2_{\mathcal{H}^1_\mu(\mathcal{T})}, \tag{4.11}$$

in particular (4.6) *holds for all n.*

Proof. As usual, we split up φ in the form

$$\varphi = \varphi(v_{0,0})\chi + \psi,$$

with $\psi \in \mathcal{H}^1_\mu(\mathcal{T})$ real valued with $\psi(v_{0,0}) = 0$. Hence by (4.10), we have

$$\begin{aligned}\int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|\varphi(\mathbf{x})|^2\mathrm{d}\mathbf{x} &\leqslant 2(\varphi(v_{0,0}))^2\int_{e_{0,0}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|\chi(\mathbf{x})|^2\mathrm{d}\mathbf{x} + 2\int_{\mathcal{T}} \mu(\mathbf{x})|u'_{\mathrm{D}}(\mathbf{x})|^2|\psi(\mathbf{x})|^2\mathrm{d}\mathbf{x}\\ &\lesssim (\varphi(v_{0,0}))^2 + \int_{\mathcal{T}} \mu(\mathbf{x})(\psi')^2(\mathbf{x})\mathrm{d}\mathbf{x}.\end{aligned}$$

Using that $\psi' = \varphi' - \varphi(v_{0,0})\chi'$ on each edge, one concludes that (4.11) holds. □

Altogether we have proved the next trace theorem.

Theorem 4.10 (Trace theorem)**.** *Given a weighted p-adic tree $\mathcal{T} = (\mathcal{V}, \mathcal{E}, \mu)$, and assume that* (4.2) *holds. Then there exists a p-recursive partition γ of the unit interval such that T^∞_γ is continuous from $\mathcal{H}^1_\mu(\mathcal{T})$ to $\mathrm{L}^2(]0,1[)$.*

Proof. Take

$$\ell^n_{p,j} = -R\mu_{n-1,j}u'_{\mathrm{D},n-1,j}, \tag{4.12}$$

and set $\gamma^n_{p,0} = 0$ and for $j \geqslant 1$

$$\gamma^n_{p,j} = \sum_{k<j} \ell^n_{p,k}. \tag{4.13}$$

We now check that relations of Definition 4.1 hold. Indeed the sole nontrivial one is

$$\gamma^{n+1}_{p,pj} = \gamma^n_{p,j}, \tag{4.14}$$

for any $n \geqslant 1$ and for any $0 \leqslant j \leqslant p^{n-1}$. But it actually follows from Kirchoff law satisfied by u_{D} that

$$\mu_{\ell,j} u'_{\mathrm{D},\ell,j} = \sum_{k=0}^{p-1} \mu_{\ell+1,pj+k} u'_{\mathrm{D},\ell+1,pj+k}.$$

Using this identity we find that

$$\begin{aligned}\gamma^n_{p,j} &= -R \sum_{k'=0}^{j-1} \mu_{n-1,k'} u'_{\mathrm{D},n-1,k'} \\ &= -R \sum_{k'=0}^{j-1} \sum_{k_1=0}^{p-1} \mu_{n,pk'+k_1} u'_{\mathrm{D},n,pk'+k_1}.\end{aligned}$$

By the change of unknown $k = pk' + k_1$, we find that

$$\gamma^n_{p,j} = -R \sum_{k=0}^{pj-1} \mu_{n,k} u'_{\mathrm{D},n,k} = \gamma^{n+1}_{p,pj},$$

which is nothing else than (4.14).

With the choice (4.12), the estimate (4.7) becomes

$$\sum_{j=0}^{p^{n-1}-1} \ell^n_{p,j} |u(v_{n,j})|^2 \lesssim \|u\|^2_{\mathcal{H}^1_\mu(\mathcal{T})},$$

or equivalently

$$\|T^n_\gamma u\|_{L^2(]0,1[)} \lesssim \|u\|_{\mathcal{H}^1_\mu(\mathcal{T})}.$$

Hence up to a subsequence, it admits a (weak) limit $T^\infty_\gamma u$ such that

$$\|T^\infty_\gamma u\|_{L^2(]0,1[)} \lesssim \|u\|_{\mathcal{H}^1_\mu(\mathcal{T})}.$$ □

Since our trace operator brings close to each other points which are far away from each other in the tree, the continuity of the trace of a function $u \in \mathcal{H}^1_\mu(\mathcal{T})$ is not guaranteed, even if u is smooth on all the edges. Let us illustrate this fact by the following example.

Proposition 4.11. *Under the assumptions of Theorem 4.10, and assume that $p = 2$. Then, by eventually changing the indices (ℓ, j) into $(\ell, 2^\ell - 1 - j)$, the quantity $\gamma^2_{2,1}$ (see (4.13)) is positive and there exists a function $u \in \mathcal{H}^1_\mu(\mathcal{T})$ such that*

$$T^\infty_\gamma u = \mathbb{1}_{]\gamma^2_{2,1},1[}.$$

Proof. First we notice that by (4.13) and (4.12), one has

$$\gamma^2_{2,1} = -R\mu_{1,0} u'_{\mathrm{D},1,0}.$$

As we have already seen that

$$-R(\mu_{1,0}u'_{\mathrm{D},1,0} + \mu_{1,1}u'_{\mathrm{D},1,1}) = 1,$$

and as $u'_{\mathrm{D},1,j} \leqslant 0$, we deduce that either $u'_{\mathrm{D},1,0} < 0$ or $u'_{\mathrm{D},1,1} < 0$. In the first case, we keep the tree as it is, otherwise we will continue with the tree that consists in exchanging the indices (ℓ, j) into $(\ell, 2^\ell - 1 - j)$ (which has the same global resistivity).

Now we define the function u on $\mathcal{T}$ as follows:

1. on the subtree $\mathcal{T}_{e_{1,1}}$ and on $e_{0,0}$, $u = 1$,
2. on the subtree $\mathcal{T}_{e_{1,0}}$, u is equal to the solution of the Dirichlet problem on $\mathcal{T}_{e_{1,0}}$.

As on $\mathcal{T}_{e_{1,0}}$, u is the limit of compactly supported functions u_n in $\mathcal{T}^{n-1}_{e_{1,0}}$, $n \geqslant 1$ with $u_n(v_{1,0}) = 1$, we deduce by the definition of our operator T^m_γ that

$$T^m_\gamma(u_n) = \mathbb{1}_{]\gamma^2_{2,1},1[}, \forall m \geqslant n,$$

recalling that $\gamma^n_{2,2^{n-2}} = \gamma^2_{2,1}$, for all $n \geqslant 2$. Hence passing to the limit in $m \to \infty$, we deduce that

$$T^\infty_\gamma(u_n) = \mathbb{1}_{]\gamma^2_{2,1},1[}, \forall n \geqslant 1.$$

The limit in n yields the result. □

5. Perspectives

Beyond the fundamental results obtained here, some open questions remain.

1. The first one concerns the characterization of the space $\mathcal{H}^1_{\mu,0}(\mathcal{T})$, namely under the assumption of Theorem 4.10, do we have

$$\mathcal{H}^1_{\mu,0}(\mathcal{T}) = \ker T^\infty_\gamma?$$

2. The second question concerns the characterisation of the trace space of $\mathcal{H}^1_\mu(\mathcal{T})$. More precisely, in the setting of Theorem 4.10, we can define the trace space

$$T = \{T^\infty_\gamma u : u \in \mathcal{H}^1_\mu(\mathcal{T})\},$$

that we equipped with the induced norm

$$\|f\|_T := \inf_{u \in \mathcal{H}^1_\mu(\mathcal{T}) : T^\infty_\gamma u = f} \|u\|_{\mathcal{H}^1_\mu(\mathcal{T})}.$$

Hence the question is the characterization of this induced norm? In particular is it equivalent to the (Sobolev/Besov) norm of $H^s(]0,1])$, for some $s \in]0, \frac{1}{2}[$? In the discrete setting, we refer to [19].

3. Once points 1 and 2 are fixed, we can define the Dirichlet to Neumann maps $\Lambda_{\mathsf{DtN},\ell}$, $\ell \in \mathbb{N}$ as follows. Given $f \in T$, we look for a harmonic function $u \in \mathcal{H}^1_\mu(\mathcal{T})$ such that

$$T^\infty_\gamma u = f \text{ and } u(v_{0,0}) = 1.$$

Such a solution exists and is unique. Indeed we first fix a lifting w of v, namely one $w \in \mathcal{H}^1_\mu(\mathcal{T})$ such that

$$T^\infty_\gamma w = f \text{ and } w(v_{0,0}) = 1.$$

Then by point 1, $d := u - w$ belongs to $\mathcal{H}^1_{\mu,0}(\mathcal{T})$ and satisfies $d(v_{0,0}) = 0$. To guarantee the harmonicity of u, we then impose that d satisfies

$$\int_{\mathcal{T}} \mu(\mathbf{x}) d'(\mathbf{x}) \overline{\varphi'(\mathbf{x})} \mathrm{d}\mathbf{x} = - \int_{\mathcal{T}} \mu(\mathbf{x}) w'(\mathbf{x}) \overline{\varphi'(\mathbf{x})} \mathrm{d}\mathbf{x}, \quad \forall \varphi \in \mathcal{H}^1_{\mu,0}(\mathcal{T}),\ \varphi(v_{0,0}) = 0.$$

Since this problem has a unique solution d, the existence and uniqueness of u is then guaranteed.

We finally define

$$\Lambda_{\mathsf{DtN},0} : f \mapsto u'(v_{0,0}),$$

which is a linear and continuous operator from T into $\mathbb{R}$, while for $\ell \geqslant 1$, we set

$$\Lambda_{\mathsf{DtN},\ell} : f \mapsto (u'(v_{\ell,j}))_{j=0}^{p^{\ell-1}-1},$$

which is a linear and continuous operator from T into $\mathbb{R}^{p^{\ell-1}}$. In practical applications, such mappings can be used to reconstruct f from the measurement of $\Lambda_{\mathsf{DtN},\ell} f$, for different values of ℓ.

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