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ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF A DISTRIBUTED OPTIMAL CONTROL PROBLEM GOVERNED BY THE VON KÁRMÁN EQUATIONS

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Abstract. In this paper, we discuss the numerical approximation of a distributed optimal control problem governed by the von Kármán equations, defined in polygonal domains with point-wise control constraints. Conforming finite elements are employed to discretize the state and adjoint variables. The control is discretized using piece-wise constant approximations. *A priori* error estimates are derived for the state, adjoint and control variables. Numerical results that justify the theoretical results are presented.

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1. Introduction

Consider the distributed control problem governed by the von Kármán equations defined by:

$$\min_{u\in U_{ad}} J(\Psi,u) \qquad \text{subject to} \tag{1.1a}$$

$$\Delta^2\psi_1 = [\psi_1,\psi_2] + f + \mathcal{C}u \quad \text{in } \Omega, \tag{1.1b}$$

$$\Delta^2\psi_2 = -\frac{1}{2}[\psi_1,\psi_1] \quad \text{in } \Omega, \tag{1.1c}$$

$$\psi_1 = 0,\ \frac{\partial\psi_1}{\partial\nu} = 0 \text{ and } \psi_2 = 0,\ \frac{\partial\psi_2}{\partial\nu} = 0 \text{ on } \partial\Omega, \tag{1.1d}$$

where $\Psi = (\psi_1,\psi_2)$ and the components ψ_1 and ψ_2 denote the displacement and Airy-stress respectively, $\Delta^2\varphi := \varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy}$, the von Kármán bracket $[\eta,\chi] := \eta_{xx}\chi_{yy} + \eta_{yy}\chi_{xx} - 2\eta_{xy}\chi_{xy}$ and ν is the unit outward normal to the boundary $\partial\Omega$ of the polygonal domain $\Omega\subset\mathbb{R}^2$. The load function $f\in H^{-1}(\Omega)$, for $\omega\subset\Omega$, $\mathcal{C}\in\mathcal{L}(L^2(\omega),L^2(\Omega))$ is the extension operator defined by

$$\mathcal{C}u(x) = u(x) \quad \text{if } x\in\omega, \quad \text{and} \quad \mathcal{C}u(x) = 0 \quad \text{if } x\notin\omega.$$

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The cost functional $J(\Psi, u)$ is defined by

$$J(\Psi, u) := \frac{1}{2}\|\Psi - \Psi_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\int_\omega |u|^2 \,\mathrm{d}x, \tag{1.2}$$

with a fixed regularization parameter $\alpha > 0$, $\Psi_d = (\psi_{1d}, \psi_{2d})$ is the given observation for Ψ and $U_{ad} \subset L^2(\omega)$ is a non-empty, convex and bounded admissible space of controls defined by

$$U_{ad} = \{u \in L^2(\omega) : u_a \le u(x) \le u_b \quad \text{for almost every } x \text{ in } \omega\}, \tag{1.3}$$

$u_a,\ u_b \in \mathbb{R},\ u_a \le u_b$ are given.

Let us first discuss the results available for the state equations, for a given $u \in L^2(\omega)$. For results regarding the existence of solutions, regularity and bifurcation phenomena of the von Kármán equations we refer to [1–4, 12, 22] and the references therein. It is well known [4] that in a polygonal domain Ω, for given $f \in H^{-1}(\Omega)$, the solution of the biharmonic problem belongs to $H^2_0(\Omega) \cap H^{2+\gamma}(\Omega)$, where $\gamma \in (\frac{1}{2}, 1]$, referred to as the index of elliptic regularity, is determined by the interior angles of Ω. Note that when Ω is convex, $\gamma = 1$ and the solution belongs to $H^2_0(\Omega) \cap H^3(\Omega)$. It is also stated in [4] that similar regularity results hold true for von Kármán equations, that is, solutions ψ_1, ψ_2 belong to $H^2_0(\Omega) \cap H^{2+\gamma}(\Omega)$. However, we give the details of the arguments of this proof in this paper.

Due to the importance of the problem in application areas, several numerical approaches have been attempted in the past for the state equation. The major challenges of the problem from the numerical analysis point of view are the facts that the system under consideration is semilinear and the higher order nature of the equations. In [7, 27, 28], the authors consider the state equation with *homogeneous* boundary conditions and study the approximation and error bounds for *nonsingular* solutions. The convergence analysis has been studied using conforming finite element methods in [7, 24], nonconforming finite element methods in [25], C^0 interior penalty method [6], the Hellan-Hermann-Miyoshi mixed finite element method in [27, 29] and a stress-hybrid method in [28], respectively.

For the numerical approximation of the distributed control problem defined in (1.1a)–(1.1d), not many results are available in literature. The paper [21] discusses conforming finite element approximation of the problem defined in convex domains *without* control constraints and when the control is defined over the whole domain Ω, whereas [18] discusses an abstract framework for a class of nonlinear optimization problems using a Lagrange multiplier approach. For results on optimal control problems of the steady-state Navier-Stokes equations and quasi-linear equations with and without control constraints, many references are available, see for example, [8, 10, 13, 19, 20] to mention a few. Employing a post processing of the discretized control u, [17, 26] establish a super convergence result for the control variable for the second-order and fourth-order linear elliptic problems. In this paper, we discuss the existence of solutions of conforming finite element approximations of *local nonsingular* solutions of the control problem and establish *a priori* error estimates. The contributions of this paper are

(i) error estimates for the state and adjoint variables in the energy norm and that for the control variable in the L^2 norm, under realistic regularity assumptions for the exact solution of the problem defined in polygonal domains under the assumption that the source function $f \in H^{-1}(\Omega)$,
(ii) a guaranteed quadratic convergence result in convex domains for a post processed control [26] constructed by projecting the discrete adjoint state into the admissible set of controls,
(iii) error estimates for state and adjoint variables in lower order H^1 and L^2 norms,
(iv) numerical results that illustrate all the theoretical estimates.

Throughout the paper, standard notations on Lebesgue and Sobolev spaces and their norms are employed. The standard semi-norm and norm on $H^s(\Omega)$ (resp. $W^{s,p}(\Omega)$) for $s > 0$ are denoted by $|\cdot|_s$ and $\|\cdot\|_s$ (resp. $|\cdot|_{s,p}$ and $\|\cdot\|_{s,p}$). The standard L^2 inner product is denoted by $(\cdot, \cdot)$. We use the notation $\mathbf{H}^s(\Omega)$ (resp. $\mathbf{L}^p(\Omega)$) to denote the product space $H^s(\Omega) \times H^s(\Omega)$ (resp. $L^p(\Omega) \times L^p(\Omega)$). The notation $a \lesssim b$ means there exists a generic mesh independent constant C such that $a \le Cb$. The positive constants C appearing in the inequalities denote generic constants which do not depend on the mesh-size.

The rest of the paper is organized as follows. The weak formulation and some auxiliary results needed for the analysis are described in Section 2. The state and adjoint variables are discretized and some intermediate discrete problems along with error estimates are established in Section 3. In Section 4, the discretization of the control variable is described and the existence and convergence results for the fully discrete problem are established. This is followed by derivation of the error estimates for the state, and adjoint and control variables in Section 5. The post processing of control for improved error estimates and lower order estimates for state and adjoint variables are also derived. Section 6 deals with the results of numerical experiments. The discrete optimization problem is solved using the Primal-dual active set strategy, see for example [30]. The state and adjoint variables are discretized using Bogner-Fox-Schmit finite elements and the control variable is discretized using piecewise constant functions. The post-processed control is also computed.

The analysis that we do in Sections 2 and 3 and several results stated there are very close to what is done in [10] for the Navier-Stokes system. However, many of the proofs in our paper are based on results specific to the von Kármán equations. This is why we have to adapt several results from [10] to our setting.

2. Weak formulation and auxiliary results

In this section, we state the weak formulation corresponding to (1.1a)–(1.1d) in the first subsection and present some auxiliary results in the second subsection. This will be followed by the derivation of first order and second order optimality conditions for the control problem in the third subsection.

2.1. Weak formulation

The weak formulation corresponding to (1.1a)–(1.1d) reads:

$$\min_{(\Psi,u)\in \mathbf{V}\times U_{ad}} J(\Psi,u) \quad \text{subject to} \tag{2.1a}$$

$$A(\Psi,\Phi)+B(\Psi,\Psi,\Phi)=(F+\mathbf{Cu},\Phi), \tag{2.1b}$$

for all $\Phi\in\mathbf{V}$, where $\mathbf{V}:=V\times V$ with $V:=H_0^2(\Omega)$. For all $\boldsymbol{\xi}=(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \boldsymbol{\lambda}=(\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2)$, $\Phi=(\varphi_1,\varphi_2)\in\mathbf{V}$,

$$\begin{aligned}
&A(\boldsymbol{\lambda},\Phi):=a(\boldsymbol{\lambda}_1,\varphi_1)+a(\boldsymbol{\lambda}_2,\varphi_2),\\
&B(\boldsymbol{\xi},\boldsymbol{\lambda},\Phi):=b(\boldsymbol{\xi}_1,\boldsymbol{\lambda}_2,\varphi_1)+b(\boldsymbol{\xi}_2,\boldsymbol{\lambda}_1,\varphi_1)-b(\boldsymbol{\xi}_1,\boldsymbol{\lambda}_1,\varphi_2),\\
&F=\begin{pmatrix} f\\ 0\end{pmatrix},\ \mathbf{Cu}=\begin{pmatrix}\mathcal{C}u\\ 0\end{pmatrix},\ \mathbf{u}=\begin{pmatrix}u\\ 0\end{pmatrix}\ \text{and}\ (F+\mathbf{Cu},\Phi):=(f+\mathcal{C}u,\varphi_1),
\end{aligned}$$

and for all $\eta,\chi,\varphi\in V$,

$$a(\eta,\chi):=\int_\Omega D^2\eta : D^2\chi \mathrm{d}x,\ \ b(\eta,\chi,\varphi):=\frac{1}{2}\int_\Omega \operatorname{cof}(D^2\eta)D\chi\cdot D\varphi \mathrm{d}x.$$

For consistency, when scalar controls $\bar{u}$, v, v_k, and so on, are involved in calculations with vector functions we shall use, without necessarily mentioning that, $\bar{\mathbf{u}}=\begin{pmatrix}\bar{u}\\ 0\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}v\\ 0\end{pmatrix}$, $\mathbf{v}_k=\begin{pmatrix}v_k\\ 0\end{pmatrix}$, and so on.

Remark 2.1. Note that

$$\int_\Omega [\eta,\chi]\varphi \mathrm{d}x=-\int_\Omega \operatorname{cof}(D^2\eta)D\chi\cdot D\varphi \mathrm{d}x,\quad \forall \eta,\chi\in H^2(\Omega) \text{ and } \varphi\in H_0^2(\Omega). \tag{2.2}$$

The relation also holds true under assumptions, say when (a) $\eta \in H^{2+\gamma}(\Omega)$, $\chi \in H^2(\Omega)$, $\varphi \in H^1(\Omega)$; (b) $\eta \in H^2(\Omega)$, $\chi \in H^2(\Omega)$, $\varphi \in H^{1+\epsilon}(\Omega)$, $0 < \epsilon < 1/2$. This can be established by proceeding similar to the proof of Lemma 2.2 [6].

The form $b(\cdot,\cdot,\cdot)$ is derived using the divergence-free rows property [14]. Since the Hessian matrix $D^2\eta$ is symmetric, $\mathrm{cof}(D^2\eta)$ is symmetric. Consequently, $b(\cdot,\cdot,\cdot)$ is symmetric with respect to the second and third variables, that is, $b(\eta,\boldsymbol{\xi},\varphi) = b(\eta,\varphi,\boldsymbol{\xi})$. Moreover, since $[\cdot,\cdot]$ is symmetric, $b(\cdot,\cdot,\cdot)$ is symmetric with respect to all the variables. Also, $B(\cdot,\cdot,\cdot)$ is symmetric in the first and second variables due to the symmetry of $b(\cdot,\cdot,\cdot)$.

The Sobolev space $\mathbf{V}$ is equipped with the norm defined by

$$\|\Phi\|_2 := (|\varphi_1|^2_{2,\Omega} + |\varphi_2|^2_{2,\Omega})^{\frac{1}{2}} \qquad \forall \Phi = (\varphi_1,\varphi_2) \in \mathbf{V},$$

where $|\varphi|^2_{2,\Omega} = \int_\Omega D^2\varphi : D^2\varphi \mathrm{d}x$, for all $\varphi \in V$.

In the sequel, for $s > 0$, the product norms defined on $\mathbf{H}^s(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by $\|\cdot\|_s$ and $\|\cdot\|$, respectively.

The following properties of boundedness and coercivity of $A(\cdot,\cdot)$ and boundedness of $B(\cdot,\cdot,\cdot)$ hold true: $\forall \boldsymbol{\xi}, \boldsymbol{\lambda}, \Phi \in \mathbf{V}$,

$$|A(\boldsymbol{\xi},\Phi)| \leq \|\boldsymbol{\xi}\|_2 \ \|\Phi\|_2\,, \tag{2.3}$$

$$|A(\boldsymbol{\xi},\boldsymbol{\xi})| \geq \|\boldsymbol{\xi}\|^2_2\,, \quad \text{and} \tag{2.4}$$

$$|B(\boldsymbol{\xi},\boldsymbol{\lambda},\Phi)| \leq C_b \|\boldsymbol{\xi}\|_2 \ \|\boldsymbol{\lambda}\|_2 \ \|\Phi\|_2\,. \tag{2.5}$$

The definition of $B(\cdot,\cdot,\cdot)$, the symmetry of $b(\cdot,\cdot,\cdot)$ and the Sobolev imbedding yields [24]

$$|B(\Xi,\Theta,\Phi)| \lesssim \begin{cases} \|\Xi\|_{2+\gamma} \|\Theta\|_2 \|\Phi\|_1 & \forall\, \Xi \in \mathbf{H}^{2+\gamma}(\Omega) \text{ and } \Theta,\Phi \in \mathbf{V}, \\ \|\Xi\|_{2+\gamma} \|\Theta\|_1 \|\Phi\|_2 & \forall\, \Xi \in \mathbf{H}^{2+\gamma}(\Omega) \text{ and } \Theta,\Phi \in \mathbf{V}, \\ \|\Xi\|_1 \|\Theta\|_{2+\gamma} \|\Phi\|_2 & \forall\, \Xi \in \mathbf{V}, \Theta \in \mathbf{H}^{2+\gamma}(\Omega) \text{ and } \Phi \in \mathbf{V}, \end{cases} \tag{2.6}$$

where $\gamma \in (\frac{1}{2},1]$ denotes the elliptic regularity index. The above estimates are also valid if γ is replaced by any $\gamma_0 \in (1/2,\gamma)$, that is

$$|B(\Xi,\Theta,\Phi)| \leq C_{\gamma_0} \|\Xi\|_{2+\gamma_0} \|\Theta\|_2 \|\Phi\|_1\,, \tag{2.7}$$

for all $\gamma_0 \in (1/2,\gamma)$.

We now prove another boundedness result which will be also needed later.

Lemma 2.2. *For $\Xi,\Theta,\Phi \in \mathbf{V}$, there holds*

$$|B(\Xi,\Theta,\Phi)| \leq C_\epsilon \|\Xi\|_2 \|\Theta\|_2 \|\Phi\|_{1+\epsilon}\,, \quad 0 < \epsilon < 1/2. \tag{2.8}$$

Proof. It is enough to prove that

$$\int_\Omega \mathrm{cof}(D^2\xi) D\theta \cdot D\varphi \mathrm{d}x \leq C_\epsilon \|\xi\|_2 \|\theta\|_2 \|\varphi\|_{1+\epsilon} \quad \forall \xi,\theta,\varphi \in V.$$

For $0 < \epsilon < 1/2$, we have

$$\int_\Omega \mathrm{cof}(D^2\xi) D\theta \cdot D\varphi \mathrm{d}x \leq \left\|\mathrm{cof}(D^2\xi)\right\| \|D\theta\|_{L^{2/\epsilon}(\Omega)} \|D\varphi\|_{L^{2/(1-\epsilon)}(\Omega)}$$

$$\leq C_\epsilon \left\|\xi\right\|_2 \left\|\theta\right\|_2 \left\|\varphi\right\|_{1+\epsilon}.$$

The last inequality follows from the imbeddings

$$H^1(\Omega) \hookrightarrow L^{2/\epsilon}(\Omega), \quad H^\epsilon(\Omega) \hookrightarrow L^{2/(1-\epsilon)}(\Omega) \text{ for } 0 < \epsilon < 1/2.$$

The proof is complete. □

2.2. Some auxiliary results

Define the operator $\mathcal{A} \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ by

$$\left\langle \mathcal{A}\Psi, \Phi \right\rangle_{\mathbf{V}',\mathbf{V}} = A(\Psi, \Phi) \quad \forall \Psi, \Phi \in \mathbf{V},$$

and the nonlinear operator $\mathcal{B}$ from $\mathbf{V}$ to $\mathbf{V}'$ by

$$\left\langle \mathcal{B}(\Psi), \Phi \right\rangle_{\mathbf{V}',\mathbf{V}} = B(\Psi, \Psi, \Phi) \quad \forall \Psi, \Phi \in \mathbf{V}.$$

For simplicity of notation, the duality pairing $\left\langle \cdot, \cdot \right\rangle_{\mathbf{V}',\mathbf{V}}$ is denoted by $\left\langle \cdot, \cdot \right\rangle$.

In the sequel, the weak formulation (2.1b) will also be written as

$$\Psi \in \mathbf{V}, \quad \mathcal{A}\Psi + \mathcal{B}(\Psi) = F + \mathbf{Cu} \text{ in } \mathbf{V}'. \tag{2.9}$$

Note that the nonlinear operator $\mathcal{B}(\Psi)$ can also be expressed in the form

$$\mathcal{B}(\Psi) := \begin{pmatrix} -[\psi_1, \psi_2] \\ \frac{1}{2}[\psi_1, \psi_1] \end{pmatrix}.$$

It is easy to verify that, for all $\Psi \in \mathbf{V}$, the operator $\mathcal{B}'(\Psi) \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$[1] and its adjoint operator $\mathcal{B}'(\Psi)^* \in \mathcal{L}(\mathbf{V}, \mathbf{V}')$ satisfy

$$\left\langle \mathcal{B}'(\Psi)\boldsymbol{\xi}, \Phi \right\rangle = 2B(\Psi, \boldsymbol{\xi}, \Phi) \qquad \forall \boldsymbol{\xi}, \Phi \in \mathbf{V}, \tag{2.10}$$

$$\left\langle \mathcal{B}'(\Psi)^*\boldsymbol{\xi}, \Phi \right\rangle = 2B(\Psi, \Phi, \boldsymbol{\xi}) \qquad \forall \boldsymbol{\xi}, \Phi \in \mathbf{V}. \tag{2.11}$$

Moreover, $\mathcal{B}'' \in \mathcal{L}(\mathbf{V} \times \mathbf{V}, \mathbf{V}')$ satisfies

$$\left\langle \mathcal{B}''(\Psi, \boldsymbol{\xi}), \Phi \right\rangle = 2B(\Psi, \boldsymbol{\xi}, \Phi) \qquad \forall \Psi, \boldsymbol{\xi}, \Phi \in \mathbf{V}. \tag{2.12}$$

Theorem 2.3 (Existence [12, 22])**.** *For given $u \in L^2(\omega)$, the problem (2.1b) possesses at least one solution.*

The linearization of (2.1b) around Ψ in the direction $\boldsymbol{\xi}$ is given by

$$\boldsymbol{L}\boldsymbol{\xi} := \mathcal{A}\boldsymbol{\xi} + \mathcal{B}'(\Psi)\boldsymbol{\xi}.$$

[1] The same notation ′ is used either to denote the dual of a space or the Fréchet derivative of an operator, but the context helps to clarify its precise meaning.

Definition 2.4 (Nonsingular solution)**.** For a given $u \in L^2(\omega)$, a solution Ψ of (2.1b) is said to be regular if the linearized form is nonsingular. That is, if $\langle \boldsymbol{L\xi}, \Phi \rangle = 0$ for all $\Phi \in \mathbf{V}$, then $\boldsymbol{\xi} = \mathbf{0}$. In that case, we will also say that the pair (Ψ, u) is a nonsingular solution of (1.1b)–(1.1c).

Remark 2.5. The dependence of Ψ with respect to u is made explicit with the notation Ψ_u whenever it is necessary to do so.

Lemma 2.6 (*Properties of* $\mathcal{A}^{-1}$)**.** *The following properties hold true:*

(i) $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{V}', \mathbf{V})$.
(ii) $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H}^{-1}(\Omega), \mathbf{H}^{2+\gamma}(\Omega) \cap \mathbf{V}), \gamma \in (1/2, 1]$ *is the elliptic regularity index.*
(iii) $\mathcal{A}^{-1} \in \mathcal{L}(\mathbf{H}^{-1-\epsilon}(\Omega), \mathbf{H}^{2+\gamma(1-\epsilon)}(\Omega))$ *for all* $0 < \epsilon < 1/2$.

Proof. The statement (i) follows from the Lax-Milgram Lemma. The statement (ii) follows from the regularity result for biharmonic problem (see [4]). Now (iii) follows from (i) and (ii) by interpolation. □

In the next lemma, we obtain *a priori* bounds for any solution Ψ of (2.1b).

Lemma 2.7 (*An a priori estimate*)**.** *For* $f \in H^{-1}(\Omega)$ *and* $u \in L^2(\omega)$*, any solution* Ψ *of* (2.1b) *belongs to* $\mathbf{H}^{2+\gamma}(\Omega)$*,* $\gamma \in (1/2, 1]$ *being the elliptic regularity index, and satisfies the a priori bounds*

$$\|\Psi\|_2 \leq C(\|f\|_{-1} + \|u\|_{L^2(\omega)}), \tag{2.13a}$$

$$\|\Psi\|_{2+\gamma} \leq C \left(\|f\|_{-1}^3 + \|u\|_{L^2(\omega)}^3 + \|f\|_{-1}^2 + \|u\|_{L^2(\omega)}^2 + \|f\|_{-1} + \|u\|_{L^2(\omega)} \right). \tag{2.13b}$$

Proof. From the scalar form of (2.1b), we obtain,

$$a(\psi_1, \varphi_1) = \int_\Omega [\psi_1, \psi_2]\varphi_1 \mathrm{d}x + (f + \mathcal{C}u, \varphi_1) \quad \forall \varphi_1 \in V, \tag{2.14}$$

$$a(\psi_2, \varphi_2) = -\frac{1}{2} \int_\Omega [\psi_1, \psi_1]\varphi_2 \mathrm{d}x \quad \forall \varphi_2 \in V. \tag{2.15}$$

Choose $\varphi_1 = \psi_1$ in (2.14) and $\varphi_2 = \psi_2$ in (2.15), use the result $\int_\Omega [\psi_1, \psi_2]\psi_1 \mathrm{d}x = \int_\Omega [\psi_1, \psi_1]\psi_2 \mathrm{d}x$ and the definition of $a(\cdot, \cdot)$ to obtain

$$|\psi_1|_2^2 + 2|\psi_2|_2^2 \leq \|f\|_{-1}\|\psi_1\|_1 + \|u\|_{L^2(\omega)}\|\psi_1\|_0.$$

An application of Poincaré inequality leads to (2.13a).

It is already proved in [4] that (2.1b) admits a solution in $\mathbf{H}^2(\Omega)$. From (2.8), it follows that

$$|B(\Psi, \Psi, \Phi)| \leq C_\epsilon \|\Psi\|_2^2 \|\Phi\|_{1+\epsilon} \text{ for } 0 < \epsilon < 1/2.$$

Thus $\mathcal{B}(\Psi)$ belongs to $\mathbf{H}^{-1-\epsilon}(\Omega)$ and $\|\mathcal{B}(\Psi)\|_{-1-\epsilon} \leq C_\epsilon \|\Psi\|_2^2$. From Lemma 2.6 (iii), it follows that

$$\|\Psi\|_{2+\gamma(1-\epsilon)} \leq C_\epsilon \left(\|\Psi\|_2^2 + \|u\| + \|f\|_{-1} \right).$$

Next using (2.7), we obtain

$$\|\mathcal{B}(\Psi)\|_{-1} \leq C \|\Psi\|_{2+\gamma(1-\epsilon)} \|\Psi\|_2.$$

Combining this estimate with Lemma 2.6(ii), we finally obtain the required result (2.13b). □

Note that $\Psi \in \mathbf{H}^{2+\gamma}(\Omega)$ is already observed in [4], but the arguments are not completely given there and hence we have given a complete proof for clarity.

The implicit function theorem yields the following result, see [10].

Theorem 2.8. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.1b). Then there exist a neighbourhood $\mathcal{O}(\bar{u})$ of $\bar{u}$ in $L^2(\omega)$, a neighbourhood $\mathcal{O}(\bar{\Psi})$ of $\bar{\Psi}$ in $\mathbf{V}$, and a mapping G from $\mathcal{O}(\bar{u})$ to $\mathcal{O}(\bar{\Psi})$ of class C^∞, such that, for all $u \in \mathcal{O}(\bar{u})$, $\Psi_u = G(u)$ is the unique solution in $\mathcal{O}(\bar{\Psi})$ to (2.9). The operator $G'(u) = (\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}$ is uniformly bounded from a smaller neighbourhood into a smaller neighbourhood. (These smaller neighbourhoods are still denoted by $\mathcal{O}(\bar{u})$ and $\mathcal{O}(\bar{\Psi})$ for notational simplicity.) Moreover, if $G'(u)v =: \mathbf{z}_v \in \mathbf{V}$ and $G''(u)v^2 =: \mathbf{w} \in \mathbf{V}$, then $\mathbf{z}_v$ and $\mathbf{w}$ satisfy the equations*

$$\mathcal{A}\mathbf{z}_v + \mathcal{B}'(\Psi_u)\mathbf{z}_v = \mathbf{C}\mathbf{v} \quad \text{in } \mathbf{V}', \tag{2.16}$$

$$\mathcal{A}\mathbf{w} + \mathcal{B}'(\Psi_u)\mathbf{w} + \mathcal{B}''(\mathbf{z}_v, \mathbf{z}_v) = 0 \quad \text{in } \mathbf{V}', \tag{2.17}$$

and $(\mathcal{A} + \mathcal{B}'(\Psi_u))$ is an isomorphism from $\mathbf{V}$ into $\mathbf{V}'$ for all $u \in \mathcal{O}(\bar{u})$.

Also, the following holds true:

$$\|\mathcal{A} + \mathcal{B}'(\Psi_u)\|_{\mathcal{L}(\mathbf{V},\mathbf{V}')} \leq C, \quad \|(\mathcal{A} + \mathcal{B}'(\Psi_u))^{-1}\|_{\mathcal{L}(\mathbf{V}',\mathbf{V})} \leq C \ \ \forall u \in \mathcal{O}(\bar{u}),$$
$$\|\mathbf{z}_v\|_2 \leq \|G'(u)\|_{\mathcal{L}(L^2(\omega), H^2(\Omega))} \|v\|_{L^2(\omega)}.$$

Lemma 2.9 (*A priori bounds for the linearized problem*)**.** *The solution $\mathbf{z}_v$ of the linearized problem (2.16) belongs to $\mathbf{H}^{2+\gamma}(\Omega)$, $\gamma \in (1/2, 1]$ being the elliptic regularity index, and satisfies the a priori bound*

$$\|\mathbf{z}_v\|_{2+\gamma} \leq C\|v\|_{L^2(\omega)}.$$

Proof. From Theorem 2.8, we know that there exists $C > 0$ such that $\|\mathbf{z}_v\|_2 \leq C$ for $u \in \mathcal{O}(\bar{u})$. Now rewriting (2.16) in the form

$$\mathcal{A}\mathbf{z}_v = \mathbf{C}\mathbf{v} - \mathcal{B}'(\Psi_u)\mathbf{z}_v, \tag{2.18}$$

and using Theorem 2.8 and (2.13b), we obtain, for $u \in \mathcal{O}(\bar{u})$

$$\|\mathcal{B}'(\Psi_u)\mathbf{z}_v\|_{-1} \ \leq C \,\|\Psi_u\|_{2+\gamma} \,\|\mathbf{z}_v\|_2 \leq C\|v\|_{L^2(\omega)}.$$

Since $A(\cdot, \cdot)$ is bounded and coercive, a use of Lemma 2.6(ii) and the above result in (2.18) leads to the required regularity result [4]. □

The next lemma is an easy consequence of the *a priori* bounds in Lemma 2.7.

Lemma 2.10. *Let $(\bar{\Psi}, \bar{u})$ be a nonsingular solution of (2.1b), as defined in Theorem 2.8. Let $(u_k)_k$ be a sequence in $\mathcal{O}(\bar{u})$ weakly converging to $\bar{u}$ in $L^2(\omega)$. Let Ψ_{u_k} be the solution to equation (2.1b) in $\mathcal{O}(\bar{\Psi})$ corresponding to u_k. Then, $(\Psi_{u_k})_k$ converges to $\bar{\Psi}$ in $\mathbf{V}$.*

2.3. Optimality conditions

In this subsection, we discuss the first order and second order optimality conditions for the optimal control problem.

Definition 2.11 (Local solution of the optimal control problem [10])**.** The pair $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times U_{ad}$ is a local solution of (2.1) if and only if $(\bar{\Psi}, \bar{u})$ satisfies (2.1b) and there exist neighbourhoods $\mathcal{O}(\bar{\Psi})$ of $\bar{\Psi}$ in $\mathbf{V}$ and $\mathcal{O}(\bar{u})$ of $\bar{u}$ in $L^2(\omega)$ such that $J(\bar{\Psi}, \bar{u}) \leq J(\Psi, u)$ for all pairs $(\Psi, u) \in \mathcal{O}(\bar{\Psi}) \times (U_{ad} \cap \mathcal{O}(\bar{u}))$ satisfying (2.1b).

The existence of a solution of (2.1) can be obtained using standard arguments of considering a minimizing sequence, which is bounded in $\mathbf{V} \times L^2(\omega)$, and passing to the limit [21, 23, 30].

For the purpose of numerical approximations, we consider only local solutions $(\bar{\Psi}, \bar{u})$ of (2.1) such that the pair is a nonsingular solution of (2.9). For a *local nonsingular solution* chosen in this fashion, we can apply Theorem 2.8 and modify the control problem (2.1) to

$$\inf_{u \in U_{ad} \cap \mathcal{O}(\bar{u})} j(u), \tag{2.19}$$

where $j : U_{ad} \cap \mathcal{O}(\bar{u}) \to \mathbb{R}$ is the reduced cost functional defined by $j(u) := J(G(u), u)$ and $G(u) = \Psi_u = (\psi_{1u}, \psi_{2u}) \in \mathbf{V}$ is the unique solution to (2.1b) as defined in Theorem 2.8. Then, $\bar{u}$ is a local solution of (2.19).

Since G is of class C^∞ in $\mathcal{O}(\bar{u})$, j is of class C^∞ and for every $u \in \mathcal{O}(\bar{u})$ and $v \in L^2(\omega)$, it is easy to compute

$$j'(u)v = \int_\omega (\mathcal{C}^* \theta_{1u} + \alpha u)\, v \mathrm{d}x, \tag{2.20a}$$

$$j''(u)v^2 = \int_\Omega \left(|\mathbf{z}_v|^2 + [[\mathbf{z}_v, \mathbf{z}_v]] \cdot \Theta_u \right) \mathrm{d}x + \alpha \int_\omega |v|^2 \, \mathrm{d}x, \tag{2.20b}$$

where $\mathbf{z}_v = (z_{1v}, z_{2v})$ is the solution of (2.16),

$$[[\mathbf{z}_v, \mathbf{z}_v]] := -2\, \mathcal{B}(\mathbf{z}_v) = \begin{pmatrix} 2\big([z_{1v}, z_{2v}] \\ -[z_{1v}, z_{1v}] \end{pmatrix},$$

$[\cdot, \cdot]$ being the von Kármán bracket, $\Theta_u = (\theta_{1u}, \theta_{2u}) \in \mathbf{V}$ is the solution of the adjoint system and

$$[[\mathbf{z}_v, \mathbf{z}_v]] \cdot \Theta_u := 2[z_{1v}, z_{2v}]\theta_{1u} - [z_{1v}, z_{1v}]\theta_{2u}.$$

The adjoint system is given by

$$\Delta^2 \theta_1 - [\psi_{2u}, \theta_{1u}] + [\psi_{1u}, \theta_{2u}] = \psi_{1u} - \psi_{1d} \quad \text{in } \Omega, \tag{2.21a}$$

$$\Delta^2 \theta_2 - [\psi_{1u}, \theta_{1u}] = \psi_{2u} - \psi_{2d} \quad \text{in } \Omega, \tag{2.21b}$$

$$\theta_1 = 0,\ \frac{\partial \theta_1}{\partial \nu} = 0 \text{ and } \theta_2 = 0,\ \frac{\partial \theta_2}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.21c}$$

As for the case of the state equations, the adjoint equations in (2.21) can also be written equivalently in an operator form as

$$\Theta_u \in \mathbf{V} \quad \mathcal{A}^* \Theta_u + \mathcal{B}'(\Psi_u)^* \Theta_u = \Psi_u - \Psi_d \quad \text{in} \quad \mathbf{V}', \tag{2.22}$$

with the operator $(\mathcal{A}^* + \mathcal{B}'(\Psi_u)^*)$ being an isomorphism from $\mathbf{V}$ into $\mathbf{V}'$ (see Thm. 2.8). The first order optimality condition $j'(\bar{u})(u - \bar{u}) \geq 0$ for all $u \in U_{ad}$ translates to

$$\int_\omega \left(\mathbf{C}^* \bar{\Theta} + \alpha \bar{\mathbf{u}} \right) \cdot (\mathbf{u} - \bar{\mathbf{u}})\, \mathrm{d}x \ \geq 0 \quad \forall \mathbf{u} = (u, 0),\ u \in U_{ad},$$

where $\bar{\mathbf{u}} = (\bar{u}, 0)$ and $\bar{\Theta} = (\bar{\theta}_1, \bar{\theta}_2)$ being the adjoint state corresponding to a local nonsingular solution $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times U_{ad}$ of (2.1), or equivalently in a scalar form as

$$\int_\omega \left(\mathcal{C}^* \bar{\theta}_1 + \alpha \bar{u}\right)(u - \bar{u})\, \mathrm{d}x \geq 0 \quad \forall u \in U_{ad}.$$

The *optimality system* for the optimal control problem (2.1) can be stated as follows:

$$A(\bar{\Psi}, \Phi) + B(\bar{\Psi}, \bar{\Psi}, \Phi) = (F + \mathbf{C}\bar{\mathbf{u}}, \Phi) \quad \forall \Phi \in \mathbf{V} \;\; (\textit{State equations}) \tag{2.23a}$$

$$A(\Phi, \bar{\Theta}) + 2B(\bar{\Psi}, \Phi, \bar{\Theta}) = (\bar{\Psi} - \Psi_d, \Phi) \quad \forall \Phi \in \mathbf{V} \;\; (\textit{Adjoint equations}) \tag{2.23b}$$

$$\left(\mathbf{C}^* \bar{\Theta} + \alpha \bar{\mathbf{u}}, \mathbf{u} - \bar{\mathbf{u}}\right)_{\mathbf{L}^2(\omega)} \geq 0 \quad \forall\, \mathbf{u} = (u, 0),\; u \in U_{ad}, \;\; (\textit{First order optimality condition}). \tag{2.23c}$$

The optimal control $\bar{u}$ in (2.23c) has the representation for a.e. $x \in \Omega$:

$$\bar{u}(x) = \pi_{[u_a, u_b]}\left(-\frac{1}{\alpha}\mathcal{C}^* \bar{\theta}_1(x)\right), \tag{2.24}$$

where the projection operator $\pi_{[a,b]}$ is defined by $\pi_{[a,b]}(g) := \min\{b, \max\{a, g\}\}$.

Remark 2.12. Since $\mathcal{C}$ is the extension operator by zero to $\Omega \setminus \omega$, $\mathcal{C}^*$ appearing in (2.24) is nothing but the multiplication by χ_ω. Thus $\mathbf{C}^* \bar{\Theta}|_\omega$ belongs to $C^{0,1}(\bar{\omega})$, but in general $\mathbf{C}^* \bar{\Theta}$ does not belong to $C^{0,1}(\bar{\Omega})$, unless $\Omega = \omega$.

Remark 2.13. The optimality conditions in (2.23) can also be derived with the help of a Lagrangian for the constrained optimization problem (2.1) defined by

$$L(\Psi, u, \Theta) = J(\Psi, u) - (A(\Psi, \Theta) + B(\Psi, \Psi, \Theta) - (F + \mathbf{C}\mathbf{u}, \Theta)) \quad \forall (\Psi, u, \Theta) \times \mathbf{V} \times U_{ad} \times \mathbf{V}.$$

For the error analysis for this nonlinear control problem, second order sufficient optimality conditions are required. We now proceed to discuss the second order optimality conditions.

Define the *tangent cone* at $\bar{u}$ to U_{ad} as

$$\mathscr{C}_{U_{ad}}(\bar{u}) := \left\{u \in L^2(\omega) : u \text{ satisfies } (2.25)\right\},$$

with

$$\begin{cases} u(x) \in \mathbb{R} & \text{if } \bar{u}(x) \in (u_a, u_b), \\ u(x) \geq 0 & \text{if } \bar{u}(x) = u_a, \\ u(x) \leq 0 & \text{if } \bar{u}(x) = u_b. \end{cases} \tag{2.25}$$

The function $\mathcal{C}^* \bar{\theta}_1 + \alpha \bar{u}$ or $\mathbf{C}^* \bar{\Theta} + \alpha \bar{\mathbf{u}}$ in the vector form, is used frequently in the analysis. Introduce the notation

$$\bar{d}(x) = \mathcal{C}^* \bar{\theta}_1 + \alpha \bar{u}, \quad x \in \omega.$$

Associated with $\bar{d}$, we introduce another cone $\mathscr{C}_{\bar{u}} \subset \mathscr{C}_{U_{ad}}(\bar{u})$ defined by

$$\mathscr{C}_{\bar{u}} := \left\{u \in L^2(\omega) : u \text{ satisfies } (2.26)\right\},$$

with

$$\begin{cases} u(x) = 0 \text{ if } \bar{d}(x) \neq 0, \\ u(x) \geq 0 \text{ if } \bar{d}(x) = 0 \text{ and } \bar{u}(x) = u_a, \\ u(x) \leq 0 \text{ if } \bar{d}(x) = 0 \text{ and } \bar{u}(x) = u_b. \end{cases} \tag{2.26}$$

By the definition of $\bar{d}$, we have

$$j'(u)v = \int_\omega \bar{d}(x)v(x)\mathrm{d}x \quad \forall v \in L^2(\omega).$$

Moreover, if we choose $v \in \mathscr{C}_{\bar{u}}$, the optimality condition (2.23c) yields $\bar{d}(x)v(x) = 0$ for almost all $x \in \omega$.

The following theorem is on second order necessary optimality conditions. The proof is on similar lines of the proof of Theorem 3.6 in [10] and hence skipped.

Theorem 2.14. *Let $(\bar{\Psi}, \bar{u})$ be a nonsingular local solution of (2.1). Then*

$$j''(\bar{u})v^2 \geq 0 \quad \forall v \in \mathscr{C}_{\bar{u}}. \tag{2.27}$$

The optimality condition (2.27) is equivalent to

$$\int_\Omega \left(|\bar{\mathbf{z}}_v|^2 + [[\bar{\mathbf{z}}_v, \bar{\mathbf{z}}_v]] \cdot \bar{\Theta}\right) \mathrm{d}x + \alpha \int_\omega |v|^2 \mathrm{d}x \geq 0,$$

for all $v \in \mathscr{C}_{\bar{u}}$, where $\bar{\Theta} = \Theta(\bar{u})$ is the associated adjoint state and $\bar{\mathbf{z}}_v = \mathbf{z}_v(\bar{u})$ is the solution to (2.16).

Theorem 2.15 (Second order sufficient condition)**.** *Let $(\bar{\Psi}, \bar{u})$ be a nonsingular local solution of (2.1) and let $\bar{\Theta} = \Theta(\bar{u})$ be the associated adjoint state. Assume that*

$$\int_\Omega \left(|\bar{\mathbf{z}}_v|^2 + [[\bar{\mathbf{z}}_v, \bar{\mathbf{z}}_v]] \cdot \bar{\Theta}\right) \mathrm{d}x + \alpha \int_\omega |v|^2 \mathrm{d}x > 0$$

for all $v \in \mathscr{C}_{\bar{u}}$, $v \neq 0$. Then, there exist $\epsilon > 0$ and $\mu > 0$ such that, for all $u \in U_{ad}$ satisfying, together with Ψ_u,

$$\|u - \bar{u}\|^2_{L^2(\omega)} + \left|\!\left|\!\left|\Psi_u - \bar{\Psi}\right|\!\right|\!\right|^2 \leq \epsilon^2,$$

we have

$$J(\bar{\Psi}, \bar{u}) + \frac{\mu}{2}\left(\|u - \bar{u}\|^2_{L^2(\omega)} + \left|\!\left|\!\left|\Psi_u - \bar{\Psi}\right|\!\right|\!\right|^2\right) \leq J(\Psi_u, u).$$

Remark 2.16. Theorem 2.15 is a result of similar type as in ([9], Thm. 2.3). However seeing that it is an immediate consequence of ([9], Thm. 2.3) is not obvious. Indeed, when ([9], Thm. 2.3) is applied to optimal control of PDE, as it is done in ([9], Sects. 3–5), this requires further developments.

Proof of Theorem 2.15. Here, we follow the lines of the proof in ([10], Thm. 3.8). However, since we need the $H^{2+\gamma}(\Omega)$ regularity for the state variables and some passages to the limit in equations are different from those in ([10], Thm. 3.8), we repeat the main steps of the proof for the convenience of the reader.

We argue by contradiction. Let $((\Psi_k, u_k))_k$ be a sequence satisfying (2.1b) with $u_k \in U_{ad}$, such that

$$\|u_k - \bar{u}\|^2_{L^2(\omega)} + \left|\!\left|\!\left|\Psi_k - \bar{\Psi}\right|\!\right|\!\right|^2 \leq \frac{1}{k^2} \tag{2.28}$$

and

$$J(\bar{\Psi}, \bar{u}) + \frac{1}{k}\left(\|u_k - \bar{u}\|^2_{L^2(\omega)} + \left\|\!\left|\Psi_k - \bar{\Psi}\right|\!\right\|^2\right) > J(\Psi_k, u_k). \tag{2.29}$$

Set

$$\rho_k = \sqrt{\|u_k - \bar{u}\|^2_{L^2(\omega)} + \left\|\!\left|\Psi_k - \bar{\Psi}\right|\!\right\|^2}, \quad v_k = \frac{u_k - \bar{u}}{\rho_k},\ \mathbf{z}_k = \frac{\Psi_k - \bar{\Psi}}{\rho_k}.$$

Note that $(\Psi_k)_k$ is bounded in $\mathbf{H}^{2+\gamma}(\Omega)$, see (2.13b). Clearly, $\|v_k\|^2_{L^2(\omega)} + \left\|\!\left|\mathbf{z}_k\right|\!\right\|^2 = 1$ and the pair $(\mathbf{z}_k, v_k)$ satisfies the equation

$$\mathcal{A}\mathbf{z}_k + \frac{1}{2}\mathcal{B}'(\bar{\Psi})\mathbf{z}_k + \frac{1}{2}\mathcal{B}'(\Psi_k)\mathbf{z}_k = \mathbf{C}\mathbf{v}_k \ \text{ in } \mathbf{V}'. \tag{2.30}$$

Following the proof of Lemma 2.9, we can verify that $\left\|\!\left|\mathbf{z}_k\right|\!\right\|_{2+\gamma} \le C\|v_k\|_{L^2(\omega)}$, with a constant C independent of k. By passing to the limit (up to a subsequence) in (2.30), we can prove that

$$\mathbf{z}_k \rightharpoonup \mathbf{z} \text{ in } \mathbf{H}^{2+\gamma}(\Omega),\ \mathbf{z}_k \to \mathbf{z} \text{ in } \mathbf{V}, v_k \rightharpoonup v \text{ in } L^2(\omega),$$

and $\mathbf{z} = \bar{\mathbf{z}}_v$, that is, $\mathbf{z}$ is the solution of (2.16) associated with v, and for $\Psi_u = \Psi_{\bar{u}}$.

Now we verify that $v \in \mathscr{C}_{\bar{u}}$. With (2.29), we have

$$\begin{aligned}\frac{\rho_k}{k} &> \frac{J(\bar{\Psi} + \rho_k\mathbf{z}_k, \bar{u} + \rho_k v_k) - J(\bar{\Psi}, \bar{u})}{\rho_k}\\ &= \frac{1}{2}\int_\Omega \left(2(\bar{\Psi} - \Psi_d) + \rho_k\mathbf{z}_k\right)\cdot\mathbf{z}_k \mathrm{d}x + \frac{\alpha}{2}\int_\omega \left(2\bar{u} + \rho_k v_k\right) v_k \mathrm{d}x.\end{aligned}$$

By passing to the limit as $k \to \infty$ and using (2.28), we obtain

$$\int_\Omega (\bar{\Psi} - \Psi_d)\cdot\mathbf{z}_v \mathrm{d}x + \alpha\int_\omega \bar{u}v\mathrm{d}x \le 0,$$

which yields $\int_\omega \bar{d}(x)v(x)\mathrm{d}x \le 0$. The last condition implies that $v \in \mathscr{C}_{\bar{u}}$.

Making a second order Taylor expansion of J at $(\bar{\Psi}, \bar{u})$, we have

$$\begin{aligned}J(\Psi_k, u_k) =& J(\bar{\Psi}, \bar{u}) + \partial_\Psi J(\bar{\Psi}, \bar{u})\,\rho_k\mathbf{z}_k + \partial_u J(\bar{\Psi}, \bar{u})\,\rho_k v_k\\ &+ \frac{1}{2}\int_\Omega |\Psi_k - \bar{\Psi}|^2\mathrm{d}x + \frac{\alpha}{2}\int_\omega |u_k - \bar{u}|^2\mathrm{d}x.\end{aligned}$$

Thus with (2.29), we can write

$$\frac{1}{\rho_k}\left(\partial_\Psi J(\bar{\Psi}, \bar{u})\mathbf{z}_k + \partial_u J(\bar{\Psi}, \bar{u})v_k\right) + \frac{1}{2}\int_\Omega |\mathbf{z}_k|^2\mathrm{d}x + \frac{\alpha}{2}\int_\omega |v_k|^2\mathrm{d}x < \frac{1}{k}. \tag{2.31}$$

Also,

$$\partial_\Psi J(\bar{\Psi}, \bar{u})\mathbf{z}_k + \partial_u J(\bar{\Psi}, \bar{u})v_k = \int_\Omega (\bar{\Psi} - \Psi_d)\cdot\mathbf{z}_k\mathrm{d}x + \alpha\int_\omega \bar{u}v_k\mathrm{d}x,$$

and using the adjoint state $\bar{\Theta}$, we obtain

$$\int_\Omega (\bar{\Psi} - \Psi_d) \cdot \mathbf{z}_k \mathrm{d}x = \int_\Omega \bar{\Theta} \cdot \mathbf{C}\mathbf{v}_k \mathrm{d}x - \frac{1}{2}\langle \mathcal{B}'(\mathbf{z}_k)(\Psi_k - \bar{\Psi}), \bar{\Theta}\rangle.$$

Thus,

$$\frac{1}{\rho_k}\left(\partial_\Psi J(\bar{\Psi}, \bar{u})\mathbf{z}_k + \partial_u J(\bar{\Psi}, \bar{u})v_k\right) = \frac{1}{\rho_k}\int_\Omega \bar{d}(x) v_k \mathrm{d}x - \frac{1}{2}\langle \mathcal{B}'(\mathbf{z}_k)\mathbf{z}_k, \bar{\Theta}\rangle.$$

Since $\bar{d}(x)v_k(x) \geq 0$, with (2.31), we have

$$-\langle \mathcal{B}'(\mathbf{z}_k)\mathbf{z}_k, \bar{\Theta}\rangle + \int_\Omega |\mathbf{z}_k|^2 \mathrm{d}x + \alpha \int_\omega |v_k|^2 \mathrm{d}x < \frac{2}{k}.$$

By passing to the inferior limit, we have

$$\int_\Omega \left([[\mathbf{z}, \mathbf{z}]] \cdot \bar{\Theta} + |\mathbf{z}|^2\right) \mathrm{d}x + \alpha \int_\omega |v|^2 \mathrm{d}x \leq 0.$$

Since $v \in \mathscr{C}_{\bar{u}}$ and due to our assumption about the **sufficient** second order optimality condition, we have $v = 0$. Since $(v_k)_k$ converges to 0 weakly in $L^2(\omega)$, $\mathbf{z} = 0$. By passing to the superior limit in the inequality satisfied by $(\mathbf{z}_k, v_k)$, we have

$$\alpha \limsup_{k\to\infty} \int_\omega |v_k|^2 \mathrm{d}x \leq 0.$$

Thus $\lim_{k\to\infty} \int_\omega |v_k|^2 \mathrm{d}x = 0$, and we have a contradiction with $\|v_k\|^2_{L^2(\omega)} + \|\mathbf{z}_k\|^2_{L^2(\Omega)} = 1$. The proof is complete. □

Note that the second order optimality condition

$$\int_\Omega \left(|\bar{\mathbf{z}}_v|^2 + [[\bar{\mathbf{z}}_v, \bar{\mathbf{z}}_v]] \cdot \bar{\Theta}\right) \mathrm{d}x + \alpha \int_\omega |v|^2 \mathrm{d}x > 0,$$

for all $v \in \mathcal{C}_{\bar{u}}$ is equivalent to $j''(\bar{u})v^2 > 0 \quad \forall v \in \mathscr{C}_{\bar{u}}$.

As in [10], we reinforce the above condition by assuming that

$$j''(\bar{u})v^2 > \delta\left(\|v\|^2_{L^2(\omega)} + \|\bar{\mathbf{z}}_v\|^2_{L^2(\Omega)}\right) \quad \forall v \in \mathscr{C}^\tau_{\bar{u}}, \tag{2.32}$$

where

$$\mathscr{C}^\tau_{\bar{u}} := \left\{v \in L^2(\omega) : (2.33) \text{ is satisfied}\right\},$$

with

$$\begin{cases} v(x) = 0 \text{ if } |d(x)| > \tau, \\ v(x) \geq 0 \text{ if } |d(x)| \leq \tau \text{ and } \bar{u}(x) = u_a, \\ v(x) \leq 0 \text{ if } |d(x)| \leq \tau \text{ and } \bar{u}(x) = u_b, \end{cases} \tag{2.33}$$

and $\bar{\mathbf{z}}_v$ is the solution of (2.16) with $u = \bar{u}$.

Theorem 2.17 (Thm. 3.10 of [10]). *The condition* (2.27) *is equivalent to* (2.32).

3. Discretization of state and adjoint variables

In this section, first of all, we describe the discretization of the state variable using conforming finite elements. This is followed by definition of an auxiliary discrete problem corresponding to the state equation for a given control $u \in U_{ad}$. We establish the existence of a unique solution and error estimates for this problem under suitable assumptions. Similar results for an auxiliary problem corresponding to the adjoint variable are proved next.

3.1. Conforming finite elements

Let $\mathcal{T}_h$ be a regular, conforming and quasi-uniform triangulation of $\overline{\Omega}$ into closed triangles, rectangles or quadrilaterals. Set $h_T = diam(T)$, $T \in \mathcal{T}_h$ and define the discretization parameter $h := \max_{T\in\mathcal{T}_h} h_T$. We now provide examples of two conforming finite elements defined on a triangle and a rectangle, namely the Argyris and Bogner-Fox-Schmit elements (see Fig. 1).

Definition 3.1 (Argyris element [5, 11]). The Argyris element is a triplet $(T, P_5(T), \Sigma_T)$ where T is a triangle, $P_5(T)$ denotes polynomials of degree ≤ 5 in both the variables and the set of 21 degrees of freedom Σ_T is determined by the values of the unknown functions, its first order and second order derivatives at the three vertices and the normal derivatives at the midpoints of the three edges of T (see Fig. 1A).

Definition 3.2 (Bogner-Fox-Schmit element [11]). Let $T \in \mathcal{T}_h$ be a rectangle with vertices $a_i = (x_i, y_i)$, $i = 1, 2, 3, 4$. The Bogner-Fox-Schmit element is a triplet $(T, Q_3(T), \Sigma_T)$, where $Q_3(T)$ denotes polynomials of degree ≤ 3 in both the variables and the set of degrees of freedom Σ_T is defined by $\Sigma_T = \{p(a_i), \frac{\partial p}{\partial x}(a_i), \frac{\partial p}{\partial y}(a_i), \frac{\partial^2 p}{\partial x \partial y}(a_i),\ 1 \leq i \leq 4\}$ (see Fig. 1B).

The conforming C^1 finite element spaces associated with Argyris and Bogner-Fox-Schmit elements are contained in $C^1(\overline{\Omega}) \cap H^2(\Omega)$. Define

$$V_h = \left\{ v \in C^1(\overline{\Omega}) : v|_T \in P_T \quad \forall\, T \in \mathcal{T}_h \text{ with } v|_{\partial\Omega} = 0, \frac{\partial v}{\partial \nu}\big|_{\partial\Omega} = 0 \right\} \subset H_0^2(\Omega),$$

where

$$P_T = \begin{cases} P_5(T) & \text{for Argyris element,} \\ Q_3(T) & \text{for Bogner-Fox-Schmit element.} \end{cases}$$

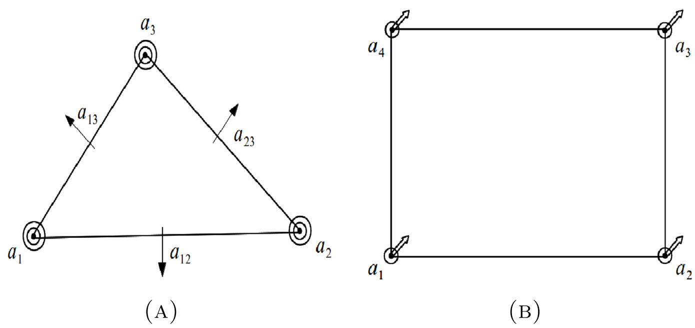

Figure 1. (A) Argyris element and (B) Bogner-Fox-Schmit element.

The discrete state and adjoint variables are sought in the finite dimensional space defined by $\mathbf{V}_h := V_h \times V_h$.

Lemma 3.3 (Interpolant [11])**.** *Let $\Pi_h : V \longrightarrow V_h$ be the Argyris or Bogner-Fox-Schmit nodal interpolation operator. Then for $\varphi \in H^{2+\gamma}(\Omega)$, with $\gamma \in (\frac{1}{2}, 1]$ denoting the index of elliptic regularity, it holds:*

$$\|\varphi - \Pi_h \varphi\|_m \leq C h^{2+\gamma-m} \|\varphi\|_{2+\gamma} \quad \text{for } m = 0, 1, 2; \tag{3.1}$$

Also, if $\varphi \in H^l(\Omega)$ for $l = 4, 5, 6$,

$$\|\varphi - \Pi_h \varphi\|_m \leq C h^{\min\{k+1,\, l\}-m} \|\varphi\|_l \quad \text{for } m = 0, 1, 2,$$

where $k = 5$ (resp. 3) for the Argyris element (resp. Bogner-Fox-Schmit element).

3.2. Auxiliary problems for the state equations

Define an auxiliary continuous problem associated with the state equation as follows:
Seek $\Psi_u \in \mathbf{V}$ such that

$$A(\Psi_u, \Phi) + B(\Psi_u, \Psi_u, \Phi) = (F + \mathbf{C}\mathbf{u}, \Phi) \quad \forall \Phi \in \mathbf{V}, \tag{3.2}$$

where $\mathbf{u} = (u, 0)$, $u \in L^2(\omega)$ is given.

A discrete conforming finite element approximation for this problem can be defined as:
Seek $\Psi_{u,h} \in \mathbf{V}_h$ such that

$$A(\Psi_{u,h}, \Phi_h) + B(\Psi_{u,h}, \Psi_{u,h}, \Phi_h) = (F + \mathbf{C}\mathbf{u}, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h. \tag{3.3}$$

For a given $u \in L^2(\omega)$, (3.3) is not well-posed in general. The main results of this subsection are stated now.

Theorem 3.4. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9). Then, there exist $\rho_1, \rho_2 > 0$ and $h_1 > 0$ such that, for all $0 < h < h_1$ and $u \in B_{\rho_2}(\bar{u})$, (3.3) admits a unique solution in $B_{\rho_1}(\bar{\Psi})$.*

Remark 3.5. For $\rho > 0$, $u \in B_\rho(\bar{u})$ means that $\|u - \bar{u}\|_{L^2(\omega)} \leq \rho$. Similarly, $\Psi \in B_\rho(\bar{\Psi}) \Longrightarrow \left\|\!\left|\Psi - \bar{\Psi}\right|\!\right\|_2 \leq \rho$.

Theorem 3.6. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9). Let h_1 and ρ_2 be defined as in Theorem 3.4. Then, for $u \in B_{\rho_2}(\bar{u})$ and $0 < h < h_1$, the solutions Ψ_u and $\Psi_{u,h}$ of (3.2) and (3.3) satisfy the error estimates:*

$$(a)\ \left\|\!\left|\Psi_u - \Psi_{u,h}\right|\!\right\|_2 \leq C h^\gamma \quad (b)\ \left\|\!\left|\Psi_u - \Psi_{u,h}\right|\!\right\|_1 \leq C h^{2\gamma}, \tag{3.4}$$

where $\gamma \in (1/2, 1]$ denotes the index of elliptic regularity.

We proceed to establish several results which will be essential to prove Theorem 3.4. The proof of Theorem 3.6 follows from the error estimates for the approximation of von Kármán equations using conforming finite element methods; see [7, 24].

3.2.1. An auxiliary linear problem and discretization

For a given $\mathbf{g} = (g_1, g_2) \in \mathbf{V}'$, let $T \in \mathcal{L}(\mathbf{V}', \mathbf{V})$ be defined by $T\mathbf{g} := \boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbf{V}$ where $\boldsymbol{\xi}$ solves the system of biharmonic equations given by:

$$\Delta^2 \xi_1 = g_1 \quad \text{in } \Omega, \tag{3.5a}$$

$$\Delta^2 \xi_2 = g_2 \quad \text{in } \Omega, \tag{3.5b}$$

$$\xi_1 = 0,\ \frac{\partial \xi_1}{\partial \nu} = 0 \text{ and } \xi_2 = 0,\ \frac{\partial \xi_2}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{3.5c}$$

Equivalently, $\boldsymbol{\xi} \in \mathbf{V}$ solves $A(\boldsymbol{\xi}, \Phi) = \langle \mathbf{g}, \Phi \rangle_{\mathbf{V}',\mathbf{V}} \quad \forall \Phi \in \mathbf{V}$.

Also, let $T_h \in \mathcal{L}(\mathbf{V}', \mathbf{V}_h)$ be defined by $T_h\mathbf{g} := \boldsymbol{\xi}_h$, where $\boldsymbol{\xi}_h \in \mathbf{V}_h$ solves the discrete problem

$$A(\boldsymbol{\xi}_h, \Phi_h) = \langle \mathbf{g}, \Phi_h \rangle_{\mathbf{V}',\mathbf{V}} \quad \forall \Phi_h \in \mathbf{V}_h. \tag{3.6}$$

Lemma 3.7 (A bound for T_h)**.** *There exists a constant $C > 0$, independent of h, such that*

$$\|T_h\|_{\mathcal{L}(\mathbf{V}',\mathbf{V})} \leq C.$$

Proof. The definition of $T_h\mathbf{g}$ along with coercivity property of the bilinear form $A(\cdot,\cdot)$ lead to the required result. □

Lemma 3.8 (Error estimates [5])**.** *Let $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_h$ solve* (3.5) *and* (3.6) *respectively. Then it holds:*

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_2 \leq Ch^{\gamma} \|\mathbf{g}\|_{-1} \quad \forall \mathbf{g} \in \mathbf{H}^{-1}(\Omega), \tag{3.7a}$$

$$\|\boldsymbol{\xi} - \boldsymbol{\xi}_h\|_1 \leq Ch^{2\gamma} \|\mathbf{g}\|_{-1} \quad \forall \mathbf{g} \in \mathbf{H}^{-1}(\Omega), \tag{3.7b}$$

$\gamma \in (1/2, 1]$ being the index of elliptic regularity. That is, $\|(T - T_h)\mathbf{g}\|_2 \leq Ch^{\gamma} \|\mathbf{g}\|_{-1}$ and $\|(T - T_h)\mathbf{g}\|_1 \leq Ch^{2\gamma} \|\mathbf{g}\|_{-1}$.

Remark 3.9. When $\mathbf{g} = \begin{pmatrix} g \\ 0 \end{pmatrix}$, we denote $T\mathbf{g}$ (resp. $T_h\mathbf{g}$) as Tg (resp. T_hg), purely for notational convenience.

3.2.2. A nonlinear mapping and its properties

Define a nonlinear mapping $\mathscr{N} : \mathbf{V} \times L^2(\omega) \to \mathbf{V}$ by

$$\mathscr{N}(\Psi, u) := \Psi + T[\mathcal{B}(\Psi) - (F + \mathbf{Cu})], \quad \mathbf{u} = (u, 0).$$

Now $\mathscr{N}(\Psi, u) = 0$ if and only if (Ψ, u) solves (2.9); that is, $\mathcal{A}\Psi + \mathcal{B}(\Psi) = F + \mathbf{Cu}$ in $\mathbf{V}'$.

Similarly, define a nonlinear mapping $\mathscr{N}_h : \mathbf{V} \times L^2(\omega) \to \mathbf{V}$ by

$$\mathscr{N}_h(\Psi, u) := \Psi + T_h[\mathcal{B}(\Psi) - (F + \mathbf{Cu})], \quad \mathbf{u} = (u, 0).$$

Note that, $\mathscr{N}_h(\Psi, u) = 0$ if and only if $\Psi \in \mathbf{V}_h$ and $\Psi = \Psi_{u,h}$ solves (3.3).

The derivative mapping $\partial_\Psi \mathscr{N}(\Psi, u)$ (resp. $\partial_\Psi \mathscr{N}_h(\Psi, u)) \in \mathcal{L}(\mathbf{V})$ is defined by

$$\partial_\Psi \mathscr{N}(\Psi, u)(\Phi) = \Phi + T[\mathcal{B}'(\Psi)\Phi] \quad \forall \Phi \in \mathbf{V}.$$

$$\big(\text{resp. } \partial_\Psi \mathscr{N}_h(\Psi, u)(\Phi) = \Phi + T_h[\mathcal{B}'(\Psi)\Phi] \quad \forall \Phi \in \mathbf{V}.\big)$$

With definitions of nonsingular solution (see Def. 2.4), the linear mapping T and the derivative mapping $\partial_\Psi \mathscr{N}(\Psi, u)$, we obtain the following result, the proof of which is skipped.

Lemma 3.10 ([10])**.** *If $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ is a nonsingular solution of* (2.1b)*, then $\partial_\Psi \mathscr{N}(\bar{\Psi}, \bar{u})$ is an automorphism in $\mathbf{V}$. The converse also holds true.*

We want to establish that if $(\bar{\Psi},\bar{u})$ is a nonsingular solution, then the derivative mapping $\partial_\Psi \mathscr{N}_h(\cdot,\cdot)$ is an automorphism in $\mathbf{V}$, with respect to small perturbations of its arguments. That is, if $\left\|\Psi-\bar{\Psi}\right\|_2$ and $\|u-\bar{u}\|_{L^2(\omega)}$ are small enough, then $\partial_\Psi \mathscr{N}_h(\Psi,u)$ is an automorphism in $\mathbf{V}$. The next two lemmas will be useful in proving this result.

Lemma 3.11. *Let $\bar{\Psi}\in\mathbf{V}$ be a nonsingular solution of (2.9). Then, $\forall\epsilon>0$, $\exists h_\epsilon>0$ such that*

$$\|T[\mathcal{B}'(\bar{\Psi})]-T_h[\mathcal{B}'(\Psi)]\|_{\mathcal{L}(\mathbf{V})}<\epsilon \quad \forall\Psi\in B_{\rho_\epsilon}(\bar{\Psi}), \tag{3.8}$$

whenever $0<h<h_\epsilon$.

Proof. For a fixed $\mathbf{z}\in\mathbf{V}$, let $T[\mathcal{B}'(\bar{\Psi})\mathbf{z}]=:\boldsymbol{\theta}(\bar{\Psi})\in\mathbf{V}$ and $T_h[\mathcal{B}'(\Psi)\mathbf{z}]=:\boldsymbol{\theta}_h(\Psi)\in\mathbf{V}_h$. Then $\boldsymbol{\theta}(\bar{\Psi})$ and $\boldsymbol{\theta}_h(\Psi)$, respectively solve

$$A(\boldsymbol{\theta}(\bar{\Psi}),\Phi)=\left\langle\mathcal{B}'(\bar{\Psi})\mathbf{z},\Phi\right\rangle_{\mathbf{V}',\mathbf{V}}=2B(\bar{\Psi},\mathbf{z},\Phi) \quad \forall\Phi\in\mathbf{V}, \tag{3.9}$$

$$A(\boldsymbol{\theta}_h(\Psi),\Phi_h)=\left\langle\mathcal{B}'(\Psi)\mathbf{z},\Phi_h\right\rangle_{\mathbf{V}',\mathbf{V}}=2B(\Psi,\mathbf{z},\Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h. \tag{3.10}$$

Let $\boldsymbol{\theta}_h(\bar{\Psi})\in\mathbf{V}_h$ be the solution to the intermediate problem defined by

$$A(\boldsymbol{\theta}_h(\bar{\Psi}),\Phi_h)=\left\langle\mathcal{B}'(\bar{\Psi})\mathbf{z},\Phi_h\right\rangle_{\mathbf{V}',\mathbf{V}}=2B(\bar{\Psi},\mathbf{z},\Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h. \tag{3.11}$$

The triangle inequality yields

$$\left\|\boldsymbol{\theta}(\bar{\Psi})-\boldsymbol{\theta}_h(\Psi)\right\|_2\le\left\|\boldsymbol{\theta}(\bar{\Psi})-\boldsymbol{\theta}_h(\bar{\Psi})\right\|_2+\left\|\boldsymbol{\theta}_h(\bar{\Psi})-\boldsymbol{\theta}_h(\Psi)\right\|_2. \tag{3.12}$$

To estimate the first term in the right hand side of (3.12), consider (3.9) and (3.11); use the facts that $\mathbf{V}_h\subset\mathbf{V}$, the error $(\boldsymbol{\theta}(\bar{\Psi})-\boldsymbol{\theta}_h(\bar{\Psi}))$ is orthogonal to $\mathbf{V}_h$ in the energy norm, the coercivity of $A(\cdot,\cdot)$, the interpolation estimate given in Lemma 3.3 and the fact that $\bar{\Psi}\in\mathbf{H}^{2+\gamma}(\Omega)$ to obtain

$$\left\|\boldsymbol{\theta}(\bar{\Psi})-\boldsymbol{\theta}_h(\bar{\Psi})\right\|_2\le Ch^\gamma\left\|\boldsymbol{\theta}(\bar{\Psi})\right\|_{2+\gamma}\le Ch^\gamma\left\|\mathcal{B}'(\bar{\Psi})\mathbf{z}\right\|_{-1}. \tag{3.13}$$

From definition of $\mathcal{B}'(\bar{\Psi})\mathbf{z}$, (2.6) and the fact that $B(\cdot,\cdot,\cdot)$ is symmetric in first and second variables, it follows that

$$\begin{aligned}\left\|\mathcal{B}'(\bar{\Psi})\mathbf{z}\right\|_{-1}&=\sup\frac{\left|\left\langle\mathcal{B}'(\bar{\Psi})\mathbf{z},\Phi\right\rangle\right|}{\|\Phi\|_1}\\&\le\sup\frac{|2B(\bar{\Psi},\mathbf{z},\Phi)|}{\|\Phi\|_1}\\&\lesssim\left\|\bar{\Psi}\right\|_{2+\gamma}\|\mathbf{z}\|_2.\end{aligned} \tag{3.14}$$

A substitution of (3.14) in (3.13) leads to

$$\left\|\boldsymbol{\theta}(\bar{\Psi})-\boldsymbol{\theta}_h(\bar{\Psi})\right\|_2\le Ch^\gamma\left\|\bar{\Psi}\right\|_2\|\mathbf{z}\|_2. \tag{3.15}$$

To estimate the second term on the right hand side of (3.12), subtract (3.10) and (3.11), choose $\Phi_h=\boldsymbol{\theta}_h(\bar{\Psi})-\boldsymbol{\theta}_h(\Psi)$, use (2.4) and (2.5) to obtain

$$\left\|\boldsymbol{\theta}_h(\bar{\Psi})-\boldsymbol{\theta}_h(\Psi)\right\|_2\le C\left\|\bar{\Psi}-\Psi\right\|_2\|\mathbf{z}\|_2. \tag{3.16}$$

A use of (3.15) and (3.16) in (3.12) leads to the required result, when h_ϵ and ρ_ϵ are chosen sufficiently small. □

The next lemma is a standard result in Banach spaces and hence we refrain from providing a proof.

Lemma 3.12. *Let X be a Banach space, $A \in \mathcal{L}(X)$ be invertible and $B \in \mathcal{L}(X)$. If $\|A - B\|_{\mathcal{L}(X)} < 1/\|A^{-1}\|_{\mathcal{L}(X)}$, then B is invertible. If $\|A - B\|_{\mathcal{L}(X)} < 1/(2\|A^{-1}\|_{\mathcal{L}(X)})$ then $\|B^{-1}\|_{\mathcal{L}(X)} \leq 2\|A^{-1}\|_{\mathcal{L}(X)}$.*

The following theorem is a consequence of Lemmas 3.11 and 3.12.

Theorem 3.13 (Invertibility of $\partial_\Psi \mathscr{N}_h(\Psi, u)$)**.** *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.1b). Then, there exist $h_0 > 0$ and $\rho_0 > 0$ such that, for all $0 < h < h_0$ and all $\Psi \in B_{\rho_0}(\bar{\Psi})$, $\mathscr{N}_h(\Psi, u)$ is an automorphism in $\mathbf{V}$ and*

$$\|\partial_\Psi \mathscr{N}_h(\Psi, u)^{-1}\|_{\mathcal{L}(\mathbf{V})} \leq 2\|\partial_\Psi \mathscr{N}(\bar{\Psi}, \bar{u})^{-1}\|_{\mathcal{L}(\mathbf{V})}.$$

Proof. Choose $h_0 := h_\epsilon,\ \rho_0 := \rho_\epsilon$ and $\epsilon = \frac{1}{2\|\partial_\Psi \mathscr{N}(\bar{\Psi}, \bar{u})^{-1}\|_{\mathcal{L}(\mathbf{V})}}$ in Lemma 3.11.

For every $0 < h < h_0$ and for all $\Psi \in B_{\rho_0}(\bar{\Psi})$, the definitions of the derivatives of $\partial_\Psi \mathscr{N}$, $\partial_\Psi \mathscr{N}_h$ and (3.8) yield

$$\|\partial_\Psi \mathscr{N}(\bar{\Psi}, \bar{u}) - \partial_\Psi \mathscr{N}_h(\Psi, u)\|_{\mathcal{L}(\mathbf{V})} = \|T[\mathcal{B}'(\bar{\Psi})] - T_h[\mathcal{B}'(\Psi)]\|_{\mathcal{L}(\mathbf{V})} < \epsilon.$$

Now, an application of Lemma 3.12 yields the required result. □

We now proceed to provide a proof Theorem 3.4, which is the main result of this subsection.

Proof of Theorem 3.4. We follow the lines of the proof in ([10], Thm. 4.8). But due to our estimates the factor h in the majorizations is replaced by h^γ.

Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9).

We need to establish that there exist $\rho_1, \rho_2 > 0$ and $h_1 > 0$ such that, for all $0 < h < h_1$ and $u \in B_{\rho^2}(\bar{u})$, $\mathscr{N}_h(\Psi, u) = 0$ admits a unique solution in $B_{\rho_1}(\bar{\Psi})$.

Let ρ_0 and h_0 be the positive constants as defined in Theorem 3.13. For $\rho \leq \rho_0,\ h \leq h_0$ and $u \in B_{\rho^2}(\bar{u})$, define a mapping $\mathscr{G}(\cdot, u) : B_\rho(\bar{\Psi}) \to \mathbf{V}$ by

$$\mathscr{G}(\Psi, u) = \Psi - [\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})]^{-1} \mathscr{N}_h(\Psi, u).$$

Any fixed point of $\mathscr{G}(\cdot, u)$ is a solution of the discrete nonlinear problem $\mathscr{N}_h(\Psi, u) = 0$. In the next two steps, we establish that (i) $\mathscr{G}(\cdot, u)$ maps $B_\rho(\bar{\Psi})$ into itself; and (ii) $\mathscr{G}(\cdot, u)$ is a strict contraction, if ρ is small enough.

Step 1: The definition of $\mathscr{G}(\cdot, u)$, an addition of the zero term $\mathscr{N}(\bar{\Psi}, \bar{u})$, an addition and subtraction of an intermediate term and the Taylor's Theorem yield

$$\begin{aligned}
\left|\!\left|\!\left|\mathscr{G}(\Psi, u) - \bar{\Psi}\right|\!\right|\!\right|_2 &= \left|\!\left|\!\left|(\Psi - \bar{\Psi}) - [\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})]^{-1} \mathscr{N}_h(\Psi, u)\right|\!\right|\!\right|_2 \\
&\leq \left|\!\left|\!\left|[\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})]^{-1} \left\{[\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})](\Psi - \bar{\Psi}) - [\mathscr{N}_h(\Psi, u) - \mathscr{N}_h(\bar{\Psi}, u)]\right\}\right|\!\right|\!\right|_2 \\
&\quad + \left|\!\left|\!\left|[\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})]^{-1} [\mathscr{N}(\bar{\Psi}, \bar{u}) - \mathscr{N}_h(\bar{\Psi}, u)]\right|\!\right|\!\right|_2.
\end{aligned} \tag{3.17}$$

A use of Theorem 3.13, Taylor formula for the second expression in the first term of the right hand side of (3.17) along with the fact that the expression for the derivative ∂_Ψ is independent of u yields for $\Psi_t = \bar{\Psi} + t(\Psi - \bar{\Psi}),\ 0 < t < 1$,

$$\begin{aligned}
\left|\!\left|\!\left|\mathscr{G}(\Psi, u) - \bar{\Psi}\right|\!\right|\!\right|_2 &\leq C \left|\!\left|\!\left|\partial_\Psi \mathscr{N}_h(\bar{\Psi}, \bar{u})](\Psi - \bar{\Psi}) - \int_0^1 \partial_\Psi \mathscr{N}_h(\Psi_t, u)(\Psi - \bar{\Psi})\right|\!\right|\!\right|_2 \\
&\quad + C \left|\!\left|\!\left|[\mathscr{N}(\bar{\Psi}, \bar{u}) - \mathscr{N}_h(\bar{\Psi}, u)]\right|\!\right|\!\right|_2.
\end{aligned}$$

With definitions of $\mathscr{N}(\cdot,\cdot)$ and $\mathscr{N}_h(\cdot,\cdot)$, and the triangle inequality in the above expression, we obtain

$$\begin{aligned}\left\|\left\|\mathscr{G}(\Psi,u)-\bar{\Psi}\right\|\right\|_2 \leq & C\int_0^1 \left\|\left\|\partial_\Psi \mathscr{N}_h(\bar{\Psi},\bar{u})] - \partial_\Psi \mathscr{N}_h(\Psi_t,u)\right\|\right\|_2 \mathrm{d}t \times \left\|\left\|\Psi-\bar{\Psi}\right\|\right\|_2 \\ & + C\left\|\left\|(T-T_h)(\mathcal{B}(\bar{\Psi})-F)\right\|\right\|_2 + C\left\|\left\|(T-T_h)(\mathbf{C}\bar{\mathbf{u}})\right\|\right\|_2 \\ & + C\left\|\left\|T_h(\mathbf{C}\bar{\mathbf{u}}-\mathbf{C}\mathbf{u})\right\|\right\|_2 =: T_1+T_2+T_3+T_4. \end{aligned} \tag{3.18}$$

We now estimate the terms T_1 to T_4. With the definition of $\partial_\Psi \mathscr{N}_h(\cdot,\cdot)$, Lemma 3.7, the definition of $\mathcal{B}'(\cdot)$ and (2.5), it yields

$$\begin{aligned}\|\partial_\Psi \mathscr{N}_h(\bar{\Psi},\bar{u}) - \partial_\Psi \mathscr{N}_h(\bar{\Psi}+t(\Psi-\bar{\Psi}),u)\|_{\mathcal{L}(V)} & = \|T_h(\mathcal{B}'(\bar{\Psi}+t(\Psi-\bar{\Psi}))) - T_h(\mathcal{B}'(\bar{\Psi}))\|_{\mathcal{L}(V)} \\ & \leq C\left\|\left\|\Psi-\bar{\Psi}\right\|\right\|_2. \end{aligned} \tag{3.19}$$

A use of the facts that $\bar{\Psi}\in \mathbf{H}^{2+\gamma}(\Omega)$, $\mathcal{B}(\bar{\Psi})\in \mathbf{H}^{-1}(\Omega)$, $f\in H^{-1}(\Omega)$, and $F=\begin{pmatrix} f \\ 0\end{pmatrix}$, along with (3.7) lead to an estimate for T_2 as

$$\left\|\left\|(T-T_h)(\mathcal{B}(\bar{\Psi})-F)\right\|\right\|_2 \leq Ch^\gamma(\left\|\left\|\bar{\Psi}\right\|\right\|_2^2 + \|f\|_{H^{-1}(\Omega)})), \tag{3.20}$$

where $\gamma\in(1/2,1]$ is the elliptic regularity index. Since $\bar{u}\in L^2(\omega)$, T_3 can also be estimated using (3.7) as

$$\left\|\left\|(T-T_h)(\mathbf{C}\bar{\mathbf{u}})\right\|\right\|_2 \leq Ch^\gamma\|\bar{u}\|_{L^2(\omega)}. \tag{3.21}$$

The boundedness of T_h from Lemma 3.7 leads to

$$\left\|\left\|T_h(\mathbf{C}\bar{\mathbf{u}}-\mathbf{C}\mathbf{u})\right\|\right\|_2 \leq C\left\|\left\|\mathcal{C}\bar{u}-\mathcal{C}u\right\|\right\| \leq C\|\bar{u}-u\|_{L^2(\omega)}. \tag{3.22}$$

The substitutions of (3.19)–(3.22) in (3.18) yield

$$\left\|\left\|\mathscr{G}(\Psi,u)-\bar{\Psi}\right\|\right\|_2 \leq \hat{C}(h^\gamma+\rho^2).$$

Choose $\hat{\rho}_1 \leq \min\left\{\rho_0, \frac{1}{2\hat{C}}\right\}$, $\hat{\rho}_2=\hat{\rho}_1^2$, and $\hat{h}_1=\min\left\{h_0^{1/\gamma}, \left(\frac{\hat{\rho}_1}{2\hat{C}}\right)^{1/\gamma}\right\}$. For all $0<h<\hat{h}_1$ and all $u\in B_{\hat{\rho}_2}(\bar{u})$, $\mathscr{G}(\Psi,u)$ is a mapping from $B_{\hat{\rho}_1}(\bar{\Psi})$ into itself.

Step 2: Let $\Psi_1,\Psi_2\in B_{\hat{\rho}_1}(\bar{\Psi})$, $0<h<\hat{h}_1$ and $u\in B_{\hat{\rho}_2}(\bar{u})$. The definition of the mapping $\mathscr{G}(\cdot,u)$ and standard calculations lead to

$$\begin{aligned}&\left\|\left\|\mathscr{G}(\Psi_1,u)-\mathscr{G}(\Psi_2,u)\right\|\right\|_2 = \left\|\left\|\Psi_1-\Psi_2-[\partial_\Psi \mathscr{N}_h(\bar{\Psi},\bar{u})]^{-1}\left\{\mathscr{N}_h(\Psi_1,u)-\mathscr{N}_h(\Psi_2,u)\right\}\right\|\right\|_2 \\ &= \left\|\left\|[\partial_\Psi \mathscr{N}_h(\bar{\Psi},\bar{u})]^{-1}\left\{\partial_\Psi \mathscr{N}_h(\bar{\Psi},\bar{u})(\Psi_1-\Psi_2) - \int_0^1 \partial_\Psi \mathscr{N}_h(\Psi_2+t(\Psi_1-\Psi_2),u)(\Psi_1-\Psi_2)\mathrm{d}t\right\}\right\|\right\|_2. \end{aligned} \tag{3.23}$$

Now a use of Theorem 3.13 and a repetition of arguments used in (3.19) lead to the result that, there exists a positive constant $\tilde{C}$ independent of $\hat{\rho}_1$ and h such that

$$\left\|\left\|\mathscr{G}(\Psi_1,u)-\mathscr{G}(\Psi_2,u)\right\|\right\|_2 \leq \tilde{C}\hat{\rho}_1^2. \tag{3.24}$$

A choice of $\rho_1 = \min\left\{\rho_0, \frac{1}{2\hat{C}}, \frac{1}{\sqrt{2\tilde{C}}}\right\}$, $\rho_2 = \rho_1^2$, and $h_1 = \min\left\{h_0^{1/\gamma}, \left(\frac{\rho_1}{2\hat{C}}\right)^{1/\gamma}\right\}$ leads to the result that for all $0 < h < h_1$ and $u \in B_{\rho_2}(\bar{u})$, $\mathscr{G}(\cdot, u)$ is a strict contraction in $B_{\rho_1}(\bar{\Psi})$.

This concludes the proof of Theorem 3.4. □

We have established that, for $0 < h < h_1$, $u \in B_{\rho_2}(\bar{u})$, $\mathscr{N}_h(\Psi_h, u) = 0$ admits a unique solution $\Psi_{u,h} \in B_{\rho_1}(\bar{\Psi}) \cap \mathbf{V}_h$. Also, $\partial_\Psi \mathscr{N}_h(\Psi_{u,h}, u)$ is an automorphism in $\mathbf{V}$. Hence, the mapping $G_h : B_{\rho_2}(\bar{u}) \to B_{\rho_1}(\bar{\Psi}) \cap \mathbf{V}_h$ defined by $G_h(u) = \Psi_{u,h}$ satisfies $\mathscr{N}_h(G_h(u), u) = 0$. The implicit theorem yields that G_h is of class C^∞ in the interior of the ball.

This fact, along with Theorem 3.6 yields the following lemma.

Lemma 3.14. *For u, $\hat{u} \in B_{\rho_2}(\bar{u})$, $0 < h < h_1$, the solutions Ψ_u and $\Psi_{\hat{u},h}$ to (3.2) and (3.3), with controls chosen as u and $\hat{u}$ respectively, satisfy*

$$\|\Psi_u - \Psi_{\hat{u},h}\|_2 \leq C(h^\gamma + \|u - \hat{u}\|_{L^2(\omega)}),$$

$\gamma \in (1/2, 1]$ *being the elliptic regularity index.*

Proof. The triangle inequality yields

$$\|\Psi_u - \Psi_{\hat{u},h}\|_2 \leq \|\Psi_u - \Psi_{u,h}\|_2 + \|\Psi_{u,h} - \Psi_{\hat{u},h}\|_2 . \tag{3.25}$$

Theorem 3.6 yields the estimate for the first term on the right hand side of (3.25) as

$$\|\Psi_u - \Psi_{u,h}\|_2 \leq Ch^\gamma.$$

From the expression $\mathscr{N}_h(G_h(u), u) = 0$ and the definition of $\mathscr{N}_h$, we obtain

$$G_h'(u)(v) = [\partial_\Psi \mathscr{N}_h(\Psi_{u,h}, u)]^{-1} T_h(\mathbf{C}\mathbf{v}),$$

where $\mathbf{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}$ and u belongs to the interior of $B_{\rho_2}(\bar{u})$.

Hence Lemma 3.7 and Theorem 3.13 yield

$$\begin{aligned} \|\Psi_{u,h} - \Psi_{\hat{u},h}\|_2 &= \|G_h(u) - G_h(\hat{u})\|_2 \\ &= \left\| \int_0^1 [\partial_\Psi \mathscr{N}_h(\Psi_h(u_t), u_t)]^{-1} T_h(\mathbf{C}(\mathbf{u} - \hat{\mathbf{u}})) \right\|_2 \\ &\leq C\|u - \hat{u}\|_{L^2(\omega)}, \end{aligned} \tag{3.26}$$

with $u_t = \hat{u} + t(u - \hat{u})$. A substitution of the estimate in (3.25) yields the required result. □

3.3. Auxiliary discrete problem for the adjoint equations

Define an auxiliary continuous problem associated with the adjoint equations as follows:

Seek $\Theta_u \in \mathbf{V}$ such that

$$A(\Phi, \Theta_u) + 2B(\Psi_u, \Phi, \Theta_u) = (\Psi_u - \Psi_d, \Phi) \quad \forall \Phi \in \mathbf{V}, \tag{3.27}$$

where $\Psi_u \in \mathbf{V}$ is the solution of (3.2) and Ψ_d is given. A conforming finite element discretization for (3.27) is defined as:

Seek $\Theta_{u,h} \in \mathbf{V}_h$ such that

$$A(\Phi_h, \Theta_{u,h}) + 2B(\Psi_{u,h}, \Phi_h, \Theta_{u,h}) = (\Psi_{u,h} - \Psi_d, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h. \tag{3.28}$$

The main results of this subsection will be on the existence of solution of the discrete adjoint problem in (3.28) and its error estimates. They are stated now.

Theorem 3.15. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9). Then, there exist $0 < \rho_3 \leq \rho_2$ and $h_3 > 0$ such that, for all $0 < h \leq h_3$ and $u \in B_{\rho_3}(\bar{u})$, (3.28) admits a unique solution.*

Theorem 3.16. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9). Then, for $u \in B_{\rho_3}(\bar{u})$ and $0 < h < h_3$, the solutions Θ_u and $\Theta_{u,h}$ of (3.27) and (3.28) satisfy the error estimates:*

$$(a)\ \ \|\Theta_u - \Theta_{u,h}\|_2 \leq Ch^{\gamma} \quad (b)\ \ \|\Theta_u - \Theta_{u,h}\|_1 \leq Ch^{2\gamma}, \tag{3.29}$$

where $\gamma \in (1/2, 1]$ denotes the index of elliptic regularity.

3.3.1. A linear mapping and its properties

As in the case of the derivative mapping defined in the previous subsection for state equations, define the linear mapping $\mathscr{F}_{\Psi}$(resp. $\mathscr{F}_{\Psi,h}) \in \mathcal{L}(\mathbf{V})$ by

$$\mathscr{F}_{\Psi}(\Phi) = \Phi + T[\mathcal{B}'(\Psi)^*\Phi] \quad \forall \Phi \in \mathbf{V},$$

$$(\text{resp. } \mathscr{F}_{\Psi,h}(\Phi) = \Phi + T_h[\mathcal{B}'(\Psi)^*\Phi] \quad \forall \Phi \in \mathbf{V})$$

where $\mathcal{B}'(\Psi)^*$ is the adjoint operator corresponding to $\mathcal{B}'(\Psi)$ (see (2.11)).

The next lemma is easy to establish and hence the proof is skipped.

Lemma 3.17. *The mapping $\mathscr{F}_{\Psi}$ is an automorphism in $\mathbf{V}$ if and only if $\Psi \in \mathbf{V}$ is a nonsingular solution of (2.1b).*

Proof of Theorem 3.15. Proceeding as in the proof of Theorem 3.13, we can assume that h_0 is chosen so that, for all $0 < h < h_0$ and all $\Psi \in B_{\rho_0}(\bar{\Psi})$, $\mathscr{F}_{\Psi,h}$ is an automorphism in $\mathbf{V}$. In particular, by using Lemma 3.14, there exist $0 < h_3 \leq h_2$ and $0 < \rho_3 \leq \rho_2$ such that, for all $0 < h \leq h_3$ and all $u \in B_{\rho_3}(\bar{u})$, $\mathscr{F}_{\Psi_{u,h},h}$ is an automorphism in $\mathbf{V}$ and

$$\|\mathscr{F}^{-1}_{\Psi_{u,h},h}\|_{\mathcal{L}(\mathbf{V})} \leq 2\|\mathscr{F}^{-1}_{\bar{\Psi}}\|_{\mathcal{L}(\mathbf{V})}.$$

We can also assume that $\mathscr{F}_{\Psi_u}$ is an automorphism in $\mathbf{V}$ for all $u \in B_{\rho_3}(\bar{u})$.

Now we establish that $\Theta_{u,h} \in \mathbf{V}_h$ is a solution of (3.28) if and only if $\mathscr{F}_{\Psi_{u,h},h}(\Theta_{u,h}) = \boldsymbol{\eta}_h$, where $\boldsymbol{\eta}_h \in \mathbf{V}_h$ satisfies $T_h(\Psi_{u,h} - \Psi_d) = \boldsymbol{\eta}_h$.

With definitions of $\mathscr{F}_{\Psi_{u,h},h}$ and the operator T_h, it yields

$$\begin{aligned}\mathscr{F}_{\Psi_{u,h},h}(\Theta_{u,h}) = \boldsymbol{\eta}_h &\Longleftrightarrow \Theta_{u,h} + T_h[\mathcal{B}'(\Psi_{u,h})^*\Theta_{u,h}] = \boldsymbol{\eta}_h \\ &\Longleftrightarrow A(\Theta_{u,h}, \Phi_h) + \left\langle \mathcal{B}'(\Psi_{u,h})^*\Theta_{u,h}, \Phi_h \right\rangle_{\mathbf{V}',\mathbf{V}} = (\Psi_{u,h} - \Psi_d, \Phi_h)\ \forall \Phi_h \in \mathbf{V}_h.\end{aligned}$$

That is, $\Theta_{u,h} \in \mathbf{V}_h$ solves (3.28). □

Proof of Theorem 3.16. The problem under consideration being linear, it is straight forward to obtain the required estimates. We will sketch the steps of the proof.

The space $\mathbf{V}_h$ is a subspace of $\mathbf{V}$ and hence (3.27) holds true for test functions in $\mathbf{V}_h$.

The definition of $\mathscr{F}_{\Psi_{u,h},h}$, and of the continuous and discrete adjoint problems lead to

$$\begin{aligned}\mathscr{F}_{\Psi_{u,h},h}(\Theta_u-\Theta_{u,h}) &= \mathscr{F}_{\Psi_{u,h},h}(\Theta_u)-\mathscr{F}_{\Psi_{u,h},h}(\Theta_{u,h})\\ &= \Theta_u+T_h[\mathcal{B}'(\Psi_{u,h})^*\Theta_u]-T_h(\Psi_{u,h}-\Psi_d)\\ &= T(\Psi_u-\Psi_d)-T[\mathcal{B}'(\Psi_u)^*\Theta_u]+T_h[\mathcal{B}'(\Psi_{u,h})^*\Theta_u]-T_h(\Psi_{u,h}-\Psi_d)\\ &= T(\Psi_u-\Psi_{u,h})+(T-T_h)(\Psi_{u,h}-\Psi_d-\mathcal{B}'(\Psi_{u,h})^*\Theta_u)+T[\mathcal{B}'(\Psi_{u,h})^*\Theta_u-\mathcal{B}'(\Psi_u)^*\Theta_u].\end{aligned}$$

Since $F_{\Psi_{u,h},h}$ is an automorphism in $\mathbf{V}_h$, the boundedness of T leads to

$$\begin{aligned}|\!|\!|\Theta_u-\Theta_{u,h}|\!|\!|_2 \lesssim |\!|\!|\Psi_u-\Psi_{u,h}|\!|\!|_2 &+ \|T-T_h\|\,|\!|\!|\Psi_{u,h}-\Psi_d-\mathcal{B}'(\Psi_{u,h})^*\Theta|\!|\!|_2\\ &+\|\mathcal{B}'(\Psi_{u,h}-\Psi_u)^*\Theta_u\|_{\mathbf{V}'}.\end{aligned}$$

A use of Theorem 3.6(a) and Lemma 3.8 leads to the first estimate in (3.29).

To establish the second estimate in (3.29), define an auxiliary problem and its discretization.

For all $\mathbf{g}\in\mathbf{H}^{-1}(\Omega)$, let $\boldsymbol{\chi}_{\mathbf{g}}\in\mathbf{V}$ and $\boldsymbol{\chi}_{\mathbf{g},h}\in\mathbf{V}_h$ be the solutions to the equations

$$A(\boldsymbol{\chi}_{\mathbf{g}},\Phi)+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g}},\Phi)=\langle\mathbf{g},\Phi\rangle\quad\forall\Phi\in\mathbf{V}. \tag{3.30}$$

$$A(\boldsymbol{\chi}_{\mathbf{g},h},\Phi_h)+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g},h},\Phi_h)=\langle\mathbf{g},\Phi_h\rangle\quad\forall\Phi_h\in\mathbf{V}_h. \tag{3.31}$$

The well-posedness of (3.30) implies that $|\!|\!|\boldsymbol{\chi}_{\mathbf{g}}|\!|\!|_2\lesssim|\!|\!|\mathbf{g}|\!|\!|_{-1}$ and $|\!|\!|\boldsymbol{\chi}_{\mathbf{g}}|\!|\!|_{2+\gamma}\lesssim|\!|\!|\mathbf{g}|\!|\!|_{-1}$. By proceeding as in the proof of (a), we can establish that

$$|\!|\!|\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h}|\!|\!|_2\le Ch^{\gamma}\,|\!|\!|\mathbf{g}|\!|\!|_{-1}, \tag{3.32}$$

where the constant C depends on $|\!|\!|\Psi|\!|\!|_2$, and $\gamma\in(1/2,1]$ is the index of elliptic regularity.

From (3.27) and (3.28), it follows that

$$A(\Phi_h,\Theta_u-\Theta_{u,h})+2B(\Psi_u,\Phi_h,\Theta_u)-2B(\Psi_{u,h},\Phi_h,\Theta_{u,h})=(\Psi_u-\Psi_{u,h},\Phi_h)\quad\text{for all }\Phi_h\in\mathbf{V}_h. \tag{3.33}$$

Choose $\Phi=\Theta_u-\Theta_{u,h}$ in (3.30) and an adjustment of terms yield

$$\begin{aligned}(\mathbf{g},\Theta_u-\Theta_{u,h})&=A(\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})\\ &\quad+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})+A(\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h}).\end{aligned} \tag{3.34}$$

Choose $\Phi_h=\boldsymbol{\chi}_{\mathbf{g},h}$ in (3.33) and combine with (3.34) to obtain

$$\begin{aligned}(\mathbf{g},\Theta_u-\Theta_{u,h})&=A(\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})\\ &\quad+2B(\Psi_{u,h}-\Psi_u,\boldsymbol{\chi}_{\mathbf{g},h},\Theta_{u,h})+(\Psi_u-\Psi_{u,h},\boldsymbol{\chi}_{\mathbf{g},h})\\ &=A(\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})+2B(\Psi_u,\boldsymbol{\chi}_{\mathbf{g}}-\boldsymbol{\chi}_{\mathbf{g},h},\Theta_u-\Theta_{u,h})\\ &\quad+2B(\Psi_{u,h}-\Psi_u,\boldsymbol{\chi}_{\mathbf{g},h}-\boldsymbol{\chi}_{\mathbf{g}},\Theta_{u,h})+2B(\boldsymbol{\chi}_{\mathbf{g},h}-\boldsymbol{\chi}_{\mathbf{g}},\Psi_{u,h}-\Psi_u,\Theta_{u,h})\\ &\quad+(\Psi_u-\Psi_{u,h},\boldsymbol{\chi}_{\mathbf{g},h})\end{aligned} \tag{3.35}$$

A choice of $\mathbf{g} = -\Delta(\Theta_u - \Theta_{u,h})$ in the above equation (3.35) and then integration by parts, and a use of boundedness properties (2.3), (2.5) and (2.6) lead to

$$\begin{aligned}\|\Theta_u - \Theta_{u,h}\|_1^2 &\lesssim \|\boldsymbol{\chi}_{\mathbf{g}} - \boldsymbol{\chi}_{\mathbf{g},h}\|_2 \|\Theta_u - \Theta_{u,h}\|_2 + \|\Psi_{u,h} - \Psi_u\|_1 \|\boldsymbol{\chi}_{\mathbf{g}}\|_{2+\gamma} + \|\Psi_{u,h} - \Psi_u\| \|\boldsymbol{\chi}_{\mathbf{g},h}\| \\ &\lesssim \|\boldsymbol{\chi}_{\mathbf{g}} - \boldsymbol{\chi}_{\mathbf{g},h}\|_2 \|\Theta_u - \Theta_{u,h}\|_2 + \|\Psi_{u,h} - \Psi_u\|_1 \|\boldsymbol{\chi}_{\mathbf{g}}\|_{2+\gamma} \\ &\quad + \|\Psi_{u,h} - \Psi_u\| \|\boldsymbol{\chi}_{\mathbf{g}} - \boldsymbol{\chi}_{\mathbf{g},h}\| + \|\Psi_{u,h} - \Psi_u\| \|\boldsymbol{\chi}_{\mathbf{g}}\|. \end{aligned} \tag{3.36}$$

Note that $\|\mathbf{g}\|_{-1} = \|\Theta_u - \Theta_{u,h}\|_1$, and the well posedness of (3.30) implies $\|\boldsymbol{\chi}_{\mathbf{g}}\|_2 \leq \|\Theta_u - \Theta_{u,h}\|_1$ and $\|\boldsymbol{\chi}_{\mathbf{g}}\|_{2+\gamma} \leq \|\Theta_u - \Theta_{u,h}\|_1$. This, and estimates (3.32), part (a) of (3.29) and part (b) of (3.4) lead to part (b) estimate of (3.29). ☐

As for the case of the state equations (see Lem. 3.14), we have the following result.

Lemma 3.18. *For* $u, \hat{u} \in B_{\rho_3}(\bar{u})$, $0 < h < h_3$, *the solutions* Θ_u *and* $\Theta_{\hat{u},h}$ *to* (3.27) *and* (3.28) *with corresponding controls chosen as* u *and* $\hat{u}$ *respectively, satisfy*

$$\|\Theta_u - \Theta_{\hat{u},h}\|_2 \leq C(h^\gamma + \|u - \hat{u}\|_{L^2(\omega)}),$$

$\gamma \in (1/2, 1]$ *being the elliptic regularity index.*

4. Control discretization

First we describe the discretization of the control variable and then formulate the fully discrete problem. This is followed by existence and convergence results for the discrete problem. We make the following assumptions:

(A1) Let $\omega \subset \Omega$ be a polygonal domain.
(A2) Assume that $\mathcal{T}_h$ restricted to ω yields a triangulation for $\bar{\omega}$.

Note that the above assumptions are not very restrictive in practical situations. In case ω is not a polygonal domain, it can be approximated by a polygonal domain. The second assumption can be realized easily by choosing an initial triangulation appropriately.

Set

$$U_{h,ad} = \left\{ u \in L^2(\omega) : u|_T \in P_0(T),\ u_a \leq u \leq u_b \text{ for all } T \in \mathcal{T}_h \right\}.$$

The discrete control problem associated with (2.1) is defined as follows:

$$\min_{(\Psi_h, u_h) \in \mathbf{V}_h \times U_{h,ad}} J(\Psi_h, u_h) \quad \text{subject to} \tag{4.1a}$$

$$A(\Psi_h, \Phi_h) + B(\Psi_h, \Psi_h, \Phi_h) = (F + \mathbf{C}\mathbf{u}_h, \Phi_h), \tag{4.1b}$$

for all $\Phi_h \in \mathbf{V}_h$.

Recall that (Ψ_h, u_h) satisfies (4.1b) if and only if

$$\mathscr{N}_h(\Psi_h, u_h) = 0. \tag{4.2}$$

We aim to study the existence of local minima of (4.1) which approximate the local minima of (2.1). This can be established for nonsingular local solutions of (4.1).

The following lemma is crucial in establishing the existence of solution of (4.1) in Theorem 4.3.

Lemma 4.1. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.1). If $u_h \in B_{\rho_2}(\bar{u})$ and $u_h \rightharpoonup u$ weakly, then $\Psi_{u_h,h}$ converges to Ψ_u in $\mathbf{H}^2(\Omega)$, where $\rho_2 > 0$, defined in the proof of Theorem 3.4 denotes the radius of the ball $B_{\rho_2}(\bar{u})$ such that the discrete state equation (3.3) admits a unique solution, when the mesh parameter h, is chosen sufficiently small.*

Proof. Let $(u_h)_h$ be a sequence in $B_{\rho_2}(\bar{u}) \cap U_{ad}$ converging weakly to u. The result (2.13b) in Lemma 2.7 yields that Ψ_u and Ψ_{u_h} belong to $\mathbf{H}^{2+\gamma}(\Omega)$ and are bounded in $\mathbf{H}^{2+\gamma}(\Omega)$. Thus, there exists a subsequence (still denoted by the same notation) such that

$$\begin{aligned} &\Psi_{u_h} \rightharpoonup \tilde{\Psi} \text{ in } \mathbf{H}^{2+\gamma}(\Omega), \\ &\Psi_{u_h} \to \tilde{\Psi} \text{ in } \mathbf{H}^2(\Omega). \end{aligned}$$

Note that Ψ_{u_h} satisfies

$$A(\Psi_{u_h}, \Phi) + B(\Psi_{u_h}, \Psi_{u_h}, \Phi) = \left\langle F + \mathbf{C}u_h, \Phi \right\rangle_{\mathbf{V}',\mathbf{V}} \quad \forall \Phi \in \mathbf{V}.$$

By passing to the limit, we have $\tilde{\Psi} = \Psi_u$. That is, $\Psi_{u_h} \to \Psi_u$ in $\mathbf{H}^2(\Omega)$.

Now a combination of this convergence result with Theorem 3.6, along with the triangle inequality and the fact that u_h is bounded yield that $\Psi_{u_h,h}$ converges to Ψ_u in $\mathbf{H}^2(\Omega)$. □

Corollary 4.2. *A result analogous to Lemma 4.1 holds true for the convergence of the solutions of the continuous and discrete adjoint problems as well. That is, for a nonsingular solution $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ of (2.9), if $u_h \in B_{\rho_3}(\bar{u})$ and $u_h \rightharpoonup u$ weakly in $L^2(\omega)$, then $\Theta_h(u_h)$ converges to Θ_u in $\mathbf{H}^2(\Omega)$, where $\rho_3 > 0$ is as defined in Theorem 3.15.*

The next theorem states the existence of at least one solution of the discrete control problem stated in (4.1) and the convergence results for the control and state variables. The proof is skipped as it can be derived by following the proof of Theorem 4.11 in [10]).

Theorem 4.3. *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.1). Then there exists $h_2 > 0$ such that, for all $0 < h < h_2$, (4.1) has at least one solution. If furthermore $(\bar{\Psi}, \bar{u})$ is a strict local minimum of (2.1), then for all $0 < h < h_2$, (4.1) has a local minimum $(\bar{\Psi}_h, \bar{u}_h)$ in a neighborhood of $(\bar{\Psi}, \bar{u})$ and the following results hold:*

$$\lim_{h\to 0} j_h(\bar{u}_h) = j(\bar{u}), \ \lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} = 0 \ \text{and} \ \lim_{h\to 0} \left|\!\left|\!\left| \bar{\Psi} - \bar{\Psi}_h \right|\!\right|\!\right|_2 = 0,$$

where $j_h(\bar{u}_h) = \mathscr{N}(\bar{\Psi}_h, \bar{u}_h)$.

Let $(\bar{\Psi}, \bar{u})$ be a nonsingular strict local minimum of (2.1) and $\{(\bar{\Psi}_h, \bar{u}_h)\}_{h \le h_3}$ be a sequence of local minima of problems (4.1) converging to $(\bar{\Psi}, \bar{u})$ in $\mathbf{V} \times L^2(\omega)$, with $\bar{u}_h \in B_{\rho_3}(\bar{u})$, where h_3 and ρ_3 are given by Theorem 3.15. Then every element $\bar{u}_h$ from a sequence $\{\bar{u}_h\}_{h \le h_3}$ is a local solution of the problem with a discrete reduced cost functional

$$\min_{u \in U_{h,ad}} j_h(u) := \mathscr{N}_h(\Psi_{u,h}, u), \tag{4.3}$$

where $\Psi_{u,h} = G_h(u)$.

In the next lemma, we establish the optimality condition for the discrete control problem and the uniform convergence of the controls.

Lemma 4.4. *Let $\bar{u}_h$ be a solution to problem (4.3), and let $\bar{\Psi}_h$, $\bar{\Theta}_h \in \mathbf{V}_h$ denote the corresponding discrete state and adjoint state. Then $\bar{\mathbf{u}}_h = \begin{pmatrix} \bar{u}_h \\ 0 \end{pmatrix}$ satisfies*

$$\int_\omega (\mathbf{C}^* \bar{\Theta}_h + \alpha \bar{\mathbf{u}}_h) \cdot (\mathbf{u}_h - \bar{\mathbf{u}}_h) \mathrm{d}x \geq 0 \text{ for all } \mathbf{u}_h = \begin{pmatrix} u_h \\ 0 \end{pmatrix}, \ u_h \in U_{h,ad}. \tag{4.4}$$

Also, $\lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^\infty(\omega)} = 0.$

Proof. For the first part, we use the optimality condition for the reduced discrete cost functional. That is,

$$j_h'(\bar{u}_h)(u_h - \bar{u}_h) = \int_\omega (\mathcal{C}^* \bar{\theta}_{1h} + \alpha \bar{u}_h)(u_h - \bar{u}_h) \mathrm{d}x \geq 0,$$

from which the required result (4.4) follows.

From (4.4), we can express the discrete control as the projection of the adjoint variable on $[u_a, u_b]$. That is,

$$\bar{u}_h|_T = \pi_{[u_a,u_b]} \left(-\frac{1}{\alpha|T|} \int_T (\mathcal{C}^* \theta_{1h})(x) \mathrm{d}x \right).$$

For $x \in T$, the projection formula for the continuous control in (2.24), the mean value theorem and the Lipschitz continuity of the projection operator yield

$$\begin{aligned}
|\bar{u}_h(x) - \bar{u}(x)| &\leq \left| \frac{1}{\alpha|T|} \int_T (\mathcal{C}^* \theta_{1h})(s) \mathrm{d}s - \frac{1}{\alpha} (\mathcal{C}^* \bar{\theta}_{1h})(x) \right| \\
&= \frac{1}{\alpha} \left| (\mathcal{C}^* \bar{\theta}_{1h})(x_T) - (\mathcal{C}^* \bar{\theta}_1)(x) \right| \\
&\leq \left| (\mathcal{C}^* \bar{\theta}_{1h})(x_T) - (\mathcal{C}^* \bar{\theta}_1)(x_T) \right| + \left| (\mathcal{C}^* \bar{\theta}_1)(x_T) - (\mathcal{C}^* \bar{\theta}_1)(x) \right| \\
&\leq C \left(\left\|\!\left| \bar{\Theta}_h - \bar{\Theta} \right|\!\right\|_\infty + |x_T - x| \right) \\
&\leq C \left(\left\|\!\left| \bar{\Theta}_h - \bar{\Theta} \right|\!\right\|_\infty + h \right),
\end{aligned}$$

for some $x_T \in T$, and the result follows from the Sobolev imbedding result together with Lemma 3.18 and Theorem 4.3. □

5. Error estimates

In this section, we develop error estimates for the state, adjoint and control variables.

Let $(\bar{\Psi}, \bar{u})$ be a nonsingular strict local minimum of (2.1) satisfying the second order optimality condition in Theorem 2.15 (or equivalently (2.32)). Let $\{(\bar{\Psi}_h, \bar{u}_h)\}_{h \leq h_3}$ be a sequence of local minima of problems (4.1) converging to $(\bar{\Psi}, \bar{u})$ in $\mathbf{V} \times L^2(\omega)$, with $\bar{u}_h \in B_{\rho_3}(\bar{u})$, where h_3 and ρ_3 are given by Theorem 3.15. Since $h \leq h_3$ and $\bar{u}_h \in B_{\rho_3}(\bar{u})$, $\bar{u}_h$ is a local minimum of (4.3).

First we state a lemma which is essential for the proof of the main convergence result in Theorem 5.2. For a proof see ([10], Lems. 4.16 and 4.17).

Lemma 5.1.

(a) Let the second order optimality condition (2.32) hold true. Then, there exists a mesh size h_4 with $0 < h_4 \leq h_3$ such that

$$\frac{\delta}{2} \|\bar{u} - \bar{u}_h\|^2_{L^2(\omega)} \leq \left(j'(\bar{u}) - j'(\bar{u}_h) \right) (\bar{u} - \bar{u}_h) \ \ \forall\, 0 < h < h_4. \tag{5.1}$$

(b) There exist a mesh size h_5 with $0 < h_5 \leq h_4$ and a constant $C > 0$ such that, for every $0 < h \leq h_5$, there exists $u_h^* \in U_{h,ad}$ satisfying

$$(i)\ j'(\bar{u})(\bar{u} - u_h^*) = 0 \quad (ii)\ \|\bar{u} - u_h^*\|_{L^\infty(\omega)} \leq Ch. \tag{5.2}$$

The following theorem establishes the convergence rates for control, state and adjoint variables.

Theorem 5.2. *Let $(\bar{\Psi}, \bar{u})$ be a nonsingular strict local minimum of (2.1) and $\{(\bar{\Psi}_h, \bar{u}_h)\}_{h \leq h_3}$ be a solution to (4.1) converging to $(\bar{\Psi}, \bar{u})$ in $\mathbf{V} \times L^2(\omega)$, for a sufficiently small mesh-size h with $\bar{u}_h \in B_{\rho_3}(\bar{u})$, where ρ_3 is given in Theorem 3.15. Let $\bar{\Theta}$ and $\bar{\Theta}_h$ be the corresponding continuous and discrete adjoint state variables, respectively. Then, there exists a constant $C > 0$ such that, for all $0 < h \leq h_5$, we have*

$$(i)\ \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \leq Ch \quad (ii)\ \left\|\!\left|\bar{\Psi} - \bar{\Psi}_h\right|\!\right\|_2 \leq Ch^\gamma \quad (iii)\ \left\|\!\left|\bar{\Theta} - \bar{\Theta}_h\right|\!\right\|_2 \leq Ch^\gamma,$$

$\gamma \in (1/2, 1]$ being the index of elliptic regularity.

Proof. For $0 < h \leq h_5$, from (5.1), we have

$$\begin{aligned}\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\omega)} &\leq (j'(\bar{u}) - j'(\bar{u}_h))(\bar{u} - \bar{u}_h) \\ &= (j'(\bar{u}) - j_h'(\bar{u}_h))(\bar{u} - \bar{u}_h) + (j_h'(\bar{u}_h) - j'(\bar{u}_h))(\bar{u} - \bar{u}_h).\end{aligned} \tag{5.3}$$

We now proceed to estimate the two terms in the right hand side of (5.3). From first order optimality conditions for continuous and discrete problems, we have

$$j'(\bar{u})(\bar{u}_h - \bar{u}) \geq 0, \quad j_h'(\bar{u}_h)(u_h^* - \bar{u}_h) \geq 0.$$

Also, $0 \leq j_h'(\bar{u}_h)(u_h^* - \bar{u}_h) = j_h'(\bar{u}_h)(u_h^* - \bar{u}) + j_h'(\bar{u}_h)(\bar{u} - \bar{u}_h)$ holds.

For $\bar{u}_h \in B_{\rho_3}(\bar{u})$, the above expressions, (3.29), stability of the continuous adjoint solution and (5.2) lead to

$$\begin{aligned}j'(\bar{u})(\bar{u} - \bar{u}_h) - j_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) &\leq j_h'(\bar{u}_h)(u_h^* - \bar{u}) \\ &= j_h'(\bar{u}_h)(u_h^* - \bar{u}) - j'(\bar{u})(u_h^* - \bar{u}) \\ &= \int_\omega (\mathcal{C}^*(\bar{\theta}_{1h} - \bar{\theta}_1) + \alpha(\bar{u}_h - \bar{u}))(u_h^* - \bar{u})\mathrm{d}x \\ &\leq C\left(\left\|\!\left|\bar{\Theta}_h - \bar{\Theta}\right|\!\right\| + \|\bar{u}_h - \bar{u}\|_{L^2(\omega)}\right)\|u_h^* - \bar{u}\|_{L^2(\omega)} \\ &\leq Ch\left(\left\|\!\left|\bar{\Theta}_h - \Theta_{\bar{u}_h}\right|\!\right\| + \left\|\!\left|\Theta_{\bar{u}_h} - \bar{\Theta}\right|\!\right\| + \|\bar{u}_h - \bar{u}\|_{L^2(\omega)}\right) \\ &\leq Ch\left(h^{2\gamma} + \|\bar{u}_h - \bar{u}\|_{L^2(\omega)}\right).\end{aligned} \tag{5.4}$$

The estimate (3.29) yields

$$\begin{aligned}(j_h'(\bar{u}_h) - j'(\bar{u}_h))(\bar{u} - \bar{u}_h) &= \int_\omega (\mathcal{C}^*(\bar{\theta}_{1h} - \theta_1(\bar{u}_h)) + \alpha(\bar{u}_h - \bar{u}_h))(\bar{u} - \bar{u}_h)\mathrm{d}x \\ &\leq C\left\|\!\left|\bar{\Theta}_h - \Theta_{\bar{u}_h}\right|\!\right\| \|\bar{u} - \bar{u}_h\|_{L^2(\omega)} \\ &\leq Ch^{2\gamma}\|\bar{u} - \bar{u}_h\|_{L^2(\omega)}.\end{aligned} \tag{5.5}$$

A substitution of the expression (5.4)–(5.5) in (5.3) along with the Young's inequality yields the first required estimate.

A use of the control estimate (i) in Lemmas 3.14 and 3.18 yield the required estimates for state and adjoint variables in (ii) and (iii) respectively. This concludes the proof. □

5.1. Post processing for control

A post processing of control helps us to obtain improved error estimates for control. Also, error estimates for the state and adjoint variables in H^1 and L^2 norms are derived. Recall the assumptions **(A1)** and **(A2)** on ω and $\mathcal{T}_h$ as described in Section 4.

Definition 5.3 (Interpolant)**.** The projection $\mathcal{P}_h$ is the bounded linear operator from the space of piecewise-continuous functions over $\mathcal{T}_h$ to the space of piecewise-constant functions over $\mathcal{T}_h$ defined by

$$(\mathcal{P}_h\chi)(x) = \chi(S_i) \qquad \forall\, x \in T_i \in \mathcal{T}_h,$$

where S_i denotes the centroid of the triangle T_i. When $\mathcal{P}_h$ is applied to a vector valued function, the image is understood component-wise.

Definition 5.4 (Post processed control)**.** The post processed control $\widetilde{\bar{u}}_h$ is defined as:

$$\widetilde{\bar{u}}_h(x) = \pi_{[u_a,u_b]}\left(-\frac{1}{\alpha}(\mathcal{C}^*\bar{\theta}_{h,1})(x)\right), \tag{5.6}$$

where $\bar{\Theta}_h$ is the discrete adjoint variable corresponding to the control $\bar{u}_h$.

Let $\mathcal{T}_h^1 = \mathcal{T}_h^{1,1} \cup \mathcal{T}_h^{1,2}$ denote the union of active and inactive set of triangles contained in ω, where $\bar{u}(x)$ satisfies

$$\begin{aligned} &\bar{u} \equiv u_a \;\text{ on } T;\;\; \bar{u} \equiv u_b \;\text{ on } T \;\text{ (in the active part } \mathcal{T}_h^{1,1}),\\ &u_a < \bar{u} < u_b \;\text{ on } T \;\text{ (in the inactive part } \mathcal{T}_h^{1,2}), \end{aligned}$$

and $\mathcal{T}_h^2 := \mathcal{T}_h \setminus \mathcal{T}_h^1$, the set of triangles, where $\bar{u}$ takes the value u_a (resp. u_b) as well as values greater than u_a (resp. lesser than u_b). Let $\Omega_h^1 = int\left(\cup_{T\in\mathcal{T}_h^1} T\right)$ (where the notation int denotes the interior) be the uncritical part and let $\Omega_h^{1,1}$ and $\Omega_h^{1,2}$ be the union of the triangles in the active and inactive parts, respectively. That is, $\Omega_h^1 = int\left(\overline{\Omega_h^{1,1} \cup \Omega_h^{1,2}}\right)$ with $\Omega_h^{1,1} = int\left(\cup_{T\in\mathcal{T}_h^{1,1}} T\right)$, $\Omega_h^{1,2} = int\left(\cup_{T\in\mathcal{T}_h^{1,2}} T\right)$. Define $\Omega_h^2 = int\left(\cup_{T\in\mathcal{T}_h^2} T\right)$ as the critical part of $\mathcal{T}_h$. We make an assumption on Ω_h^2, the set of critical triangles which is fulfilled in practical cases [15]:

(A3) Assume $|\Omega_h^2| = \sum_{T\in\mathcal{T}_h^2} |T| < Ch$, for some positive constant C independent of h.

$$|\Omega_h^2| = \sum_{T\in\mathcal{T}_h^2} |T| < Ch, \tag{5.7}$$

for some positive constant C independent of h. This implies that the mesh domain of the critical cells is sufficiently small.

Use the splitting $\bar{\Omega} = \bar{\Omega}_h^1 \cup \bar{\Omega}_h^2$, to define a discrete norm $\|\cdot\|_{\bar{h}}$ for the control as

$$\|\bar{u}\|_{\bar{h}} := |\!|\!|\bar{u}|\!|\!|_{H^2(\Omega_h^1)} + |\!|\!|\bar{u}|\!|\!|_{W^{1,\infty}(\Omega_h^2)},$$

where $|\!|\!|\bar{u}|\!|\!|^2_{H^2(\Omega_h^1)} := \sum_{T\subset\Omega_h^1} \|\bar{u}\|^2_{H^2(T)}$ and $|\!|\!|\bar{u}|\!|\!|^2_{W^{1,\infty}(\Omega_h^2)} := \sum_{T\subset\Omega_h^2} \|\bar{u}\|^2_{W^{1,\infty}(T)}$.

Lemma 5.5 (Numerical integration estimate)**.** *Lemma 3.2 [26] Let g be a function belonging to $H^2(T_i)$ for all i in a certain index set I. Then, there holds :*

$$\left|\int_{T_i}(g(x)-g(S_i))\mathrm{d}x\right| \le Ch^2\sqrt{|T_i|}\,|g|_{H^2(T_i)} \tag{5.8}$$

where S_i denotes the centroid of T_i.

Also, the following result can be established using scaling arguments.

Lemma 5.6 (Scaling results)**.** *For $\bar{u} \in W^{1,\infty}(T_i)$ (resp. $H^2(T_i)$) with $T_i \in \mathcal{T}_h$,*

$$\|\bar{u}-\mathcal{P}_h\bar{u}\|_{L^\infty(T_i)} \le Ch\|\bar{u}\|_{W^{1,\infty}(T_i)},\ (\textit{resp.}\ \|\bar{u}-\mathcal{P}_h\bar{u}\|_{0,T_i} \le Ch\|\bar{u}\|_{H^2(T_i)}). \tag{5.9}$$

Theorem 5.7. *Let $\Psi_{\bar{u},h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ be solutions of (3.3) with respect to control $\bar{u}$ and post processed control $\mathcal{P}_h\bar{u}$, respectively. Then the following error estimate holds true:*

$$|||\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}||| \le Ch^2\|\bar{u}\|_{\bar{h}}.$$

Proof. Consider the perturbed auxiliary problem:
Seek $\boldsymbol{\xi} \in \mathbf{V}$ that solves

$$A(\mathbf{z},\boldsymbol{\xi}) + B(\Psi_{\bar{u},h},\mathbf{z},\boldsymbol{\xi}) + B(\mathbf{z},\Psi_{\mathcal{P}_h\bar{u},h},\boldsymbol{\xi}) = (\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h},\mathbf{z}) \quad \forall \mathbf{z}\in\mathbf{V}. \tag{5.10}$$

Its discretization is given by:
Seek $\boldsymbol{\xi}_h \in \mathbf{V}_h$ that solves

$$A(\mathbf{z}_h,\boldsymbol{\xi}_h) + B(\Psi_{\bar{u},h},\mathbf{z}_h,\boldsymbol{\xi}_h) + B(\mathbf{z}_h,\Psi_{\mathcal{P}_h\bar{u},h},\boldsymbol{\xi}_h) = (\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h},\mathbf{z}_h) \quad \forall \mathbf{z}_h\in\mathbf{V}_h. \tag{5.11}$$

The above equation (5.11) can be written in the operator form as

$$\mathcal{A}^*\boldsymbol{\xi}_h + \mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\xi}_h + \frac{1}{2}\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h}-\Psi_{\bar{u},h})^*\boldsymbol{\xi}_h = \Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h} \text{ in } \mathbf{V}_h. \tag{5.12}$$

Note that (3.26) and (5.9) lead to

$$|||\Psi_{\mathcal{P}_h\bar{u},h}-\Psi_{\bar{u},h}|||_2 \le C\|\mathcal{P}_h\bar{u}-\bar{u}\|_{L^2(\omega)} \le Ch. \tag{5.13}$$

The invertibility of $\mathcal{A}^*+\mathcal{B}'(\Psi_{\bar{u},h})^*$, Lemma 3.12 and (5.13) lead to well-posedness of (5.11). Choose $\mathbf{z}_h = \Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}$ in (5.11) and simplify the terms to obtain

$$\begin{aligned}|||\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}|||^2 &= A(\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h},\boldsymbol{\xi}_h) + B(\Psi_{\bar{u},h},\Psi_{\bar{u},h},\boldsymbol{\xi}_h)\\ &\quad - B(\Psi_{\mathcal{P}_h\bar{u},h},\Psi_{\mathcal{P}_h\bar{u},h},\boldsymbol{\xi}_h).\end{aligned} \tag{5.14}$$

Note that $\Psi_{\bar{u},h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ satisfy the following discrete problems:

$$\begin{aligned}&A(\Psi_{\bar{u},h},\Phi_h) + B(\Psi_{\bar{u},h},\Psi_{\bar{u},h},\Phi_h) = (F+\mathbf{C}\bar{\mathbf{u}},\Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h,\\ &A(\Psi_{\mathcal{P}_h\bar{u},h},\Phi_h) + B(\Psi_{\mathcal{P}_h\bar{u},h},\Psi_{\mathcal{P}_h\bar{u},h},\Phi_h) = (F+\mathbf{C}(\mathcal{P}_h\bar{\mathbf{u}}),\Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h.\end{aligned}$$

Subtract the above two equations to obtain

$$A(\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h},\Phi_h)+B(\Psi_{\bar{u},h},\Psi_{\bar{u},h},\Phi_h)-B(\Psi_{\mathcal{P}_h\bar{u},h},\Psi_{\mathcal{P}_h\bar{u},h},\Phi_h)=(\mathbf{C}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}}),\Phi_h).$$

Choose $\Phi_h=\boldsymbol{\xi}_h$ in the above equation and use (5.14) to obtain

$$\left\|\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}\right\|^2=(\mathbf{C}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}}),\boldsymbol{\xi}_h). \tag{5.15}$$

Consider

$$\int_{\Omega_h^1}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h\mathrm{d}x=\sum_{T_i\subset\Omega_h^1}\left((\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}},\boldsymbol{\xi}_h(S_i))_{T_i}+(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}},\boldsymbol{\xi}_h-\boldsymbol{\xi}_h(S_i))_{T_i}\right).$$

A use of (5.8) along with the result $|\boldsymbol{\xi}_h(S_i)|\sqrt{T_i}=\|\boldsymbol{\xi}_h(S_i)\|_{0,T_i}\le\|\boldsymbol{\xi}_h\|_{0,T_i}$ for the first term leads to

$$\begin{aligned}\int_{\Omega_h^1}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h\mathrm{d}x&\le Ch^2\sum_{T_i\subset\Omega_h^1}\|\bar{u}\|_{H^2(T_i)}\left(\|\boldsymbol{\xi}_h\|_{0,T_i}+\|\boldsymbol{\xi}_h\|_{H^2(T_i)}\right)\\&\le Ch^2\left\|\bar{u}\right\|_{H^2(\Omega_h^1)}\left\|\boldsymbol{\xi}_h\right\|_{H^2(\Omega_h^1)}.\end{aligned} \tag{5.16}$$

Also, consider

$$\begin{aligned}\int_{\Omega_h^2}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h\mathrm{d}x&=\sum_{T_i\subset\Omega_h^2}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}},\boldsymbol{\xi}_h)_{T_i}\\&\le\|\bar{u}-\mathcal{P}_h\bar{u}\|_{L^\infty(\Omega_h^2)}\|\boldsymbol{\xi}_h\|_{L^\infty(\Omega_h^2)}\sum_{T_i\subset\mathcal{T}_h^2}|T_i|.\end{aligned}$$

The assumption **(A3)**, the estimate (5.9) and Sobolev imbedding result in the above equation lead to

$$\int_{\Omega_h^2}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h\mathrm{d}x\le Ch^2\left\|\bar{u}\right\|_{W^{1,\infty}(\Omega_h^2)}\left\|\boldsymbol{\xi}_h\right\|_{H^2(\Omega_h^2)}. \tag{5.17}$$

A combination of (5.16) and (5.17) yields

$$\int_{\Omega_h^1\cup\Omega_h^2}(\bar{\mathbf{u}}-\mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h\mathrm{d}x\le Ch^2\|\bar{u}\|_{\bar{h}}\left\|\boldsymbol{\xi}_h\right\|_2. \tag{5.18}$$

Now we estimate $\|\boldsymbol{\xi}_h\|_2$. Let $\mathcal{L}$ (resp. $\mathcal{L}_h$) : $\mathbf{V}\to\mathbf{V}$ be defined by

$$\begin{gathered}\mathcal{L}(\boldsymbol{\chi}):=\boldsymbol{\chi}+\frac{1}{2}T[\mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\chi}]+\frac{1}{2}T[\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h})^*\boldsymbol{\chi}]\\(\text{resp. }\mathcal{L}_h(\boldsymbol{\chi}):=\boldsymbol{\chi}+\frac{1}{2}T_h[\mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\chi}]+\frac{1}{2}T_h[\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h})^*\boldsymbol{\chi}]).\end{gathered}$$

The auxiliary perturbed problem and its discretization (5.10) and (5.11) can now be expressed as

$$\begin{aligned}\mathcal{L}(\boldsymbol{\xi})&=T(\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}),\\\mathcal{L}_h(\boldsymbol{\xi}_h)&=T_h(\Psi_{\bar{u},h}-\Psi_{\mathcal{P}_h\bar{u},h}).\end{aligned}$$

From the above characterization, it follows that

$$\begin{aligned}\mathcal{L}_h(\boldsymbol{\xi}-\boldsymbol{\xi}_h) &= \mathcal{L}_h(\boldsymbol{\xi}) - \mathcal{L}_h(\boldsymbol{\xi}_h)\\ &= \boldsymbol{\xi} + \frac{1}{2}T_h[\mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\xi}] + \frac{1}{2}T_h[\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h})^*\boldsymbol{\xi}] - T_h(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h})\\ &= (T-T_h)(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}) - \frac{1}{2}(T-T_h)[\mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\xi}] - \frac{1}{2}(T-T_h)[\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h})^*\boldsymbol{\xi}].\end{aligned}$$

The invertibility of $\mathcal{L}_h$ and Lemma 3.8 lead to

$$\|\boldsymbol{\xi}-\boldsymbol{\xi}_h\|_2 \lesssim h^\gamma \left(\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\| + \|\boldsymbol{\xi}\|_{2+\gamma}\right). \tag{5.19}$$

Combine (5.18) and (5.19), and use triangle inequality together with the estimate for $\|\boldsymbol{\xi}\|_2$ and $\|\boldsymbol{\xi}\|_{2+\gamma}$ to obtain

$$(\mathbf{C}(\bar{\mathbf{u}} - \mathcal{P}_h\bar{\mathbf{u}}), \boldsymbol{\xi}_h) = \int_{\Omega_h^1\cup\Omega_h^2} \mathbf{C}(\bar{\mathbf{u}} - \mathcal{P}_h\bar{\mathbf{u}})\cdot\boldsymbol{\xi}_h \mathrm{d}x \le Ch^2\|\bar{u}\|_{\bar{h}}\,\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\|. \tag{5.20}$$

This and (5.15) lead to the required estimate

$$\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\| \le Ch^2\|\bar{u}\|_{\bar{h}}.$$

□

Following the proof of the above theorem, the next result holds immediately.

Corollary 5.8. *Let $\Psi_{\bar{u}_h,h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ be the solutions of (3.3) with the control $\bar{u}_h$ and the post processed control $\mathcal{P}_h\bar{u}$, respectively. Then the following error estimate holds true:*

$$\|\Psi_{\bar{u}_h,h} - \Psi_{\mathcal{P}_h\bar{u},h}\| \le Ch^2\|\bar{u}\|_{\bar{h}}.$$

The discrete post processed adjoint problem can be stated as:

Find $\Theta_{\mathcal{P}_h\bar{u},h}\in\mathbf{V}_h$ such that

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\mathcal{P}_h\bar{u},h}) := A(\Phi_h,\Theta_{\mathcal{P}_h\bar{u},h}) + 2B(\Psi_{\bar{u},h},\Phi_h,\Theta_{\mathcal{P}_h\bar{u},h}) = (\Psi_{\mathcal{P}_h\bar{u},h} - \Psi_d, \Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h. \tag{5.21}$$

Lemma 5.9. *Let $\Theta_{\bar{u},h}$ be solution of (3.28) with the control $\bar{u}$ and $\Theta_{\mathcal{P}_h\bar{u},h}$ be the solution of (5.21). Then the following error estimate holds true:*

$$\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\| \le Ch^2\|\bar{u}\|_{\bar{h}}.$$

Proof. The discrete adjoint problem (3.28) can be written as

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\bar{u},h}) := A(\Phi_h,\Theta_{\bar{u},h}) + 2B(\Psi_{\bar{u},h},\Phi_h,\Theta_{\bar{u},h}) = (\Psi_{\bar{u},h} - \Psi_d, \Phi_h) \quad \forall\Phi_h\in\mathbf{V}_h. \tag{5.22}$$

The subtraction of (5.22) and (5.21) leads to

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}) = (\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h). \tag{5.23}$$

We consider a well-posed auxiliary problem:

Find $\boldsymbol{\chi}_h \in \mathbf{V}_h$ such that

$$\tilde{\mathcal{L}}_h(\boldsymbol{\chi}_h, \Phi_h) = (\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h \tag{5.24}$$

with the *a priori* bound $\|\boldsymbol{\chi}_h\|_2 \leq C \,\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\|$. Choose $\Phi_h = \boldsymbol{\chi}_h$ in (5.23) and $\Phi_h = \Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}$ in (5.24) to obtain

$$\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\|^2 = (\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}, \boldsymbol{\chi}_h). \tag{5.25}$$

The Cauchy-Schwarz inequality, Poincaré inequality, well-posedness of (5.24) and Theorem 5.7 lead to

$$\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\| \leq C \,\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\| \leq Ch^2 \|\bar{u}\|_{\bar{h}}.$$

This completes the proof. □

Choose the load function in (5.22) as $\Psi_{\bar{u}_h,h} - \Psi_d$, proceed as in the proof of Lemma 5.9 and use Corollary 5.8 to obtain the next result.

Corollary 5.10. *Let $\bar{\Theta}_h$ be solution of (3.28) with respect to control $\bar{u}_h$ and $\Theta_{\mathcal{P}_h\bar{u},h}$ be the solution of (5.21). Then the following error estimate holds true:*

$$\left\|\bar{\Theta}_h - \Theta_{\mathcal{P}_h\bar{u},h}\right\| \leq Ch^2 \|\bar{u}\|_{\bar{h}}.$$

Lemma 5.11 (A variational inequality [26], (3.15))**.** *The post processed control $\mathcal{P}_h\bar{u}$ satisfies the variational inequality*

$$\|\mathcal{P}_h\bar{u} - \bar{u}_h\|^2_{L^2(\omega)} \leq C(\mathcal{P}_h\bar{\Theta} - \bar{\Theta}_h, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}). \tag{5.26}$$

The proof of the next lemma is standard (for example [17, 26]). However, we provide a proof for the sake of completeness.

Theorem 5.12 (Convergence rate at centroids)**.** *Under the assumption* **(A3)***, the estimate*

$$\|\bar{u}_h - \mathcal{P}_h\bar{u}\|_{L^2(\omega)} \leq Ch^\beta$$

holds true with $\beta = \min\{2\gamma, 2\}$, $\gamma \in (1/2, 1]$ being the index of elliptic regularity.

Proof. A use of (5.26) and simple manipulations lead to

$$\begin{aligned} \|\mathcal{P}_h\bar{u} - \bar{u}_h\|^2_{L^2(\omega)} &\lesssim (\mathcal{P}_h\bar{\Theta} - \bar{\Theta}_h, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) \\ &= (\mathcal{P}_h\bar{\Theta} - \bar{\Theta}, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) + (\bar{\Theta} - \Theta_{\mathcal{P}_h\bar{u},h}, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) + (\bar{\Theta}_{\mathcal{P}_h\bar{u},h} - \bar{\Theta}_h, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}). \end{aligned} \tag{5.27}$$

The first term is estimated using the fact that $\bar{u}_h - \mathcal{P}_h\bar{u}$ is a constant in each $T \in \mathcal{T}_h$ and hence,

$$(\mathcal{P}_h\bar{\Theta} - \bar{\Theta}, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) = \sum_{T_i \in \mathcal{T}_h} (\bar{\mathbf{u}}(S_i) - \mathcal{P}_h\bar{\mathbf{u}}(S_i)) \int_{T_i} (\mathcal{P}_h\bar{\Theta} - \bar{\Theta})\mathrm{d}x.$$

Since $\bar{\Theta}|_{T_i} \in \mathbf{H}^2(T_i)$, a use of (5.8) in the above equation and *a priori* bound of $\bar{\Theta}$ from (2.23a)–(2.23b) as $\left\|\bar{\Theta}\right\|_2 \leq C(\|f\| + \|\bar{u}\| + \|y_d\|)$ lead to

$$(\mathcal{P}_h\bar{\Theta} - \bar{\Theta}, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) \leq Ch^2 \|\bar{u}_h - \mathcal{P}_h\bar{u}\|_{L^2(\omega)} \left\|\bar{\Theta}\right\|_2. \tag{5.28}$$

The triangle inequality, Lemma 5.9, Poincaré inequality and (3.29) yield

$$\left\|\bar{\Theta} - \Theta_{\mathcal{P}_h\bar{u},h}\right\| \leq Ch^{\beta} \tag{5.29}$$

with $\beta = \min\{2\gamma, 2\}$. The equation (5.29), and Cauchy-Schwarz inequality leads to the estimate for the second term of (5.27) as

$$|(\bar{\Theta} - \Theta_{\mathcal{P}_h\bar{u},h}, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}})| \leq Ch^{\beta}\|\bar{u}_h - \mathcal{P}_h\bar{u}\|_{L^2(\omega)}. \tag{5.30}$$

The Cauchy-Schwarz inequality and Corollary 5.10 lead to the estimate for the last term of (5.27) as

$$(\bar{\Theta}_{\mathcal{P}_h\bar{u},h} - \bar{\Theta}_h, \bar{\mathbf{u}}_h - \mathcal{P}_h\bar{\mathbf{u}}) \leq Ch^{\beta}\|\bar{u}_h - \mathcal{P}_h\bar{u}\|_{L^2(\omega)}. \tag{5.31}$$

A use of the estimates (5.28)–(5.31) in (5.27) leads to the required estimate. □

Theorem 5.13 (Estimate for post-processed control)**.** *The following estimate for post-processed control holds true:*

$$\|\bar{u} - \tilde{\bar{u}}_h\|_{L^2(\omega)} \leq Ch^{\beta},$$

where $\bar{u}$ is the optimal control and $\tilde{\bar{u}}_h$ is the post-processed control defined in (5.6)*, and* $\beta = \min\{2\gamma, 2\}$.

Proof. The Lipschitz continuity of the projection operator $\pi_{[u_a,u_b]}$ and triangle inequality yield

$$\begin{aligned}\|\bar{u} - \tilde{\bar{u}}_h\|_{L^2(\omega)} &\leq \|\pi_{[u_a,u_b]}\big(-\frac{1}{\alpha}\mathcal{C}^*\bar{\theta}_1\big) - \pi_{[u_a,u_b]}\big(-\frac{1}{\alpha}\mathcal{C}^*\bar{\theta}_{h,1}\big)\| \leq C\left\|\bar{\Theta} - \bar{\Theta}_h\right\| \\ &\leq C(\left\|\bar{\Theta} - \Theta_{\mathcal{P}_h\bar{u},h}\right\| + \left\|\Theta_{\mathcal{P}_h\bar{u},h} - \bar{\Theta}_h\right\|).\end{aligned}$$

Now (5.29) and Corollary 5.10 lead to the required result. □

Theorem 5.14. *Let $\Psi_{\bar{u},h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ be solution of* (3.3) *with respect to control $\bar{u}$ and post processed control $\mathcal{P}_h\bar{u}$, respectively. Then the following error estimate holds true:*

$$\left\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\right\|_1 \leq Ch^2\|\bar{u}\|_{\bar{h}}.$$

Proof. Consider the perturbed auxiliary problem: Seek $\boldsymbol{\xi} \in \mathbf{V}$ that solves

$$A(\mathbf{z}, \boldsymbol{\xi}) + B(\Psi_{\bar{u},h}, \mathbf{z}, \boldsymbol{\xi}) + B(\mathbf{z}, \Psi_{\mathcal{P}_h\bar{u},h}, \boldsymbol{\xi}) = -(\Delta(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}), \mathbf{z}) \quad \forall \mathbf{z} \in \mathbf{V}. \tag{5.32}$$

Its discretization is given by: Seek $\boldsymbol{\xi}_h \in \mathbf{V}_h$ that solves

$$A(\mathbf{z}_h, \boldsymbol{\xi}_h) + B(\Psi_{\bar{u},h}, \mathbf{z}_h, \boldsymbol{\xi}_h) + B(\mathbf{z}_h, \Psi_{\mathcal{P}_h\bar{u},h}, \boldsymbol{\xi}_h) = -(\Delta(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}), \mathbf{z}_h) \quad \forall \mathbf{z}_h \in \mathbf{V}_h. \tag{5.33}$$

The above equation (5.33) can be written in the operator form as

$$\mathcal{A}^*\boldsymbol{\xi}_h + \mathcal{B}'(\Psi_{\bar{u},h})^*\boldsymbol{\xi}_h + \frac{1}{2}\mathcal{B}'(\Psi_{\mathcal{P}_h\bar{u},h} - \Psi_{\bar{u},h})^*\boldsymbol{\xi}_h = -T_h(\Delta(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h})) \text{ in } \mathbf{V}_h. \tag{5.34}$$

Note that (3.26) and (5.9) lead to

$$\left\|\Psi_{\mathcal{P}_h\bar{u},h} - \Psi_{\bar{u},h}\right\|_2 \leq C\|\mathcal{P}_h\bar{u} - \bar{u}\|_{L^2(\omega)} \leq Ch. \tag{5.35}$$

The invertibility of $\mathcal{A}^* + \mathcal{B}'(\Psi_{\bar{u},h})^*$, Lemma 3.12 and (5.35) lead to well-posedness of (5.33). Choose $\mathbf{z}_h = \Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}$ in (5.33) and simplify the terms to obtain

$$\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\|_1^2 = A(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}, \boldsymbol{\xi}_h) + B(\Psi_{\bar{u},h}, \Psi_{\bar{u},h}, \boldsymbol{\xi}_h) - B(\Psi_{\mathcal{P}_h\bar{u},h}, \Psi_{\mathcal{P}_h\bar{u},h}, \boldsymbol{\xi}_h). \tag{5.36}$$

Note that $\Psi_{\bar{u},h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ satisfy the following discrete problems:

$$A(\Psi_{\bar{u},h}, \Phi_h) + B(\Psi_{\bar{u},h}, \Psi_{\bar{u},h}, \Phi_h) = (F + \mathbf{C}\bar{\mathbf{u}}, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h,$$
$$A(\Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h) + B(\Psi_{\mathcal{P}_h\bar{u},h}, \Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h) = (F + \mathbf{C}\,\mathcal{P}_h\bar{\mathbf{u}}, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h.$$

Subtract the above two equations to obtain

$$A(\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h) + B(\Psi_{\bar{u},h}, \Psi_{\bar{u},h}, \Phi_h) - B(\Psi_{\mathcal{P}_h\bar{u},h}, \Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h) = (\mathbf{C}(\bar{\mathbf{u}} - \mathcal{P}_h\bar{\mathbf{u}}), \Phi_h).$$

Choose $\Phi_h = \boldsymbol{\xi}_h$ in the above equation and use (5.36) to obtain

$$\|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\|_1^2 = (\mathbf{C}(\bar{\mathbf{u}} - \mathcal{P}_h\bar{\mathbf{u}}), \boldsymbol{\xi}_h). \tag{5.37}$$

Now proceed as in the proof of Theorem 5.7 to obtain the required estimate. □

Following the proof of the above theorem, the next result holds immediately.

Corollary 5.15. *Let $\Psi_{\bar{u}_h,h}$ and $\Psi_{\mathcal{P}_h\bar{u},h}$ be the solutions of (3.3) with the control $\bar{u}_h$ and the post processed control $\mathcal{P}_h\bar{u}$, respectively. Then the following error estimate holds true:*

$$\|\Psi_{\bar{u}_h,h} - \Psi_{\mathcal{P}_h\bar{u},h}\|_1 \leq Ch^2\|\bar{u}\|_{\bar{h}}.$$

The discrete post processed adjoint problem can be stated as:
Find $\Theta_{\mathcal{P}_h\bar{u},h} \in \mathbf{V}_h$ such that

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\mathcal{P}_h\bar{u},h}) := A(\Phi_h, \Theta_{\mathcal{P}_h\bar{u},h}) + 2B(\Psi_{\bar{u},h}, \Phi_h, \Theta_{\mathcal{P}_h\bar{u},h}) = (\Psi_{\mathcal{P}_h\bar{u},h} - \Psi_d, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h. \tag{5.38}$$

Lemma 5.16. *Let $\Theta_{\bar{u},h}$ be solution of (3.28) with the control $\bar{u}$ and $\Theta_{\mathcal{P}_h\bar{u},h}$ be the solution of (5.38). Then the following error estimate holds true:*

$$\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\|_1 \leq Ch^2\|\bar{u}\|_{\bar{h}}.$$

Proof. The discrete adjoint problem (3.28) can be written as

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\bar{u},h}) := A(\Phi_h, \Theta_{\bar{u},h}) + 2B(\Psi_{\bar{u},h}, \Phi_h, \Theta_{\bar{u},h}) = (\Psi_{\bar{u},h} - \Psi_d, \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h. \tag{5.39}$$

The subtraction of (5.39) and (5.38) leads to

$$\tilde{\mathcal{L}}_h(\Phi_h, \Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}) = (\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}, \Phi_h). \tag{5.40}$$

The proof is similar to that of Lemma 5.9 except for the change that in place of (5.24), we consider the following well-posed auxiliary problem: Find $\boldsymbol{\chi}_h \in \mathbf{V}_h$ such that

$$\tilde{\mathcal{L}}_h(\boldsymbol{\chi}_h, \Phi_h) = (-\Delta(\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}), \Phi_h) \quad \forall \Phi_h \in \mathbf{V}_h \tag{5.41}$$

with the *a priori* bound $\|\boldsymbol{\chi}_h\|_2 \leq C\,\|-\Delta(\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h})\|_{-1} \leq C\,\|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\|_1$. □

Corollary 5.17. *Let $\bar{\Theta}_h$ be solution of (3.28) with respect to control $\bar{u}_h$ and $\Theta_{\mathcal{P}_h\bar{u},h}$ be the solution of (5.38). Then the following error estimate holds true:*

$$\left\|\!\left|\bar{\Theta}_h - \Theta_{\mathcal{P}_h\bar{u},h}\right|\!\right\|_1 \leq Ch^2\|\bar{u}\|_{\bar{h}}.$$

Theorem 5.18 (H^1 and L^2-estimates for state and adjoint variables)**.** *Let $(\bar{\Psi}, \bar{u}) \in \mathbf{V} \times L^2(\omega)$ be a nonsingular solution of (2.9). Let $\bar{\Psi}$ and $\bar{\Psi}_h$ be solutions of (2.1) and (4.1) respectively, and $\bar{\Theta}$ and $\bar{\Theta}_h$ be the solutions of the corresponding adjoint problems. For sufficiently small h, the following estimates hold true:*

$$\begin{aligned}
&(a)\ \left\|\!\left|\bar{\Psi} - \bar{\Psi}_h\right|\!\right\|_1 \leq Ch^{2\gamma}, \quad \left\|\!\left|\bar{\Theta} - \bar{\Theta}_h\right|\!\right\|_1 \leq Ch^{2\gamma}.\\
&(b)\ \left\|\!\left|\bar{\Psi} - \bar{\Psi}_h\right|\!\right\| \leq Ch^{2\gamma}, \quad \left\|\!\left|\bar{\Theta} - \bar{\Theta}_h\right|\!\right\| \leq Ch^{2\gamma}.
\end{aligned}$$

Proof. The triangle inequality, Theorems 3.6 and 5.14 and Corollary 5.15 lead to

$$\left\|\!\left|\bar{\Psi} - \bar{\Psi}_h\right|\!\right\|_1 \leq \left\|\!\left|\bar{\Psi} - \Psi_{\bar{u},h}\right|\!\right\|_1 + \left\|\!\left|\Psi_{\bar{u},h} - \Psi_{\mathcal{P}_h\bar{u},h}\right|\!\right\|_1 + \left\|\!\left|\Psi_{\mathcal{P}_h\bar{u},h} - \bar{\Psi}_h\right|\!\right\|_1 \leq Ch^{2\gamma}.$$

Similarly, the triangle inequality, Theorems 3.16 and 5.16 and Corollary 5.17 lead to

$$\left\|\!\left|\bar{\Theta} - \bar{\Theta}_h\right|\!\right\|_1 \leq \left\|\!\left|\bar{\Theta} - \Theta_{\bar{u},h}\right|\!\right\|_1 + \left\|\!\left|\Theta_{\bar{u},h} - \Theta_{\mathcal{P}_h\bar{u},h}\right|\!\right\|_1 + \left\|\!\left|\Theta_{\mathcal{P}_h\bar{u},h} - \bar{\Theta}_h\right|\!\right\|_1 \leq Ch^{2\gamma}.$$

This completes the proof of part (a). Part (b) follows easily. □

6. Numerical results

In this section, we present two numerical examples to illustrate the theoretical estimates obtained in this paper. The discrete optimization problem (4.1) is solved using the primal-dual active set strategy [30]. The state and adjoint variables are discretized using Bogner-Fox-Schmit finite elements and the control variable is discretized using piecewise constants. Further, the post-processed control is computed with the help of the discrete adjoint variable. Let the lth level error and mesh parameter be denoted by e_l and h_l, respectively. The lth level experimental order of convergence is defined by

$$\delta_l := log(e_l/e_{l-1})/log(h_l/h_{l-1}).$$

The errors and numerical orders of convergence are presented for both the examples.

Example 6.1. Let the computational domain be $\Omega = (0,1)^2$ and $\mathcal{C} = \mathrm{I}$, $\omega = \Omega$. A slightly modified version of (1.1a)–(1.1d) is constructed in such a way that that the exact solution is known. This is done by choosing the state variables $\bar{\psi}_1, \bar{\psi}_2$ and the adjoint variables $\bar{\theta}_1, \bar{\theta}_2$ as

$$\bar{\psi}_1 = \bar{\psi}_2 = \sin^2(\pi x)\sin^2(\pi y), \quad \bar{\theta}_1 = \bar{\theta}_2 = x^2y^2(1-x)^2(1-y)^2,$$

and the control $\bar{u}$ as $\bar{u}(x) = \pi_{[-750,-50]}\left(-\frac{1}{\alpha}\bar{\theta}_1(x)\right)$, where the regularization parameter α is chosen as 10^{-5}.

The source terms $f, \tilde{f}$ and observation $\bar{\Psi}_d = (\bar{\psi}_{1d}, \bar{\psi}_{2d})$ are then computed using

$$\begin{aligned}
&f = \Delta^2\bar{\psi}_1 - [\bar{\psi}_1, \bar{\psi}_2] - \bar{u}, \quad \tilde{f} = \Delta^2\bar{\psi}_2 + \frac{1}{2}[\bar{\psi}_1, \bar{\psi}_1] \quad \text{and}\\
&\bar{\psi}_{1d} = \bar{\psi}_1 - \Delta^2\bar{\theta}_1, \quad \bar{\psi}_{2d} = \bar{\psi}_2 - \Delta^2\bar{\theta}_2 + [\bar{\psi}_1, \bar{\theta}_1].
\end{aligned}$$

The errors and orders of convergence for the numerical approximations to state, adjoint and control variables are shown in Tables 1 and 2.

TABLE 1. Errors and orders of convergence for the state, adjoint, control and post processed control variables in Example 6.1.

N	h/h_0	$\left\lVert\bar{\Psi}-\bar{\Psi}_h\right\rVert_2$	δ_l	$\left\lVert\bar{\Theta}-\bar{\Theta}_h\right\rVert_2$	δ_l	$\lVert\bar{u}-\bar{u}_h\rVert$	δ_l	$\lVert\bar{u}-\tilde{\bar{u}}_h\rVert$	δ_l
36	2^{-1}	1.60389465	–	0.00479710	–	46.72538744	–	0.68424095	–
196	2^{-2}	0.41295628	1.957	0.00121897	1.976	25.52270587	0.872	0.37526702	0.866
900	2^{-3}	0.10369078	1.993	0.00030420	2.002	12.92074925	0.982	0.11011474	1.768
3844	2^{-4}	0.02592309	1.999	0.00007602	2.000	6.53425879	0.983	0.02846417	1.951
15876	2^{-5}	0.00648563	1.998	0.00001900	2.000	3.27120390	0.998	0.00717641	1.987
64516	2^{-6}	0.00161877	2.002	0.00000475	1.999	1.63710571	0.998	0.00179764	1.997

TABLE 2. H^1 and L^2 errors and orders of convergence for the state and adjoint variables in Example 6.1.

N	h/h_0	$\left\lVert\bar{\Psi}-\bar{\Psi}_h\right\rVert_1$	δ_l	$\left\lVert\bar{\Psi}-\bar{\Psi}_h\right\rVert$	δ_l	$\left\lVert\bar{\Theta}-\bar{\Theta}_h\right\rVert_1$	δ_l	$\left\lVert\bar{\Theta}-\bar{\Theta}_h\right\rVert$	δ_l
36	2^{-1}	0.06403252	–	0.80432845E−2	–	0.24927604E−3	–	0.37036191E−4	–
196	2^{-2}	0.01290311	2.311	0.17586110E−2	2.193	0.05657690E−3	2.139	0.08441657E−4	2.133
900	2^{-3}	0.00315079	2.033	0.04899818E−2	1.843	0.01386737E−3	2.028	0.02134139E−4	1.983
3844	2^{-4}	0.00077657	2.020	0.01226971E−2	1.997	0.00345918E−3	2.003	0.00537248E−4	1.989
15876	2^{-5}	0.00019514	1.992	0.00312879E−2	1.971	0.00086273E−3	2.003	0.00134258E−4	2.000
64516	2^{-6}	0.00004781	2.029	0.00075095E−2	2.058	0.00021657E−3	1.994	0.00033748E−4	1.992

In all the tables, $h_0 = 1/\sqrt{2}$ is the initial mesh size and N denotes the number of degrees of freedom. Since Ω is convex, we have the index of elliptic regularity $\gamma = 1$. The numerical convergence rates with respect H^1 and L^2 norms for the state and adjoint variables are quadratic as predicted theoretically. Linear orders of convergence for the control variable and quadratic order of convergence for the post-processed control are obtained and this confirms the theoretical results established in Theorem 5.13.

Example 6.2. Let Ω be the non-convex L-shaped domain $\Omega = (-1,1)^2 \setminus \big([0,1)\times(-1,0]\big)$ and $\mathcal{C} = \mathrm{I}$, $\omega = \Omega$. We consider a problem with the exact singular solution borrowed from [16] in polar coordinates. The state and adjoint variables $\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2)$ and $\bar{\Theta} = (\bar{\theta}_1, \bar{\theta}_2)$ are given by

$$\bar{\psi}_1 = \bar{\psi}_2 = \bar{\theta}_1 = \bar{\theta}_2 = (r^2\cos^2\theta - 1)^2(r^2\sin^2\theta - 1)^2 r^{1+\gamma} g_{\gamma,\omega}(\theta)$$

where $\omega = \frac{3\pi}{2}$, and $\gamma \approx 0.5444837367$ is a non-characteristic root of $\sin^2(\gamma\omega) = \gamma^2\sin^2(\omega)$ with

$$\begin{aligned} g_{\gamma,\omega}(\theta) = {} & \left(\frac{1}{\gamma-1}\sin\big((\gamma-1)\omega\big) - \frac{1}{\gamma+1}\sin\big((\gamma+1)\omega\big)\right) \times \Big(\cos\big((\gamma-1)\theta\big) - \cos\big((\gamma+1)\theta\big)\Big) \\ & - \left(\frac{1}{\gamma-1}\sin\big((\gamma-1)\theta\big) - \frac{1}{\gamma+1}\sin\big((\gamma+1)\theta\big)\right) \times \Big(\cos\big((\gamma-1)\omega\big) - \cos\big((\gamma+1)\omega\big)\Big). \end{aligned}$$

The exact control $\bar{u}$ is chosen as $\bar{u}(x) = \pi_{[-600,-50]}\left(-\frac{1}{\alpha}\bar{\theta}_1(x)\right)$, where $\alpha = 10^{-3}$. The source terms $f, \tilde{f}$ and the observation $\Psi_d = (\psi_{1d}, \psi_{2d})$ are computed as in the previous example. The errors and orders of convergence for the numerical approximations to state, adjoint and control variables are shown in Tables 3 and 4. Since Ω is non-convex, we expect only $1/2 < \gamma < 1$ as predicted by the theoretical results. Note that only suboptimal orders of convergence are attained for the state and adjoint variables in the energy, H^1 and L^2 norms. However, we

TABLE 3. Errors and orders of convergence for the state, adjoint, control and post-processed control variables in Example 6.2.

N	h/h_0	$\lVert\bar{\Psi}-\bar{\Psi}_h\rVert_2$	δ_l	$\lVert\bar{\Theta}-\bar{\Theta}_h\rVert_2$	δ_l	$\lVert\bar{u}-\bar{u}_h\rVert$	δ_l	$\lVert\bar{u}-\tilde{\bar{u}}_h\rVert$	δ_l
36	2^{-1}	9.81990941	–	7.47714482	–	242.759537	–	34.2926324	–
164	2^{-2}	2.95442143	1.732	2.82045689	1.406	116.204133	1.062	9.72134519	1.818
708	2^{-3}	1.41082575	1.066	1.35893052	1.053	61.137057	0.926	5.29868866	0.875
2948	2^{-4}	0.82993102	0.765	0.82022205	0.728	31.226881	0.969	2.54226401	1.059
12036	2^{-5}	0.54373393	0.610	0.54214544	0.597	15.691309	0.992	1.17813556	1.109
48644	2^{-6}	0.36837935	0.561	0.36796971	0.559	7.860646	0.997	0.55275176	1.091

TABLE 4. H^1 and L^2 errors and orders of convergence for the state and adjoint variables in Example 6.2.

N	h/h_0	$\lVert\bar{\Psi}-\bar{\Psi}_h\rVert_1$	δ_l	$\lVert\bar{\Psi}-\bar{\Psi}_h\rVert$	δ_l	$\lVert\bar{\Theta}-\bar{\Theta}_h\rVert_1$	δ_l	$\lVert\bar{\Theta}-\bar{\Theta}_h\rVert$	δ_l
36	2^{-1}	1.10962279	–	0.26602624	–	0.46789881	–	0.05277374	–
164	2^{-2}	0.15147063	2.872	0.02879033	3.207	0.11399788	2.037	0.01813025	1.5414
708	2^{-3}	0.06196779	1.289	0.01416231	1.023	0.04533146	1.330	0.00982844	0.883
2948	2^{-4}	0.02244196	1.465	0.00484927	1.546	0.02080671	1.123	0.00479517	1.035
12036	2^{-5}	0.00895880	1.324	0.00178338	1.443	0.00970478	1.100	0.00226287	1.083
48644	2^{-6}	0.00405435	1.143	0.00080248	1.152	0.00455562	1.091	0.00106533	1.086

observe a linear order of convergence for the control variable and 2γ rate of convergence for the post-processed control and this confirms the theoretical results established in Theorem 5.13.

7. CONCLUSIONS

In this paper, an attempt has been made to establish error estimates for state, adjoint and control variables for distributed optimal control problems governed by the von Kármán equations defined over polygonal domains. The convergence results in energy, H^1 and L^2 norms for state and adjoint variables are derived under realistic regularity assumptions on the exact solution of the problem. Also, the convergence results in L^2 norm for the control variable and a post processed control are established. The results of the numerical experiments confirm the theoretical error estimates. The extension of the analysis to nonconforming finite element methods, say piecewise quadratic Morley finite elements or C^0 interior penalty methods is quite attractive from the implementation perspective. However, for the control problem, the nonconformity of the Morley finite element space or C^0 interior penalty methods offers a lot of challenges in a straight forward extension of the theoretical error estimates. We are currently working on this problem.

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