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UNIFORM REGULARITY IN THE RANDOM SPACE AND SPECTRAL ACCURACY OF THE STOCHASTIC GALERKIN METHOD FOR A KINETIC-FLUID TWO-PHASE FLOW MODEL WITH RANDOM INITIAL INPUTS IN THE LIGHT PARTICLE REGIME[☆]

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Abstract. We consider a kinetic-fluid model with random initial inputs which describes disperse two-phase flows. In the light particle regime, using energy estimates, we prove the uniform regularity in the random space of the model for random initial data near the global equilibrium in some suitable Sobolev spaces, with the randomness in the initial particle distribution and fluid velocity. By hypocoercivity arguments, we prove that the energy decays exponentially in time, which means that the long time behavior of the solution is insensitive to such randomness in the initial data. Then we consider the generalized polynomial chaos stochastic Galerkin method (gPC-sG) for the same model. For initial data near the global equilibrium and smooth enough in the physical and random spaces, we prove that the gPC-sG method has spectral accuracy, uniformly in time and the Knudsen number, and the error decays exponentially in time.

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1. Introduction

In this paper we consider a kinetic-fluid model for disperse two-phase flows, known as the Navier-Stokes-Vlasov-Fokker-Planck system, first proposed in [9, 10]. Similar two-phase flow models appear in combustion theory [5, 8, 29], the dynamic of sprays [14, 15, 27] and granular flow [1, 7], to name a few. The model we consider describes a mixture of two types of material, called the primary phase and the secondary phase. They are assumed to satisfy the following physical assumptions:

(1) The primary phase is liquid or dilute gas, and therefore modeled by the incompressible Navier-Stokes equations.

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(2) The secondary phase is small particles (or droplets, bubbles), scattered in the fluid, and it is modeled by a kinetic equation.
(3) The interaction between the two phases is assumed to be the Stokes drag force, *i.e.*, a particle is subject to a force proportional to the relative velocity between it and the fluid.
(4) The particles are assumed to be subject to the Brownian motions.

There are two scalings that are physically important: one is the light particle regime [9], which assumes:

(1) The velocity of the fluid is small compared to the typical molecular velocity of the particles.
(2) The particles are light, and thus its effect on the fluid is small.
(3) The relaxation time is much smaller than the typical time scale.

Another one is the fine particle regime [10], which assumes:

(1) The particle size is small compared to the typical length scale.
(2) The density of the fluid and particles are of the same order.
(3) The relaxation time is much smaller than the typical time scale.

In this paper we focus on the light particle regime. For simplicity the space is taken as $\mathbb{T}^3 = [-\pi,\pi]^3$ with periodic boundary condition. The equations for the model are given by

$$\begin{cases} u_t + u\cdot\nabla_x u + \nabla_x p - \Delta_x u = \dfrac{1}{\epsilon}\displaystyle\int (v-\epsilon u)F\,\mathrm{d}v, \\ \nabla_x\cdot u = 0, \\ F_t + \dfrac{1}{\epsilon} v\cdot\nabla_x F = \dfrac{1}{\epsilon^2}\nabla_v\cdot(\nabla_v F + (v-\epsilon u)F), \end{cases} \tag{1.1}$$

with initial data

$$u|_{t=0} = u_0, \quad \nabla_x\cdot u_0 = 0, \quad F|_{t=0} = F_0, \tag{1.2}$$

where $t\in\mathbb{R}^+$ is the time variable, $x\in\mathbb{T}^3$ is the space variable, and $v\in\mathbb{R}^3$ is the velocity variable. $u=u(t,x)$ is the velocity field of the fluid, and $F=F(t,x,v)$ is the distribution function of the particles. ϵ is the Knudsen number, which satisfies $0<\epsilon\le 1$. $\epsilon=O(1)$ corresponds to the kinetic regime, while $\epsilon\to 0$ corresponds to the fluid regime.

This system satisfies the following conservation properties:

$$\begin{aligned} &\text{Mass conservation: } \frac{\mathrm{d}}{\mathrm{d}t}\int\int F\,\mathrm{d}v\,\mathrm{d}x = 0, \\ &\text{Momentum conservation: } \frac{\mathrm{d}}{\mathrm{d}t}\left(\int u\,\mathrm{d}x + \epsilon\int\int vF\,\mathrm{d}v\,\mathrm{d}x\right) = 0, \\ &\text{Energy/Entropy dissipation: } \frac{\mathrm{d}}{\mathrm{d}t}\left(\int \frac{|u|^2}{2}\,\mathrm{d}x + \int\int (F\ln F + \frac{|v|^2}{2}F)\,\mathrm{d}v\,\mathrm{d}x\right) \\ &\qquad + \frac{1}{\epsilon^2}\int\int \frac{|(\epsilon u - v)F - \nabla_v F|^2}{F}\,\mathrm{d}v\,\mathrm{d}x + \int |\nabla_x u|^2\,\mathrm{d}x = 0. \end{aligned} \tag{1.3}$$

As $\epsilon\to 0$, it is shown in [9] that (1.1) has a hydrodynamic limit

$$\begin{cases} u_t + u\cdot\nabla_x u + \nabla_x p - \Delta_x u = 0, \\ \nabla_x\cdot u = 0, \\ \partial_t\rho + \nabla_x\cdot(u\rho - \nabla_x\rho) = 0, \end{cases} \tag{1.4}$$

with $\rho(x) = \int F(x,v)\,\mathrm{d}v$ being the particle density, which is self-consistent Navier-Stokes equations for u, and a convection-diffusion equation for ρ with drift velocity u.

Goudon *et al.* [11] proved the first existence result of (1.1), in the case of kinetic regime ($\epsilon = O(1)$) and initial data near the global equilibrium, which means that F is close enough to the global Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{3/2}|\mathbb{T}^3|} e^{-|v|^2/2}, \tag{1.5}$$

and u is close to 0, in some suitable Sobolev spaces. In fact their method also works for small ϵ. They first write

$$F = \mu + \sqrt{\mu} f. \tag{1.6}$$

Then (1.1) becomes the following system for (u, f):

$$\begin{cases} u_t + u \cdot \nabla_x u + \nabla_x p - \Delta_x u + u + \displaystyle\int \sqrt{\mu} u f \,\mathrm{d}v - \frac{1}{\epsilon} \int v \sqrt{\mu} f \,\mathrm{d}v = 0, \\ \nabla_x \cdot u = 0, \\ f_t + \dfrac{1}{\epsilon} v \cdot \nabla_x f + \dfrac{1}{\epsilon}\left(\nabla_v - \dfrac{v}{2}\right) \cdot (uf) - \dfrac{1}{\epsilon} u \cdot v \sqrt{\mu} = \dfrac{1}{\epsilon^2}\left(\dfrac{-|v|^2}{4} + \dfrac{3}{2} + \Delta_v\right) f, \end{cases} \tag{1.7}$$

with initial data

$$u|_{t=0} = u_0, \quad f|_{t=0} = f_0. \tag{1.8}$$

They assume that (u_0, f_0), the perturbation of initial data, satisfies the conditions

$$\int u_0 \,\mathrm{d}x + \int\int v\sqrt{\mu} f_0 \,\mathrm{d}v\,\mathrm{d}x = 0, \quad \nabla_x \cdot u_0 = 0, \tag{1.9}$$

$$\int\int \sqrt{\mu} f_0 \,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.10}$$

which mean that the perturbation does not affect the total momentum and mass, and the perturbation of the fluid velocity is divergence-free. Then, combining with a relation for the mean fluid velocity

$$\bar{u}(t) = \frac{1}{|\mathbb{T}^3|} \int u(t,x)\,\mathrm{d}x, \tag{1.11}$$

$$\bar{u}_t + 2\bar{u} + \frac{1}{|\mathbb{T}^3|} \int\int \sqrt{\mu}(uf)\,\mathrm{d}v\,\mathrm{d}x = 0, \tag{1.12}$$

which is a consequence of (1.9), using energy estimates, they proved the decay of an energy functional, defined as the summation of some suitable Sobolev norms, under the assumption that it is small enough initially. Then, by using hypocoercivity arguments, they proved that the L^2 norms of u and f decay exponentially in time, under some smoothness assumptions.

On the numerical aspect, an Asymptotic-Preserving (AP) scheme was developed by Goudon *et al.* [12] for the model with the fine particle regime. The AP property, first introduced by Jin [16] for time-dependent kinetic problems, means that a numerical scheme for a kinetic model, as the Knudsen number ϵ goes to zero, automatically becomes a numerical scheme for the hydrodynamic limit of the kinetic model, with a numerical stability independent of ϵ. The AP property enables one to capture the hydrodynamic limit without resolving the small Knudsen number. Simply speaking, the AP scheme for this model uses a combination of the projection method for the Navier-Stokes equations and an implicit treatment of the stiff Fokker-Planck operator.

Most of the works on kinetic-fluid two-phase flow models are deterministic. However, there are many sources of uncertainties in these models. For example, the initial data and boundary data usually come from experiments, and thus have measurement error. Uncertainty could also arise from the modeling of drag forces, particle diffusions, etc. It is important to quantify these uncertainties, because such quantification can help us understand how the uncertainties affect the solution, and therefore make reliable predictions.

For simplicity, for the model (1.1) we only consider the uncertainty from initial data. To model the uncertainty, we use the same equations, but let the functions $u = u(t, x, z)$ and $F = F(t, x, v, z)$ depend on a random variable z, which lives in the random space I_z with probability distribution $\pi(z)\,\mathrm{d}z$. Then the uncertainty from initial data is described by letting the initial data u_0 and F_0 depend on z. We will assume the random space I_z is one-dimensional, for simplicity of notation. Our results can be extend to the case multi-dimensional random spaces. See more discussions at the end of Section 2.

We summarize some popular numerical methods for uncertainty quantification (UQ) [6, 13, 24, 30, 31]: the first one is Monte-Carlo (MC) methods [25], which take random samples in I_z, solve the deterministic problem on these samples, and then get the statistical moments by taking the average on these samples. MC methods are half-order accurate for any dimensional random spaces, and thus they are not accurate enough for low dimensional random spaces, but very efficient for high-dimensional random spaces. The second method is stochastic collocation (sC) methods [3, 4, 26, 32], which take sample points on a well-designed grid (quadrature points, sparse grids, or by some optimization procedure), compute the deterministic solutions on the samples, and then reconstruct the solution in the whole random domain by some interpolation rules. SC methods can achieve good accuracy in low dimensional random spaces, but the efficiency drops as the dimension becomes high. The third method is stochastic Galerkin (sG) methods [2, 4, 33], which takes an orthonormal basis in the random domain, approximate the functions by a truncated Fourier series, and then obtain a deterministic system of equations on the Fourier coefficients via the Galerkin projection. SG methods are as accurate as sC methods for low dimensional random spaces, and behave better than sC for moderately high dimensional random spaces if one wants to achieve high accuracy [4].

For sG methods for kinetic equations with a hydrodynamic limit, it is important to have a property called "stochastic asymptotic-preserving" (s-AP), first proposed by Jin *et al.* [20]. The s-AP property means that as the small parameter ϵ goes to zero, the sG method for the kinetic equation automatically becomes an sG method for the limiting hydrodynamic system. Similar to the AP property, the s-AP property enables one to choose all numerical parameters, including the number of basis functions K in polynomial chaos approximations, independent of ϵ. In [18] the authors proposed an s-AP method for the model with the fine particle regime. We followed the idea of the AP scheme in [12], and overcame the difficulty of the implicit treatment of the vectorized Fokker-Planck operator by proving a structure theorem of this operator.

In order to analyze the accuracy of the sC and sG methods, it is very important to analyze the regularity of the exact solution in the random space. In fact, in order to achieve a high accuracy order for the interpolations in sC, and the truncated series approximations in sG, one usually needs such regularity. For the sG methods, it is not straightforward to prove accuracy from the z-regularity, due to the Galerkin projection error. Instead, one has to derive the evolution equations for the error, and then conduct estimates based on the z-regularity of the exact solution. Recently there have been several attempts to prove the uniform-in-ϵ random space regularity for kinetic equations, including Jin *et al.* [21] for linear transport equations, Jin–Zhu [19] for the Vlasov–Poisson–Fokker–Planck equation, Jin–Liu [17] and Liu [23] for the linear semiconductor Boltzmann equation, and Li–Wang [22] for general linear kinetic equations that conserve mass. [17, 21, 23] also proves the spectral accuracy for the sG method.

In this paper, we first analyze the z-regularity of (1.7) for random initial data near the global equilibrium in some suitable Sobolev spaces (with derivatives with respect to x and z). We use energy estimates and hypocoercivity arguments similar to [11] on the z-derivatives of u and f. Our result implies that for near equilibrium initial data with regular dependence on x and z, the solution depends regularly on z for all time, and is insensitive to random perturbations on the initial data for large time. Then for the sG method, we consider the most popular choice of basis functions, the generalized polynomial chaos (gPC) [33], *i.e.*, the orthonormal polynomials with respect to $\pi(z)\,\mathrm{d}z$. We write the equations for the gPC coefficients and do energy estimates, in which we manage to make this estimate independent of K, the number of basis functions. This difficulty will be explained in detail in the next paragraph. Finally we write the equations for the error of the gPC-sG method and do energy and hypocoercivity estimates. Our result implies that if the random initial data (u_0, f_0) is small enough in some suitable Sobolev spaces, then the gPC-sG method has spectral accuracy, uniformly in time and ϵ, and captures the exponential decay in time of the exact solution. An important feature of our results is that all the constants involved are *independent of* ϵ.

As mentioned in the previous paragraph, the biggest difficulty is that a naive energy estimates for the gPC coefficients require a small initial data condition depending on K, the number of basis functions, since the nonlinear terms in (1.7) produce a large number (K^3) of terms in the equations of the gPC coefficients. But it is desirable to have a small initial data condition independent of the numerical parameter K, which means that the accuracy results are true for this set of initial data, for all K. To overcome this difficulty, we introduce a weighted sum of the Sobolev norm of the gPC coefficients (Lem. 5.1), which enables us to combine some of the terms together as part of a convergent series, and control the nonlinear terms with an estimate independent of K.

This paper is organized as follows: in Section 2, we introduce some notations and state the main results; in Section 3 we prove the energy estimates for the z-derivatives of u and f; in Section 4 we use hypocoercivity arguments to prove the exponential decay of these derivatives; in Section 5 we prove the spectral accuracy of the sG method; in Section 6 we conclude the paper.

2. Notations and statements of main results

2.1. Notations

Due to the extra variable z (compared to [11]), our notation is slightly different from that in [11]. All the norms or inner products with a single bound (like $|\cdot|, \langle\cdot,\cdot\rangle$) are only integrated in x, v and pointwise in z (thus the result is a function in z). All the norms or inner products with a double bound (like $\|\cdot\|, \langle\langle\cdot,\cdot\rangle\rangle$) are integrated in all variables, thus the result is a number.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a multi-index. Then define

$$\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}. \tag{2.1}$$

The z-derivative of order γ of a function f is denoted by

$$f^\gamma = \partial_z^\gamma f. \tag{2.2}$$

There will not be any Sobolev norm with v-derivatives, so we do not give a short notation for them.

For function $u = u(x)$, $f = f(x, v)$, define the Sobolev norm (with x-derivatives)

$$\|u\|_s^2 = \sum_{|\alpha|\le s} \|\partial^\alpha u\|_{L_x^2}^2, \quad \|f\|_s^2 = \sum_{|\alpha|\le s} \|\partial^\alpha f\|_{L_{x,v}^2}^2. \tag{2.3}$$

In particular, $\|u\|_0$ denote the L_x^2 norm of u. For function $u = u(x, z)$, $f = f(x, v, z)$, define the sum of Sobolev norms

$$|u|_{s,r}^2 = \sum_{|\gamma|\le r} \|u^\gamma(\cdot, z)\|_s^2, \quad |f|_{s,r}^2 = \sum_{|\gamma|\le r} \|f^\gamma(\cdot,\cdot, z)\|_s^2, \tag{2.4}$$

where $|u|_{s,r}$ and $|f|_{s,r}$ are functions of z. Then define the expected value of the total Sobolev norm by

$$\|u\|_{s,r}^2 = \int |u|_{s,r}^2 \pi(z)\,\mathrm{d}z, \quad \|f\|_{s,r}^2 = \int |f|_{s,r}^2 \pi(z)\,\mathrm{d}z. \tag{2.5}$$

For function $\bar{u} = \bar{u}(z)$, define the sum of derivatives and the Sobolev norm by

$$|\bar{u}|_r^2 = \sum_{|\gamma|\le r} |\bar{u}^\gamma|^2, \quad \|\bar{u}\|_r^2 = \int |\bar{u}|_r^2 \pi(z)\,\mathrm{d}z. \tag{2.6}$$

In all these notations, the sub-index r is omitted when $r = 0$.

The L^2 inner product of functions defined on x-space of x, v-space will be denoted by $\langle\cdot,\cdot\rangle$, *i.e.*,

$$\langle f, g\rangle = \int fg\,\mathrm{d}x, \quad \text{or} \quad \langle f, g\rangle = \int\int fg\,\mathrm{d}v\,\mathrm{d}x. \tag{2.7}$$

In case the inputs also depend on z, $\langle f, g\rangle$ only integrates in x or (x, v), and the result is a function in z. For example,

$$\langle f, g\rangle(z) = \int f(x,z)g(x,z)\,\mathrm{d}x. \tag{2.8}$$

Then we introduce the inner products related to the hypocoercivity arguments. Define

$$\mathcal{K} = \nabla_v + \frac{v}{2}, \quad \mathcal{P} = v\cdot\nabla_x, \quad \mathcal{S}_i = [\mathcal{K}_i, \mathcal{P}] = \mathcal{K}_i\mathcal{P} - \mathcal{P}\mathcal{K}_i = \partial_{x_i}, \quad \mathcal{K}^* = -\nabla_v + \frac{v}{2}, \tag{2.9}$$

where $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$, in the sense that $\langle \mathcal{K}f, g\rangle = \langle f, \mathcal{K}^*\cdot g\rangle$, where f has one component and g has three components.

For functions $f = f(x,v)$, $g = g(x,v)$, define

$$\begin{aligned}(f,g) &= 2\langle \mathcal{K}f, \mathcal{K}g\rangle + \epsilon\langle \mathcal{K}f, \mathcal{S}g\rangle + \epsilon\langle \mathcal{S}f, \mathcal{K}g\rangle + \epsilon^2\langle \mathcal{S}f, \mathcal{S}g\rangle,\\ [f,g] &= \langle \mathcal{K}f, \mathcal{K}g\rangle + \epsilon^2\langle \mathcal{S}f, \mathcal{S}g\rangle + \langle \mathcal{K}^2 f, \mathcal{K}^2 g\rangle + \epsilon^2\langle \mathcal{K}\mathcal{S}f, \mathcal{K}\mathcal{S}g\rangle,\end{aligned} \tag{2.10}$$

where we denote $\langle \mathcal{K}\mathcal{S}f, \mathcal{K}\mathcal{S}g\rangle := \sum_{i,j=1}^3 \langle \mathcal{K}_i\mathcal{S}_j f, \mathcal{K}_i\mathcal{S}_j g\rangle$.

For functions $f = f(x,v,z)$, $g = g(x,v,z)$, define

$$(f,g)_{s,r} = \sum_{|\gamma|\le r}\sum_{|\alpha|\le s} (\partial^\alpha f^\gamma(\cdot,\cdot,z), \partial^\alpha g^\gamma(\cdot,\cdot,z)), \tag{2.11}$$

where $(f,g)_{s,r}$ is a function of z. Similarly define $[f,g]_{s,r}$.

Then we introduce the inner product in the (x,v,z) space:

$$\langle\langle f, g\rangle\rangle = \int \langle f, g\rangle \pi(z)\,\mathrm{d}z, \tag{2.12}$$

and similarly define $((f,g))$, $((f,g))_{s,r}$, $[[f,g]]$, $[[f,g]]_{s,r}$ as the corresponding inner products integrated in z. We also define the following norms in the (x,v,z) space:

$$\begin{aligned} \|u\|_{W^{s,\infty}} &= \max_{|\alpha|\le s} \|\partial^\alpha u\|_{L^\infty_{x,z}}, \\ \|f\|_{W^{s,\infty}} &= \max_{|\alpha|\le s} \|\partial^\alpha f\|_{L^\infty_{x,z}(L^2_v)}. \end{aligned} \tag{2.13}$$

2.2. Regularity in the random space

Now we focus on the system (1.7) with the random variable z. In all of our results, the constants involved are *independent of* ϵ.

Both results in this subsection can be viewed as generalization of those in [11]. Our first main result is the following energy estimate assuming near equilibrium initial data:

Theorem 2.1. *Assume (u,f) solves (1.7) with initial data verifying (1.9). Fix a point z. Define the energy*

$$E(t;z) = E_{s,r}(t;z) = |u|^2_{H^{s,r}} + |f|^2_{s,r} + |\bar{u}|^2_r, \tag{2.14}$$

with integers $s \ge 2$ and $r \ge 0$. Then there exists a constant $c_1 = c_1(s,r) > 0$, such that $E(0;z) \le c_1$ implies that $E(t;z)$ is non-increasing in t.

This theorem is proved by an energy estimate on $\partial^\alpha f^\gamma$. This theorem means that for initial data near the global equilibrium, in the sense that $E(0;z)$ is small, the solution depends regularly in z for all time and all ϵ, and the z-derivatives are bounded uniformly in t and ϵ.

From now on we will omit the dependence on z of E, in case there is no confusion.

Next, by a standard hypocoercivity argument, we strengthen the above theorem into the following one:

Theorem 2.2. *Assume (u,f) solves (1.7) with initial data verifying (1.9) and (1.10). There exists a constant $c_1'(s,r)$ such that, if we assume $s \ge 0$, $E_{s+3,r}(0) \le c_1'(s,r)$, and that $C^h_{s,r} = (f,f)_{s,r}|_{t=0}$ (defined by (2.11)) is finite, then there exists a constant $\lambda > 0$ such that*

$$E_{s,r}(t) \le C(E_{s,r}(0) + C^h_{s,r})e^{-\lambda t}, \tag{2.15}$$

where $C = C(s,r)$.

This theorem implies that as long as the random perturbation (u_0, f_0) on the initial data is small in suitable Sobolev spaces and has vanishing total mass and momentum, the long-time behavior of the solution is not sensitive to the random initial data. The smallness condition is independent of ϵ.

2.3. Error estimate for the gPC-sG method

We then introduce the gPC-sG method for the two-phase flow model (1.7). We start by taking the basis functions $\{\phi_k(z)\}_{k=1}^\infty$ as the gPC basis, *i.e.*, the set of polynomials defined on I_z, orthonormal with respect to the given probability measure $\pi(z)\,\mathrm{d}z$, with ϕ_k being a polynomial of degree $k-1$.

We expand the functions u, f, p into

$$u(t,x,z) = \sum_{k=1}^\infty u_k(t,x)\phi_k(z), \quad f(t,x,v,z) = \sum_{k=1}^\infty f_k(t,x,v)\phi_k(z), \quad p(t,x,z) = \sum_{k=1}^\infty p_k(t,x)\phi_k(z), \tag{2.16}$$

and approximate them by truncated series up to order K:

$$u \approx u^K = \sum_{k=1}^{K} u_k \phi_k(z), \quad f \approx f^K = \sum_{k=1}^{K} f_k \phi_k(z), \quad p \approx p^K = \sum_{k=1}^{K} p_k \phi_k(z). \tag{2.17}$$

Then substitute into (1.7) and conduct the Galerkin projection, one gets the following deterministic system for $(u_k, f_k)_{k=1}^K$:

$$\begin{cases} \partial_t u_k + (u \cdot \nabla_x u)_k + \nabla_x p_k - \Delta_x u_k + u_k + \int \sqrt{\mu}(uf)_k \,\mathrm{d}v - \int v\sqrt{\mu} f_k \,\mathrm{d}v = 0, \\ \nabla_x \cdot u_k = 0, \\ \partial_t f_k + v \cdot \nabla_x f_k + (\nabla_v - \frac{v}{2}) \cdot (uf)_k - u_k \cdot v\sqrt{\mu} = (\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) f_k, \end{cases} \tag{2.18}$$

with initial data

$$u_k|_{t=0} = (u_0)_k = \int u_0 \phi_k(z) \pi(z) \,\mathrm{d}z, \quad f_k|_{t=0} = (f_0)_k. \tag{2.19}$$

Here the gPC coefficient of a product is given by

$$(uw)_k = \sum_{i,j=1}^{K} S_{ijk} u_i w_j, \tag{2.20}$$

where

$$S_{ijk} = \int \phi_i \phi_j \phi_k \pi(z) \,\mathrm{d}z, \tag{2.21}$$

is the triple product coefficient.

The goal is to show that under smallness assumptions on the initial data, the gPC-sG method (2.18) has uniform-in-ϵ spectral accuracy for all K. We start from an energy estimate for (2.18). Although being similar to the original system (1.7), it indeed requires some new idea to obtain an estimate *independent of* K, *i.e.*, the smallness requirement on the initial data is independent of K. The difficulty comes from the K^2 nonlinear terms appeared in the gPC product (2.20).

To overcome this difficulty, we introduce the technical condition (2.22), and introduce the weighted Sobolev norm $\sum_{k=1}^K \|k^q u_k\|_s^2$ (see the theorem below for detail). This idea originates from the analog between the gPC series and Fourier series. If one takes a function $\Phi = \Phi(y)$ defined on $y \in [-1, 1]$, then $\|\Phi\|_{H^r}^2 = \sum_k |\hat{\Phi}_k (1 + |k|^2)^{r/2}|^2$ where $\hat{\Phi}_k$ denotes the k-th Fourier coefficient. Therefore our weighted Sobolev norm is almost a Sobolev norm in both x and z spaces. With this viewpoint, it is natural to expect a nonlinear estimate with this norm (Lem. 5.1, the key nonlinear estimate), being *independent of* K, similar to the nonlinear estimate $\|uw\|_{H^s} \le C\|u\|_{H^s}\|w\|_{H^s}$ in the x-space. With the aid of this new technique, we prove

Theorem 2.3. *Assume the technical condition*

$$\|\phi_k\|_{L^\infty} \le C k^p, \quad \forall k, \tag{2.22}$$

with a parameter $p > 0$*. Let* $q > p + 2$ *and* $s \geq 2$*. Let* (u_k, f_k)*,* $k = 1, \ldots, K$*, solve (2.18) with initial data verifying (1.9), and define the energy* E^K *by*

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} (\|k^q u_k\|_s^2 + \|k^q f_k\|_s^2 + |k^q \bar{u}_k|^2). \tag{2.23}$$

Then there exists a constant $c_2 = c_2(s, q) > 0$*, independent of* K*, such that* $E^K(0) \leq c_2$ *implies that* $E^K(t)$ *is decreasing in* t*.*

Next we give a sufficient condition on the initial data, under which the assumption $E^K(0) \leq c_2$ in Theorem 2.3 holds:

Proposition 2.4. *With the same assumptions as Theorem 2.3, the condition* $E^K_{s,q}(0) \leq c_2(s, q)$ *holds if* $\|E_{s,r}(0)\|_{L^1_z} \leq Cc_2(s, q)$ *with* $r > q + \frac{1}{2}$*, and* $C = C(s, q, r)$*.*

Theorem 2.3 is proved by the same type of energy estimate as Theorem 2.1, with the aid of the nonlinear estimate Lemma 5.1. Notice that c_2 being independent of K is important, because it implies that the condition $E^K(0) \leq c_2$ is in fact, in view of Proposition 2.4, a consequence of a smoothness condition on (u_0, f_0), for all K. This means for such initial data, the gPC-sG method is stable *for all* K.

We remark that (2.22) holds for gPC basis with respect to a large class of probability measures supported on a finite interval. To be precise, we have

Proposition 2.5. *Suppose* $I_z = [-R, R]$*,* $R < +\infty$ *with* $\pi(z)$ *satisfying* $1/\pi(z) \in L^{p_1}$ *for some* $p_1 > 0$*. Then (2.22) holds with* $p = 1 + 1/p_1$*.*

This proposition gives (2.22) for the uniform distribution on $[-1, 1]$ (with normalized Legendre polynomials as gPC basis), the distribution $\pi(z) = \frac{2}{\pi\sqrt{1-z^2}}$ on $[-1, 1]$ (with normalized Chebyshev polynomials as gPC basis), and all piecewise polynomial probability distributions on a finite interval with isolated zeros. More details about (2.22) can be found in Section 5.

Finally, by a combination of the above results, we obtain the spectral accuracy of the gPC-sG method, uniformly in t and ϵ, with a small initial data assumption on (u_0, f_0), independent of K and ϵ:

Theorem 2.6. *Assume (2.22) holds. Let* (u_k, f_k)*,* $k = 1, \ldots, K$*, solve (2.18) with initial data verifying (1.9)(1.10). There exists a constant* $c_1''(s, r)$ *such that the following holds: Assume* $s \geq 0$*,* $r > p + \frac{5}{2}$*,* $\|E_{s+3,r}(0)\|_{L^\infty_z} \leq c_1''(s, r)$*, and* $C^h_{s,r}$ *is finite. Then* E^e*, the energy of the gPC approximation error, defined by*

$$E^e = \|u^e\|_s^2 + \|f^e\|_s^2 + \|\bar{u}^e\|^2, \quad u^e = u - u^K, \quad f^e = f - f^K, \tag{2.24}$$

satisfies

$$E^e \leq \frac{C}{K^{2r}}, \tag{2.25}$$

for all time, i.e., *the gPC-sG method has* r*-th order accuracy uniformly in time.*

This theorem is proved by an energy estimate in the (x, v, z) space on (u^e, f^e) with the aid of the previous theorems.

Finally we prove that the error also decays exponentially in time, by a hypocoercivity argument:

Theorem 2.7. *Assume (2.22) holds. Let* (u_k, f_k)*,* $k = 1, \ldots, K$*, solves (2.18) with initial data verifying (1.9)(1.10). There exists a constant* $c_2''(s, r)$ *such that the following holds: Assume* $s \geq 0$*,* $r > p + \frac{5}{2}$*,*

$\|E_{s+6,r}(0)\|_{L^\infty_z} \le c_2''(s,r)$, *and* $C^h_{s+3,r}$ *is finite. Then there exists a constant* $\lambda^e > 0$ *such that*

$$E^e \le \frac{C}{K^{r-p-1/2}} e^{-\lambda^e t}. \tag{2.26}$$

These theorems imply that for random initial data near the global equilibrium, in the sense that (u_0, f_0) is small in some suitable Sobolev spaces, the gPC-sG method has spectral accuracy, uniformly in time and ϵ, and it captures the long-time behavior of (1.7) with random initial data.

Remark 2.8. In cases where the random space I_z has dimension $d > 1$, the proof of Theorems 2.1 and 2.2 stays valid, but the results of other theorems may deteriorate due to:

(1) The spectral accuracy of gPC approximation deteriorates. To be precise, suppose one takes the multi-dimensional gPC basis as the tensor product of one-dimensional ones, then the approximation error becomes $\frac{C}{K^{r/d}}$, where K is the number of basis functions.
(2) The constant p in (2.22) will become pd in the case of tensor product basis.

To investigate how d affects the estimate for the gPC-sG method is left as our future work.

3. Basic energy estimate: proof of Theorem 2.1

We first state some lemmas on nonlinear estimates. Denote the space of functions with finite $\|\cdot\|_s$ norm as

$$H^s = \{u(x) : \|u\|_s < \infty\}, \quad \tilde{H}^s = \{f(x,v) : \|f\|_s < \infty\}. \tag{3.1}$$

The following lemma is from [11]:

Lemma 3.1. *Let* $u = u(x) \in H^s, w = w(x) \in H^s, f = f(x,v) \in \tilde{H}^s$. *Then for* $s > 3/2$,

$$\|uw\|_s \le C\|u\|_s\|w\|_s, \tag{3.2}$$
$$\|uf\|_s \le C\|u\|_s\|f\|_s, \tag{3.3}$$

where $C = C(s)$.

It follows that

Lemma 3.2. *Let* $u = u(x,z) \in W^{r,\infty}_z(H^s), w = w(x,z) \in W^{r,\infty}_z(H^s), f = f(x,v,z) \in W^{r,\infty}_z(\tilde{H}^s)$. *Let* $|\gamma| \le r$. *Then for* $s > 3/2$ *and all* z,

$$|(uw)^\gamma|_s \le C|u|_{s,r}|w|_{s,r}, \tag{3.4}$$
$$|(uf)^\gamma|_s \le C|u|_{s,r}|f|_{s,r}, \tag{3.5}$$

where $C = C(s,r)$.

Proof. By the Leibniz rule,

$$(uw)^\gamma = \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} u^\beta w^{\gamma-\beta}. \tag{3.6}$$

Then

$$|(uw)^\gamma|_s \le \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} |u^\beta w^{\gamma-\beta}|_s \le C(s) \sum_{\beta=0}^{\gamma} \binom{\gamma}{\beta} |u^\beta|_s |w^{\gamma-\beta}|_s \le C(s,r)|u|_{s,r}|w|_{s,r}, \tag{3.7}$$

where the second inequality uses (3.2). This finishes the proof of (3.4). The proof of (3.5) is similar, in view of (3.3). ☐

And then a bilinear version follows:

Lemma 3.3. *Let* $u = u(x,z) \in W_z^{r,\infty}(H^s), w = w(x,z) \in W_z^{r,\infty}(H^s), y = y(x,z) \in W_z^{r,\infty}(H^s), f = f(x,v,z) \in W_z^{r,\infty}(\tilde{H}^s), g = g(x,v,z) \in W_z^{r,\infty}(\tilde{H}^s)$. *Let* $|\gamma| \le r$, $|\alpha| \le s$. *Then for* $s > 3/2$ *and all* z,

$$|\langle \partial^\alpha (uw)^\gamma, y^\gamma \rangle| \le C(\delta, s, r)|u|_{s,r}^2 |w|_{s,r}^2 + \delta |y|_{0,r}^2, \tag{3.8}$$

$$|\langle \partial^\alpha (uf)^\gamma, g^\gamma \rangle| \le C(\delta, s, r)|u|_{s,r}^2 |f|_{s,r}^2 + \delta |g|_{0,r}^2, \tag{3.9}$$

where δ *is any positive number.*

Proof. To prove (3.8),

$$\begin{aligned} ||\langle \partial^\alpha (uw)^\gamma, y^\gamma \rangle| &\le \frac{1}{4\delta}|\partial^\alpha (uw)^\gamma|_{L^2}^2 + \delta |y^\gamma|_{L^2}^2 \le \frac{1}{4\delta}|(uw)^\gamma|_s^2 + \delta |y|_{0,r}^2 \\ &\le C(\delta, s, r)|u|_{s,r}^2 |w|_{s,r}^2 + \delta |y|_{0,r}^2, \end{aligned} \tag{3.10}$$

where the first inequality uses Young's inequality, and the last inequality uses (3.4). The proof of (3.9) is similar. ☐

Proof of Theorem 2.1. Taking z-derivative of order γ and x-derivative of order α of (1.7), and taking z-derivative of order γ of (1.12) gives

$$\begin{aligned} &\partial_t \partial^\alpha u^\gamma + \partial^\alpha (u \cdot \nabla_x u)^\gamma + \nabla_x \partial^\alpha p^\gamma \underline{- \Delta_x \partial^\alpha u^\gamma} \underbrace{+ \partial^\alpha u^\gamma} + \int \sqrt{\mu}\, \partial^\alpha (uf)^\gamma \, \mathrm{d}v \underbrace{- \frac{1}{\epsilon} \int v \sqrt{\mu}\, \partial^\alpha f^\gamma \, \mathrm{d}v} = 0, \\ &\nabla_x \cdot \partial^\alpha u^\gamma = 0, \\ &\partial_t \partial^\alpha f^\gamma + \frac{1}{\epsilon} v \cdot \nabla_x \partial^\alpha f^\gamma + \frac{1}{\epsilon}(\nabla_v - \frac{v}{2}) \cdot \partial^\alpha (uf)^\gamma \underbrace{- \frac{1}{\epsilon} \partial^\alpha u^\gamma \cdot v \sqrt{\mu}} = \underbrace{\frac{1}{\epsilon^2}(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) \partial^\alpha f^\gamma}, \\ &\partial_t \bar{u}^\gamma \underline{+ 2\bar{u}^\gamma} + \frac{1}{|\mathbb{T}^3|} \int \int \sqrt{\mu} (uf)^\gamma \, \mathrm{d}v \, \mathrm{d}x = 0. \end{aligned} \tag{3.11}$$

Now do L^2 estimate on each equation above (except the second one), *i.e.*, multiply the first equation by $\partial^\alpha u^\gamma$ and integrate in x; multiply the third equation by $\partial^\alpha f^\gamma$ and integrate in (v, x); multiply the fourth equation by $\bar{u}^\gamma$. And then add the results together and sum over $|\gamma| \le r, |\alpha| \le s$. Then one gets the following equation (at each z):

$$\frac{1}{2} \partial_t E + G + B = 0, \tag{3.12}$$

where the energy E is given by (2.14). The good terms G are given by

$$G = \underline{G_1} + \underbrace{G_2} = \sum_{|\gamma| \le s} G_{1,\gamma} + \sum_{|\gamma| \le s} G_{2,\gamma}, \tag{3.13}$$

with

$$G_{1,\gamma} = |\nabla_x u^\gamma|_s^2 + 2|\bar{u}^\gamma|^2 \geq C|u^\gamma|_{s+1}^2,$$
$$G_{2,\gamma} = \left| u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma \right|_s^2, \tag{3.14}$$

where the above inequality is by the Poincare-Wirtinger inequality. G_1 and G_2 come from the underlined terms and the underbraced terms in (3.11), respectively. To verify the G_2 term, we provide the following calculation:

$$\begin{aligned}
&\langle \partial^\alpha u^\gamma, \partial^\alpha u^\gamma\rangle - \frac{1}{\epsilon}\langle v\sqrt{\mu}\partial^\alpha u^\gamma, \partial^\alpha f^\gamma\rangle - \frac{1}{\epsilon}\langle \partial^\alpha u^\gamma \cdot v\sqrt{\mu}, \partial^\alpha f^\gamma\rangle - \frac{1}{\epsilon^2}\left\langle \left(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v\right)\partial^\alpha f^\gamma, \partial^\alpha f^\gamma \right\rangle \\
&= \langle \partial^\alpha u^\gamma\sqrt{\mu}, \partial^\alpha u^\gamma\sqrt{\mu}\rangle - 2\frac{1}{\epsilon}\langle \partial^\alpha u^\gamma\sqrt{\mu}, \frac{v}{2}\partial^\alpha f^\gamma\rangle - 2\frac{1}{\epsilon}\langle \partial^\alpha u^\gamma\sqrt{\mu}, \nabla_v\partial^\alpha f^\gamma\rangle \\
&\quad + \frac{1}{\epsilon^2}\langle \nabla_v\partial^\alpha f^\gamma + \frac{v}{2}\partial^\alpha f^\gamma, \nabla_v\partial^\alpha f^\gamma + \frac{v}{2}\partial^\alpha f^\gamma\rangle \\
&= \langle A_1, A_1\rangle - 2\langle A_1, A_3\rangle - 2\langle A_1, A_2\rangle + \langle A_2 + A_3, A_2 + A_3\rangle \\
&= |A_1 - A_2 - A_3|_0^2 = \left| \partial^\alpha \left(u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma \right) \right|_0^2,
\end{aligned} \tag{3.15}$$

where we used integration by parts in v, $\nabla_v\sqrt{\mu} = -\frac{v}{2}\sqrt{\mu}$, and the notations

$$A_1 = \partial^\alpha u^\gamma\sqrt{\mu}, \quad A_2 = \frac{1}{\epsilon}\nabla_v\partial^\alpha f^\gamma, \quad A_3 = \frac{1}{\epsilon}\frac{v}{2}\partial^\alpha f^\gamma. \tag{3.16}$$

The notation $|\cdot|_0$ is interpreted by (2.4) with $s = r = 0$, *i.e.*, taking $L^2_{x,v}$ norm for a fixed z.

The bad terms B are given by

$$B = B_1 + B_2 + B_3 = \sum_{|\gamma|\leq r, |\alpha|\leq s} B_{1,\alpha,\gamma} + \sum_{|\gamma|\leq r, |\alpha|\leq s} B_{2,\alpha,\gamma} + \sum_{|\gamma|\leq r} B_{3,\gamma}, \tag{3.17}$$

with

$$B_{1,\alpha,\gamma} = \langle \partial^\alpha (u\cdot\nabla_x u)^\gamma, \partial^\alpha u^\gamma\rangle,$$
$$B_{2,\alpha,\gamma} = \left\langle \partial^\alpha (uf)^\gamma, \partial^\alpha\left[u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma \right] \right\rangle,$$
$$B_{3,\gamma} = \frac{1}{|\mathbb{T}^3|}\langle (uf)^\gamma, \bar{u}^\gamma\sqrt{\mu}\rangle, \tag{3.18}$$

coming from the nonlinear terms.

By using Lemma 3.3, the bad terms are controlled by

$$|B_{1,\alpha,\gamma}| \leq C(\delta)|u|_{s+1,r}^2|u|_{s,r}^2 + \delta|u|_{s,r}^2 \leq C(\delta)EG_1 + \delta G_1,$$
$$|B_{2,\alpha,\gamma}| \leq C(\delta)|u|_{s,r}^2|f|_{s,r}^2 + \delta\left| u^\gamma\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^\gamma - \frac{1}{\epsilon}\frac{v}{2}f^\gamma \right|_s \leq C(\delta)EG_1 + \delta G_2, \tag{3.19}$$
$$|B_{3,\gamma}| \leq C(\delta)|u|_{s,r}^2|f|_{s,r}^2 + \delta|\bar{u}^\gamma|^2 \leq C(\delta)EG_1 + \delta G_1. \tag{3.20}$$

In conclusion, we have the energy estimate

$$\frac{1}{2}\partial_t E \leq -(1 - C(\delta)E - C\delta)G. \tag{3.21}$$

Take $\delta = \frac{1}{4C}$ where C is the constant in (3.21), and $c_1(s,r) = \frac{1}{4C(\delta)}$. Then we will show that $E(t) \leq c_1$ for all t. In fact, let

$$T^* = \sup\{\tilde{T} \geq 0 : \sup_{0\leq t<\tilde{T}} E(t) \leq c_1\}. \tag{3.22}$$

Then it follows that $E(t) \leq c_1$ for $0 \leq t \leq T^*$. Then by our choice of δ and c_1,

$$1 - C(\delta)E - C\delta \geq 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}, \tag{3.23}$$

and therefore (3.21) implies

$$\partial_t E + G \leq 0, \tag{3.24}$$

for $0 \leq t \leq T^*$. This prevents T^* from being finite. Thus we proved $E(t) \leq c_1$ for all t, and as a result, (3.24) holds for all t. Thus $E(t)$ is decreasing in t. □

4. Hypocoercivity estimates: proof of Theorem 2.2

We will use the following lemma, which is Proposition 4.2 in [11]:

Lemma 4.1. *There exisits a constant $C > 0$ such that for $f(x,v) \in L^2_{x,v}$ orthogonal to $\sqrt{\mu}$, one has*

$$\|f\|^2_{L^2} \leq C(\|\mathcal{K}f\|^2_{L^2} + \|\mathcal{S}f\|^2_{L^2}). \tag{4.1}$$

We begin by proving the following lemma, which is indeed a modification of part of the proof of Proposition 4.1 in [11]:

Lemma 4.2. *For f and g orthogonal to $\sqrt{\mu}$,*

$$|(u \cdot \mathcal{K}^* f, g)| \leq C\frac{1}{\epsilon}\|u\|_{H^3}([f,f] + [g,g]). \tag{4.2}$$

Proof. Using the commutator relation

$$\mathcal{K}(u \cdot \mathcal{K}^* f) = (u \cdot \mathcal{K}^*)\mathcal{K}f + uf, \tag{4.3}$$

i.e.,

$$\mathcal{K}_i\left(\sum_{j=1}^3 u_j \mathcal{K}_j^* f\right) = \sum_{j=1}^3 u_j \mathcal{K}_j^* \mathcal{K}_i f + u_i f, \tag{4.4}$$

one gets

$$
\begin{aligned}
(u\cdot\mathcal{K}^*f,g) &= 2\langle \mathcal{K}(u\cdot\mathcal{K}^*f),\mathcal{K}g\rangle + \epsilon\langle \mathcal{K}(u\cdot\mathcal{K}^*f),\mathcal{S}g\rangle + \epsilon\langle \mathcal{S}(u\cdot\mathcal{K}^*f),\mathcal{K}g\rangle + \epsilon^2\langle \mathcal{S}(u\cdot\mathcal{K}^*f),\mathcal{S}g\rangle \\
&= 2\langle u\mathcal{K}f,\mathcal{K}^2g\rangle + 2\langle uf,\mathcal{K}g\rangle + \epsilon\langle u\mathcal{K}f,\mathcal{K}\mathcal{S}g\rangle + \epsilon\langle uf,\mathcal{S}g\rangle \\
&\quad +\epsilon\langle \mathcal{S}(uf),\mathcal{K}^2g\rangle + \epsilon^2\langle \mathcal{S}(uf),\mathcal{S}\mathcal{K}g\rangle \\
&= 2\langle u\mathcal{K}f,\mathcal{K}^2g\rangle + 2\langle uf,\mathcal{K}g\rangle + \epsilon\langle u\mathcal{K}f,\mathcal{K}\mathcal{S}g\rangle + \epsilon\langle uf,\mathcal{S}g\rangle \\
&\quad +\epsilon\langle (\mathcal{S}u)f,\mathcal{K}^2g\rangle + \epsilon\langle u(\mathcal{S}f),\mathcal{K}^2g\rangle \\
&\quad +\epsilon^2\langle (\mathcal{S}u)f,\mathcal{S}\mathcal{K}g\rangle + \epsilon^2\langle u(\mathcal{S}f),\mathcal{K}\mathcal{S}g\rangle.
\end{aligned} \tag{4.5}
$$

where the term $\langle \mathcal{S}(uf),\mathcal{S}\mathcal{K}g\rangle := \sum_{i,j=1}^3\langle \mathcal{S}_i(u_jf),\mathcal{S}_i\mathcal{K}_jg\rangle$. Now use the Cauchy-Schwarz inequality, Lemma 4.1, and the Sobolev inequality

$$
\|u\|_{L^\infty} + \|\nabla_x u\|_{L^\infty} \le C\|u\|_{H^3}, \tag{4.6}
$$

on each term. We provide the details for two of them and omit the others:

$$
\begin{aligned}
\langle uf,\mathcal{K}g\rangle &\le \|u\|_{L^\infty}\|f\|_{L^2}\|\mathcal{K}g\|_{L^2} \le C\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2}+\|\mathcal{S}f\|_{L^2})\|\mathcal{K}g\|_{L^2} \\
&\le C\|u\|_{L^\infty}\left(\|\mathcal{K}f\|_{L^2}^2 + \|\mathcal{K}g\|_{L^2}^2 + \epsilon\|\mathcal{S}f\|_{L^2}^2 + \frac{1}{\epsilon}\|\mathcal{K}g\|_{L^2}^2\right), \\
\epsilon\langle uf,\mathcal{S}g\rangle &\le \epsilon\|u\|_{L^\infty}\|f\|_{L^2}\|\mathcal{S}g\|_{L^2} \le C\epsilon\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2}+\|\mathcal{S}f\|_{L^2})\|\mathcal{S}g\|_{L^2} \\
&\le C\epsilon\|u\|_{L^\infty}(\|\mathcal{K}f\|_{L^2}^2 + \|\mathcal{S}g\|_{L^2}^2 + \|\mathcal{S}f\|_{L^2}^2 + \|\mathcal{S}g\|_{L^2}^2).
\end{aligned} \tag{4.7}
$$

Then one gets the conclusion, in view of the definition of $[\cdot,\cdot]$. ☐

Now we prove the following lemma, which is an analog to Proposition 4.1 of [11]:

Lemma 4.3. *Let the assumptions of Theorem 2.2 be fulfilled. Then there exists a constant $c_1'(s,r) \le c_1(s+3,r)$ such that, if we assume that $E_{s+3,r}(0) \le c_1'(s,r)$ is small enough, then there exists a constant $\lambda_1 > 0$ such that (at each z)*

$$
\partial_t(f,f)_{s,r} + \lambda_1\frac{1}{\epsilon^2}[f,f]_{s,r} \le C(\lambda_1)\left(|u|_{s,r}^2 + |\nabla_x u|_{s,r}^2 + \frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2\right). \tag{4.8}
$$

Proof. One can write the evolution equation of $\partial^\alpha f^\gamma$ as

$$
\begin{aligned}
\partial_t\partial^\alpha f^\gamma + \frac{1}{\epsilon}\mathcal{P}\partial^\alpha f^\gamma + \frac{1}{\epsilon^2}(\mathcal{K}^*\cdot\mathcal{K})\partial^\alpha f^\gamma &= \frac{1}{\epsilon}\partial^\alpha u^\gamma\cdot v\sqrt{\mu} \\
&\quad + \frac{1}{\epsilon}\sum_{0\le\eta\le\alpha}\sum_{0\le\beta\le\gamma}\binom{\gamma}{\beta}\binom{\alpha}{\eta}\partial^\eta u^\beta\cdot\mathcal{K}^*\partial^{\alpha-\eta}f^{\gamma-\beta}.
\end{aligned} \tag{4.9}
$$

We will take the $(\cdot,\cdot)$ inner product of (4.9) with $\partial^\alpha f^\gamma$. For the linear terms, by the same argument as the proof of Proposition 4.1 of [11], one gets

$$\begin{aligned}
\frac{1}{\epsilon}(\mathcal{P}\partial^\alpha f^\gamma,\partial^\alpha f^\gamma) &= 2\frac{1}{\epsilon}\langle \mathcal{S}\partial^\alpha f^\gamma,\mathcal{K}\partial^\alpha f^\gamma\rangle + |\mathcal{S}\partial^\alpha f^\gamma|_0^2 \geq \frac{3}{4}|\mathcal{S}\partial^\alpha f^\gamma|_0^2 - 4\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2,\\
\frac{1}{\epsilon^2}(\mathcal{K}^*\cdot\mathcal{K}\partial^\alpha f^\gamma,\partial^\alpha f^\gamma) &= 2\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2 + 2\frac{1}{\epsilon^2}|\mathcal{K}^2\partial^\alpha f^\gamma|_0^2 + |\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma|_0^2\\
&\quad + \frac{1}{\epsilon}\langle \mathcal{K}\partial^\alpha f^\gamma,\mathcal{S}\partial^\alpha f^\gamma\rangle + 2\frac{1}{\epsilon}\langle \mathcal{K}^2\partial^\alpha f^\gamma,\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma\rangle\\
&\geq \frac{3}{2}\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2 + \frac{1}{2}\frac{1}{\epsilon^2}|\mathcal{K}^2\partial^\alpha f^\gamma|_0^2 + \frac{1}{3}|\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma|_0^2 - \frac{1}{2}|\mathcal{S}\partial^\alpha f^\gamma|_0^2,\\
\frac{1}{\epsilon}|(\partial^\alpha u^\gamma\cdot v\sqrt{\mu},\partial^\alpha f^\gamma)| &= |2\frac{1}{\epsilon}\langle \mathcal{K}(\partial^\alpha u^\gamma\cdot v\sqrt{\mu}),\mathcal{K}\partial^\alpha f^\gamma\rangle + \langle \mathcal{K}(\partial^\alpha u^\gamma\cdot v\sqrt{\mu}),\mathcal{S}\partial^\alpha f^\gamma\rangle\\
&\quad + \langle \mathcal{S}(\partial^\alpha u^\gamma\cdot v\sqrt{\mu}),\mathcal{K}\partial^\alpha f^\gamma\rangle + \epsilon\langle \mathcal{S}(\partial^\alpha u^\gamma\cdot v\sqrt{\mu}),\mathcal{S}\partial^\alpha f^\gamma\rangle|\\
&\leq \delta(\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2 + |\mathcal{S}\partial^\alpha f^\gamma|_0^2) + C(\delta)(|u|_{s,r}^2 + |\nabla_x u|_{s,r}^2).
\end{aligned}\tag{4.10}$$

The notation $|\cdot|_0$ is interpreted by (2.4) with $s=r=0$, *i.e.*, taking $L^2_{x,v}$ norm for a fixed z. For the nonlinear term (the summation), we apply Lemma 4.2 and get

$$\begin{aligned}
\frac{1}{\epsilon}|(\partial^\eta u^\beta\cdot\mathcal{K}^*\partial^{\alpha-\eta}f^{\gamma-\beta},\partial^\alpha f^\gamma)| &\leq C\frac{1}{\epsilon^2}|\partial^\eta u^\beta|_{3,0}([\partial^{\alpha-\eta}f^{\gamma-\beta},\partial^{\alpha-\eta}f^{\gamma-\beta}] + [\partial^\alpha f^\gamma,\partial^\alpha f^\gamma])\\
&\leq C\frac{1}{\epsilon^2}|u|_{s+3,r}[f,f]_{s,r},
\end{aligned}\tag{4.11}$$

where we used the fact that the x and z derivatives commute with the operators $\mathcal{K}$ and $\mathcal{S}$. With these estimates, we get

$$\begin{aligned}
&\frac{1}{2}\partial_t(\partial^\alpha f^\gamma,\partial^\alpha f^\gamma) + \frac{1}{2}\frac{1}{\epsilon^2}|\mathcal{K}^2\partial^\alpha f^\gamma|_0^2 + \frac{1}{3}|\mathcal{S}\mathcal{K}\partial^\alpha f^\gamma|_0^2 + \frac{1}{4}|\mathcal{S}\partial^\alpha f^\gamma|_0^2 - \frac{5}{2}\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2\\
&\quad \leq \delta\left(\frac{1}{\epsilon^2}|\mathcal{K}\partial^\alpha f^\gamma|_0^2 + |\mathcal{S}\partial^\alpha f^\gamma|_0^2\right) + C(\delta)(|u|_{s,r}^2 + |\nabla_x u|_{s,r}^2) + C\frac{1}{\epsilon^2}|u|_{s+3,r}[f,f]_{s,r}.
\end{aligned}\tag{4.12}$$

Then we choose $\delta = 1/8$ to absorb the term $|\mathcal{S}\partial^\alpha f^\gamma|_0^2$ on the RHS by the same term on the LHS. Summing over α,γ, we get

$$\partial_t(f,f)_{s,r} + (\frac{1}{8} - C_1|u|_{s+3,r})\frac{1}{\epsilon^2}[f,f]_{s,r} \leq C_2(|u|_{s,r}^2 + |\nabla_x u|_{s,r}^2 + \frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2),\tag{4.13}$$

where $C_1 = NC$, $C_2 = \max\{3, NC(\delta)\}$, N being the number of possible pairs (α,γ).

Thus if one chooses $c_1' = \min\{c_1(s+3,r), \frac{1}{16C_1}\}$, then by Theorem 2.1, $E_{s+3,r}(t)$ is decreasing, so $E_{s+3,r}(t) \leq c_1'$ for all t. Thus $|u|_{s+3,r} \leq E_{s+3,r} \leq c_1'$ for all t, and one gets the conclusion, with $\lambda_1 = 1/16$.

□

Proof of Theorem 2.2. To obtain the energy decay estimate, we write

$$\begin{aligned}
G &= |\nabla_x u|_{s,r}^2 + 2|\bar{u}|_r^2 + |u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2\\
&\geq |\bar{u}|_r^2 + 2\lambda_2|u|_{s+1,r}^2 + |u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2\\
&\geq |\bar{u}|_r^2 + \lambda_2|u|_{s+1,r}^2 + \frac{1}{2}|u\sqrt{\mu} - \frac{1}{\epsilon}\mathcal{K}f|_{s,r}^2 + \lambda_3\frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2,
\end{aligned}\tag{4.14}$$

where $\lambda_3 = \min\{\frac{\lambda_2}{2}, \frac{1}{4}\}$. The first inequality is by the Poincare-Wirtinger inequality. The second inequality is because

$$\begin{aligned}\|\frac{1}{\epsilon}\mathcal{K}f\|_{s,r}^2 &= \|\left(\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}\right) + u\sqrt{\mu}|_{s,r}^2 \le 2\left(\|\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}\|_{s,r}^2 + |u\sqrt{\mu}|_{s,r}^2\right)\\ &= 2\left(\|\frac{1}{\epsilon}\mathcal{K}f - u\sqrt{\mu}\|_{s,r}^2 + |u|_{s,r}^2\right).\end{aligned} \tag{4.15}$$

Thus, by adding to (3.24) some positive constant λ_4 (to be chosen) times (4.8), we have

$$\partial_t \tilde{E} + \tilde{G} \le \lambda_4 \tilde{B}, \tag{4.16}$$

where

$$\tilde{E} = E + \lambda_4 (f,f)_{s,r}, \quad \tilde{G} = G + \lambda_4\lambda_1 \frac{1}{\epsilon^2}[f,f]_{s,r}, \tag{4.17}$$

$$\tilde{B} = C(\lambda_1)(|u|_{s,r}^2 + |\nabla_x u|_{s,r}^2 + \frac{1}{\epsilon^2}|\mathcal{K}f|_{s,r}^2). \tag{4.18}$$

It is clear from (4.14) that $\tilde{B} \le CG \le C\tilde{G}$. Thus by choosing $\lambda_4 = \min\{\frac{1}{2C}, 1\}$, C being the previous constant, we get

$$\partial_t \tilde{E} + \frac{1}{2}\tilde{G} \le 0. \tag{4.19}$$

Notice that Lemma 4.1 implies that

$$|f|_{s,r}^2 \le C(|\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2), \tag{4.20}$$

and by definition one also has

$$(f,f)_{s,r} \le C(|\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2) \le C\frac{1}{\epsilon^2}(f,f)_{s,r}. \tag{4.21}$$

Thus

$$\tilde{E} \le C(G + |f|_{s,r}^2) + \lambda_4((f,f))_{s,r} \le C(G + |\mathcal{K}f|_{s,r}^2 + |\mathcal{S}f|_{s,r}^2) \le C\tilde{G}. \tag{4.22}$$

This together with (4.19) implies

$$\tilde{E}(t) \le \tilde{E}(0)e^{-\lambda t}, \tag{4.23}$$

where $\lambda = \frac{1}{2C}$, C being the constant in (4.22).

Finally, the proof of Theorem 2.2 is finished by noticing that

$$E(t) \le \tilde{E}(t) \le \tilde{E}(0)e^{-\lambda t} \le (E(0) + C^h)e^{-\lambda t}. \tag{4.24}$$

□

5. Proof of spectral accuracy of the gPC-sG approximation

In order to prove the accuracy of the gPC-sG method, we first prove Theorem 2.3, which is an energy estimate for the gPC coefficients (u_k, f_k).

5.1. Estimate of the gPC coefficients: proof of Theorem 2.3

In this section, all the norms and inner products acting on ϕ_k are taken on the random space I_z, and with respect to the measure $\pi(z)\,\mathrm{d}z$ if not stated otherwise. In order to prove the estimate for the gPC coefficients, we need the extra assumption (2.22) on the basis functions.

We mention some special cases where (2.22) is satisfied [28]. For the case $I_z = [-1, 1]$ with uniform distribution, ϕ_k are the normalized Legendre polynomials, and (2.22) holds with $p = 1/2$. For the case $I_z = [-1, 1]$ with the distribution $\pi(z) = \frac{2}{\pi\sqrt{1-z^2}}$, ϕ_k are the normalized Chebyshev polynomials, and (2.22) holds with $p = 0$.

Now we prove Proposition 2.5, which guarantees (2.22) for gPC basis with respect to a large class of probability measures supported on a finite interval.

Proof of Proposition 2.5. First, if $\{\phi_k\}$ is the gPC basis for the probability measure $\pi(z)\,\mathrm{d}z$ on $[-R, R]$, then $\{\phi_k(R\cdot)\}$ is the gPC basis for the probability measure $R\pi(Rz)\,\mathrm{d}z$ on $[-1, 1]$. Therefore we can assume $R = 1$ without loss of generality.

Let $\Phi(z)$ be any degree k polynomial on $I_z = [-1, 1]$ with $\int_{I_z} \Phi(z)^2 \frac{1}{2}\,\mathrm{d}z = 1$, *i.e.*, has norm 1 in the L^2 space with uniform distribution $\frac{1}{2}\,\mathrm{d}z$. Then one can expand it into series of normalized Legendre polynomials $\{\psi_j\}$:

$$\Phi(z) = \sum_{j=1}^{k+1} \Phi_j \psi_j(z), \quad \sum_{j=1}^{k+1} \Phi_j^2 = 1. \tag{5.1}$$

Then it follows that

$$|\Phi(z)| \le \left(\sum_{j=1}^{k+1} \Phi_j^2\right)^{1/2} \left(\sum_{j=1}^{k+1} \psi_j(z)^2\right)^{1/2} \le Ck, \tag{5.2}$$

by the fact that $\{\psi_j\}$ satisfies (2.22) with $p = 1/2$. Thus $\|\Phi\|_{L^\infty} \le Ck$.

Take $\Phi = \frac{\phi_k}{\|\phi_k\|_{L^2(1/2\,\mathrm{d}z)}}$, we obtain

$$\|\phi_k\|_{L^\infty} \le Ck\|\phi_k\|_{L^2(1/2\,\mathrm{d}z)}. \tag{5.3}$$

Next writing $p_2 = p_1 + 1 > 1$, $p_2' = p_2/(p_2 - 1) = 1 + 1/p_1$, we estimate

$$\begin{aligned}
\|\phi_k\|_{L^2(1/2\,\mathrm{d}z)} &= \left(\int \phi_k(z)^2 \frac{1}{2\pi(z)} \pi(z)\,\mathrm{d}z\right)^{1/2} \\
&\le \left(\int |\phi_k(z)|^{2p_2'} \pi(z)\,\mathrm{d}z\right)^{1/(2p_2')} \left(\int \frac{1}{(2\pi(z))^{p_2}} \pi(z)\,\mathrm{d}z\right)^{1/(2p_2)} \\
&\le C\|\phi_k\|_{L^\infty}^{(p_2'-1)/p_2'} \|\phi_k\|_{L^2(\pi(z)\,\mathrm{d}z)}^{1/p_2'} \\
&= C\|\phi_k\|_{L^\infty}^{(p_2'-1)/p_2'},
\end{aligned} \tag{5.4}$$

where we use the assumption that $\int \pi(z)^{1-p_2}\,\mathrm{d}z < \infty$ in the second inequality, and $\|\phi_k\|_{L^2(\pi(z)\,\mathrm{d}z)} = 1$ in the last equality. Combining with (5.3) we conclude the proof. □

This proposition gives (2.22) for a large class of probability measures on a finite interval. For example, if $\pi(z)$ is continuous and has only finite number of zeros, with $\pi(z-z_0) \geq c|z-z_0|^{p_3}$ for some $p_3>0$, $c>0$ near any zero $z=z_0$, then the condition of Lemma 2.5 is satisfied with any $p_1 < 1/p_3$. This already includes all piecewise polynomial weights with separated zeros.

It follows from (2.22) that

$$|S_{ijk}| \leq Ci^p, \tag{5.5}$$

since

$$|S_{ijk}| \leq \|\phi_i\|_{L^\infty} \langle |\phi_j|, |\phi_k| \rangle \leq \|\phi_i\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_k\|_{L^2} \leq Ci^p. \tag{5.6}$$

Also, note that ϕ_k is a polynomial of degree $k-1$, orthogonal to all lower order polynomials. If $(i-1)+(j-1) < k-1$, then $S_{ijk}=0$. Thus S_{ijk} may be nonzero only when the triangle inequality

$$i + j \geq k+1, \tag{5.7}$$

holds.

Note that due to the symmetry in i,j,k of S_{ijk}, (5.5) and (5.7) also hold if i,j,k are permuted.

Then we have the following lemma, which is the key nonlinear estimate:

Lemma 5.1. *Assume condition (5.5). Let $q > p+2$. Let $s > \frac{3}{2}$, α be a multi-index with $|\alpha| \leq s$. Let $u_k = u_k(x) \in H^s, w_k = w_k(x) \in H^s, y_k = y_k(x) \in L^2, f_k = f_k(x,v) \in \tilde{H}^2, g_k = g_k(x,v) \in L^2$. Then*

$$\begin{aligned} \left|\sum_{k=1}^K k^{2q} \langle \partial^\alpha (uw)_k, y_k \rangle \right| &\leq C(\delta) \sum_{i=1}^K \|i^q u_i\|_s^2 \sum_{j=1}^K \|j^q w_j\|_s^2 + \delta \sum_{k=1}^K \|k^q y_k\|_0^2, \\ \left|\sum_{k=1}^K k^{2q} \langle \partial^\alpha (uf)_k, g_k \rangle \right| &\leq C(\delta) \sum_{i=1}^K \|i^q u_i\|_s^2 \sum_{j=1}^K \|j^q f_j\|_s^2 + \delta \sum_{k=1}^K \|k^q g_k\|_0^2, \end{aligned} \tag{5.8}$$

where the constants are independent of K, and δ is any positive constant.

Proof. We focus on the proof of the first inequality, and the second one is similar (just use (3.3) instead of (3.2)). Note (by (3.2))

$$k^{2q} \|S_{ijk} \partial^\alpha (u_i w_j)\|_0 \leq Ck^{2q} |S_{ijk}| \|u_i\|_s \|w_j\|_s = C\frac{k^{2q}}{i^q j^q} |S_{ijk}| \cdot \|i^q u_i\|_s \cdot \|j^q w_j\|_s. \tag{5.9}$$

First we consider the case $i \geq j$. Then since

$$i^q j^q \geq \frac{1}{C} \left(\frac{k+1}{2}\right)^q |S_{ijk}| j^{q-p}, \tag{5.10}$$

by (5.5) and (5.7), we conclude that

$$\frac{k^{2q}}{i^q j^q} |S_{ijk}| \leq Ck^q j^{p-q}. \tag{5.11}$$

Thus if we write the $(uw)_k$ on the LHS of (5.8) as a summation in i, j by (2.20), the $i \geq j$ terms can be estimated by

$$\begin{aligned}
\left| \sum_{k=1}^{K} k^{2q} \sum_{i,j=1;\, i\geq j}^{K} \chi_{ijk} S_{ijk} \langle \partial^\alpha (u_i w_j), y_k \rangle \right| &\leq \sum_{i,j,k=1;\, i\geq j}^{K} k^{2q} \| S_{ijk} \partial^\alpha (u_i w_j) \|_0 \cdot \| y_k \|_0 \cdot \chi_{ijk} \\
&\leq C \sum_{i,j,k=1;\, i\geq j}^{K} j^{p-q} \cdot \| i^q u_i \|_s \cdot \| j^q w_j \|_s \cdot \| k^q y_k \|_0 \cdot \chi_{ijk} \\
&\leq C \sum_{i,j,k=1}^{K} j^{p-q} \cdot \| i^q u_i \|_s \cdot \| j^q w_j \|_s \cdot \| k^q y_k \|_0 \cdot \chi_{ijk} \\
&\leq C(\delta) \sum_{i,j,k=1}^{K} j^{p-q} \cdot \| i^q u_i \|_s^2 \cdot \| j^q w_j \|_s^2 \cdot \chi_{ijk} + \delta \sum_{i,j,k=1}^{K} j^{p-q} \| k^q u_k \|_0^2 \cdot \chi_{ijk} \\
&= C(\delta) I + \delta II, \qquad (5.12)
\end{aligned}$$

where the second inequality uses (5.11), and χ_{ijk} is the indicator function of the set of indexes (i, j, k) for which $S_{ijk} \neq 0$.

Now we claim that

$$I \leq 2 \sum_{i=1}^{K} \| i^q u_i \|_s^2 \cdot \sum_{j=1}^{K} \| j^q w_j \|_s^2. \qquad (5.13)$$

In fact, fix i, then one can write

$$I = \sum_{i=1}^{K} \| i^q u_i \|_s^2 I_i, \quad I_i = \sum_{j,k=1}^{K} j^{p-q} \cdot \| j^q w_j \|_s^2 \chi_{ijk}. \qquad (5.14)$$

Notice that $\chi_{ijk} = 1$ implies that $i - j + 1 \leq k \leq i + j - 1$, by (5.7). Thus in the last summation, there is at most $2j$ terms corresponding to a fixed j. Thus

$$I_i \leq 2 \sum_{j=1}^{K} j^{p-q+1} \| j^q w_j \|_s^2 \leq 2 \sum_{j=1}^{K} \| j^q w_j \|_s^2, \qquad (5.15)$$

if $q \geq p + 1$. This proves (5.13).

II is controlled by

$$II \leq 2 \sum_{j=1}^{K} j^{p-q+1} \sum_{k=1}^{K} \| k^q y_k \|_0^2, \qquad (5.16)$$

since for each fixed (j, k) there is at most $2j$ choices for i. Thus if $q > p + 2$, one has

$$II \leq C \sum_{k=1}^{K} \| k^q y_k \|_0^2, \quad C = 2 \sum_{j=1}^{\infty} j^{p-q+1} \leq 2(1 + (p - q + 2)^{-1}). \qquad (5.17)$$

Thus we conclude that the $i \geq j$ terms can be controlled by the RHS of (5.8) (with δ replaced by $C\delta$).

For the terms of the LHS of (5.8) with $i \leq j$, we exchange the indexes i and j, and get the LHS of (5.12) with u and w exchanged. Thus one proceeds as before and get the same conclusion, since the RHS of (5.8) is invariant if u and w are exchanged. □

Remark 5.2. The weight k^q appeared in the above lemma is essential. Suppose one uses a summation $\sum_{k=1}^K \langle \partial^\alpha (uw)_k, y_k\rangle$, then one ends up with the estimate

$$\begin{aligned}\left|\sum_{k=1}^K \langle \partial^\alpha (uw)_k, y_k\rangle\right| &= \left|\sum_{i,j,k=1}^K S_{ijk}\langle \partial^\alpha (u_i w_j), y_k\rangle\right| \\ &\leq \sum_{i,j,k=1}^K \min(i,j,k)^p [C(\delta)\|u_i\|_s^2 \|w_j\|_s^2 + \delta \|y_k\|_0^2] \\ &\leq C(\delta)C_1(K)\sum_{i=1}^K \|u_i\|_s^2 \sum_{j=1}^K \|w_j\|_s^2 + \delta C_2(K)\sum_{k=1}^K \|y_k\|_0^2, \end{aligned} \tag{5.18}$$

where $C_1(K) = \sum_{k=1}^K k^p = O(K^{p+1}), \quad C_2(K) = K\sum_{i=1}^K i^p = O(K^{p+2})$. Thus in this way one gets an estimate with the coefficient depending on K. If one uses this estimate to prove an analog of Theorem 2.3, then one will get a constant c_2 depending on K.

In view of Proposition 2.4, c_2 being independent of K implies that the conclusion of Theorem 2.3 holds if the initial data satisfies a smoothness condition independent of K. If c_2 depends on K, then the initial data needs to satisfy a K-dependent condition to make the conclusion of Theorem 2.3 true. This is not good, since it is desirable that the gPC-sG method is stable for a class of initial data, for all K.

Remark 5.3. For gPC basis with respect to a probability measure supported on $\mathbb{R}$, for example, the Gaussian distribution, numerical evidence suggests that (5.5) may fail. In this case one might require a weaker condition, for example, (5.5) with k^p replaced by λ^k for some $\lambda > 1$, and prove results similar to Lemma 5.1 with different weights. This is out of the scope of this paper.

Due to the similarity of Lemma 3.3 and Lemma 5.1, it is straightforward to modify the proof of Theorem 2.1 into a proof of Theorem 2.3:

Proof of Theorem 2.3. We take ∂^α on the first and third equations of (2.18), and do L^2 estimates on them as well as on the fourth equation, and then sum over k and α with the kth equation multiplied by k^{2q}. Then we get

$$\frac{1}{2}\partial_t E^K + G^K + B^K = 0, \tag{5.19}$$

where

$$\begin{aligned} E^K(t) &= \sum_{k=1}^K (\|k^q u_k\|_s^2 + \|k^q f_k\|_s^2 + |k^q \bar{u}_k|^2), \\ G^K &= G_1^K + G_2^K = \sum_{k=1}^K (\|\nabla_x k^q u_k\|_s^2 + 2|k^q \bar{u}_k|^2) + \sum_{k=1}^K \left\| k^q \left(u_k\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f_k - \frac{1}{\epsilon}\frac{v}{2} f_k\right)\right\|_s^2, \\ B^K &= B_1^K + B_2^K + B_3^K = \sum_{|\alpha|\leq s} B_{1,\alpha}^K + \sum_{|\alpha|\leq s} B_{2,\alpha}^K + B_3^K, \end{aligned} \tag{5.20}$$

with

$$
\begin{aligned}
B_{1,\alpha}^K &= \sum_{k=1}^K k^{2q}\langle \partial^\alpha (u\cdot\nabla_x u)_k, \partial^\alpha u_k\rangle,\\
B_{2,\alpha}^K &= \sum_{k=1}^K k^{2q}\left\langle \partial^\alpha (uf)_k, \partial^\alpha\left[u_k\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f_k - \frac{1}{\epsilon}\frac{v}{2}f_k\right]\right\rangle,\\
B_3^K &= \frac{1}{|\mathbb{T}^3|}\sum_{k=1}^K k^{2q}\langle (uf)_k, \bar{u}_k\sqrt{\mu}\rangle.
\end{aligned}
\tag{5.21}
$$

Now apply Lemma 5.1 to get

$$
\begin{aligned}
|B_{1,\alpha}^K| &\le C(\delta)\sum_{k=1}^K \|k^q u_k\|_{s+1}^2 \sum_{k=1}^K \|k^q u_k\|_s^2 + \delta\sum_{k=1}^K \|k^q u_k\|_{s+1}^2 \le C(\delta)E^K G_1^K + \delta G_1^K,\\
|B_{2,\alpha}^K| &\le C(\delta)\sum_{k=1}^K \|k^q u_k\|_s^2 \sum_{k=1}^K \|k^q f_k\|_s^2 + \delta G_2^K \le C(\delta)E^K G_1^K + \delta G_2^K,\\
|B_3^K| &\le C(\delta)\sum_{k=1}^K \|k^q u_k\|_s^2 \sum_{k=1}^K \|k^q f_k\|_s^2 + \sum_{k=1}^K \delta |k^q \bar{u}_k|^2 \le C(\delta)E^K G_1^K + \delta G_1^K.
\end{aligned}
\tag{5.22}
$$

And then one concludes

$$
\frac{1}{2}\partial_t E^K \le -(1 - C(\delta)E^K - C\delta)G^K. \tag{5.23}
$$

Assuming $\delta = \frac{1}{4C}$ where C is the constant in (5.23), and $c_2(s,r) = \frac{1}{4C(\delta)}$, then by the same argument as in the proof of Theorem 2.1, if $E^K(0) \le c_2(s,r)$, then one has

$$
\partial_t E^K + G^K \le 0, \tag{5.24}
$$

and E^K is non-increasing.

□

Proof of Proposition 2.4. Note that $(u_0)_k$ is the kth gPC coefficient of the initial data u_0, and thus satisfies the spectral accuracy estimate

$$
|(u_0)_k(x)| \le C\frac{\|u_0(x,\cdot)\|_{H_z^r}}{k^r}, \tag{5.25}
$$

at each fixed x. By integrating (5.25) in x and replacing u by $\partial^\alpha u$ and summing over α, one gets

$$
\|k^q (u_k)_0\|_s \le C k^{q-r}\|u_0\|_{s,r}. \tag{5.26}
$$

Thus if $r > q + \frac{1}{2}$, one has

$$
\sum_{k=1}^K \|k^q (u_k)_0\|_s^2 \le C\|u_0\|_{s,r}^2. \tag{5.27}
$$

Similar estimate holds for f and $\bar{u}$. Thus one has

$$E^K_{s,q}(0) \le C\|E_{s,r}(0)\|_{L^1_z}, \tag{5.28}$$

and the proof is finished. □

5.2. Accuracy analysis: proof of Theorem 2.6

Recall the reconstructed gPC solution

$$u^K(x,z) = \sum_{k=1}^K u_k(x)\phi_k(z). \tag{5.29}$$

Then at a fixed x point one has

$$\|u^K(x,\cdot)\|^2_{L^2_z} = \sum_{k=1}^K |u_k(x)|^2 \le \sum_{k=1}^K |k^q u_k(x)|^2. \tag{5.30}$$

Thus

$$\|u^K\|_0^2 \le E^K_{0,q}. \tag{5.31}$$

for any $q \ge 0$.

Furthermore, with the assumption (2.22), one has the estimate

$$\|u^K(x)\|^2_{L^\infty_z} \le C\left(\sum_{k=1}^K |u_k(x)|k^p\right)^2 \le C\left(\sum_{k=1}^K |k^q u_k(x)|^2\right)\left(\sum_{k=1}^K k^{2(p-q)}\right) \le C\left(\sum_{k=1}^K |k^q u_k(x)|^2\right), \tag{5.32}$$

since $q > p+2$. Thus

$$\|u^K\|^2_{L^\infty_z(L^2_x)} \le \|u^K\|^2_{L^2_x(L^\infty_z)} \le CE^K_{0,q}. \tag{5.33}$$

Similar estimates hold for f and $\bar{u}$ and their x derivatives.

Proof of Theorem 2.6. The gPC coefficients of the mean fluid velocity satisfies

$$\partial_t \bar{u}_k + 2\bar{u}_k + C\int\int \sqrt{\mu}(uf)_k \,\mathrm{d}v\,\mathrm{d}x = 0. \tag{5.34}$$

Denote the projection operator onto the span of $\{\phi_k\}_{k=1}^K$ by P_K. Multiplying (2.18) and (5.34) by $\phi_k(z)$ and summing in k, one gets the equations for (u^K, f^K)

$$\begin{aligned}
&\partial_t u^K + P_K(u^K\cdot\nabla_x u^K) + \nabla_x p^K - \Delta_x u^K + u^K + \int \sqrt{\mu}P_K(u^K f^K)\,\mathrm{d}v - \frac{1}{\epsilon}\int v\sqrt{\mu} f^K\,\mathrm{d}v = 0,\\
&\nabla_x\cdot u^K = 0,\\
&\partial_t f^K + \frac{1}{\epsilon}v\cdot\nabla_x f^K + \frac{1}{\epsilon}\left(\nabla_v - \frac{v}{2}\right)\cdot P_K(u^K f^K) - \frac{1}{\epsilon}u^K\cdot v\sqrt{\mu} = \frac{1}{\epsilon^2}\left(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v\right) f^K,\\
&\partial_t \bar{u}^K + 2\bar{u}^K + \frac{1}{|\mathbb{T}^3|}\int\int \sqrt{\mu}P_K(u^K f^K)\,\mathrm{d}v\,\mathrm{d}x = 0.
\end{aligned} \tag{5.35}$$

Then subtracting from (1.7) and (1.12), one gets

$$
\begin{aligned}
&\partial_t u^e + [(I-P_K)(u\cdot\nabla_x u) + P_K(u^e\cdot\nabla_x u + u^K\cdot\nabla_x u^e)] + \nabla_x p^e - \Delta_x u^e + u^e \\
&\qquad + \int \sqrt{\mu}[(I-P_K)(uf) + P_K(u^e f + u^K f^e)]\,\mathrm{d}v - \frac{1}{\epsilon}\int v\sqrt{\mu} f^e\,\mathrm{d}v = 0, \\
&\nabla_x\cdot u^e = 0, \\
&\partial_t f^e + \frac{1}{\epsilon} v\cdot\nabla_x f^e + \frac{1}{\epsilon}(\nabla_v - \frac{v}{2})\cdot[(I-P_K)(uf) + P_K(u^e f + u^K f^e)] \\
&\qquad - \frac{1}{\epsilon} u^e\cdot v\sqrt{\mu} = \frac{1}{\epsilon^2}(\frac{-|v|^2}{4} + \frac{3}{2} + \Delta_v) f^e, \\
&\partial_t \bar{u}^e + 2\bar{u}^e + \frac{1}{|\mathbb{T}^3|}\int\int \sqrt{\mu}[(I-P_K)(uf) + P_K(u^e f + u^K f^e)]\,\mathrm{d}v\,\mathrm{d}x = 0,
\end{aligned}
\tag{5.36}
$$

where (u^e, f^e) is the approximation error

$$
u^e = u - u^K, \quad f^e = f - f^K. \tag{5.37}
$$

Notice that (5.36) is linear in (u^e, f^e).

Now take ∂^α on the first and third equations of (5.36), and do L^2 estimates on the first, third, fourth equations in (x,z), (x,v,z), z, respectively. First notice that P_K commutes with x-derivatives, and has operator norm 1 on L^2_z. Thus one has

$$
|\langle\langle \partial^\alpha P_K(u^e\cdot\nabla_x u + u^K\cdot\nabla_x u^e), \partial^\alpha u^e\rangle\rangle| \le C(\|u\|_{W^{s+1,\infty}} + \|u^K\|_{W^{s,\infty}})\|u^e\|^2_{s+1}, \tag{5.38}
$$

where the W norms mean the Sobolev norms with power index ∞, as defined in (2.13), and the sub-index $r=0$ is omitted. By estimating the terms $P_K(u^e f + u^K f^e)$ in the same manner, *i.e.*,

$$
|\langle\langle \partial^\alpha P_K(u^e f + u^K f^e), \partial^\alpha u^e\rangle\rangle| \le C(\|f\|_{W^{s,\infty}} + \|u^K\|_{W^{s,\infty}})(\|u^e\|^2_s + \|f^e\|^2_s), \tag{5.39}
$$

one gets the energy estimate

$$
\frac{1}{2}\partial_t E^e \le -\left(\frac{2}{3} - CH\right) G^e + CS, \tag{5.40}
$$

where

$$
\begin{aligned}
&E^e = \|u^e\|^2_s + \|f^e\|^2_s + \|\bar{u}^e\|^2, \\
&G^e = \|\nabla_x u^e\|^2_s + 2\|\bar{u}^e\|^2 + \left\| u^e\sqrt{\mu} - \frac{1}{\epsilon}\nabla_v f^e - \frac{1}{\epsilon}\frac{v}{2} f^e\right\|^2_s, \\
&S = (\|(I-P_K)(u\cdot\nabla_x u)\|^2_s + \|(I-P_K)(uf)\|^2_s), \\
&H = \|u\|_{W^{s+1,\infty}} + \|u^K\|_{W^{s,\infty}} + \|f\|_{W^{s,\infty}}.
\end{aligned}
\tag{5.41}
$$

First notice that by Sobolev embedding,

$$
\|u\|_{W^{s+1,\infty}} \le C\|u\|_{L^\infty_z(H^{s+3}_x)}, \quad \|f\|_{W^{s,\infty}} \le C\|f\|_{L^\infty_z(H^{s+2}_x(L^2_v))}, \tag{5.42}
$$

and by (5.33)

$$\|u^K\|^2_{W^{s,\infty}} \le C E^K_{s+2,q}. \tag{5.43}$$

Thus H can be controlled by

$$H \le C(\|E_{s+3,0}\|_{L^\infty_z} + E^K_{s+2,q})^{1/2}. \tag{5.44}$$

In view of Lemma 2.4, for $r > p + \frac{5}{2}$ one has

$$H \le C\|E_{s+3,r}\|^{1/2}_{L^\infty_z}, \tag{5.45}$$

which implies that

$$CH \le \frac{1}{6}, \tag{5.46}$$

in (5.40) for all time if $\|E_{s+3,r}(0)\|_{L^\infty_z} \le c''_1(s,r) \le \min\{\frac{1}{4C}, c_1(s,r), c_2(s,q)\}$, in view of Theorem 2.1 and Theorem 2.3.

To estimate the source term S, notice that at each fixed x, v,

$$\|(I - P_K)\partial^\alpha(uf)(x,v)\|_{L^2_z} \le C\frac{\|\partial^\alpha(uf)(x,v)\|_{H^r_z}}{K^r}. \tag{5.47}$$

Integrating in x, v and summing over α,

$$\|(I - P_K)(uf)\|_s \le C\frac{\|uf\|_{s,r}}{K^r}. \tag{5.48}$$

Notice that at each z,

$$|uf|_{s,r} \le \max_{|\alpha|\le s,\, |\gamma|\le r} \|\partial^\alpha u^\gamma\|_{L^\infty_x} |f|_{s,r} \le C|u|_{s+2,r}|f|_{s,r}. \tag{5.49}$$

Thus

$$\|uf\|_{s,r} \le \||uf|_{s,r}\|_{L^\infty_z} \le C\||u|_{s+2,r}\|_{L^\infty_z}\||f|_{s,r}\|_{L^\infty_z} \le C\|E_{s+2,r}\|^{1/2}_{L^\infty_z}\|E_{s,r}\|^{1/2}_{L^\infty_z}. \tag{5.50}$$

Then by Theorems 2.1 and 2.2 (suppress the dependence on C^h), taking $c''_1 \le c'_1(s,r)$,

$$E_{s+2,r}(t) \le C, \quad E_{s,r}(t) \le Ce^{-\lambda t}. \tag{5.51}$$

Thus we finally get

$$\|(I - P_K)(uf)\|_s \le \frac{Ce^{-\frac{\lambda}{2}t}}{K^r}. \tag{5.52}$$

The term $\|(I - P_K)(u \cdot \nabla_x u)\|_s$ can be estimated similarly, by using $|u \cdot \nabla_x u|_{s,r} \le C|u|_{s+3,r}|u|_{s,r}$, and we get

$$S \le \frac{Ce^{-\lambda t}}{K^{2r}}. \tag{5.53}$$

In conclusion, we have the estimate

$$\partial_t E^e + G^e \leq \frac{C}{K^{2r}} e^{-\lambda t}. \tag{5.54}$$

Finally, combining (5.40), (5.46) and (5.53), noticing that $\int_0^\infty e^{-2\lambda t}\,dt$ converges, one concludes that $E^e \leq \frac{C}{K^{2r}}$ uniformly in time and ϵ.

□

5.3. Hypocoercivity estimates for the error: proof of Theorem 2.7

Proof of Theorem 2.7. In order to get a hypocoercivity estimate for (u^e, f^e), we write the equation of $\partial^\alpha f^e$ as

$$\begin{aligned}\partial_t\partial^\alpha f^e + \frac{1}{\epsilon}\mathcal{P}\partial^\alpha f^e + \frac{1}{\epsilon^2}\mathcal{K}^*\mathcal{K}\partial^\alpha f^e = \frac{1}{\epsilon}\partial^\alpha u^e \cdot v\sqrt{\mu} + \frac{1}{\epsilon}[(I-P_K)\partial^\alpha(u\cdot\mathcal{K}^* f) + P_K\partial^\alpha(u^e\cdot\mathcal{K}^* f)\\ + P_K\partial^\alpha(u^K\cdot\mathcal{K}^* f^e)].\end{aligned} \tag{5.55}$$

and then do energy estimate in (x, v, z). The linear terms can be handled in the same way as Lemma 4.2. The first nonlinear term is estimated by

$$\left|\frac{1}{\epsilon}(((I-P_K)\partial^\alpha(u\cdot\mathcal{K}^* f), \partial^\alpha f^e))\right| \leq \frac{C}{K^r}\frac{1}{\epsilon^2}\|u\|_{L_z^\infty(H^{s+3,r})}([[f,f]]_{s,r} + [[f^e,f^e]]_s). \tag{5.56}$$

In fact, by modifying the proof of Lemma 4.2, one can get an expression like (4.5):

$$(((I-P_K)\partial^\alpha(u\cdot\mathcal{K}^* f), \partial^\alpha f^e)) = 2\langle\langle(I-P_K)\partial^\alpha(u\mathcal{K}f), \mathcal{K}^2\partial^\alpha f^e\rangle\rangle + \text{similar terms}. \tag{5.57}$$

The first term in (5.57) is estimated by

$$\begin{aligned}|\langle\langle(I-P_K)\partial^\alpha(u\mathcal{K}f), \mathcal{K}^2\partial^\alpha f^e\rangle\rangle| &\leq \|(I-P_K)\partial^\alpha(u\mathcal{K}f)\|_0\|\mathcal{K}^2\partial^\alpha f^e\|_0\\ &\leq \frac{C}{K^r}\|\partial^\alpha(u\cdot\mathcal{K}f)\|_{0,r}\|\mathcal{K}^2 f^e\|_s\\ &\leq \frac{C}{K^r}\max_{|\gamma|\leq r, |\beta|\leq s}\|\partial^\beta u^\gamma\|_{L^\infty}\|\mathcal{K}f\|_{s,r}\|\mathcal{K}^2 f^e\|_s\\ &\leq \frac{C}{K^r}\|u\|_{L_z^\infty(H^{s+3,r})}(\|\mathcal{K}f\|_{s,r}^2 + \|\mathcal{K}^2 f^e\|_s^2),\end{aligned} \tag{5.58}$$

and other terms in (5.57) can be estimated similarly.

The second nonlinear term in (5.55) is estimated by Lemma 4.2 as follows:

$$\begin{aligned}\left|\frac{1}{\epsilon}((P_K\partial^\alpha(u^e\cdot\mathcal{K}^* f), \partial^\alpha f^e))\right| &\leq \left|\frac{1}{\epsilon}((\partial^\alpha(u^e\cdot\mathcal{K}^* f), \partial^\alpha f^e))\right|\\ &\leq C\frac{1}{\epsilon^2}\max_{|\beta|\leq s}\|\partial^\beta u^e\|_{L^\infty}(C(\delta)[[f,f]]_s + \delta[[f^e,f^e]]_s).\end{aligned} \tag{5.59}$$

The third nonlinear term is estimated by Lemma 4.2 as follows:

$$\left|\frac{1}{\epsilon}((P_K\partial^\alpha(u^K\cdot\mathcal{K}^* f^e), \partial^\alpha f^e))\right| \leq \left|\frac{1}{\epsilon}((\partial^\alpha(u^K\cdot\mathcal{K}^* f^e), \partial^\alpha f^e))\right| \leq C\frac{1}{\epsilon^2}\|u^K\|_{L_z^\infty(H^{s+3})}[[f^e,f^e]]_s. \tag{5.60}$$

Now by assumption, $\|E_{s+3,r}(t)\|_{L^\infty_z}$ is small enough at $t = 0$ (which implies that they are small enough for all time, by Theorem 2.1). Similar result holds for $E^K_{s+3,q} \le C\|E_{s+3,r}\|_{L^\infty_z}$, by Theorem 2.3. As a result, $\|u\|_{L^\infty_z(H^{s+3,r})}$ and $\|u^K\|_{L^\infty_z(H^{s+3})}$ are small enough, see (5.33) for the latter.

To bound the term $\max_{|\beta|\le s}\|\partial^\beta u^e\|_{L^\infty}$ appeared in (5.59), we estimate

$$\|u^e\|_{L^\infty_z} = \left\|\sum_{k=1}^K (u^e)_k\phi_k(z)\right\|_{L^\infty_z} \le C\left(\sum_{k=1}^K |(u^e)_k|^2\right)^{1/2}\left(\sum_{k=1}^K k^{2p}\right)^{1/2} \le C\|u^e\|_{L^2_z}K^{p+1/2}, \tag{5.61}$$

at any fixed x. Then taking L^∞ in x we obtain

$$\|u^e\|_{L^\infty} \le CK^{p+1/2}\|u^e\|_{L^\infty_x(L^2_z)} \le CK^{p+1/2}\|u^e\|_{L^2_z(L^\infty_x)} \le CK^{p+1/2}\|u^e\|_{L^2_z(H^2_x)}. \tag{5.62}$$

By Theorem 2.6, $\|u^e\|_{L^2_z(H^{s+3})}$ is bounded by $\frac{C}{K^r}$. Doing the same estimate for the x-derivatives of u^e, we obtain

$$\max_{|\beta|\le s}\|\partial^\beta u^e\|_{L^\infty} \le \frac{C}{K^{r-p-1/2}}. \tag{5.63}$$

Then by choosing δ in (5.59) small enough, all the $[[f^e, f^e]]_s$ terms from the nonlinear terms can be absorbed by the corresponding term from the linear terms, and then one concludes the estimate

$$\partial_t((f^e, f^e))_s + \lambda_1^e\frac{1}{\epsilon^2}[[f^e, f^e]]_s \le C(\lambda_1^e)(\|u^e\|_s^2 + \|\nabla_x u^e\|_s^2 + \frac{1}{\epsilon^2}\|\mathcal{K}f^e\|_s^2) + \frac{C}{K^{r-p-1/2}}\frac{1}{\epsilon^2}[[f, f]]_{s,r}. \tag{5.64}$$

Finally, similar to the proof of Theorem 2.2, by taking a suitable linear combination of (5.64)(5.54) and (4.19) integrated in z (where the appearance of (4.19) is to control the term $[[f, f]]_{s,r}$ in (5.64)), we get

$$\partial_t\tilde{E}^e + \frac{1}{2}\tilde{G}^e \le \lambda_4^e\frac{C}{K^{r-p-1/2}}\frac{1}{\epsilon^2}[[f, f]]_{s,r} + \frac{C}{K^r}e^{-\lambda t}, \tag{5.65}$$

where

$$\tilde{E}^e = E^e + \lambda_4^e((f^e, f^e))_s + \frac{1}{K^{r-p-1/2}}\lambda_5^e\|\tilde{E}\|_{L^1_z}, \tag{5.66}$$

and

$$\tilde{G}^e = G^e + \lambda_4^e\lambda_1^e\frac{1}{\epsilon^2}[[f^e, f^e]]_s + \frac{1}{2K^{r-p-1/2}}\lambda_5^e\|\tilde{G}\|_{L^1_z}. \tag{5.67}$$

The choice of λ_4^e is in the same way as the choice of λ_4. To choose λ_5^e, one wants the $\tilde{G}$ term to control the first RHS term in (5.65), and thus choose

$$\lambda_5^e = 4\frac{C\lambda_4^e}{\lambda_4\lambda_1}, \tag{5.68}$$

where the C is the first constant in (5.65). Then

$$\partial_t\tilde{E}^e + \frac{1}{4}\tilde{G}^e \le \frac{C}{K^r}e^{-\lambda t}. \tag{5.69}$$

Then since $\tilde{E}^e \le C\tilde{G}^e$ (which can be proved similarly as the proof of $\tilde{E} \le C\tilde{G}$, see (4.22)), and $\tilde{E}^e(0) \le \frac{C}{K^{r-p-1/2}}$, one concludes that

$$\tilde{E}^e \le \frac{C}{K^{r-p-1/2}} e^{-\lambda^e t}, \tag{5.70}$$

where $\lambda^e = \min\{\lambda, \frac{1}{4C}\} - \delta$ for some $\delta > 0$ small enough, in view of the lemma below. □

Lemma 5.4. *Let $\Phi = \Phi(t)$ satisfy*

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + a_1\Phi \le a_2 e^{-a_3 t}. \tag{5.71}$$

Then

$$\Phi(t) \le e^{-at}(\Phi(0) + a_2 C(\delta)), \tag{5.72}$$

with $a = \min\{a_1, a_3\} - \delta$, δ being any positive constant.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{a_1 t}\Phi) \le a_2 e^{(a_1 - a_3)t}, \tag{5.73}$$

$$e^{a_1 t}\Phi \le \Phi(0) + \int_0^t a_2 e^{(a_1-a_3)s}\,\mathrm{d}s, \tag{5.74}$$

$$\Phi(t) \le e^{-a_1 t}\Phi(0) + a_2 \frac{e^{-a_3 t} - e^{-a_1 t}}{a_1 - a_3} = e^{-a_1 t}\Phi(0) + a_2 t e^{-\xi t}, \tag{5.75}$$

for some ξ between a_1 and a_3, by the mean value theorem. Then the conclusion follows since

$$te^{-\xi t} \le e^{-at}(te^{-\delta t}) \le C(\delta)e^{-at}, \tag{5.76}$$

where $C(\delta) = (\delta e)^{-1}$. □

6. Conclusion

In this paper we first prove the uniform regularity in the random space for a kinetic-fluid two-phase flow model with the light particle regime for random initial data near the global equilibrium, using an energy estimate in suitable Sobolev spaces. By hypocoercivity arguments we prove the energy $E(t)$ decays exponentially in time. This result implies that for random initial data near the global equilibrium, the long time behavior of the solution is insensitive to the random perturbation on initial data. Then we prove a result on the time decay of the solution of the generalized polynomial chaos stochastic Galerkin (gPC-sG) method, in which the requirement of the random initial data is independent of K, the number of basis functions. The key idea in this proof is the usage of E^K, a weighted sum of Sobolev norms of the gPC coefficients. Finally we prove the uniform spectral accuracy of the sG method for random initial data near the global equilibrium, by doing energy and hypocoercivity estimates on the sG error (u^e, f^e). All the constants involved in the results are *independent of* ϵ, the Knudsen number.

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References

[1] M.J. Andrews and P.J. O'Rourke, The multiphase particle-in-cell (mp-pic) method for dense particulate flows. *Int. J. Multiph. Flow* **22** (1996) 379–402.

[2] I. Babuska, R. Tempone and G.E. Zouraris, Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.* **42** (2004) 800–825.

[3] I. Babuska, F. Nobile and R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.* **45** (2007) 1005–1034.

[4] J. Back, F. Nobile, L. Tamellini and R. Tempone, Stochastic spectral galerkin and collocation methods for pdes with random coefficients: a numerical comparison, in Spectral and High Order Methods for Partial Differential Equations, edited by E.M. Rønquist and J.S. Hesthaven. Springer-Verlag, Berlin, Heidelberg (2011).

[5] R. Caflisch and G. Papanicolaou, Dynamic theory of suspensions with Brownian effects. *SIAM J. Appl. Math.* **43** (1983).

[6] R.G. Ghanem and P.D. Spanos, Stochastic Finite Elements: A Spectral Approach. Springer-Verlag, New York (1991).

[7] D. Gidaspow, R. Bezburuah and J. Ding, Hydrodynamics of Circulating Fluidized Beds: Kinetic Theory Approach. Illinois Institute of Technology, Department of Chemical Engineering, Chicago, IL, USA (1991).

[8] A.D. Gosman and E. Loannides, Aspects of computer simulation of liquid-fueled combustors. *J. Energy* **7** (1983) 482–490.

[9] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic limit for the Vlasov-Navier-Stokes equations I. Light particles regime. *Indiana Univ. Math. J.* **53** (2004) 1495–1516.

[10] T. Goudon, P.-E. Jabin and A. Vasseur, Hydrodynamic limit for the Vlasov-Navier-Stokes equations II. Fine particles regime. *Indiana Univ. Math. J.* **53** (2004) 1517–1536.

[11] T. Goudon, L. He, A. Moussa and P. Zhang, The Navier-Stokes-Vlasov-Fokker-Planck system near equilibrium. *SIAM J. Math. Anal.* **42** (2010) 2177–2202.

[12] T. Goudon, S. Jin, J.-G. Liu and B. Yan, Asymptotic-preserving schemes for kinetic-fluid modeling of disperse two-phase flows. *J. Comput. Phys.* **246** (2013) 145–164

[13] M.D. Gunzburger, C.G. Webster and G. Zhang, Stochastic finite element methods for partial differential equations with random input data. *Acta Numer.* **23** (2014) 521–650.

[14] P.-E. Jabin and B. Perthame, Notes on mathematical problems on the dynamics of dispersed particles interacting through a fluid, in Modeling in Applied Sciences, A Kinetic Theory Approach, edited by N. Bellomo and M. Pulvirenti. Birkhauser (2000) 111–147.

[15] X. Jiang, G.A. Siamas, K. Jagus and T.G. Karayiannis, Physical modelling and advanced simulations of gas–liquid two-phase jet flows in atomization and sprays. *Prog. Energy Combust. Sci.* **36** (2010) 131–167.

[16] S. Jin, Efficient asymptotic-preserving (AP) schemes for some multiscale kinetic equations. *SIAM J. Sci. Comput.* **21** (1999) 441–454.

[17] S. Jin and L. Liu, An asymptotic-preserving stochastic Galerkin method for the semiconductor Boltzmann equation with random inputs and diffusive scalings. *SIAM Multiscale Model. Simul.* **15** (2017) 157–183.

[18] S. Jin and R. Shu, A stochastic asymptotic-preserving scheme for a kinetic-fluid model for disperse two-phase flows with uncertainty. *J. Comput. Phys.* **335** (2017) 905–924.

[19] S. Jin and Y. Zhu, Hypocoercivity and uniform regularity for the Vlasov-Poisson-Fokker-Planck system with uncertainty and multiple scales. *SIAM J. Math. Anal.* **50** (2018) 1790–1816.

[20] S. Jin, D. Xiu and X. Zhu, Asymptotic-preserving methods for hyperbolic and transport equations with random inputs and diffusive scalings. *J. Comput. Phys.* **289** (2015) 35–52.

[21] S. Jin, J.-G. Liu and Z. Ma, Uniform spectral convergence of the stochastic Galerkin method for the linear transport equations with random inputs in diffusive regime and a micro-macro decomposition based asymptotic preserving method. *Res. Math. Sci.* **4** (2017) 15.

[22] Q. Li and L. Wang, Uniform regularity for linear kinetic equations with random input based on hypocoercivity. *SIAM Uncertain. Quantif.* **5** (2017) 1193–1219.

[23] L. Liu, Uniform spectral convergence of the stochastic Galerkin method for the linear semiconductor Boltzmann equation with random inputs and diffusive scalings. *Kinet. Relat. Model.* **11** (2018) 1139–1156.

[24] O.P. Le Maître and O.M. Knio, Spectral Methods for Uncertainty Quantification, Scientific Computation, With Applications to Computational Fluid Dynamics. Springer, New York (2010).

[25] H. Niederreiter, P. Hellekalek, G. Larcher and P. Zinterhof, Monte Carlo and Quasi-Monte Carlo Methods 1996. Springer-Verlag (1998).

[26] F. Nobile, R. Tempone and C. G. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.* **46** (2008) 2309–2345.

[27] P.J. O'Rourke, Collective drop effects on vaporizing liquid sprays. Los Alamos National Lab., NM (USA) (1981).

[28] G. Szegő, Orthogonal Polynomials. American Mathematical Society (1939).

[29] F.A. Williams, Combustion Theory, 2nd edition. Benjamin Cummings Publ. (1985).

[30] D. Xiu, Fast numerical methods for stochastic computations: a review. *Commun. Comput. Phys.* **5** (2009) 242–272.

[31] D. Xiu, Numerical Methods for Stochastic Computation. Princeton University Press, Princeton, New Jersey (2010).

[32] D. Xiu and J.S. Hesthaven, High-order collocation methods for differential equations with random inputs. *SIAM J. Sci. Comput.* **27** (2005) 1118–1139.

[33] D. Xiu and G.E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM J. Sci. Comput.* **24** (2002) 619–644.