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STABILITY OF THE ALE SPACE-TIME DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR CONVECTION-DIFFUSION PROBLEMS IN TIME-DEPENDENT DOMAINS[†]

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Abstract. The paper is concerned with the analysis of the space-time discontinuous Galerkin method (STDGM) applied to the numerical solution of nonstationary nonlinear convection-diffusion initial-boundary value problem in a time-dependent domain. The problem is reformulated using the arbitrary Lagrangian–Eulerian (ALE) method, which replaces the classical partial time derivative by the so-called ALE derivative and an additional convective term. The problem is discretized with the use of the ALE-space time discontinuous Galerkin method (ALE-STDGM). In the formulation of the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The main attention is paid to the proof of the unconditional stability of the method. An important step is the generalization of a discrete characteristic function associated with the approximate solution and the derivation of its properties.

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1. Introduction

Most of the results on the solvability and numerical analysis of nonstationary partial differential equations (PDEs) are obtained under the assumption that a space domain Ω is independent of time t. However, problems in time-dependent domains Ω_t are important in a number of areas of science and technology. We can mention, for example, problems with moving boundaries, when the motion of the boundary $\partial\Omega_t$ is prescribed, or free boundary problems, when the motion of the boundary $\partial\Omega_t$ should be determined together with the solution of the PDEs in consideration. This is particularly the case of fluid-structure interaction (FSI), when the flow is solved in a domain deformed due to the coupling with an elastic structure.

There are various approaches to the solution of problems in time-dependent domains as, for example, fictitious domain method, see [43], or immersed boundary method, see [10]. A very popular technique is the arbitrary Lagrangian–Eulerian (ALE) method based on a suitable one-to-one ALE mapping of the reference domain Ω_{ref} onto the current configuration Ω_t. It is usually applied in connection with conforming finite element space

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discretization and combined with the time discretization by the use of a backward difference formula (BDF). From a wide literature we mention, *e.g.*, the works [22, 39, 41, 42]. This method is analyzed theoretically for linear parabolic convection-diffusion initial-boundary value problems. Paper [35] investigates the stability of the ALE-conforming finite element method. In [4, 36] error estimates for the ALE-conforming finite element method are derived.

In the numerical solution of compressible flow, it is suitable to apply the discontinuous Galerkin method (DGM) for the space discretization. It is based on piecewise polynomial approximations over finite element meshes, in general discontinuous on interfaces between neighbouring elements. This method was applied to the solution of compressible flow first in [8] and then in [9]. It enables us to get a good resolution of boundary and internal layers (including shock waves and contact discontinuities) and has been used for the solution of various types of flow problems, see [19, 26, 32]. Theory of the space discretization by the discontinuous Galerkin method is a subject of a number of works. We cite only some of them: [2, 3, 13, 18, 21, 34, 38, 40, 46, 47, 52]. It is also possible to refer to the monograph [20] containing a number of references.

In the cited works, the time discretization is carried out with the aid of the BDF of the first or second order. One possibility to construct a higher order method in time is the application of the DGM in time. This technique uses a piecewise polynomial approximation in time, in general discontinuous at discrete time instants that form a partition in a time interval. This method was used for time discretization combined with conforming finite elements for the space discretization of linear parabolic equations in [1, 17, 23–25, 48–50].

By the combination of the DGM in space and time we get the space-time discontinuous Galerkin method (STDGM). This method was theoretically analyzed in [7, 14, 20, 29, 33, 53]. In [28, 44], the BDF-DGM and STDGM is applied to linear and nonlinear dynamic elasticity problems. One of the advantages of the STDGM is the possibility to use different meshes on different time levels.

The mentioned methods have also been extended to the numerical solution of initial-boundary value problems in time-dependent domains using the ALE method. The ALE method combined with the space DGM and BDF in time (ALE-DGM-BDF) was applied with success to interaction of compressible flow with elastic structures in [15, 30, 37, 44]. In [16], the ALE-STDGM is applied to the simulation of flow induced airfoil vibrations and the results are compared with the ALE-DGM-BDF approach. It appears that the ALE-STDGM is more robust and accurate. Here we can cite the important work [51] dealing with the space-time DGM to the solution of inviscid compressible flows. The approach in this paper consideres the time variable equivalent to the space variables and uses meshes formed by space-time four-dimensional elements. It allows to use different meshes in different time slabs. This paper also discusses the relation of the presented technique with the ALE method. The method analyzed in the following parts of our paper consideres time and space variables separately in contrast to [51]. Moreover, we deal with a problem containing diffusion, which should be analogy to the compressible Navier-Stokes equations.

The ALE-time discontinuous Galerkin semidiscretization of a linear parabolic convection-diffusion problem is analyzed in [11, 12]. Both papers assume that the transport velocity is divergence free and consider homogeneous Dirichlet boundary condition. In [11], the stability of the ALE-time DGM is proved and [12] is devoted to the error estimation. Papers [5, 6] are concerned with the stability analysis of the ALE-STDGM applied to a linear convection-diffusion initial-boundary value problem, and to the case with nonlinear convection and diffusion, respectively. In both cases nonhomogeneous Dirichlet boundary conditions and piecewise linear DG time discretization are used.

In the present paper we extend the results from [5]. We deal with the stability analysis of the ALE-STDGM with arbitrary polynomial degree in space as well as in time, applied to a scalar nonstationary nonlinear convection-diffusion problem equipped with initial condition and nonhomogeneous Dirichlet boundary condition. This problem can be considered as a simplified prototype of the compressible Navier-Stokes system. The ALE-STDGM analyzed here corresponds to the technique used in [16, 28] for the numerical simulation of airfoil vibrations induced by compressible flow. (The construction of the ALE mapping is described very briefly. It is hidden in the computer program.)

We present here a new formulation of the problem and technique of theoretical analysis in contrast to [5]. In [5] we proved the unconditional stability of the ALE-STDGM with arbitrary polynomial degree in space, but only linear approximation in time. Moreover, in [5] the standard ALE method prescribed globally in the whole time interval was used (see also [11, 12, 22, 35, 36, 39, 41]). In the present paper we apply a different ALE technique that can use different meshes with different numbers of elements in different time levels. We assume that the ALE mapping is prescribed for each time slab separately.

In the analysis presented in this paper it was necessary to overcome a number of various difficult obstacles. An important tool in our theory is the concept of the discrete characteristic function introduced in [17] in the framework of the time DGM applied to a linear parabolic problem. In [7, 14] the discrete characteristic function was generalized in connection with the STDGM for nonlinear parabolic problems in fixed domains. An important new and original result contained in the present paper is the extension of the discrete characteristic function and the proof of its properties in the case of the ALE-STDGM in time-dependent domains. On the basis of a technical analysis we obtain an unconditional stability of this method represented by a bound of the approximate solution in terms of data without any limitation of the time step in dependence on the size of the triangulations.

In Section 2 we formulate the continuous problem. Section 3 is devoted to the ALE space-time discretization. We describe here triangulations, ALE mappings and introduce important function spaces and concepts. Then an approximate solution is defined. Section 4 deals with the stability analysis. First some auxiliary results are presented. Then we introduce important estimates and the generalized concept of the discrete characteristic function. An important part is devoted to the derivation of its properties. Finally, the last part presents the proof of unconditional stability of the ALE-STDGM.

2. Formulation of the continuous problem

In what follows, we shall use the standard notation $L^2(\omega)$ for the Lebesgue space, $W^{k,p}(\omega)$, $H^k(\omega) = W^{k,2}(\omega)$ for the Sobolev spaces over a bounded domain $\omega \subset I\!R^d$, $d = 2, 3$, and the Bochner spaces $L^\infty(0,T;X)$ with a Banach space X and

$$W^{1,\infty}(0,T;W^{1,\infty}(\Omega_t)) = \left\{ f \in L^\infty(0,T;W^{1,\infty}(\Omega_t));\ \mathrm{d}f/\mathrm{d}t \in L^\infty(0,T;W^{1,\infty}(\Omega_t)) \right\},$$

where $\mathrm{d}f/\mathrm{d}t$ denotes here the distributional derivative.

If X is a Banach (Hilbert) space, then its norm (scalar product) will be denoted by $\|\cdot\|_X$ ($(\cdot,\cdot)_X$). By $|\cdot|_X$ we denote a seminorm in X. For simplicity we use the notation $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_\omega$, $(\cdot,\cdot)_{L^2(\omega)} = (\cdot,\cdot)_\omega$ and $\|\cdot\|_{L^2(\partial\omega)} = \|\cdot\|_{\partial\omega}$.

We shall be concerned with an initial-boundary value nonlinear convection-diffusion problem in a time-dependent bounded domain $\Omega_t \subset I\!R^d$, where $t \in [0,T]$, $T > 0$: Find a function $u = u(x,t)$ with $x \in \Omega_t$, $t \in (0,T)$ such that

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \mathrm{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0,T), \tag{2.1}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0,T), \tag{2.2}$$

$$u(x,0) = u^0(x), \quad x \in \Omega_0. \tag{2.3}$$

We assume that $f_s \in C^1(I\!R)$, $f_s(0) = 0$,

$$|f_s'| \le L_f, \quad s = 1, \dots, d, \tag{2.4}$$

where the constant L_f does not depend on u. Moreover we assume that function β is bounded and Lipschitz-continuous:

$$\beta : \mathbb{R} \to [\beta_0, \beta_1], \quad 0 < \beta_0 < \beta_1 < \infty, \tag{2.5}$$

$$|\beta(u_1) - \beta(u_2)| \le L_\beta |u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}. \tag{2.6}$$

Problem (2.1)–(2.3) can be reformulated with the aid of the so-called arbitrary Lagrangian–Eulerian (ALE) method. A standard ALE formulation is based on an ALE mapping prescribed globally in the whole time interval $[0, T]$. It is based on a regular one-to-one ALE mapping of the reference configuration Ω_{ref} onto the current configuration Ω_t:

$$\mathcal{A}_t : \overline{\Omega}_{\text{ref}} \to \overline{\Omega}_t, \quad X \in \overline{\Omega}_{\text{ref}} \to x = \mathcal{A}_t(X) \in \overline{\Omega}_t, \quad t \in [0, T]. \tag{2.7}$$

Usually it is assumed that $\Omega_{\text{ref}} = \Omega_0$, as in *cf.*, *e.g.*, [5, 11, 12, 22, 35, 36, 39, 41]. However, in this case it is impossible to use different space partitions in different time slabs, which allows the STDGM. Therefore, we shall proceed as is described in the next section.

The transformation of the partial differential equation (2.1) into the ALE form is based on the following concepts. We introduce the domain velocity

$$\tilde{\boldsymbol{z}}(X, t) = \frac{\partial}{\partial t}\mathcal{A}_t(X),\ \boldsymbol{z}(x, t) = \tilde{\boldsymbol{z}}(\mathcal{A}_t^{-1}(x), t),\ t \in [0, T],\ X \in \Omega_{\text{ref}},\ x \in \Omega_t, \tag{2.8}$$

and define the ALE derivative $D_t f = Df/Dt$ of a function $f = f(x, t)$ for $x \in \Omega_t$ and $t \in [0, T]$ as

$$D_t f(x, t) = \frac{D}{Dt} f(x, t) = \frac{\partial \tilde{f}}{\partial t}(X, t), \tag{2.9}$$

where $\tilde{f}(X, t) = f(\mathcal{A}_t(X), t)$, $X \in \Omega_{\text{ref}}$, and $x = \mathcal{A}_t(X) \in \Omega_t$. The use of the chain rule yields the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \boldsymbol{z} \cdot \nabla f, \tag{2.10}$$

which allows us to reformulate problem (2.1)–(2.3) in the ALE form: Find $u = u(x, t)$ with $x \in \Omega_t,\ t \in (0, T)$ such that

$$\frac{Du}{Dt} + \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} - \boldsymbol{z} \cdot \nabla u - \operatorname{div}(\beta(u)\nabla u) = g \quad \text{in } \Omega_t,\ t \in (0, T), \tag{2.11}$$

$$u = u_D \quad \text{on } \partial\Omega_t,\ t \in (0, T), \tag{2.12}$$

$$u(x, 0) = u^0(x), \quad x \in \Omega_0. \tag{2.13}$$

3. ALE-space time discretization

In the time interval $[0, T]$ we consider a partition $0 = t_0 < t_1 < \cdots < t_M = T$ and set $\tau_m = t_m - t_{m-1}$, $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$ for $m = 1, \ldots, M$, $\tau = \max_{m=1,\ldots,M} \tau_m$. We assume that $\tau \in (0, \overline{\tau})$, where $\overline{\tau} > 0$. The space-time discontinuous Galerkin method (STDGM) has an advantage that on every time interval $\overline{I}_m = [t_{m-1}, t_m]$ it is possible to consider a different space partition (*i.e.* triangulation) – see, *e.g.* [14, 20]. Here we also use this possibility for the application of the STDGM in the framework of the ALE method. It allows us to consider an ALE mapping separately on each time interval $[t_{m-1}, t_m)$ for $m = 1, \ldots, M$ and the resulting ALE mapping in $[0, T]$ may be discontinuous at time instants t_m, $m = 1, \ldots, M-1$. This means that one-sided limits $\mathcal{A}_{(t_m-)} \neq \mathcal{A}_{(t_m+)}$ in general. Similarly the same may hold for the approximate solution. This means that we deal with a new generalized ALE technique based on the STDGM. To this end, we introduce the following notation.

3.1. ALE mappings and triangulations

For every $m = 1, \dots, M$ we consider a standard conforming triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$ in $\Omega_{t_{m-1}}$, where $h \in (0, \overline{h})$ and $\overline{h} > 0$. This triangulation is formed by a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$) with disjoint interiors. We assume that the domain $\Omega_{t_{m-1}}$ is polygonal (polyhedral). Further, for each $m = 1, \dots, M$ we introduce a one-to-one ALE mapping

$$\mathcal{A}_{h,t}^{m-1} : \overline{\Omega}_{t_{m-1}} \xrightarrow{\text{onto}} \overline{\Omega}_t \text{ for } t \in [t_{m-1}, t_m),\ h \in (0, \overline{h}). \tag{3.1}$$

We assume that $\mathcal{A}_{h,t}^{m-1}$ is in space a piecewise affine mapping on the triangulation $\hat{\mathcal{T}}_{h,t_{m-1}}$, continuous in space variable $X \in \Omega_{t_{m-1}}$ and in time $t \in [t_{m-1}, t_m)$ and $\mathcal{A}_{h,t_{m-1}}^{m-1} = \mathrm{Id}$ (identical mapping). Hence, we assume that all domains Ω_t are polygonal (polyhedral). For every $t \in [t_{m-1}, t_m)$ we define the conforming triangulation

$$\mathcal{T}_{h,t} = \left\{ K = \mathcal{A}_{h,t}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\} \text{ in } \Omega_t. \tag{3.2}$$

This means that every domain Ω_{t_m-1} represents a reference configuration for the ALE mapping $\mathcal{A}_t^{m-1}$ with $t \in I_m$. It is important that this mapping is not an approximation of some regular mapping of Ω_0 onto Ω_t, as is standard in other works.

At $t = t_m$ we define the one-sided limit $\mathcal{A}_{h,t_m-}^{m-1}$, introduce the triangulation

$$\mathcal{T}_{h,t_m-} = \{\mathcal{A}_{h,t_m-}^{m-1}(\hat{K});\ \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}\} \text{ in } \overline{\Omega}_{t_m}$$

and suppose that

$$\mathcal{A}_{h,t_m}^{m-1}\left(\overline{\Omega}_{t_{m-1}}\right) = \overline{\Omega}_{t_m}. \tag{3.3}$$

We have $\mathcal{T}_{h,t_{m-1}} = \hat{\mathcal{T}}_{h,t_{m-1}}$, but in general, $\mathcal{T}_{h,t_m-} \neq \hat{\mathcal{T}}_{h,t_m}$.

As we see, for every $t \in [0, T]$ we may have a family $\{\mathcal{T}_{h,t}\}_{h \in (0,\overline{h})}$ of triangulations of the domain Ω_t. Triangulations $\hat{\mathcal{T}}_{h,t_{m-1}}$ and $\hat{\mathcal{T}}_{h,t_m}$ have different structure and, in general, different number of cells. Triangulations $\mathcal{T}_{h,t}$ and $\mathcal{T}_{h,t_m-}$ have the same structure as $\hat{\mathcal{T}}_{h,t_{m-1}}$ for $t \in [t_{m-1}, t_m]$, but starting from $\hat{\mathcal{T}}_{h,t_m}$ the structure of $\mathcal{T}_{h,t}$ for $t \in [t_m, t_{m+1}]$, may be different from the structure of $\mathcal{T}_{h,t}$ for $t \in [t_{m-1}, t_m]$.

In what follows, for the sake of simplicity, we use the notation $\mathcal{A}_t$ for the ALE mapping defined in $\bigcup_{m=1}^{M} I_m$ so that

$$\mathcal{A}_t(X) = \mathcal{A}_{h,t}^{m-1}(X) \quad \text{for } X \in \overline{\Omega}_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \dots, M,\ h \in (0, \overline{h}). \tag{3.4}$$

The symbol $\mathcal{A}_t^{-1}$ will denote the inverse to $\mathcal{A}_t$. This means that $\mathcal{A}_t^{-1} : \overline{\Omega}_t \xrightarrow{\text{onto}} \overline{\Omega}_{t_{m-1}}$ for $t \in \overline{I}_m,\ m = 1, \dots, M$.

3.2. Discrete function spaces

In what follows, for every $m = 1, \dots, M$ we consider the space

$$S_h^{p,m-1} = \left\{ \varphi \in L^2(\Omega_{t_{m-1}});\ \varphi|_{\hat{K}} \in P^p(\hat{K})\ \forall\, \hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}} \right\}, \tag{3.5}$$

where $p \geq 1$ is an integer and $P^p(\hat{K})$ is the space of all polynomials on $\hat{K}$ of degree $\leq p$. Now for every $t \in \overline{I}_m$ we define the space

$$S_h^{t,p,m-1} = \left\{ \varphi \in L^2(\Omega_t);\ \varphi \circ \mathcal{A}_{h,t}^{m-1} \in S_h^{p,m-1} \right\}. \tag{3.6}$$

It is possible to see that

$$S_h^{t,p,m-1} = \left\{ \varphi \in L^2(\Omega_t);\ \varphi|_K \in P^p(K)\ \forall\, K \in \mathcal{T}_{h,t} \right\}. \tag{3.7}$$

Of course, $S_h^{t_m,p,m-1} \neq S_h^{p,m}$ in general.

Further, let p, $q \geq 1$ be integers. By $P^q(I_m; S_h^{p,m-1})$ we denote the space of mappings of the time interval I_m into the space $S_h^{p,m-1}$ which are polynomials of degree $\leq q$ in time. We set

$$S_{h,\tau}^{p,q} = \left\{ \varphi;\ \varphi\left(\mathcal{A}_{h,t}^{m-1}(X), t\right) = \sum_{i=0}^{q} \vartheta_i(X)\, t^i, \quad \vartheta_i \in S_h^{p,m-1},\ X \in \Omega_{t_{m-1}},\ t \in \overline{I}_m,\ m = 1, \ldots, M \right\}. \tag{3.8}$$

An approximate solution of problem (2.11)–(2.13) and test functions will be elements of the space $S_{h,\tau}^{p,q}$. By D_t we denote the ALE derivative defined by (2.9) for $t \in \bigcup_{m=1}^{M} I_m$.

3.3. Some notation and important concepts

Over a triangulation $\mathcal{T}_{h,t}$, for each positive integer k, we define the broken Sobolev space

$$H^k(\Omega_t, \mathcal{T}_{h,t}) = \{v;\ v|_K \in H^k(K) \quad \forall K \in \mathcal{T}_{h,t}\},$$

equipped with the seminorm

$$|v|_{H^k(\Omega_t, \mathcal{T}_{h,t})} = \left(\sum_{K \in \mathcal{T}_{h,t}} |v|^2_{H^k(K)} \right)^{1/2},$$

where $|\cdot|_{H^k(K)}$ denotes the seminorm in the space $H^k(K)$.

By $\mathcal{F}_{h,t}$ we denote the system of all faces of all elements $K \in \mathcal{T}_{h,t}$. It consists of the set of all inner faces $\mathcal{F}_{h,t}^I$ and the set of all boundary faces $\mathcal{F}_{h,t}^B$: $\mathcal{F}_{h,t} = \mathcal{F}_{h,t}^I \cup \mathcal{F}_{h,t}^B$. Each $\Gamma \in \mathcal{F}_{h,t}$ will be associated with a unit normal vector $\mathbf{n}_\Gamma$. By $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)} \in \mathcal{T}_{h,t}$ we denote the elements adjacent to the face $\Gamma \in \mathcal{F}_{h,t}^I$. Moreover, for $\Gamma \in \mathcal{F}_{h,t}^B$ the element adjacent to this face will be denoted by $K_\Gamma^{(L)}$. We shall use the convention, that $\mathbf{n}_\Gamma$ is the outer normal to $\partial K_\Gamma^{(L)}$.

If $v \in H^1(\Omega_t, \mathcal{T}_{h,t})$ and $\Gamma \in \mathcal{F}_{h,t}$, then $v_\Gamma^{(L)}$ and $v_\Gamma^{(R)}$ will denote the traces of v on Γ from the side of elements $K_\Gamma^{(L)}$ and $K_\Gamma^{(R)}$, respectively. We set $h_K = \operatorname{diam} K$ for $K \in \mathcal{T}_{h,t}$, $h(\Gamma) = \operatorname{diam} \Gamma$ for $\Gamma \in \mathcal{F}_{h,t}$ and $\langle v \rangle_\Gamma = \frac{1}{2}\left(v_\Gamma^{(L)} + v_\Gamma^{(R)}\right)$, $[v]_\Gamma = v_\Gamma^{(L)} - v_\Gamma^{(R)}$, for $\Gamma \in \mathcal{F}_{h,t}^I$. Moreover, by ρ_K we denote the diameter of the largest ball inscribed into $K \in \mathcal{T}_{h,t}$.

3.4. Discretization

First we introduce the space semidiscretization of problem (2.11)–(2.13). We assume that u is a sufficiently smooth solution of our problem. If we choose an arbitrary but fixed $t \in (0, T)$, multiply equation (2.11) by a test function $\varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, integrate over any element K and finally sum over all elements $K \in \mathcal{T}_{h,t}$, then for $t \in I_m$ we get

$$\begin{aligned} &\sum_{K \in \mathcal{T}_{h,t}} \int_K \frac{Du}{Dt} \varphi \,\mathrm{d}x + \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} \frac{\partial f_s(u)}{\partial x_s} \varphi \,\mathrm{d}x \\ &\qquad - \sum_{K \in \mathcal{T}_{h,t}} \int_K \sum_{s=1}^{d} z_s \frac{\partial u}{\partial x_s} \varphi \,\mathrm{d}x - \sum_{K \in \mathcal{T}_{h,t}} \int_K \operatorname{div}(\beta(u) \nabla u) \varphi \,\mathrm{d}x = \sum_{K \in \mathcal{T}_{h,t}} \int_K g \varphi \,\mathrm{d}x. \end{aligned} \tag{3.9}$$

Applying Green's theorem to the convection and diffusion terms, introducing the concept of a numerical flux and suitable expressions mutually vanishing, after some manipulation we arrive at the identity

$$(D_t u, \varphi) + A_h(u, \varphi, t) + b_h(u, \varphi, t) + d_h(u, \varphi, t) = l_h(\varphi, t), \tag{3.10}$$

where the forms appearing here are defined for $u, \varphi \in H^2(\Omega_t, \mathcal{T}_{h,t})$, $\theta \in I\!R$ and $c_W > 0$ in the following way

$$a_h(u,\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K \beta(u)\nabla u\cdot\nabla\varphi\,\mathrm{d}x \tag{3.11}$$
$$- \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma \left(\langle\beta(u)\nabla u\rangle\cdot\mathbf{n}_\Gamma\,[\varphi] + \theta\,\langle\beta(u)\nabla\varphi\rangle\cdot\mathbf{n}_\Gamma\,[u]\right)\,\mathrm{d}S$$
$$- \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma \left(\beta(u)\nabla u\cdot\mathbf{n}_\Gamma\,\varphi + \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u - \theta\beta(u)\nabla\varphi\cdot\mathbf{n}_\Gamma\,u_D\right)\,\mathrm{d}S,$$

$$J_h(u,\varphi,t) := c_W \sum_{\Gamma\in\mathcal{F}^I_{h,t}} h(\Gamma)^{-1}\int_\Gamma [u]\,[\varphi]\,\mathrm{d}S + c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S, \tag{3.12}$$

$$J^B_h(u,\varphi,t) := c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u\,\varphi\,\mathrm{d}S, \tag{3.13}$$

$$A_h(u,\varphi,t) := a_h(u,\varphi,t) + \beta_0\,J_h(u,\varphi,t), \tag{3.14}$$

$$b_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d f_s(u)\frac{\partial\varphi}{\partial x_s}\,\mathrm{d}x \tag{3.15}$$
$$+ \sum_{\Gamma\in\mathcal{F}^I_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(R)}, \mathbf{n}_\Gamma)\,[\varphi]\,\mathrm{d}S + \sum_{\Gamma\in\mathcal{F}^B_{h,t}} \int_\Gamma H(u_\Gamma^{(L)}, u_\Gamma^{(L)}, \mathbf{n}_\Gamma)\,\varphi\,\mathrm{d}S,$$

$$d_h(u,\varphi,t) := -\sum_{K\in\mathcal{T}_{h,t}} \int_K \sum_{s=1}^d z_s\frac{\partial u}{\partial x_s}\varphi\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_{h,t}} \int_K (\boldsymbol{z}\cdot\nabla u)\varphi\,\mathrm{d}x, \tag{3.16}$$

$$l_h(\varphi,t) := \sum_{K\in\mathcal{T}_{h,t}} \int_K g\varphi\,\mathrm{d}x + \beta_0\,c_W \sum_{\Gamma\in\mathcal{F}^B_{h,t}} h(\Gamma)^{-1}\int_\Gamma u_D\,\varphi\,\mathrm{d}S. \tag{3.17}$$

Let us note that in integrals over faces we omit the subscript Γ of $\langle\cdot\rangle$ and $[\cdot]$. We consider $\theta = 1$, $\theta = 0$ and $\theta = -1$ and get the symmetric (SIPG), incomplete (IIPG) and nonsymmetric (NIPG) variants of the approximation of the diffusion terms, respectively.

In (3.15), H is a numerical flux with the following properties:

(H1) $H(u, v, \mathbf{n})$ is defined in $\mathbb{R}^2 \times B_1$, where $B_1 = \{\mathbf{n} = (n_1,\dots,n_d) \in \mathbb{R}^d;\ |\mathbf{n}| = 1\}$, and is Lipschitz-continuous with respect to u, v: there exists $L_H > 0$ such that

$$|H(u,v,\mathbf{n}) - H(u^*,v^*,\mathbf{n})| \le L_H(|u-u^*| + |v-v^*|),\ \text{for all}\ u, v, u^*, v^* \in \mathbb{R}.$$

(H2) H is consistent: $H(u,u,\mathbf{n}) = \sum_{s=1}^d f_s(u)n_s$, $\quad u\in\mathbb{R}$, $\mathbf{n}\in B_1$,

(H3) H is conservative: $H(u,v,\mathbf{n}) = -H(v,u,-\mathbf{n})$, $\quad u, v\in\mathbb{R}$, $\mathbf{n}\in B_1$.

In what follows, in the stability analysis we shall use the properties **(H1)** and **(H2)**. (Assumption **(H3)** is important for error estimation, but here it is not necessary.)

For a function φ defined in $\bigcup_{m=1}^M I_m$ we denote

$$\varphi_m^\pm = \varphi(t_m\pm) = \lim_{t\to t_m\pm}\varphi(t), \quad \{\varphi\}_m = \varphi(t_m+) - \varphi(t_m-), \tag{3.18}$$

if the one-sided limits $\varphi_m^\pm$ exist.

Now we define an ALE-STDG approximate solution of problem (2.11)–(2.13).

Definition 3.1. A function U is an approximate solution of problem (2.11)–(2.13), if $U \in S^{p,q}_{h,\tau}$ and

$$\int_{I_m} \Big((D_t U, \varphi)_{\Omega_t} + A_h(U,\varphi,t) + b_h(U,\varphi,t) + d_h(U,\varphi,t) \Big)\,\mathrm{d}t \tag{3.19}$$

$$+ (\{U\}_{m-1}, \varphi^+_{m-1})_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(\varphi,t)\,\mathrm{d}t \quad \forall \varphi \in S^{p,q}_{h,\tau}, \quad m = 1,\dots,M,$$

$$U_0^- \in S_h^{p,0}, \quad (U_0^- - u^0, v_h) = 0 \quad \forall v_h \in S_h^{p,0}. \tag{3.20}$$

(For $m = 1$ we set $\{U\}_{m-1} = \{U\}_0 := U_0^+ - U_0^-$ with U_0^- given by (3.20)).

The ALE-STDG numerical method (3.19)–(3.20) was applied in [16, 44] to the numerical simulation of a compressible flow in time-dependent domains and fluid-structure interaction.

4. Analysis of the stability

In what follows we shall be concerned with the numerical solution of the ALE problem (2.11)–(2.13) by the space-time discontinuous Galerkin method. In the theoretical analysis a number of various constants will appear. Some important constants in main assertions will be denoted by $C_{L1},\ C^*_{L1},\ C^{**}_{L1}$, etc. in Lemma 4.1, C_{L2}, etc. in Lemma 4.2, etc. and $C_{T1},\ C^*_{T1},\ C_{T2},\ C^*_{T2}$, etc. in Theorems 4.1, 4.2, etc. Further, we use special notation of constants appearing in properties of various structures, *e.g.* L_f, L_β, L_H, c_R, etc. Inside proofs, constants are denoted locally by c, c_1, c_2, c^* etc. The aim of this notation is to increase the readability of the paper and to show the relations between individual theorems and lemmas.

4.1. Some auxiliary results

As was mentioned in Section 3.1, for each $t \in [0,T]$ we consider a system of triangulations $\{\mathcal{T}_{h,t}\}_{h\in(0,\overline{h})}$. We assume that these systems are uniformly shape regular. This means that there exists a positive constant c_R, independent of K, t and h such that

$$\frac{h_K}{\rho_K} \le c_R \qquad \text{for all} \quad K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}), t \in [t_{m-1}, t_m], \tag{4.1}$$
$$\tau_m \le \tau \in (0,\overline{\tau}),\ \ m = 1,\dots,M.$$

By $(\mathcal{A}^{m-1}_{h,t})^{-1}$ we denote the inverse to the mapping $\mathcal{A}^{m-1}_{h,t}$. The symbols $\frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}}{\mathrm{d}X}$ and $\frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t})^{-1}}{\mathrm{d}x}$ denote the Jacobian matrices of $\mathcal{A}^{m-1}_{h,t}$ and $(\mathcal{A}^{m-1}_{h,t})^{-1}$, respectively. The entries of $\frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}}{\mathrm{d}X}$ and $\frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t})^{-1}}{\mathrm{d}x}$ are constant on every element $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and $K \in \mathcal{T}_{h,t}$, respectively. Moreover, we define the Jacobians $J(X,t) = \det\frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}(X)}{\mathrm{d}X},\ X \in \Omega_{t_{m-1}}$, and $J^{-1}(x,t) = \det\frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t}(x))^{-1}}{\mathrm{d}x},\ x \in \Omega_t$. The Jacobians J and J^{-1} are piecewise constant over $\hat{\mathcal{T}}_{h,t_{m-1}}$ and $\mathcal{T}_{h,t}$, respectively. The constant value of J on $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and of J^{-1} on $K \in \mathcal{T}_{h,t}$ will be denoted by $J_{\hat{K}}$ and J^{-1}_K, respectively. Of course, these terms depend on t and, hence, $J_{\hat{K}} = J_{\hat{K}}(t)$ and $J^{-1}_K = J^{-1}_K(t)$.

In what follows, we assume that

$$\mathcal{A}^{m-1}_{h,t} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_{t_{m-1}})), \quad m = 1,\dots,M,\ h \in (0,\overline{h}) \tag{4.2}$$

and

$$(A^{m-1}_{h,t})^{-1} \in W^{1,\infty}(I_m; W^{1,\infty}(\Omega_t)), \quad m = 1,\dots M,\ h \in (0,\overline{h}). \tag{4.3}$$

Obviously, we have $J \in W^{1,\infty}(I_m; L^\infty(\Omega_{t_{m-1}})),\ J^{-1} \in W^{1,\infty}(I_m; L^\infty(\Omega_t))$. Since $\mathcal{A}^{m-1}_{h,t_{m-1}}$ is the identical mapping and, hence, $J(X, t_{m-1}) = 1$, we assume that there exist constants $C_J^-,\ C_J^+ > 0$ such that the Jacobians satisfy the conditions

$$
\begin{aligned}
&C_J^- \le J(X,t) \le C_J^+, \quad X \in \overline{\Omega}_{t_{m-1}},\ t \in \overline{I}_m,\ m=1,\dots,M,\ h \in (0,\overline{h}),\\
&(C_J^+)^{-1} \le J^{-1}(x,t) \le (C_J^-)^{-1}, \quad x \in \overline{\Omega}_t,\ t \in \overline{I}_m,\ m=1,\dots,M,\ h \in (0,\overline{h}).
\end{aligned} \tag{4.4}
$$

Finally, there exist constants $C_A^-, C_A^+ > 0$ such that

$$
\left\| \frac{\mathrm{d}\mathcal{A}_{h,t}^{m-1}(X)}{\mathrm{d}X} \right\| \le C_A^+,\ X \in \overline{\Omega}_{t_{m-1}},\ t \in \overline{I}_m,\ m=1,\dots,M,\ h \in (0,\overline{h}), \tag{4.5}
$$

$$
\left\| \frac{\mathrm{d}(\mathcal{A}_{h,t}^{m-1})^{-1}(x)}{\mathrm{d}x} \right\| \le C_A^-,\ x \in \overline{\Omega}_t,\ t \in \overline{I}_m,\ m=1,\dots,M,\ h \in (0,\overline{h}), \tag{4.6}
$$

where $\|\cdot\|$ is the matrix norm induced by the Euclidean norm $|\cdot|$ in $I\!R^d$.

The above assumptions imply the following properties of the domain velocity: There exists a constant $c_z > 0$ such that

$$
|\boldsymbol{z}(x,t)|,\ |\mathrm{div}\boldsymbol{z}(x,t)| \le c_z \quad \text{for } x \in \Omega_t,\ t \in (0,T). \tag{4.7}
$$

Under assumption (4.1), the multiplicative trace inequality and the inverse inequality hold: There exist constants $c_M, c_I > 0$ independent of v, h, t and K such that

$$
\begin{aligned}
&\|v\|_{L^2(\partial K)}^2 \le c_M \left(\|v\|_{L^2(K)}\, |v|_{H^1(K)} + h_K^{-1} \|v\|_{L^2(K)}^2 \right),\\
&v \in H^1(K),\ K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}),\ t \in [0,T],
\end{aligned} \tag{4.8}
$$

and

$$
|v|_{H^1(K)} \le c_I\, h_K^{-1} \|v\|_{L^2(K)}, \quad v \in P^p(K),\ K \in \mathcal{T}_{h,t},\ h \in (0,\overline{h}),\ t \in [0,T]. \tag{4.9}
$$

In the space $H^1(\Omega_t, \mathcal{T}_{h,t})$ we define the norm

$$
\|\varphi\|_{\mathrm{DG},t} = \left(\sum_{K \in \mathcal{T}_{h,t}} |\varphi|_{H^1(K)}^2 + J_h(\varphi,\varphi,t) \right)^{1/2}. \tag{4.10}
$$

Moreover, over $\partial\Omega$ we define the norm

$$
\|u_D\|_{\mathrm{DGB},t} = \left(c_W \sum_{\Gamma \in \mathcal{F}_{h,t}^B} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \,\mathrm{d}S \right)^{1/2} = \left(J_h^B(u_D,u_D,t) \right)^{1/2}. \tag{4.11}
$$

If we use $\varphi := U$ as a test function in (3.19), we get the basic identity

$$
\begin{aligned}
&\int_{I_m} \Big((D_t U, U)_{\Omega_t} + A_h(U,U,t) + b_h(U,U,t) + d_h(U,U,t) \Big) \mathrm{d}t\\
&\qquad + (\{U\}_{m-1}, U_{m-1}^+)_{\Omega_{t_{m-1}}} = \int_{I_m} l_h(U,t)\,\mathrm{d}t.
\end{aligned} \tag{4.12}
$$

In what follows we need to estimate each term in (4.12). These estimates are summarized in Section 4.2. The skipped proofs can be found in [5]. They are based on the multiplicative trace inequality (4.8), inverse inequality (4.9), Young's inequality and assumptions (2.5) on the function β.

These estimates, apart from another, produce a problematic term $\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$, which we need to estimate in terms of data. To overcome this difficulty we generalize the concept of discrete characteristic function in time-dependent domains. In Theorem 4.1 we prove the continuity of the previously defined discrete characteristic function in $\|\cdot\|_{\Omega_t}$ and $\|\cdot\|_{\mathrm{DG},t}$ norms.

Then, in Theorems 4.2 and 4.3 we apply estimates from Section 4.2 to the basic identity (4.12). In Lemmas 4.6–4.10 we estimate similar terms in Section 4.2, but the test function (second variable) is replaced by the discrete characteristic function. Using these lemmas and properties of the discrete characteristic function proved in Theorem 4.1, we finally estimate the problematic term $\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$ in terms of data in Theorem 4.4.

Using this key result from Theorem 4.4 and the discrete Gronwall inequality from Lemma 4.11, the unconditional stability of the method is proved in Theorem 4.5.

4.2. Important estimates

Here we estimate the forms from (4.12). The proofs can be carried out in a similar way as in [5]. For a sufficiently large constant c_W we obtain the coercivity of the diffusion and penalty terms.

Lemma 4.1. *Let*

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M(c_I + 1) \quad \text{for} \quad \theta = -1 \text{ (NIPG)}, \tag{4.13}$$

$$c_W \geq \frac{\beta_1^2}{\beta_0^2} c_M(c_I + 1) \quad \text{for} \quad \theta = 0 \text{ (IIPG)}, \tag{4.14}$$

$$c_W \geq \frac{16\beta_1^2}{\beta_0^2} c_M(c_I + 1) \quad \text{for} \quad \theta = 1 \text{ (SIPG)}. \tag{4.15}$$

Then

$$\int_{I_m} \left(a_h(U,U,t) + \beta_0\, J_h(U,U,t)\right) \mathrm{d}t \geq \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \,\mathrm{d}t - \frac{\beta_0}{2} \int_{I_m} \|u_D\|^2_{\mathrm{DGB},t} \,\mathrm{d}t. \tag{4.16}$$

Further, we estimate the convection terms and the right-hand side form:

Lemma 4.2. *For each $k_1, k_2, k_3 > 0$ there exists a constant $c_b, c_d > 0$ such that we have*

$$\int_{I_m} |b_h(U,U,t)| \mathrm{d}t \leq \frac{\beta_0}{2k_1} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \tag{4.17}$$

$$\int_{I_m} |d_h(U,U,t)| \,\mathrm{d}t \leq \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \,\mathrm{d}t + \frac{c_d}{2\beta_0} \int_{I_m} \|U\|^2_{\Omega_t} \,\mathrm{d}t, \tag{4.18}$$

$$\begin{aligned} \int_{I_m} |l_h(U,t)| \,\mathrm{d}t \leq{}& \frac{1}{2} \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|U\|^2_{\Omega_t}\right) \mathrm{d}t \\ &+ \frac{\beta_0 k_3}{2} \int_{I_m} \|u_D\|^2_{\mathrm{DGB},t} \,\mathrm{d}t + \frac{\beta_0}{2k_3} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \,\mathrm{d}t. \end{aligned} \tag{4.19}$$

Finally we need to estimate the term with the ALE derivative:

Lemma 4.3. *It holds that*

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t \geq \frac{1}{2}\left(\|U_m^-\|^2_{\Omega_{t_m}} - \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z \int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t\right), \tag{4.20}$$

$$(\{U\}_{m-1}, U_{m-1}^+)_{\Omega_{t_{m-1}}} = \frac{1}{2}\left(\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}\right), \tag{4.21}$$

$$\begin{aligned}&\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} \\ &\qquad \geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} + \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t - \left(U_{m-1}^-, U_{m-1}^+\right)_{\Omega_{t_{m-1}}}.\end{aligned} \tag{4.22}$$

Proof. We start with the first inequality. We have

$$\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t = \int_{I_m} \sum_{K \in \mathcal{T}_{h,t}} (D_t U, U)_K\,\mathrm{d}t. \tag{4.23}$$

By virtue of relation (3.2), the Reynolds transport theorem (see, *e.g.* [27] or [1]) and relation (2.10), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_K U^2(x,t)\,\mathrm{d}x &= \int_K \left(\frac{\partial U^2(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla(U^2(x,t)) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t)\right)\mathrm{d}x \\ &= \int_K \left(2U(x,t)\left(\frac{\partial U(x,t)}{\partial t} + \boldsymbol{z}(x,t)\cdot\nabla U(x,t)\right) + U^2(x,t)\mathrm{div}\,\boldsymbol{z}(x,t)\right)\mathrm{d}x \\ &= 2(D_t U, U)_K + (U^2, \mathrm{div}\,\boldsymbol{z})_K.\end{aligned} \tag{4.24}$$

Expressing $(D_t U, U)_K$, summing over $K \in \mathcal{T}_{h,t}$ and integrating over I_m together with assumption (4.7) yield

$$\begin{aligned}\int_{I_m} (D_t U, U)_{\Omega_t}\,\mathrm{d}t &= \frac{1}{2}\int_{I_m} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t} U^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_{I_m} (U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\,\mathrm{d}t \\ &\geq \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{c_z}{2}\int_{I_m} \|U\|^2_{\Omega_t}\,\mathrm{d}t,\end{aligned} \tag{4.25}$$

which gives (4.20).

Further, by a simple manipulation we find that

$$2(U_{m-1}^+ - U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} = \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} + \|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}} - \|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}},$$

which immediately implies (4.21).

Concerning inequality (4.22), from (4.25) we get

$$\begin{aligned}&\int_{I_m} (D_t U, U)_{\Omega_t}\mathrm{d}t + \left(\{U\}_{m-1}, U_{m-1}^+\right)_{\Omega_{t_{m-1}}} \\ &\quad = \frac{1}{2}\|U_m^-\|^2_{\Omega_{t_m}} - \frac{1}{2}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - \frac{1}{2}\int_{I_m} (U^2, \mathrm{div}\,\boldsymbol{z})_{\Omega_t}\mathrm{d}t + \|U_{m-1}^+\|_{\Omega_{t_{m-1}}} - (U_{m-1}^-, U_{m-1}^+)_{\Omega_{t_{m-1}}} \\ &\quad \geq \frac{1}{2}\left(\|U_m^-\|^2_{\Omega_{t_m}} + \|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} - c_z\int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t\right) - \left(U_{m-1}^-, U_{m-1}^+\right)_{\Omega_{t_{m-1}}},\end{aligned}$$

which proves the lemma. ∎

4.3. Discrete characteristic function

In our further considerations, the concept of a discrete characteristic function will play an important role, which is generalized to time-dependent domains.

For $m = 1, \dots, M$ we use the following notation: $U = U(x,t)$, $x \in \Omega_t$, $t \in I_m$ will denote the approximate solution in Ω_t, and $\tilde{U} = \tilde{U}(X,t) = U(\mathcal{A}_t(X),t)$, $X \in \Omega_{t_{m-1}}$ $t \in I_m$ denotes the approximate solution transformed to the reference domain $\Omega_{t_{m-1}}$.

For $s \in I_m$ we denote $\tilde{\mathcal{U}}_s = \tilde{\mathcal{U}}_s(X,t)$, $X \in \Omega_{t_{m-1}}$, $t \in I_m$, the discrete characteristic function to $\tilde{U}$ at a point $s \in I_m$. It is defined as $\tilde{\mathcal{U}}_s \in P^q(I_m; S_h^{p,m-1})$ such that

$$\int_{I_m} (\tilde{\mathcal{U}}_s, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t = \int_{t_{m-1}}^{s} (\tilde{U}, \varphi)_{\Omega_{t_{m-1}}} \,\mathrm{d}t \quad \forall \varphi \in P^{q-1}(I_m; S_h^{p,m-1}), \tag{4.26}$$

$$\tilde{\mathcal{U}}_s(X, {t_{m-1}}^+) = \tilde{U}(X, {t_{m-1}}^+), \ X \in \Omega_{t_{m-1}}. \tag{4.27}$$

The existence and uniqueness of the discrete characteristic function satisfying (4.26) and (4.27) is proved in the monograph [20]. Further, we introduce the discrete characteristic function $\mathcal{U}_s = \mathcal{U}_s(x,t)$, $x \in \Omega_t, t \in I_m$ to $U \in S_{h,\tau}^{p,q}$ at a point $s \in I_m$:

$$\mathcal{U}_s(x,t) = \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x), t), \ x \in \Omega_t, \ t \in I_m. \tag{4.28}$$

Hence, in view of (3.8), $\mathcal{U}_s \in S_{h,\tau}^{p,q}$ and for $X \in \Omega_{t_{m-1}}$ we have

$$\mathcal{U}_s(X, t_{m-1}+) = U(X, t_{m-1}+). \tag{4.29}$$

In what follows, we prove some important properties of the discrete characteristic function. Namely, we prove that the discrete characteristic function mapping $U \to \mathcal{U}_s$ is continuous with respect of the norms $\|\cdot\|_{L^2(\Omega_t)}$ and $\|\cdot\|_{\mathrm{DG},t}$. In the proof we use a result from [7] for the discrete characteristic function on a reference domain: There exists a constant $\tilde{c}_{CH}^{(1)} > 0$ depending on q only such that

$$\int_{I_m} \|\tilde{\mathcal{U}}_s\|^2_{\Omega_{t_{m-1}}} \,\mathrm{d}t \le \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \,\mathrm{d}t, \tag{4.30}$$

for all $m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Lemma 4.4. *There exist constants* $C_{L4}^*, \ C_{L4}^{**} > 0$ *such that*

$$C_{L4}^* \, h(\hat{\Gamma})^{-1} \le h(\Gamma)^{-1} \le C_{L4}^{**} \, h(\hat{\Gamma})^{-1} \tag{4.31}$$

for all $\hat{\Gamma} \in \mathcal{F}_{h,t_{m-1}}, \Gamma = \mathcal{A}_t(\hat{\Gamma}) \in \mathcal{F}_{h,t}$ *and all* $t \in \overline{I}_m, \ m = 1, \dots, M, \ h \in (0, \overline{h})$.

Proof. We use the relation between Γ and $\hat{\Gamma}$ and the properties (4.5) and (4.6) of the mappings $\mathcal{A}_t$ and $\mathcal{A}_t^{-1}$. We also take into account that $\hat{\Gamma} \subset \hat{K}$ for some $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, $\Gamma \subset K = \mathcal{A}_t(\hat{K}) \in \mathcal{T}_{h,t}$ and that the Jacobian matrices $\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}$ and $\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}$ are constant on $\hat{K}$ and K, respectively. Then we can write

$$\begin{aligned} h(\Gamma) = \mathrm{diam}(\Gamma) &= \max_{x,x^* \in \Gamma} |x - x^*| = \max_{X,X^* \in \hat{\Gamma}} |\mathcal{A}_t(X) - \mathcal{A}_t(X^*)| \\ &\le \max_{X \in \hat{\Gamma}} \left\| \frac{\mathrm{d}\mathcal{A}_t(X)}{\mathrm{d}X} \right\| \max_{X,X^* \in \hat{\Gamma}} |X - X^*| \le C_A^+ \max_{X,X^* \in \hat{\Gamma}} |X - X^*| = C_A^+ \, h(\hat{\Gamma}). \end{aligned}$$

Similarly, we get $h(\hat{\Gamma}) \le C_A^- \, h(\Gamma)$. These inequalities immediately imply (4.31) with $C_{L4}^* = (C_A^+)^{-1}$ and $C_{L4}^{**} = C_A^-$. □

Theorem 4.1. *There exist constants* $C_{T1}^{*}, C_{T1}^{**} > 0$ *such that*

$$\int_{I_m} \|\mathcal{U}_s\|_{\Omega_t}^2 \,\mathrm{d}t \le C_{T1}^{*} \int_{I_m} \|U\|_{\Omega_t}^2 \,\mathrm{d}t \tag{4.32}$$

$$\int_{I_m} \|\mathcal{U}_s\|_{\mathrm{DG},t}^2 \,\mathrm{d}t \le C_{T1}^{**} \int_{I_m} \|U\|_{\mathrm{DG},t}^2 \,\mathrm{d}t \tag{4.33}$$

for all $s \in I_m,\ m = 1, \ldots, M$ *and* $h \in (0, \overline{h})$.

Proof. We begin with the proof of the first inequality. We have

$$\begin{aligned}
\|\mathcal{U}_s(t)\|_{\Omega_t}^2 &= \int_{\Omega_t} |\mathcal{U}_s(x,t)|^2 \,\mathrm{d}x = \int_{\Omega_t} |\tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x), t)|^2 \,\mathrm{d}x \\
&= \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 J(X,t) \,\mathrm{d}X \le C_J^{+} \int_{\Omega_{t_{m-1}}} |\tilde{\mathcal{U}}_s(X,t)|^2 \,\mathrm{d}X \\
&= C_J^{+} \|\tilde{\mathcal{U}}_s(t)\|_{\Omega_{t_{m-1}}}^2
\end{aligned}$$

Integrating over I_m and using (4.30) and (4.4), we obtain

$$\begin{aligned}
\int_{I_m} \|\mathcal{U}_s(t)\|_{\Omega_t}^2 \,\mathrm{d}t &\le C_J^{+} \int_{I_m} \|\tilde{\mathcal{U}}_s(t)\|_{\Omega_{t_{m-1}}}^2 \,\mathrm{d}t \\
&\le C_J^{+} \tilde{c}_{CH}^{(1)} \int_{I_m} \|\tilde{U}(t)\|_{\Omega_{t_{m-1}}}^2 \,\mathrm{d}t \\
&= C_J^{+} \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \,\mathrm{d}X \right) \mathrm{d}t \\
&= C_J^{+} \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |U(\mathcal{A}_t(X),t)|^2 \,\mathrm{d}X \right) \mathrm{d}t \\
&= C_J^{+} \tilde{c}_{CH}^{(1)} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 J^{-1}(x,t) \,\mathrm{d}x \right) \mathrm{d}t \\
&\le C_J^{+} \tilde{c}_{CH}^{(1)} (C_J^{-})^{-1} \int_{I_m} \left(\int_{\Omega_t} |U(x,t)|^2 \,\mathrm{d}x \right) \mathrm{d}t \\
&= C_J^{+} \tilde{c}_{CH}^{(1)} (C_J^{-})^{-1} \int_{I_m} \|U(t)\|_{\Omega_t}^2 \,\mathrm{d}t.
\end{aligned}$$

Setting $C_{T1}^{*} = C_J^{+} \tilde{c}_{CH}^{(1)} (C_J^{-})^{-1}$, we get (4.32).

Now we pay our attention to the proof of the second inequality in the theorem. From the definition of the DG-norm we have

$$\begin{aligned}
\int_{I_m} ||\mathcal{U}_s||_{\mathrm{DG},t}^2 \,\mathrm{d}t &= \int_{I_m} \sum_{K \in \mathcal{T}_{h,t}} |\mathcal{U}_s|_{H^1(K)}^2 \,\mathrm{d}t + \int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}_{h,t}^I} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2 \,\mathrm{d}S \right) \mathrm{d}t \\
&\quad + \int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}_{h,t}^B} \frac{c_W}{h(\Gamma)} \int_\Gamma |\mathcal{U}_s|^2 \,\mathrm{d}S \right) \mathrm{d}t,
\end{aligned} \tag{4.34}$$

where $\mathcal{F}_{h,t}^I = \{\mathcal{A}_{h,t}^{m-1}(\hat{\Gamma});\, \hat{\Gamma} \in \mathcal{F}_{h,t_{m-1}}^I\}$ and similarly $\mathcal{F}_{h,t}^B = \{\mathcal{A}_{h,t}^{m-1}(\hat{\Gamma});\, \hat{\Gamma} \in \mathcal{F}_{h,t_{m-1}}^B\}$.

Further, we estimate each term on the right-hand side of (4.34). From [20], relation (6.161), it follows that

$$\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \le \tilde{c}^{(2)}_{CH} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} |\tilde{U}(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t, \tag{4.35}$$

with a constant $\tilde{c}^{(2)}_{CH} > 0$ depending on q only. For simplicity let us denote

$$B_t = B_t(X) = \frac{\mathrm{d}\mathcal{A}^{m-1}_{h,t}(X)}{\mathrm{d}X}, \quad B_t^{-1} = B_t^{-1}(x) = \frac{\mathrm{d}(\mathcal{A}^{m-1}_{h,t})^{-1}(x)}{\mathrm{d}x}\cdot$$

Then it follows from (4.5) and (4.6) that $\|B_t\| \le C_A^+$ and $\|B_t^{-1}\| \le C_A^-$.
Now, for $K \in \mathcal{T}_{h,t}$, $K = \mathcal{A}_t(\hat{K})$ with $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$, using that $\|B_t|_{\hat{K}}\|$ and $\|B_t^{-1}|_{\hat{K}}\|$ are constant, we have

$$\begin{aligned} |\mathcal{U}_s(t)|^2_{H^1(K)} &= \int_K |\nabla \mathcal{U}_s(x,t)|^2\,\mathrm{d}x = \int_K \left|\nabla \tilde{\mathcal{U}}_s(\mathcal{A}_t^{-1}(x),t)\right|^2 \mathrm{d}x \\ &\le \int_{\hat{K}} \left| B_t^{-1}|_K \nabla \tilde{\mathcal{U}}_s(X,t)\right|^2 J(X,t)\,\mathrm{d}X \le (C_A^-)^2 C_J^+ \,|\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}. \end{aligned} \tag{4.36}$$

The summation over all $K \in \mathcal{T}_{h,t}$, integration over I_m, the use of (4.35), (4.4), the Fubini and the substitution theorem imply that

$$\begin{aligned} \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |\mathcal{U}_s(t)|^2_{H^1(K)}\,\mathrm{d}t &\le (C_A^-)^2 C_J^+ \int_{I_m} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} |\tilde{\mathcal{U}}_s(t)|^2_{H^1(\hat{K})}\,\mathrm{d}t \\ &\le (C_A^-)^2 C_J^+ \tilde{c}^{(2)}_{CH} \int_{I_m} \left(\sum_{K\in\mathcal{T}_{h,t}} \int_K |\nabla U(t)|^2 \|B_t\|^2 J_K^{-1}\,\mathrm{d}x \right) \mathrm{d}t \\ &\le c_1 \int_{I_m} \sum_{K\in\mathcal{T}_{h,t}} |U(t)|^2_{H^1(K)}\,\mathrm{d}t \\ &= c_1 \int_{I_m} |U(t)|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})}\,\mathrm{d}t, \end{aligned} \tag{4.37}$$

where $c_1 := (C_A^-)^2 C_J^+ (C_J^-)^{-1} \tilde{c}^{(2)}_{CH} (C_A^+)^2$.
Now we turn our attention to the term

$$\int_{I_m} \left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_\Gamma [\mathcal{U}_s]^2\,\mathrm{d}S \right) \mathrm{d}t.$$

For simplicity we assume that $d = 2$. In Appendix A we briefly describe the proof for $d = 3$. We use estimate (6.162) from [20], which implies that

$$\int_{I_m} \left(\sum_{\hat{\Gamma}\in\mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat{\Gamma})} \int_{\hat{\Gamma}} [\tilde{\mathcal{U}}_s]^2\,\mathrm{d}S^{\hat{\Gamma}} \right) \mathrm{d}t \le c_2 \int_{I_m} \left(\sum_{\hat{\Gamma}\in\mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat{\Gamma})} \int_{\hat{\Gamma}} [\tilde{U}]^2\,\mathrm{d}S^{\hat{\Gamma}} \right) \mathrm{d}t. \tag{4.38}$$

(Here $\mathrm{d}S^{\hat{\Gamma}}$ denotes the element of the arc $\hat{\Gamma}$. Similarly we use the notation $\mathrm{d}S^{\Gamma}$.)
Now we consider the relation $\Gamma = \mathcal{A}_t(\hat{\Gamma})$, $\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}$, and introduce a parametrization of $\hat{\Gamma}$:

$$\hat{\Gamma} = \mathcal{B}^{\hat{\Gamma}}_{m-1}([0,1]) = \{X = \mathcal{B}^{\hat{\Gamma}}_{m-1}(\upsilon); \upsilon \in [0,1]\}.$$

Then an element of $\hat{\Gamma}$ can be expressed as

$$\mathrm{d}S^{\hat{\Gamma}} = |(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)|\,\mathrm{d}\upsilon, \quad \upsilon \in [0,1].$$

These relations imply that

$$\Gamma = \{x = \mathcal{A}_t(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon)); \upsilon \in [0,1]\}$$
$$\mathrm{d}S^{\Gamma} = \left|\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon))(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)\right|\mathrm{d}\upsilon, \quad \upsilon \in [0,1].$$

The term $(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)$ is a tangent vector to $\hat{\Gamma}$ at the point $\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of

$$\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon))(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)$$

are identical from the sides of both elements $K_{\hat{\Gamma}}^{(L)}$ and $K_{\hat{\Gamma}}^{(R)}$ adjacent to $\hat{\Gamma}$. Then we can use the above relations, inequalities (4.31), (4.5), and write

$$\begin{aligned}\int_{\Gamma}\frac{1}{h(\Gamma)}[\mathcal{U}_s]^2\mathrm{d}S^{\Gamma} &= \int_0^1 \frac{1}{h(\Gamma)}[\mathcal{U}_s(\mathcal{A}_t(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon)))]^2\left|\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon))(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)\right|\mathrm{d}\upsilon \\ &\le \int_0^1 \frac{1}{h(\Gamma)}[\tilde{\mathcal{U}}_s(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon))]^2 \underbrace{\left\|\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}_{m-1}^{\hat{\Gamma}}(\upsilon))\right\|}_{\le C_A^+}\left|(\mathcal{B}_{m-1}^{\hat{\Gamma}})'(\upsilon)\right|\mathrm{d}\upsilon \\ &\le C_A^+\int_{\hat{\Gamma}}\frac{C_{L4}^{**}}{h(\hat{\Gamma})}[\tilde{\mathcal{U}}_s]^2\mathrm{d}S^{\hat{\Gamma}}.\end{aligned} \tag{4.39}$$

From (4.38) and (4.39) we get

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}_{h,t}^I}\frac{c_W}{h(\Gamma)}\int_{\Gamma}[\mathcal{U}_s]^2\,\mathrm{d}S^{\Gamma}\right)\mathrm{d}t \le c_2 C_A^+ C_{L4}^{**}\int_{I_m}\left(\sum_{\hat{\Gamma}\in\mathcal{F}_{h,t_{m-1}}^I}\frac{c_W}{h(\hat{\Gamma})}\int_{\hat{\Gamma}}[\tilde{U}]^2\,\mathrm{d}S^{\hat{\Gamma}}\right)\mathrm{d}t. \tag{4.40}$$

Further, for $\Gamma = \mathcal{A}_t(\hat{\Gamma})$, where $\hat{\Gamma}\in\mathcal{F}_{h,t_{m-1}}^I$, we consider the parametrization

$$\Gamma = \{x = \mathcal{B}_t^{\Gamma}(\upsilon); \upsilon\in[0,1]\},$$
$$\hat{\Gamma} = \{X = \mathcal{A}_t^{-1}(\mathcal{B}_t^{\Gamma}(\upsilon)); \upsilon\in[0,1]\},$$
$$\mathrm{d}S^{\hat{\Gamma}} = \left|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}_t^{\Gamma}(\upsilon))(\mathcal{B}_t^{\Gamma})'(\upsilon)\right|\mathrm{d}\upsilon.$$

Then, by (4.6),

$$\begin{aligned}\int_{\hat{\Gamma}}[\tilde{U}]^2\,\mathrm{d}S^{\hat{\Gamma}} &= \int_0^1 \underbrace{[\tilde{U}(\mathcal{A}_t^{-1}(\mathcal{B}_t^{\Gamma}(\upsilon)))]^2}_{[U(\mathcal{B}_t^{\Gamma}(\upsilon))]^2}\left|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}_t^{\Gamma}(\upsilon))(\mathcal{B}_t^{\Gamma})'(\upsilon)\right|\mathrm{d}\upsilon \\ &\le \int_0^1 [U(\mathcal{B}_t^{\Gamma}(\upsilon))]^2 \underbrace{\left\|\frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}_t^{\Gamma}(\upsilon))\right\|}_{\le C_A^-}\left|(\mathcal{B}_t^{\Gamma})'(\upsilon)\right|\mathrm{d}\upsilon \\ &\le C_A^-\int_0^1 [U(\mathcal{B}_t^{\Gamma}(\upsilon))]^2|(\mathcal{B}_t^{\Gamma})'(\upsilon)|\,\mathrm{d}\upsilon \\ &= C_A^-\int_{\Gamma}[U]^2\,\mathrm{d}S^{\Gamma}.\end{aligned}$$

Substituting back to (4.40) and using (4.31), we find that

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[\mathcal{U}_s]^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t\le c_3\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^I_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma[U]^2\,\mathrm{d}S\right)\mathrm{d}t, \tag{4.41}$$

where $c_3 = c_2 C_A^+ C_{L4}^{**}(C_{L4}^*)^{-1}C_A^-$.

Similarly we can prove the inequality

$$\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma|\mathcal{U}_s|^2\,\mathrm{d}S^\Gamma\right)\mathrm{d}t\le c_4\int_{I_m}\left(\sum_{\Gamma\in\mathcal{F}^B_{h,t}}\frac{c_W}{h(\Gamma)}\int_\Gamma|U|^2\,\mathrm{d}S\right)\mathrm{d}t. \tag{4.42}$$

Finally, (4.37), (4.41) and (4.42) imply (4.33) with $C_{T1}^{**}=\max\{c_1,c_3,c_4\}$. □

4.4. Proof of the unconditional stability

Theorem 4.2. *There exists a constant $C_{T2}>0$ such that*

$$\begin{aligned}\|U_m^-\|^2_{\Omega_{t_m}}-\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}+\|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\frac{\beta_0}{2}\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ \le C_{T2}\left(\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t+\int_{I_m}\|u_D\|^2_{\mathrm{DGB},t}\mathrm{d}t+\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\right).\end{aligned} \tag{4.43}$$

Proof. From (4.12), by virtue of (4.20), (4.16), (4.17), (4.18), (4.21) and (4.19), after some manipulation we get

$$\begin{aligned}&\|U_m^-\|^2_{\Omega_{t_m}}-\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}+\|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\beta_0\left(1-\frac{1}{k_1}-\frac{1}{k_2}-\frac{1}{k_3}\right)\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ &\le\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t+\beta_0(1+k_3)\int_{I_m}\|u_D\|^2_{\mathrm{DGB},t}\mathrm{d}t+\left(c_z+1+\frac{c_d}{\beta_0}+2c_b\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t.\end{aligned}$$

Hence, choosing $k_1=k_2=k_3=6$, we get (4.43) with $C_{T2}=\max\{1,7\beta_0,c_z+1+c_d/\beta_0+2c_b\}$. □

Theorem 4.3. *There exist constants $C_{T3}^*, C_{T3}^{**}>0$ such that for any $\delta_1>0$ we have*

$$\begin{aligned}&\|U_m^-\|^2_{\Omega_{t_m}}+\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}+\frac{\beta_0}{2}\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ &\le C_{T3}^*\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t+C_{T3}^{**}\int_{I_m}\left(\|g\|^2_{\Omega_t}+\|u_D\|^2_{\mathrm{DGB},t}\right)\mathrm{d}t+\frac{2}{\delta_1}\|U_{m-1}^-\|^2_{\Omega_{t_{m-1}}}+4\delta_1\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}.\end{aligned} \tag{4.44}$$

Proof. From (3.19), by virtue of (4.22), (4.16), (4.17), (4.18), (4.21) and (4.19), we get

$$\begin{aligned}&\|U_m^-\|^2_{\Omega_{t_m}}+\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}}+\beta_0\left(1-\frac{1}{k_1}-\frac{1}{k_2}-\frac{1}{k_3}\right)\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ &\le\int_{I_m}\|g\|^2_{\Omega_t}\mathrm{d}t+\beta_0(1+k_3)\int_{I_m}\|u_D\|^2_{\mathrm{DGB},t}\mathrm{d}t\\ &\quad+\left(1+c_z+2c_b+\frac{c_d}{\beta_0}\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t+2\left(U_{m-1}^-,U_{m-1}^+\right)_{\Omega_{t_{m-1}}}.\end{aligned}$$

Using Young's inequality for the term $2(U_{m-1}^-,U_{m-1}^+)$ and setting $k_1=k_2=k_3=6$, we get (4.44), where $C_{T3}^*=1+c_z+2c_b+c_d/\beta_0$ and $C_{T3}^{**}=\max\{1,7\beta_0\}$. □

We introduce the following notation:

$$t_{m-1+l/q} = t_{m-1} + \tau_m \frac{l}{q},$$
$$U_{m-1+l/q} = U(t_{m-1+l/q}), \quad l = 0, \ldots, q.$$

Lemma 4.5. *There exist constants $C^*_{L5}, C^{**}_{L5} > 0$ such that for $m = 1, \ldots, M$ we have*

$$\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \geq \frac{C^*_{L5}}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \tag{4.45}$$

$$\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \leq \frac{C^{**}_{L5}}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{4.46}$$

Proof. Using the equivalence of norms in the space of polynomials of degree $\leq q$, for $p(t) = \tilde{U}(X,t),\ t \in I_m$, and any fixed $X \in \Omega_{t_{m-1}}$, we have

$$\sum_{l=0}^{q} \tilde{U}^2\left(X, t_{m-1+l/q}\right) \geq \frac{L_q}{\tau_m} \int_{I_m} \tilde{U}^2(X,t)\,\mathrm{d}t,$$
$$\tilde{U}^2\big(X, t^+_{m-1}\big) \leq \frac{M_q}{\tau_m} \int_{I_m} \tilde{U}^2(X,t)\,\mathrm{d}t,$$

where the constants L_q, $M_q > 0$ were introduced in [20], Section 6.2.3.2. Integrating over $\Omega_{t_{m-1}}$ and using Fubini's theorem, we get

$$\begin{aligned}\sum_{l=0}^{q} \int_{\Omega_{t_{m-1}}} |\tilde{U}\left(X, t_{m-1+l/q}\right)|^2 \mathrm{d}X &\geq \frac{L_q}{\tau_m} \int_{\Omega_{t_{m-1}}} \left(\int_{I_m} |\tilde{U}(X,t)|^2 \mathrm{d}t \right) \mathrm{d}X \\ &= \frac{L_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t.\end{aligned}$$

Analogously we find that

$$\int_{\Omega_{t_{m-1}}} |\tilde{U}\big(X, t^+_{m-1}\big)|^2\,\mathrm{d}X \leq \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t.$$

Now the substitution $X = \mathcal{A}_t^{-1}(x)$, where $X \in \Omega_{t_{m-1}}$, $x \in \Omega_t$, relation $\tilde{U}(\mathcal{A}_t^{-1}(x), t) = U(x,t)$ and (4.4) imply that

$$\begin{aligned}&\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} \\ &\geq C_J^- \sum_{l=0}^{q} \int_{\Omega_{t_{m-1+l/q}}} |U(x, t_{m-1+l/q})|^2 J^{-1}(x, t_{m-1+l/q})\,\mathrm{d}x \\ &\geq \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\ &= \frac{L_q}{\tau_m} C_J^- \int_{I_m} \left(\int_{\Omega_t} |\tilde{U}(\mathcal{A}_t^{-1}(x),t)|^2 J^{-1}(x,t)\,\mathrm{d}x \right) \mathrm{d}t \\ &\geq \frac{L_q}{\tau_m} (C_J^+)^{-1} C_J^- \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t.\end{aligned}$$

Hence, we get (4.45) with $C^*_{L5} = L_q (C_J^+)^{-1} C_J^-$.

Further, since $x = \mathcal{A}_{t_{m-1}}(X) = X$ and, thus, $\tilde{U}(X, t_{m-1}^+) = U(x, t_{m-1}^+)$, using the substitution theorem and (4.4), we obtain

$$\begin{aligned}\|U_{m-1}^+\|^2_{\Omega_{t_{m-1}}} &= \int_{\Omega_{t_{m-1}}} |\tilde{U}(X, t_{m-1}^+)|^2 \mathrm{d}X \\ &\le \frac{M_q}{\tau_m} \int_{I_m} \left(\int_{\Omega_{t_{m-1}}} |\tilde{U}(X,t)|^2 \mathrm{d}X \right) \mathrm{d}t \\ &\le \frac{C^{**}_{L5}}{\tau_m} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,\end{aligned}$$

where $C^{**}_{L5} = M_q (C_J^-)^{-1}$. □

In what follows, because of simplicity, we use the notation $\tilde{U}' = \frac{\partial \tilde{U}}{\partial t}$ and do not write the arguments X and t in integrals.

Lemma 4.6. *There exists a constant $C_{L6} > 0$ such that*

$$\begin{aligned}&\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t + (\{U\}_{m-1}, \mathcal{U}_s(t_{m-1}^+))_{\Omega_{t_{m-1}}} \\ &\qquad \ge \frac{1}{2}\left(\|U(s-)\|^2_{\Omega_s} + \|U(t_{m-1}^+)\|^2_{\Omega_{t_{m-1}}} \right) - C_{L6} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t - (U_{m-1}^+, U_{m-1}^-)_{\Omega_{t_{m-1}}}.\end{aligned} \tag{4.47}$$

for any $s \in I_m$, $m = 1, \dots, M$ and $h \in (0, \overline{h})$.

Proof. By virtue of the definition of the ALE derivative (2.9), the definitions of $\tilde{U}, \tilde{\mathcal{U}}_s, \mathcal{U}_s$, the fact that $\tilde{U}'$ is a polynomial of degree $\le q-1$ in time and the substitution theorem we can write

$$\begin{aligned}\int_{I_m} (D_t U, \mathcal{U}_s)_{\Omega_t} \mathrm{d}t &= \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U} J \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U}(1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \\ &= \int_{t_{m-1}}^{s} (D_t U, U)_{\Omega_t} \mathrm{d}t + \int_{t_{m-1}}^{s} \left(\tilde{U}', \tilde{U}(1-J) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t + \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t.\end{aligned} \tag{4.48}$$

Now we estimate the second and third term on the right-hand side. We begin with the third term. The fact that J is constant on each $\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}$ and the substitution theorem imply that

$$\begin{aligned}\left| \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J-1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| &= \left| \sum_{\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} (J_{\hat{K}} - 1) \left(\int_{\hat{K}} \tilde{U}' \tilde{\mathcal{U}}_s \, \mathrm{d}X \right) \mathrm{d}t \right| \\ &\le \sum_{\hat{K} \in \hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t \in I_m} |J_{\hat{K}} - 1| \int_{I_m} \left(\int_{\hat{K}} |\tilde{U}' \tilde{\mathcal{U}}_s| \, \mathrm{d}X \right) \mathrm{d}t.\end{aligned}$$

Using the relation $J_{\hat{K}}(t_{m-1}) = 1$, we have

$$\max_{t\in I_m} |J_{\hat{K}} - 1| \le \int_{t_{m-1}}^{t_m} |J'_{\hat{K}}| \, \mathrm{d}t \le c_J \tau_m,$$

where $c_J > 0$ is a constant independent of h, τ_m, m. Then we find that

$$\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \max_{t\in I_m} |J_{\hat{K}} - 1| \int_{I_m} \int_{\hat{K}} |\tilde{U}'\tilde{\mathcal{U}}_s| \, \mathrm{d}X\mathrm{d}t$$
$$\le c_J\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{U}'|^2 \, \mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \, \mathrm{d}t \right)^{1/2} \right) \mathrm{d}X.$$

Now we apply the inverse inequality in time: There exists a constant $\hat{c}_I$ such that

$$\left(\int_{I_m} |\tilde{U}'(X,t)|^2 \, \mathrm{d}t \right)^{1/2} \le \frac{\hat{c}_I}{\tau_m} \left(\int_{I_m} |\tilde{U}(X,t)|^2 \, \mathrm{d}t \right)^{1/2} \tag{4.49}$$

holds for every $X \in \Omega_{t_{m-1}}$, $\tau_m \in (0, \overline{\tau})$ and $m = 1, \dots, M$.

This inequality, Young's inequality, Fubini's theorem, (4.30), substitution theorem and (4.4) imply that

$$\tau_m \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\left(\int_{I_m} |\tilde{U}'|^2 \, \mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \, \mathrm{d}t \right)^{1/2} \right) \mathrm{d}X$$
$$\le \hat{c}_I \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} |\tilde{U}|^2 \mathrm{d}t \right)^{1/2} \left(\int_{I_m} |\tilde{\mathcal{U}}_s|^2 \mathrm{d}t \right)^{1/2} \mathrm{d}X$$
$$\le \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{\hat{K}} \left(\int_{I_m} \left(|\tilde{U}|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}t \right) \mathrm{d}X$$
$$= \frac{\hat{c}_I}{2} \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}} \int_{I_m} \left(\int_{\hat{K}} \left(|\tilde{U}|^2 + |\tilde{\mathcal{U}}_s|^2 \right) \mathrm{d}X \right) \mathrm{d}t$$
$$\le \frac{\hat{c}_I}{2} (1 + \tilde{c}_{CH}^{(1)}) \int_{I_m} \|\tilde{U}\|^2_{\Omega_{t_{m-1}}} \mathrm{d}t$$
$$\le c^* \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t,$$

where $c^* = (C_J^-)^{-1} \hat{c}_I (1 + \tilde{c}_{CH}^{(1)})/2$. Summarizing the obtained results, we see that we have proved the inequality

$$\left| \int_{I_m} \left(\tilde{U}', \tilde{\mathcal{U}}_s (J - 1) \right)_{\Omega_{t_{m-1}}} \mathrm{d}t \right| \le c^* c_J \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t. \tag{4.50}$$

Similarly as above we can estimate the second term on the right-hand side of (4.48):

$$\begin{aligned}\left|\int_{t_{m-1}}^{s}\left(\tilde{U}',\tilde{U}(1-J)\right)_{\Omega_{t_{m-1}}}\mathrm{d}t\right| &\le \int_{I_m}\left|(\tilde{U}',\tilde{U}(1-J))_{\Omega_{t_{m-1}}}\right|\mathrm{d}t\\ &\le \sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}}\max_{t\in I_m}|1-J_{\hat{K}}|\int_{I_m}\int_{\hat{K}}|\tilde{U}'\tilde{U}|\,\mathrm{d}X\,\mathrm{d}t\\ &\le c_J\tau_m\sum_{\hat{K}\in\hat{\mathcal{T}}_{h,t_{m-1}}}\int_{\hat{K}}\left(\left(\int_{I_m}|\tilde{U}'|^2\,\mathrm{d}t\right)^{1/2}\left(\int_{I_m}|\tilde{U}|^2\,\mathrm{d}t\right)^{1/2}\right)\mathrm{d}X.\end{aligned}$$

Now the inverse inequality in time, Young's inequality, Fubini's theorem, (4.30) and (4.4) yield the inequality

$$\left|\int_{t_{m-1}}^{s}\left(\tilde{U}',\tilde{U}(1-J)\right)_{\Omega_{t_{m-1}}}\mathrm{d}t\right|\le c_1\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t. \tag{4.51}$$

with $c_1=c_J(C_J^-)^{-1}\hat{c}_I/2$.

Finally, from (4.48), (4.50), (4.51) and analogy to (4.22), (4.29) putting $c_2=c^*c_J+c_1$ we find that

$$\begin{aligned}&\int_{I_m}(D_tU,\mathcal{U}_s)_{\Omega_t}\mathrm{d}t+(\{U\}_{m-1},\mathcal{U}_s(t_{m-1}+))_{\Omega_{t_{m-1}}}\\ &\quad\ge\int_{t_{m-1}}^{s}(D_tU,U)_{\Omega_t}\mathrm{d}t+\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}-(U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}}-c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\ &\quad=\frac{1}{2}\int_{t_{m-1}}^{s}\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}U^2(x,t)\mathrm{d}x\right)\mathrm{d}t-\frac{1}{2}\int_{t_{m-1}}^{s}\left(U^2\mathrm{div},\boldsymbol{z}\right)_{\Omega_t}\mathrm{d}t\\ &\quad\quad+\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}-(U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}}-c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\ &\quad=\frac{1}{2}\left(\|U(s-)\|^2_{\Omega_s}+\|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}\right)-\frac{c_z}{2}\int_{t_{m-1}}^{s}\|U\|^2_{\Omega_t}\mathrm{d}t\\ &\quad\quad-c_2\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t-(U^-_{m-1},U^+_{m-1})_{\Omega_{t_{m-1}}},\end{aligned}$$

which implies (4.47) with $C_{L6}=c_z/2+c_2$. □

In the following lemmas, for simplicity we use the notation $\mathcal{U}_l^*$ and $\tilde{\mathcal{U}}_l^*$ for the discrete characteristic functions to U and $\tilde{U}$, respectively at the time instant $t_{m-1+l/q}$.

Lemma 4.7. *There exists a constant $C_{L7}>0$ such that*

$$|a_h(U,\mathcal{U}_l^*,t)+\beta_0J_h(U,\mathcal{U}_l^*,t)|\le C_{L7}\left(\|U\|^2_{\mathrm{DG},t}+\|\mathcal{U}_l^*\|^2_{\mathrm{DG},t}+\|u_D\|^2_{\mathrm{DGB},t}\right) \tag{4.52}$$

for all $t,l\in I_m$, $m=1,\ldots,M$, $h\in(0,\overline{h})$.

Proof. Using the definition of the form a_h, the property of the function β, the Cauchy inequality and Young's inequality, we get

$$
\begin{aligned}
|a_h(U,\mathcal{U}_l^*,t)| \le \beta_1 & \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right)\,\mathrm{d}x \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right) + \frac{c_W}{h(\Gamma)}\,[\mathcal{U}_l^*]^2\right)\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}\left(|\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2\right) + \frac{c_W}{h(\Gamma)}\,[U]^2\right)\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla U|^2 + \frac{c_W}{h(\Gamma)}\,|\mathcal{U}_l^*|^2\right)\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \left(\frac{h(\Gamma)}{c_W}|\nabla \mathcal{U}_l^*|^2 + \frac{c_W}{h(\Gamma)}\,|U|^2\right)\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\,|u_D|\,\mathrm{d}S.
\end{aligned} \tag{4.53}
$$

The last term can be estimated using Young's inequality and the relation $h(\Gamma) \le h_{K_\Gamma^{(L)}}$, for each $\varepsilon > 0$ the last term can be estimated in the following way:

$$
\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\;|u_D|\;\mathrm{d}S \le \frac{\beta_1\varepsilon}{2c_W} J_h^B(u_D,u_D) + \frac{\beta_1}{2\varepsilon} \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_{\partial K_\Gamma^{(L)}} h_{K_\Gamma^{(L)}}\,|\nabla \mathcal{U}_l^*|^2\;\mathrm{d}S.
$$

Now we express the first term on the right-hand side of this inequality with the aid of the definition of the $\|\cdot\|_{\mathrm{DGB},t}$-norm and to the second term we apply the multiplicative trace inequality (4.8) and the inverse inequality (4.9). We get

$$
\beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma |\nabla \mathcal{U}_l^*|\;|u_D|\;\mathrm{d}S \le \frac{\beta_1\varepsilon}{2c_W}\|u_D\|_{\mathrm{DGB},t}^2 + \frac{\beta_1}{2\varepsilon} c_M(c_I+1)\|\mathcal{U}_l^*\|_{\mathrm{DG},t}^2. \tag{4.54}
$$

Setting $\varepsilon := \frac{\beta_1}{\beta_0} c_M(c_I+1)$ in (4.54) and substituting back to (4.53) we get

$$
\begin{aligned}
|a_h(U,\mathcal{U}_l^*,t)| \le \beta_1 & \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right)\,\mathrm{d}x \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \frac{h(\Gamma)}{c_W}\left(|\nabla U_\Gamma^{(L)}|^2 + |\nabla U_\Gamma^{(R)}|^2\right)\mathrm{d}S + \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \frac{h(\Gamma)}{c_W}|\nabla U|^2\,\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^I} \int_\Gamma \frac{h(\Gamma)}{c_W}\left(|\nabla (\mathcal{U}_l^*)_\Gamma^{(L)}|^2 + |\nabla (\mathcal{U}_l^*)_\Gamma^{(R)}|^2\right)\mathrm{d}S \\
&+ \beta_1 \sum_{\Gamma\in\mathcal{F}_{h,t}^B} \int_\Gamma \frac{h(\Gamma)}{c_W}|\nabla \mathcal{U}_l^*|^2\,\mathrm{d}S + \frac{\beta_1^2}{2\beta_0 c_W} c_M(c_I+1)\|u_D\|_{\mathrm{DGB},t}^2 \\
&+ \frac{\beta_0}{2}\|\mathcal{U}_l^*\|_{\mathrm{DG},t}^2 + \beta_1\,J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \beta_1\,J_h(U,U,t).
\end{aligned}
$$

Using the inequality $h(\Gamma) \le h_K$ for $\Gamma \subset \partial K$, we have

$$|a_h(U,\mathcal{U}_l^*,t)| \le \beta_1 \sum_{K\in\mathcal{T}_{h,t}} \int_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}x + \frac{\beta_1}{c_W} \sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}S \\ + \frac{\beta_1^2}{2\beta_0 c_W} c_M(c_I+1)\|u_D\|^2_{\mathrm{DGB},t} + \frac{\beta_0}{2}\|\mathcal{U}_l^*\|^2_{\mathrm{DG},t} \\ + \beta_1\, J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \beta_1\, J_h(U,U,t). \tag{4.55}$$

Now, applying the multiplicative inequality and the inverse inequality, we can obtain the estimate

$$\sum_{K\in\mathcal{T}_{h,t}} \int_{\partial K} h_K \left(|\nabla U|^2 + |\nabla \mathcal{U}_l^*|^2\right) \mathrm{d}S \le c_M(c_I+1) \sum_{K\in\mathcal{T}_{h,t}} \left(|U|^2_{H^1(\Omega)} + |\mathcal{U}_l^*|^2_{H^1(\Omega)}\right). \tag{4.56}$$

From (4.55) and (4.56), the definition of the $\|\cdot\|_{\mathrm{DG},t}$-norm, using the inequality

$$J_{h,}(U,\mathcal{U}_l^*,t) \le J_h(U,U,t) + J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t)$$

and putting $C_{L7} = \max\{\beta_0+\beta_1+\beta_1 c_M(c_I+1)/c_W, \beta_1^2 c_M(c_I+1)/(2\beta_0 c_W)\}$, we finally get

$$|a_h(U,\mathcal{U}_l^*,t) + \beta_0\, J_h(U,\mathcal{U}_l^*,t)| \le \left(\beta_1 + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |U|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\ + (\beta_0+\beta_1) J_h(U,U,t) + \left(\beta_1 + \frac{\beta_0}{2} + \frac{\beta_1}{c_W} c_M(c_I+1)\right) |\mathcal{U}_l^*|^2_{H^1(\Omega_t,\mathcal{T}_{h,t})} \\ + (\beta_0+\beta_1) J_h(\mathcal{U}_l^*,\mathcal{U}_l^*,t) + \frac{\beta_1^2}{2\beta_0 c_W} c_M(c_I+1)\|u_D\|^2_{\mathrm{DGB},t} \\ \le C_{L7}\left(\|U\|^2_{\mathrm{DG},t} + \|\mathcal{U}_l^*\|^2_{\mathrm{DG},t} + \|u_D\|^2_{\mathrm{DGB},t}\right).$$

□

Lemma 4.8. *For each $k_1 > 0$ there exists a constant $c_b > 0$ such that for the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ we have the inequality*

$$\int_{I_m} |b_h(U,\mathcal{U}_l^*,t)|\mathrm{d}t \le \frac{\beta_0}{2k_1} \int_{I_m} \|\mathcal{U}_l^*\|^2_{\mathrm{DG},t}\mathrm{d}t + c_b \int_{I_m} \|U\|^2_{\Omega_t}\mathrm{d}t. \tag{4.57}$$

Proof. It can be proved in a similar way as in the proof of inequality (5.18) from [7]. □

Lemma 4.9. *For each $k_2 > 0$ there exists a constant $c_d > 0$ such that the approximate solution U and the discrete characteristic function $\mathcal{U}_l^*$ satisfy the inequality*

$$\int_{I_m} |d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t \le \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t + \frac{c_d}{2\beta_0} \int_{I_m} \|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t. \tag{4.58}$$

Proof. By (3.16), (4.7) and the Cauchy and Young's inequalities,

$$\int_{I_m} |d_h(U,\mathcal{U}_l^*,t)|\,\mathrm{d}t \le \frac{\beta_0}{2k_2} \int_{I_m} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t + \frac{c_z^2 k_2}{2\beta_0} \int_{I_m} \|\mathcal{U}_l^*\|^2_{\Omega_t}\,\mathrm{d}t,$$

which is (4.58) with $c_d = c_z^2 k_2$. □

Lemma 4.10. *For the approximate solution U, the discrete characteristic function $\mathcal{U}_l^*$ and any $k_3 > 0$ we have*

$$\int_{I_m} |l_h(\mathcal{U}_l^*, t)| \,\mathrm{d}t \le \frac{1}{2} \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|\mathcal{U}_l^*\|^2_{\Omega_t}\right) \,\mathrm{d}t + \frac{\beta_0 k_3}{2} \int_{I_m} \|u_D\|^2_{\mathrm{DGB},t} \,\mathrm{d}t + \frac{\beta_0}{2k_3} \int_{I_m} \|\mathcal{U}_l^*\|^2_{\mathrm{DG},t} \,\mathrm{d}t. \tag{4.59}$$

Proof. From (3.17), using the Cauchy and Young's inequality with $k_3 > 0$, we find that

$$\begin{aligned}
&|(g, \mathcal{U}_l^*) + \beta_0\, c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma u_D\, \mathcal{U}_l^* \,\mathrm{d}S| \\
&\le \frac{1}{2}(\|g\|^2_{\Omega_t} + \|\mathcal{U}_l^*\|^2_{\Omega_t}) + \frac{\beta_0 k_3}{2} \underbrace{c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma |u_D|^2 \,\mathrm{d}S}_{= \|u_D\|^2_{\mathrm{DGB},t}} \\
&+ \frac{\beta_0}{2k_3} \underbrace{c_W \sum_{\Gamma \in \mathcal{F}^B_{h,t}} h(\Gamma)^{-1} \int_\Gamma |\mathcal{U}_l^*|^2 \,\mathrm{d}S}_{\le J_h(\mathcal{U}_l^*, \mathcal{U}_l^*, t) \le \|\mathcal{U}_l^*\|^2_{\mathrm{DG},t}},
\end{aligned}$$

from which we get (4.59) by integrating both sides over the interval I_m. ☐

Now we prove an important estimate regarding the problematic term $\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t$.

Theorem 4.4. *There exist constants $C_{T4}, C^*_{T4} > 0$ such that*

$$\int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \le C_{T4} \tau_m \left(\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t}\right) \mathrm{d}t \right) \tag{4.60}$$

provided $0 < \tau_m < C^*_{T4}$.

Proof. For $q = 1$, the proof can be carried out similarly as in [5]. Let us assume that $q \ge 2$, $l \in \{1, \ldots, q-1\}$.

From the definition of the approximate solution (3.19) and (3.20) for $\varphi := \mathcal{U}_l^*$ we get

$$\begin{aligned}
&\int_{I_m} (D_t U, \mathcal{U}_l^*)_{\Omega_t} \,\mathrm{d}t + \left(\{U\}_{m-1}, \{\mathcal{U}_l^*\}^+_{m-1}\right)_{\Omega_{t_{m-1}}} \\
&= \int_{I_m} \left(-a_h(U, \mathcal{U}_l^*, t) - \beta_0 J_h(U, \mathcal{U}_l^*, t) - b_h(U, \mathcal{U}_l^*, t)\right) \mathrm{d}t \\
&+ \int_{I_m} \left(-d_h(U, \mathcal{U}_l^*, t) + l_h(\mathcal{U}_l^*, t)\right) \mathrm{d}t.
\end{aligned} \tag{4.61}$$

This relation and Lemma 4.6 imply that

$$\begin{aligned}
&\frac{1}{2}\left(\left\| U_{m-1+l/q} \right\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \right) \\
&\le \int_{I_m} |a_h(U, \mathcal{U}_l^*, t) + \beta_0 J_h(U, \mathcal{U}_l^*, t)| \,\mathrm{d}t + \int_{I_m} |b_h(U, \mathcal{U}_l^*, t)| \,\mathrm{d}t \\
&+ \int_{I_m} |d_h(U, \mathcal{U}_l^*, t)| \,\mathrm{d}t + \int_{I_m} |l_h(\mathcal{U}_l^*, t)| \,\mathrm{d}t + \left(U^-_{m-1}, U^+_{m-1}\right)_{\Omega_{t_{m-1}}} \\
&+ C_{L6} \int_{I_m} \|U\|^2_{\Omega_t} \,\mathrm{d}t \equiv \mathrm{RHS}.
\end{aligned} \tag{4.62}$$

Now we need to estimate the right-hand side of (4.62) from above. Using (4.52), (4.57), (4.58),(4.59) with $k_1 = k_2 = k_3 = 1$, (4.47) and Young's inequality with any $\delta_2 > 0$, we get

$$\begin{aligned} \mathrm{RHS} \le c_1 \int_{I_m} \left(\|U\|^2_{\mathrm{DG},t} + \|\mathcal{U}^*_l\|^2_{\mathrm{DG},t} + \|\mathcal{U}^*_l\|^2_{\Omega_t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t} \right) \mathrm{d}t \\ + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}, \end{aligned}$$

where $c_1 = \max\{C_{L9} + \beta_0 + c_d/(2\beta_0) + 1/2, c_b + C_{L6}\}$. Now we apply Theorem 4.1 on the continuity of the discrete characteristic function:

$$\int_{I_m} \|\mathcal{U}^*_l\|^2_{\Omega_t} \mathrm{d}t \le C^*_{T1} \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t, \quad \int_{I_m} \|\mathcal{U}^*_l\|^2_{\mathrm{DG},t} \mathrm{d}t \le C^{**}_{T1} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \mathrm{d}t.$$

Hence,

$$\begin{aligned} \mathrm{RHS} \le c_2 \int_{I_m} \left(\|U\|^2_{\mathrm{DG},t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t} \right) \mathrm{d}t \\ + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}, \end{aligned}$$

with $c_2 = c_1 \max\{1 + C^*_{T1}, 1 + C^{**}_{T1}\}$. Then it follows from (4.62) that

$$\begin{aligned} &\frac{1}{2}\Big(\|U^-_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \Big) \\ &\quad \le c_2 \int_{I_m} \Big(\|U\|^2_{\mathrm{DG},t} + \|U\|^2_{\Omega_t} + \|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t} \Big) \mathrm{d}t + \frac{\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}}{\delta_2} + \delta_2 \|U^+_{m-1}\|_{\Omega_{t_{m-1}}}. \end{aligned} \tag{4.63}$$

Further, multiplying (4.63) by $\frac{\beta_0}{4c_2(q-1)}$, summing over $l = 1, \ldots, q-1$ and adding to (4.44), we find that

$$\begin{aligned} \|U^-_m\|^2_{\Omega_{t_m}} &+ \frac{\beta_0}{8c_2(q-1)} \sum_{l=1}^{q-1} \|U\|^2_{\Omega_{t_{m-1+l/q}}} + \left(\frac{\beta_0}{8c_2} + 1 \right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \mathrm{d}t \\ &\le \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \mathrm{d}t + \left(\frac{\beta_0}{4} + C^*_{T3} \right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t \\ &\quad + \left(\frac{\beta_0}{4} + C^{**}_{T3} \right) \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t} \right) \mathrm{d}t \\ &\quad + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1} \right) \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1 \right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}. \end{aligned}$$

Setting $c_3 := \min\left\{ \frac{\beta_0}{8c_2(q-1)}, \frac{\beta_0}{8c_2} + 1 \right\}$ and rearranging, we get

$$\begin{aligned} &\underbrace{c_3 \Big(\|U^-_m\|_{\Omega^2_{t_m}} + \sum_{l=1}^{q-1} \|U^2_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}} + \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}} \Big)}_{=\sum_{l=0}^{q} \|U_{m-1+l/q}\|^2_{\Omega_{t_{m-1+l/q}}}} + \frac{\beta_0}{4} \int_{I_m} \|U\|^2_{\mathrm{DG},t} \mathrm{d}t \\ &\le \left(\frac{\beta_0}{4} + C^*_{T3} \right) \int_{I_m} \|U\|^2_{\Omega_t} \mathrm{d}t + \left(\frac{\beta_0}{4} + C^{**}_{T3} \right) \int_{I_m} \left(\|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t} \right) \mathrm{d}t \\ &\quad + \left(\frac{\beta_0}{4c_2\delta_2} + \frac{2}{\delta_1} \right) \|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}} + \left(\frac{\beta_0\delta_2}{4c_2} + 4\delta_1 \right) \|U^+_{m-1}\|^2_{\Omega_{t_{m-1}}}. \end{aligned}$$

It follows from inequalities (4.45) and (4.46) that

$$\begin{aligned}\frac{c_3 L_q^*}{\tau_m}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t+\frac{\beta_0}{4}\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ \le\left(\frac{\beta_0\delta_2 M_q^*}{4c_2\tau_m}+\frac{4\delta_1 M_q^*}{\tau_m}+\frac{\beta_0}{4}+C^*_{T3}\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t\\ +\left(\frac{\beta_0}{4}+C^{**}_{T3}\right)\int_{I_m}\left(\|g\|^2_{\Omega_t}+\|u_D\|^2_{\mathrm{DG},t}\right)\mathrm{d}t+\left(\frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}\right)\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.\end{aligned}$$

Setting $\delta_1=\frac{c_3L_q^*}{16M_q^*}$, $\delta_2=\frac{c_3c_2L_q^*}{\beta_0M_q^*}$, $c_4:=\frac{\beta_0}{4c_2\delta_2}+\frac{2}{\delta_1}$, $c_5:=\frac{\beta_0}{4}+C^{**}_{T3}$ we get

$$\begin{aligned}\left(\frac{c_3L_q^*}{2\tau_m}-\frac{\beta_0}{4}-C^*_{T3}\right)\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t+\frac{\beta_0}{4}\int_{I_m}\|U\|^2_{\mathrm{DG},t}\mathrm{d}t\\ \le c_5\int_{I_m}\left(\|g\|^2_{\Omega_t}+\|u_D\|^2_{\mathrm{DGB},t}\right)\mathrm{d}t+c_4\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.\end{aligned}\tag{4.64}$$

If the condition $0<\tau_m\le C^*_{T4}:=\frac{c_3L_q^*}{4(\frac{\beta_0}{4}+C^*_{T3})}$ is satisfied, then $\frac{\beta_0}{4}+C^*_{T3}\ge\frac{c_3L_q^*}{4\tau_m}$ and from (4.64) we obtain the estimate

$$\frac{c_3L_q^*}{4\tau_m}\int_{I_m}\|U\|^2_{\Omega_t}\mathrm{d}t+\frac{\beta_0}{4}\int_{I_m}\|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t\le c_5\int_{I_m}\left(\|g\|^2_{\Omega_t}+\|u_D\|^2_{\mathrm{DGB},t}\right)\mathrm{d}t+c_4\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}},$$

which implies (4.60). □

The stability analysis will be finished by the application of the following auxiliary lemma.

Lemma 4.11. *(Discrete Gronwall inequality) Let x_m, a_m, b_m and y_m, where $m=1,2,\dots,$ be non-negative sequences and let the sequence a_m be nondecreasing. Then, if*

$$\begin{aligned}&x_0+y_0\le a_0,\\ &x_m+y_m\le a_m+\sum_{j=0}^{m-1}b_jx_j\quad\text{for}\quad m\ge 1,\end{aligned}$$

we have

$$x_m+y_m\le a_m\prod_{j=0}^{m-1}(1+b_j)\quad\text{for}\quad m\ge 0.$$

The proof can be carried out by induction, see [20].

Now, if (4.60) is substituted into (4.43), an inequality is obtained, which is a basis of the proof of our main result about the stability:

$$\begin{aligned}\|U_m\|^2_{\Omega_{t_m}}-\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}+\|\{U\}_{m-1}\|^2_{\Omega_{t_{m-1}}}+\frac{\beta_0}{2}\int_{I_m}\|U\|^2_{\mathrm{DG},m}\,\mathrm{d}t\\ \le(C_{T2}+C_{T4}\,\tau_m)\int_{I_m}(\|g\|^2_{\Omega_t}+\|u_D\|^2_{\mathrm{DGB},t})\,\mathrm{d}t+C_{T2}C_{T4}\,\tau_m\|U^-_{m-1}\|^2_{\Omega_{t_{m-1}}}.\end{aligned}\tag{4.65}$$

Theorem 4.5. *Let $0 < \tau_m \le C^*_{T4}$ for $m = 1, \dots, M$. Then there exists a constant $C_{T5} > 0$ such that*

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t \\ \le C_{T5}\left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t\right), \quad m = 1, \dots, M,\ h \in (0, \overline{h}), \tag{4.66}$$

where $R_{t,j} = (C_{T2} + C_{T4}\,\tau_j)\,(\|g\|^2_{\Omega_t} + \|u_D\|^2_{\mathrm{DGB},t})$ for $t \in I_j$.

Proof. Writing j instead of m in (4.65),we obtain

$$\|U_j^-\|^2_{\Omega_{t_j}} - \|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}} + \|\{U\}_{j-1}\|^2_{\Omega_{t_{m-1}}} + \frac{\beta_0}{2}\int_{I_j} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t \\ \le \int_{I_j} R_{t,j}\,\mathrm{d}t + C_{T2}C_{T4}\,\tau_j\|U_{j-1}^-\|^2_{\Omega_{t_{j-1}}}.$$

Let $m \ge 1$. The summation over all $j = 1, \dots, m$ yields the inequality

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U\}_{j-1}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t \\ \le \|U_0^-\|^2_{\Omega_0} + C_{T2}C_{T4}\sum_{j=0}^m \tau_{j+1}\|U_j^-\|^2_{\Omega_{t_j}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t.$$

The use of the discrete Gronwall inequality with setting

$$\begin{aligned} x_0 &= a_0 = \|U_0^-\|^2_{\Omega_{t_0}}, \quad c_0 = 0,\\ x_m &= \|U_m^-\|^2_{\Omega_{t_m}},\\ y_m &= \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t,\\ a_m &= \|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_{I_j} R_{t,j}\,\mathrm{d}t,\\ b_j &= C_{T2}C_{T4}\,\tau_{j+1}, \quad j = 0, 1, \dots, m, \end{aligned}$$

yield

$$\|U_m^-\|^2_{\Omega_{t_m}} + \sum_{j=1}^m \|\{U_{j-1}\}\|^2_{\Omega_{t_{j-1}}} + \frac{\beta_0}{2}\sum_{j=1}^m \int_{I_j} \|U\|^2_{\mathrm{DG},t}\,\mathrm{d}t \\ \le \left(\|U_0^-\|^2_{\Omega_{t_0}} + \sum_{j=1}^m \int_j R_{t,j}\,\mathrm{d}t\right)\prod_{j=0}^{m-1}\big(1 + C_{T2}C_{T4}\,\tau_{j+1}\big). \tag{4.67}$$

Finally (4.67) and the inequality $1 + \sigma < \exp(\sigma)$ valid for any $\sigma > 0$ immediately yield (4.66) with the constant $C_{T5} := \exp(C_{T2}C_{T4}T)$. □

5. Conclusion

This paper is devoted to the stability analysis of the space-time discontinuous Galerkin method applied to the numerical solution of an initial-boundary value problem for a nonlinear convection-diffusion equation in a time-dependent domain. The problem is formulated with the aid of a new version of the arbitrary Lagrangian–Eulerian (ALE) method allowing to use different meshes in different time slabs. In the numerical scheme we use the nonsymmetric, symmetric and incomplete versions of the space discretization of diffusion terms and interior and boundary penalty. The nonlinear convection terms are discretized with the aid of a numerical flux. The space discretization uses piecewise polynomial approximations of degree $\leq p$ with an integer $p \geq 1$. For the discontinuous Galerkin discretization in time we use polynomials of degree $\leq q$ with $q \geq 1$. (We are not concerned with the case $q = 0$, which yields the simple backward Euler time discretization.) Main attention is paid here to the situation when $q \geq 2$, which is much more complicated and a special technique based on the ALE-generalization of the concept of the discrete characteristic function has been applied. This approach combined with a number of various estimates results in the proof of unconditional stability of the method. The obtained results represent a theoretical support of the ALE-STDGM developed in [16] for the numerical solution of compressible Navier-Stokes equations in time-dependent domains and interaction of compressible flow with elastic structures.

Further step will be the application of derived results to the analysis of error estimates of the ALE-STDGM in time-dependent domains. Interesting, but very difficult would be the analysis of the ALE-STDGM applied to singularly perturbed nonlinear problems, generalizing results of papers [45, 54].

Appendix A. Proof of estimates (4.41) and (4.42) from the proof of Theorem 4.1 in the 3D case (by Z. Vlasáková)

We introduce a parametrization of $\hat{\Gamma}$. Let Δ^2 be a reference simplex in $\mathbb{R}^2$ (with one vertex being the origin and all of the other vertices have only one non-zero coordinate equal to 1). Now

$$
\begin{aligned}
\Gamma &= \mathcal{A}_t(\hat{\Gamma}), \quad \hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}},\\
\hat{\Gamma} &= \mathcal{B}^{\hat{\Gamma}}_{m-1}(\Delta^2) = \{X = \mathcal{B}^{\hat{\Gamma}}_{m-1}(v);\, v \in \Delta^2\},\\
\mathrm{d}S^{\hat{\Gamma}} &= \left\| \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2, \quad v \in \Delta^2,\\
\Gamma &= \{x = \mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v));\, v \in \Delta^2\},\\
\mathrm{d}S^{\Gamma} &= \left\| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2, \quad v \in \Delta^2.
\end{aligned}
$$

By the symbol $\times$ we denote the vector product. The terms $\frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^i}(v)$ are tangent vectors to $\hat{\Gamma}$ at the point $\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)$. It follows from the properties of the mapping $\mathcal{A}_t$ that the values of $\frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^i}(v)$ are identical from the sides of both elements $\hat{K}^{\hat{\Gamma}}_L$ and $\hat{K}^{\hat{\Gamma}}_R$ adjacent to $\hat{\Gamma}$.

In what follows, for the sake of simplicity, by c we denote a generic positive constant independent of h, with different values at different places. Then we can write

$$
\begin{aligned}
\int_\Gamma \frac{1}{h(\Gamma)} [U_s]^2 \mathrm{d}S^\Gamma = \int_{\Delta^2} \frac{1}{h(\Gamma)} &[U_s(\mathcal{A}_t(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)))]^2 \\
&\left\| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2
\end{aligned}
$$

$$\leq \int_{\Delta^2} \frac{1}{h(\Gamma)} [\tilde{U}_s(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v))]^2 \left\| \frac{\mathrm{d}\mathcal{A}_t}{\mathrm{d}X}(\mathcal{B}^{\hat{\Gamma}}_{m-1}(v)) \right\|^2 \left\| \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\hat{\Gamma}}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2$$
$$\leq \int_{\hat{\Gamma}} \frac{c}{h(\hat{\Gamma})} [\tilde{U}_s]^2 \mathrm{d}S^{\hat{\Gamma}}.$$

Hence,

$$\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_{\Gamma} [U_s]^2 \mathrm{d}S^{\Gamma} \right) \mathrm{d}t \leq c \int_{I_m} \left(\sum_{\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}} \frac{c_W}{h(\hat{\Gamma})} \int_{\hat{\Gamma}} [\tilde{U}]^2 \mathrm{d}S^{\hat{\Gamma}} \right) \mathrm{d}t.$$

Further for $\Gamma = \mathcal{A}_t(\hat{\Gamma})$, $\hat{\Gamma} \in \mathcal{F}^I_{h,t_{m-1}}$, we consider the parametrization

$$\Gamma = \{x = \mathcal{B}^{\Gamma}_t(v);\ v \in \Delta^2\},$$
$$\hat{\Gamma} = \{X = \mathcal{A}_t^{-1}(\mathcal{B}^{\Gamma}_t(v));\ v \in \Delta^2\},$$
$$\mathrm{d}S^{\Gamma} = \left\| \frac{\partial \mathcal{B}^{\Gamma}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\Gamma}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}v, \quad v \in \Delta^2$$
$$\mathrm{d}S^{\hat{\Gamma}} = \left\| \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v)) \frac{\partial \mathcal{B}^{\Gamma}_t}{\partial x^1}(v) \times \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v)) \frac{\partial \mathcal{B}^{\Gamma}_t}{\partial x^2}(v) \right\| \mathrm{d}v, \quad v \in \Delta^2.$$

Then

$$\int_{\hat{\Gamma}} [\tilde{U}]^2 \mathrm{d}S^{\hat{\Gamma}} = \int_{\Delta^2} [\tilde{U}(\mathcal{A}_t^{-1}(\mathcal{B}^{\Gamma}_t(v)))]^2 \left\| \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v)) \frac{\partial \mathcal{B}^{\Gamma}_t}{\partial x^1}(v) \times \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v)) \frac{\partial \mathcal{B}^{\Gamma}_t}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2$$
$$\leq \int_{\Delta^2} [U(\mathcal{B}^{\Gamma}_t(v))]^2 \left\| \frac{\mathrm{d}\mathcal{A}_t^{-1}}{\mathrm{d}x}(\mathcal{B}^{\Gamma}_t(v)) \right\|^2 \left\| \frac{\partial \mathcal{B}^{\Gamma}_{m-1}}{\partial x^1}(v) \times \frac{\partial \mathcal{B}^{\Gamma}_{m-1}}{\partial x^2}(v) \right\| \mathrm{d}x^1 \mathrm{d}x^2$$
$$\leq c \int_{\Delta^2} [U]^2 \mathrm{d}S^{\Gamma}.$$

Together we get

$$\int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_{\Gamma} [U_s]^2 \mathrm{d}S^{\Gamma} \right) \mathrm{d}t \leq c \int_{I_m} \left(\sum_{\Gamma \in \mathcal{F}^I_{h,t}} \frac{c_W}{h(\Gamma)} \int_{\Gamma} [U]^2 \mathrm{d}S^{\Gamma} \right) \mathrm{d}t,$$

which is the 3D version of (4.41). Similarly we prove (4.42) in the 3D case.

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