Mathematical Modelling and Numerical Analysis ESAIM: M2AN
Modélisation Mathématique et Analyse Numérique M2AN, Vol. 37, N° 2, 2003, pp. 209–225
DOI: 10.1051/m2an:2003023

A FINITE ELEMENT METHOD FOR DOMAIN DECOMPOSITION WITH NON-MATCHING GRIDS

Roland Becker[1], Peter Hansbo[2] and Rolf Stenberg[3]

Abstract. In this note, we propose and analyse a method for handling interfaces between non-matching grids based on an approach suggested by Nitsche (1971) for the approximation of Dirichlet boundary conditions. The exposition is limited to self-adjoint elliptic problems, using Poisson's equation as a model. *A priori* and *a posteriori* error estimates are given. Some numerical results are included.

Mathematics Subject Classification. 65N30, 65N55.

Received: September 3, 2001. Revised: October 21, 2002.

1. Introduction

In any domain decomposition method, one has to define how the continuity between the subdomains is to be enforced. Different approaches have been proposed:

- Iterative procedures, enforcing that the approximate solution or its normal derivative or combinations thereof should be continuous across interfaces. This forms the basis for the standard Schwarz alternating method as defined, *e.g.*, by Lions [16].
- Direct procedures, using Lagrange multiplier techniques to achieve continuity. Different variants have been proposed, *e.g.*, by Le Tallec and Sassi [15], and Bernadi *et al.* [8].

The multiplier method has the advantage of directly yielding a solvable global system. However, in the latter method, new unknowns (the multipliers) must be introduced and solved for. The method must then either satisfy the inf-sup condition, which necessitates special choices of multiplier spaces (such as mortar elements, *cf.* [8]), or then stabilization techniques (*cf.* [3, 4, 9]) must be used.

In this paper, we consider a third possibility, *i.e.* Nitsche's method [17], which was originally introduced for the purpose of solving Dirichlet problems without enforcing the boundary conditions in the definition of the finite element spaces. This method has later been used by Arnold [2] for the discretization of second order elliptic equations by discontinuous finite elements. In earlier papers [18, 19] we have pointed out the close connection between Nitsche's method and stabilized methods and proposed it as a mortaring method. In this paper we will give a more detailed analysis of this domain decomposition technique where independent approximations

Keywords and phrases. Nitsche's method, domain decomposition, non-matching grids.

[1] Institute of Applied Mathematics, University of Heidelberg, INF 294, 69120 Heidelberg, Germany.
[2] Department of Applied Mechanics, Chalmers University of Technology, 412 96 Göteborg, Sweden.
e-mail: `hansbo@am.chalmers.se`
[3] Institute of Mathematics, Box 1100, 02015 Helsinki University of Technology, Finland.

are used on the different subdomains. The continuity of the solution across interfaces is enforced weakly, but in such a way that the resulting discrete scheme is consistent with the original partial differential equation. Under some regularity assumptions we derive both *a priori* and *a posteriori* error estimates. We also give numerical results obtained with the method.

Although we discuss its application to domain decomposition, the same technique is also suited for other applications, *e.g.*,

- to handle diffusion terms in the discontinuous Galerkin method [2,6];
- to simplify mesh generation (different parts can be meshed independently from each other);
- finite element methods with different polynomial degree on adjacent elements;
- new finite element methods such as linear approximations on quadrilaterals.

2. The domain decomposition method

In this section we will introduce the mortaring method based on the classical method of Nitsche. We will perform a classical stability and *a priori* error analysis. For simplicity, we consider the model Poisson problem, *i.e.* of solving the partial differential equation

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in two or three space dimensions and $f \in L_2(\Omega)$.

Likewise for ease of presentation, we consider only the case where Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2, $\Omega_1 \cup \Omega_2$, with interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$. We further assume that the subdomains are polyhedral (or polygonal in $\mathbb{R}^2$) and that Γ is polygonal (or a broken line).

This equation can be written in weak form as: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad (f,v) = \int_\Omega f\, v \,\mathrm{d}x, \tag{3}$$

and $H_0^1(\Omega)$ is the space of square-integrable functions, with square-integrable first derivatives, that vanish on the boundary $\partial\Omega$ of Ω.

Our discrete method for the approximate solution of (1) is a nonconforming finite element method which is continuous within each Ω_i and discontinuous across Γ. We start by rewriting the original problem (1) as two equations and the interface conditions:

$$\begin{aligned} -\Delta u_i &= f \quad \text{in } \Omega_i, \quad i=1,2, \\ u_i &= 0 \quad \text{on } \partial\Omega \cap \Omega_i, \quad i=1,2, \\ u_1 - u_2 &= 0 \quad \text{on } \Gamma, \\ \frac{\partial u_1}{\partial n_1} + \frac{\partial u_2}{\partial n_2} &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{4}$$

Here, n_i is the outward unit normal to $\partial\Omega_i$.

We will perform our analysis under the following regularity assumption.

Assumption 2.1. *The solution of (2) satisfies $u \in H^s(\Omega)$, with $s > 3/2$.*

With this assumption it holds $\partial u_i/\partial n_i \in L_2(\Gamma)$ and the two problems (1) and (4) are equivalent (see for example [1]) with:

$$u|_{\Omega_i} = u_i, \quad i = 1, 2. \tag{5}$$

In the following we will therefore write $u = (u_1, u_2) \in V_1 \times V_2$ with the continuous spaces

$$V_i = \left\{ v_i \in H^1(\Omega_i) : \ \partial v_i/\partial n_i \in L_2(\Gamma), \ v_i|_{\partial\Omega\cap\partial\Omega_i} = 0 \right\}, \quad i = 1, 2.$$

To formulate our method, we suppose that we have regular finite element partitionings $\mathcal{T}_h^i$ of the subdomains Ω_i into shape regular simplexes. These two meshes induce "trace" meshes on the interface

$$\mathcal{G}_h^i = \{E : E = K \cap \Gamma, \ K \in \mathcal{T}_h^i\}\cdot \tag{6}$$

By h_K and h_E we denote the diameter of element $K \in \mathcal{T}_h^i$ and $E \in \mathcal{G}_h^i$, respectively. For the purpose of the *a priori* analysis, we also define

$$h = \max\{h_K, \, h_E : \ K \in \mathcal{T}_h^i, \ E \in \mathcal{G}_h^i, \, i = 1, 2\}\cdot$$

In our analysis we will need the following assumption.

Assumption 2.2. *There exists positive constants C_1, C_2 such that*

$$C_1 \, h_{E_1} \le h_{E_2} \le C_2 \, h_{E_1}$$

holds for all pairs $(E_1, \ E_2) \in \mathcal{T}_h^1 \times \mathcal{T}_h^2$, with $E_1 \cap E_2 \neq \emptyset$.

We seek the approximation $u_h = (u_{1,h}, u_{2,h})$ in the space $V^h = V_1^h \times V_2^h$, where

$$V_i^h = \left\{ v_i \in V_i : \ v_i|_K \text{ is a polynomial of degree } p \text{ for all } K \in \mathcal{T}_h^i \right\},$$

choosing for simplicity p constant on all subdomains. On the interface we will use the notation

$$[\![v]\!] := v_1 - v_2 \tag{7}$$

for the jump,

$$\{v\} := \frac{1}{2} v_1 + \frac{1}{2} v_2 \tag{8}$$

for the average and

$$\langle v, w \rangle_S := \int_S v \, w \, \mathrm{d}s, \qquad \text{for } S \in \mathcal{G}_h^i, \ \text{ or } S = \Gamma. \tag{9}$$

Finally, we denote

$$n := n_1 = -n_2. \tag{10}$$

With this notation we have

$$\left\{ \frac{\partial v}{\partial n} \right\} = \frac{1}{2} \frac{\partial v_1}{\partial n_1} - \frac{1}{2} \frac{\partial v_2}{\partial n_2} \tag{11}$$

and hence

$$\left\{ \frac{\partial u}{\partial n} \right\} = \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n_1} = -\frac{\partial u_2}{\partial n_2}\cdot \tag{12}$$

The methods of Nitsche [17] and Arnold [2] now give the following domain decomposition method.

Mortar method. *Find $u_h \in V^h$ such that*

$$a_h(u_h, v) = f_h(v) \quad \forall v \in V^h, \tag{13}$$

with

$$a_h(w,v) := \sum_{i=1}^{2}\Big[(\nabla w_i,\nabla v_i)_{\Omega_i} + \gamma \sum_{E\in\mathcal{G}_h^i} h_E^{-1}\left\langle [\![w]\!]\,,[\![v]\!]\right\rangle_E\Big] \\ -\left\langle\left\{\frac{\partial w}{\partial n}\right\},[\![v]\!]\right\rangle_\Gamma - \left\langle\left\{\frac{\partial v}{\partial n}\right\},[\![w]\!]\right\rangle_\Gamma \tag{14}$$

and

$$f_h(v) := \sum_{i=1}^{2}(f,v_i)_{\Omega_i}, \tag{15}$$

where $\gamma>0$ is chosen sufficiently large, see Lemma 2.6 below.

The first observation is that this formulation gives a consistent method.

Lemma 2.3. *The solution $u=(u_1,u_2)$ to* (4) *satisfies*

$$a_h(u,v) = f_h(v) \quad \forall v\in V. \tag{16}$$

Proof. Multiplying the first equation in (4) with v_i, integrating over Ω_i, using Greens formula and the relations (12) yields

$$f_h(v) = \sum_{i=1}^{2}(f,v_i)_{\Omega_i} = \sum_{i=1}^{2}\left[(\nabla u_i,\nabla v_i)_{\Omega_i} - \left\langle\frac{\partial u_i}{\partial n_i},v_i\right\rangle_\Gamma\right] \\ = \sum_{i=1}^{2}(\nabla u_i,\nabla v_i)_{\Omega_i} - \left\langle\left\{\frac{\partial u}{\partial n}\right\},[\![v]\!]\right\rangle_\Gamma. \tag{17}$$

Since $[\![u]\!]=0$ on Γ we have

$$0 = -\left\langle\left\{\frac{\partial v}{\partial n}\right\},[\![u]\!]\right\rangle_\Gamma + \gamma\sum_{i=1}^{2}\sum_{E\in\mathcal{G}_h^i} h_E^{-1}\left\langle[\![u]\!]\,,[\![v]\!]\right\rangle_E. \tag{18}$$

Adding (17) and (18) the gives the claim. □

For the stability analysis below we need the following mesh-dependent dual norms. To emphasize that Γ is to be considered as a part of $\partial\Omega_i$ we write Γ_i.

$$\|v\|^2_{1/2,h,\Gamma_i} = \sum_{E\in\mathcal{G}_h^i} h_E^{-1}\,\|v\|^2_{L_2(E)} \tag{19}$$

and

$$\|v\|^2_{-1/2,h,\Gamma_i} = \sum_{E\in\mathcal{G}_h^i} h_E\,\|v\|^2_{L_2(E)}, \tag{20}$$

which satisfy

$$|\left\langle v,w\right\rangle_\Gamma| \le \|v\|_{1/2,h,\Gamma_i}\,\|w\|_{-1/2,h,\Gamma_i}. \tag{21}$$

The analysis will be performed using the norms

$$\|v\|^2_{1,h} = \sum_{i=1}^{2}\Big(\|\nabla v_i\|^2_{L_2(\Omega_i)} + \|[\![v]\!]\|^2_{1/2,h,\Gamma_i}\Big) \tag{22}$$

and

$$|||v|||_{1,h}^2 = \|v\|_{1,h}^2 + \sum_{i=1}^{2} \left\| \frac{\partial v_i}{\partial n_i} \right\|_{-1/2,h,\Gamma_i}^2 . \tag{23}$$

The following estimate is readily proved by local scaling.

Lemma 2.4. *There is a positive constant C_{I} such that*

$$\left\| \frac{\partial v_i}{\partial n_i} \right\|_{-1/2,h,\Gamma_i}^2 \leq C_{\mathrm{I}} \|\nabla v_i\|_{L^2(\Omega_i)}^2 \quad \forall v_i \in V_i^h . \tag{24}$$

For linear elements ∇v is constant on each element and then it is particularly easy to give a bound for the constant C_{I}, see Remark 2.12 below.

A immediate consequence of the above lemma is the equivalence of norms in the finite element subspace.

Lemma 2.5. *The norms $\|\cdot\|_{1,h}$ and $|||\cdot|||_{1,h}$ are equivalent on the subspace V^h.*

The stability of the method can now be proved.

Lemma 2.6. *Suppose that $\gamma > C_{\mathrm{I}}/4$. Then it holds*

$$a_h(v,v) \geq C|||v|||_{1,h}^2 \qquad \forall v \in V^h .$$

Proof. From the definition (14), the relation (11), and the preceding lemmas we get

$$\begin{aligned}
a_h(v,v) &= \sum_{i=1}^{2} \left[\|\nabla v_i\|_{L_2(\Omega_i)}^2 + \gamma \, \|[\![v]\!]\|_{1/2,h,\Gamma_i}^2 \right] - 2 \left\langle [\![v]\!] , \left\{ \frac{\partial v}{\partial n} \right\} \right\rangle_\Gamma \\
&= \sum_{i=1}^{2} \left[\|\nabla v_i\|_{L_2(\Omega_i)}^2 + \gamma \, \|[\![v]\!]\|_{1/2,h,\Gamma_i}^2 \right] - \left\langle [\![v]\!] , \frac{\partial v_1}{\partial n_1} \right\rangle_\Gamma + \left\langle [\![v]\!] , \frac{\partial v_2}{\partial n_2} \right\rangle_\Gamma \\
&\geq \sum_{i=1}^{2} \left[\|\nabla v_i\|_{L_2(\Omega_i)}^2 + \gamma \, \|[\![v]\!]\|_{1/2,h,\Gamma_i}^2 - \left| \left\langle [\![v]\!] , \frac{\partial v_i}{\partial n_i} \right\rangle_\Gamma \right| \right] \\
&\geq \sum_{i=1}^{2} \left[\|\nabla v_i\|_{L_2(\Omega_i)}^2 + \gamma \, \|[\![v]\!]\|_{1/2,h,\Gamma_i}^2 - \|[\![v]\!]\|_{1/2,h,\Gamma_i} \left\| \frac{\partial v_i}{\partial n_i} \right\|_{-1/2,h,\Gamma_i} \right] \\
&\geq \sum_{i=1}^{2} \left[\left(1 - \frac{C_{\mathrm{I}}}{2\varepsilon} \right) \|\nabla v_i\|_{L_2(\Omega_i)}^2 + \left(\gamma - \frac{\varepsilon}{2} \right) \|[\![v]\!]\|_{1/2,h,\Gamma_i}^2 \right] \\
&\geq C_1 \|v\|_{1,h}^2 \\
&\geq C_2 |||v|||_{1,h}^2 ,
\end{aligned}$$

for $\gamma > \varepsilon/2$ and choosing $\varepsilon > C_{\mathrm{I}}/2$. □

The following interpolation estimates holds, *cf.* [20].

Lemma 2.7. *Suppose that $u \in H^s(\Omega)$, with $3/2 < s \leq p+1$. Then it holds*

$$\inf_{v \in V^h} \|u - v\|_{1,h} \leq C h^{s-1} \|u\|_{H^s(\Omega)} \tag{25}$$

and

$$\inf_{v \in V^h} |||u - v|||_{1,h} \leq C h^{s-1} \|u\|_{H^s(\Omega)} . \tag{26}$$

We are now able to prove the following *a priori* error estimate.

Theorem 2.8. *Suppose $\gamma > C_{\mathrm{I}}/4$. Then it holds*

$$|||u - u_h|||_{1,h} \leq C \inf_{v \in V^h} |||u - v|||_{1,h}.$$

If $u \in H^s(\Omega)$, for $3/2 < s \leq p + 1$, then

$$|||u - u_h|||_{1,h} \leq C h^{s-1} \left\| u \right\|_{H^s(\Omega)}.$$

Proof. Since the bilinear form a_h is bounded with respect to the norm $|||\cdot|||_{1,h}$, the first inequality follows from the stability estimate of Lemma 2.6 and the triangle inequality. The second estimate then follows from Lemma 2.7. □

Remark 2.9. As $[\![u]\!] = 0$ on Γ the error estimate gives $\| [\![u_h]\!] \|_{-1/2,h,\Gamma_i} = O(h^{s-1})$. For quasi-uniform meshes it then holds $\| [\![u_h]\!] \|_{L_2(\Gamma)} = O(h^{s-1/2})$.

Remark 2.10. The presented method resembles a mesh-dependent penalty method, but with added consistency terms involving normal derivatives across the interface. Note that the formulation allows us to deduce optimal order error estimates with preserved condition number of $O(h^{-2})$ for a quasiuniform mesh. Pure penalty methods, in contrast, are not consistent, and optimal error estimates require degrading the condition number for higher polynomial approximation (*cf.* [5]).

Remark 2.11. The form $a_h(\cdot,\cdot)$ in (14) is symmetric and positive definite, which is natural as the problem to be approximated has the same properties. With this there exists fast solvers, such as preconditioned conjugate gradients, for the resulting matrix problem. If the bilinear form is changed to

$$\begin{aligned} a_h(w,v) := \sum_{i=1}^{2} \Big[(\nabla w_i, \nabla v_i)_{\Omega_i} + \gamma \sum_{E \in \mathcal{G}_h^i} h_E^{-1} \left\langle [\![w]\!], [\![v]\!] \right\rangle_E \Big] \\ - \left\langle \left\{ \frac{\partial w}{\partial n} \right\}, [\![v]\!] \right\rangle_\Gamma + \left\langle \left\{ \frac{\partial v}{\partial n} \right\}, [\![w]\!] \right\rangle_\Gamma, \end{aligned} \tag{27}$$

then one obtains a method which is stable for all positive values of γ:

$$a_h(v,v) \geq C |||v|||_{1,h}^2 \qquad \forall v \in V^h,\ \forall \gamma > 0.$$

If the Laplace operator is a part of a problem which is not symmetric, then it might be practical to use this nonsymmetric bilinear form. See [10, 11] for applications to convection diffusion problems.

Remark 2.12. For linear elements ∇v_i ($v_i \in V_i^h$) is constant on each element and hence for $K \in \mathcal{T}_h^i$ with the base $E \in \mathcal{G}_h^i$ we have

$$\left\| \frac{\partial v_i}{\partial n_i} \right\|_{0,E}^2 = \mathrm{meas}(E) \left| \frac{\partial v_i}{\partial n_i} \right|^2 \tag{28}$$

and

$$\|\nabla v_i\|_{0,K}^2 \geq \left\| \frac{\partial v_i}{\partial n_i} \right\|_{0,K}^2 = \mathrm{meas}(K) \left| \frac{\partial v_i}{\partial n_i} \right|^2 \tag{29}$$

and hence it holds

$$h_E \left\| \frac{\partial v_i}{\partial n_i} \right\|_{0,E}^2 \leq \frac{h_E\, \mathrm{meas}(E)}{\mathrm{meas}(K)} \left\| \nabla v_i \right\|_{0,K}^2. \tag{30}$$

Hence, once the shape regularity of the mesh is specified, one has a bound for C_{I}. In a practical implementation easier would be to replace the weight γh_E^{-1} in (14) with a parameter

$$\alpha_K = \alpha \frac{\operatorname{meas}(E)}{\operatorname{meas}(K)}, \tag{31}$$

with $\alpha > 1/4$ fixed. With this all results of the paper hold.

Remark 2.13. We have performed our analysis assuming that the solution is sufficiently smooth, *i.e.* that $u \in H^s(\Omega)$ with $s > 3/2$. The analysis can be extended for less regular solutions if the corner and transmission singularities are carefully taken into account, *cf.* the recent papers [12, 13].

3. *A posteriori* error estimates

We will first consider control of the error $e = u - u_h$ in the mesh dependent energy norm $|||\cdot|||_{1,h}$. We define the local and global estimators as

$$E_K(u_h)^2 = h_K^2 \, \|f + \Delta u_h\|^2_{L_2(K)} + h_K \left\| \left[\!\!\left[\frac{\partial u_h}{\partial n_K} \right]\!\!\right] \right\|^2_{L_2(\partial K)} + h_K^{-1} \, \| [\![u_h]\!] \|^2_{L_2(\partial K)} \tag{32}$$

and

$$E(u_h) = \left[\sum_{K \in \mathcal{T}_h} E_K(u_h)^2 \right]^{1/2}. \tag{33}$$

To be able to control the normal derivatives across the interface, we are forced to introduce a "saturation" assumption. This assumption is consistent with the Assumption 2.1 on the regularity of the solution and the interpolation estimates of Theorem 2.7.

Assumption 3.1. *There is a constant C such that*

$$|||u - u_h|||_{1,h} \le C \|u - u_h\|_{1,h}. \tag{34}$$

We also remark that an analog assumption is used in the context of *a posteriori* error estimates for the mortar element method, see Wohlmuth [21].

The *a posteriori* estimate is now the following.

Theorem 3.2. *Suppose that the Assumption 3.1 is valid. Then there is a positive constant C such that*

$$|||u - u_h|||_{1,h} \le C E(u_h). \tag{35}$$

Proof. We denote $e = u - u_h$. By Lemma 2.3 we have

$$\begin{aligned} \|e\|^2_{1,h} &= a_h(e,e) + 2 \left\langle [\![e]\!] \, , \left\{ \frac{\partial e}{\partial n} \right\} \right\rangle_\Gamma + (1-\gamma) \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} h_E^{-1} \left\langle [\![e]\!] \, , [\![e]\!] \right\rangle_E \\ &:= R_1 + R_2, \end{aligned} \tag{36}$$

with

$$R_1 := a_h(e,e),$$

and

$$R_2 := 2 \left\langle [\![e]\!] \, , \left\{ \frac{\partial e}{\partial n} \right\} \right\rangle_\Gamma + (1-\gamma) \sum_{i=1}^{2} \| [\![e]\!] \|^2_{1/2,h,\Gamma_i} \, .$$

Since $[\![e]\!]\,|_\Gamma = [\![u_h]\!]\,|_\Gamma$ Assumption 3.1 gives

$$\begin{aligned} R_2 &= \left\langle [\![u_h]\!]\,, \frac{\partial e_1}{\partial n_1} \right\rangle_\Gamma - \left\langle [\![u_h]\!]\,, \frac{\partial e_2}{\partial n_2} \right\rangle_\Gamma + (1-\gamma) \sum_{i=1}^{2} \|[\![e]\!]\|^2_{1/2,h,\Gamma_i} \\ &\le C \sum_{i=1}^{2} \left(\|[\![u_h]\!]\|_{1/2,h,\Gamma_i} \left\| \frac{\partial e_i}{\partial n_i} \right\|_{-1/2,h,\Gamma_i} + \|[\![u_h]\!]\|_{1/2,h,\Gamma_i} \|[\![e]\!]\|_{1/2,h,\Gamma_i} \right) \\ &\le C \|e\|_{1,h} \left(\sum_{i=1}^{2} \|[\![u_h]\!]\|_{1/2,h,\Gamma_i} \right) \le C\|e\|_{1,h} E(u_h). \end{aligned} \tag{37}$$

Next, let $\pi_h e$ be the Clément interpolant to e, which satisfies

$$\left(\sum_{K\in\mathcal{T}_h} \left(h_K^{-2} \|e-\pi_h e\|^2_{L_2(K)} + h_K^{-1} \|e-\pi_h e\|^2_{L_2(\partial K)} \right) \right)^{1/2} \le C\|e\|_{1,h} \tag{38}$$

and

$$\|e-\pi_h e\|_{1,h} \le C\|e\|_{1,h}. \tag{39}$$

From the consistency (2.3) we have $a_h(e,\pi_h e) = 0$. Hence

$$\begin{aligned} R_1 \;&=\; a_h(e,e) = a_h(e, e-\pi_h e) \\ &= \sum_{i=1}^{2} \left((\nabla e_i, \nabla(e_i - \pi_h e_i))_{\Omega_i} + \gamma \sum_{E\in\mathcal{G}_h^i} h_E^{-1} \langle [\![e]\!]\,, [\![e-\pi_h e]\!] \rangle_E \right) \\ &\quad - \left\langle \left\{ \frac{\partial e}{\partial n} \right\}, [\![e-\pi_h e]\!] \right\rangle_\Gamma - \left\langle \left\{ \frac{\partial (e-\pi_h e)}{\partial n} \right\}, [\![e]\!] \right\rangle_\Gamma \\ &:= S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{40}$$

Integrating by parts on each $K \in \mathcal{T}_h$ yields

$$\begin{aligned} S_1 \;&=\; \sum_{i=1}^{2} (\nabla e_i, \nabla(e_i - \pi_h e_i))_{\Omega_i} \qquad (41) \\ &= \sum_{K\in\mathcal{T}_h} \left(-(\Delta e, e-\pi_h e)_K + \left\langle \frac{\partial e}{\partial n_K}, e - \pi_h e \right\rangle_{\partial K} \right) \\ &:= T_1 + T_2. \end{aligned}$$

Since on $K \in \mathcal{T}_h$ it holds $-\Delta e = -\Delta u + \Delta u_h = f + \Delta u_h$, the first term above is estimated using (38)

$$\begin{aligned} T_1 &= - \sum_{K\in\mathcal{T}_h} (\Delta e, e-\pi_h e)_K \qquad (42) \\ &\le \left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\Delta u_h + f\|^2_{L_2(K)} \right)^{1/2} \left(\sum_{K\in\mathcal{T}_h} h_K^{-2} \|e-\pi_h e\|^2_{L_2(K)} \right)^{1/2} \\ &\le \left(\sum_{K\in\mathcal{T}_h} h_K^2 \|\Delta u_h + f\|^2_{L_2(K)} \right)^{1/2} \|e\|_{1,h}. \end{aligned}$$

Let $\mathcal{I}_h$ be the collection of element sides in the interiors of the subdomains Ω_i. The boundary integrals in (41) above we now split into those in $\mathcal{I}_h$ and those lying on the interface Γ:

$$\begin{aligned} T_2 &= \sum_{K\in\mathcal{T}_h} \left\langle \frac{\partial e}{\partial n_K}, e - \pi_h e \right\rangle_{\partial K} \\ &= \sum_{E\in\mathcal{I}_h} \left\langle \frac{\partial e}{\partial n_E}, e - \pi_h e \right\rangle_E + \left\langle \frac{\partial e_1}{\partial n_1}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \left\langle \frac{\partial e_2}{\partial n_2}, e_2 - \pi_h e_2 \right\rangle_\Gamma . \end{aligned} \tag{43}$$

The integrals over the interior sides can now be grouped together two by two yielding the estimate

$$\begin{aligned} \left| \sum_{E\in\mathcal{I}_h} \left\langle \frac{\partial e}{\partial n_E}, e - \pi_h e \right\rangle_E \right| &\le C \left[\sum_{K\in\mathcal{T}_h} h_K \left\| \left[\!\left[\frac{\partial u_h}{\partial n_K} \right]\!\right] \right\|^2_{L_2(\partial K)} \right]^{1/2} \left[\sum_{K\in\mathcal{T}_h} h_K^{-1} \left\| e - \pi_h e \right\|^2_{L_2(\partial K)} \right]^{1/2} \\ &\le C \left[\sum_{K\in\mathcal{T}_h} h_K \left\| \left[\!\left[\frac{\partial u_h}{\partial n_K} \right]\!\right] \right\|^2_{L_2(\partial K)} \right]^{1/2} \|e\|_{1,h}, \end{aligned} \tag{44}$$

where (38) was used.

Using the definition (11) we have

$$\begin{aligned} S_3 &= -\left\langle \left\{ \frac{\partial e}{\partial n} \right\}, [\![e - \pi_h e]\!] \right\rangle_\Gamma = -\left\langle \frac{1}{2}\frac{\partial e_1}{\partial n_1} + \frac{1}{2}\frac{\partial e_2}{\partial n_2}, e_1 - \pi_h e_1 - (e_2 - \pi_h e_2) \right\rangle_\Gamma \\ &= -\left\langle \frac{1}{2}\frac{\partial e_1}{\partial n_1} - \frac{1}{2}\frac{\partial e_2}{\partial n_2}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \left\langle \frac{1}{2}\frac{\partial e_1}{\partial n_1} - \frac{1}{2}\frac{\partial e_2}{\partial n_2}, (e_2 - \pi_h e_2) \right\rangle_\Gamma . \end{aligned} \tag{45}$$

Combining these terms with two last terms in (43) gives

$$\begin{aligned} &\left\langle \frac{\partial e_1}{\partial n_1}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \left\langle \frac{\partial e_2}{\partial n_2}, e_2 - \pi_h e_2 \right\rangle_\Gamma - \left\langle \left\{ \frac{\partial e}{\partial n} \right\}, [\![e - \pi_h e]\!] \right\rangle_\Gamma \\ &= \left\langle \left(1 - \frac{1}{2}\right) \frac{\partial e_1}{\partial n_1} + \frac{1}{2}\frac{\partial e_2}{\partial n_2}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \left\langle \frac{1}{2}\frac{\partial e_1}{\partial n_1} + \left(1 - \frac{1}{2}\right) \frac{\partial e_2}{\partial n_2}, (e_2 - \pi_h e_2) \right\rangle_\Gamma \\ &= \frac{1}{2} \left\langle \frac{\partial e_1}{\partial n_1} + \frac{\partial e_2}{\partial n_2}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \frac{1}{2} \left\langle \frac{\partial e_1}{\partial n_1} + \frac{\partial e_2}{\partial n_2}, e_2 - \pi_h e_2 \right\rangle_\Gamma \\ &= \frac{1}{2} \left\langle \frac{\partial (e_1 - e_2)}{\partial n}, e_1 - \pi_h e_1 \right\rangle_\Gamma + \frac{1}{2} \left\langle \frac{\partial (e_1 - e_2)}{\partial n}, e_2 - \pi_h e_2 \right\rangle_\Gamma \\ &= \frac{1}{2} \left\langle \left[\!\left[\frac{\partial e}{\partial n} \right]\!\right], e_1 - \pi_h e_1 \right\rangle_\Gamma + \frac{1}{2} \left\langle \left[\!\left[\frac{\partial e}{\partial n} \right]\!\right], e_2 - \pi_h e_2 \right\rangle_\Gamma \\ &= \frac{1}{2} \left\langle \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right], e_1 - \pi_h e_1 \right\rangle_\Gamma + \frac{1}{2} \left\langle \left[\!\left[\frac{\partial u_h}{\partial n} \right]\!\right], e_2 - \pi_h e_2 \right\rangle_\Gamma \\ &\le \frac{1}{2} E(u_h) \left[\sum_{K\in\mathcal{T}_h} h_K^{-1} \left\| e - \pi_h e \right\|^2_{L_2(\partial K)} \right]^{1/2} \\ &\le C E(u_h) \|e\|_{1,h}. \end{aligned} \tag{46}$$

From (41) to (46) we now get

$$S_1 + S_3 \le C \|e\|_{1,h} E(u_h). \tag{47}$$

From the Clément estimate (38) have

$$S_2 = \gamma \sum_{i=1}^{2} \sum_{E \in \mathcal{G}_h^i} h_E^{-1} \left\langle [\![e]\!] \,, [\![e - \pi_h e]\!] \right\rangle_E \le C \|e\|_{1,h} E(u_h).$$

To estimate the last term S_4 we use Lemma 2.4 and the Assumption 3.1

$$\begin{aligned} S_4 &= -\left\langle \left\{ \frac{\partial (e - \pi_h e)}{\partial n} \right\}, [\![e]\!] \right\rangle_\Gamma = -\left\langle \left\{ \frac{\partial e}{\partial n} \right\}, [\![e]\!] \right\rangle_\Gamma + \left\langle \left\{ \frac{\partial \pi_h e}{\partial n} \right\}, [\![e]\!] \right\rangle_\Gamma \\ &\le C \|e\|_{1,h} E(u_h). \end{aligned} \tag{48}$$

Hence, collecting the estimates (40) to (48) yields

$$R_1 \le C \|e\|_{1,h} E(u_h), \tag{49}$$

which together with (37) proves the estimate

$$\|u - u_h\|_{1,h} = \|e\|_{1,h} \le C E(u_h). \tag{50}$$

The claim then follows from (34). $\square$

Next, we consider the error in the L_2-norm. This is measured with the estimator

$$L(u_h) = \left[\sum_{K \in \mathcal{T}_h} h_K^2 E_K(u_h)^2 \right]^{1/2}. \tag{51}$$

For the *a posteriori* estimate we as usual, need the H^2-regularity but not Assumption 3.1.

Theorem 3.3. *Let z be the solution to the problem*

$$\begin{aligned} -\Delta z &= g \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{52}$$

and suppose that the shift theorem

$$\|z\|_{H^2(\Omega)} \le C \, \|g\|_{L_2(\Omega)} \tag{53}$$

is valid. Then there is a positive constant C such that

$$\|u - u_h\|_{L_2(\Omega)} \le C L(u_h). \tag{54}$$

Proof. We again write $z = (z_1, z_2)$ and note that

$$\frac{\partial z_1}{\partial n_1} + \frac{\partial z_2}{\partial n_2} = 0 \quad \text{and} \quad \left\{ \frac{\partial z}{\partial n} \right\} = \frac{\partial z_1}{\partial n_1} = -\frac{\partial z_2}{\partial n_2} \cdot$$

Choosing $g = u - u_h$ in (52) we then get

$$
\begin{aligned}
\|u-u_h\|^2_{L_2(\Omega)} &= \sum_{i=1}^{2}\left[(\nabla z_i, \nabla(u_i-u_{i,h}))_{\Omega_i} - \left\langle \frac{\partial z_i}{\partial n_i}, u_i - u_{i,h}\right\rangle_{\Gamma_i}\right] \\
&= \sum_{i=1}^{2}\left[(\nabla z_i, \nabla(u_i-u_{i,h}))_{\Omega_i} + \left\langle \frac{\partial z_i}{\partial n_i}, u_{i,h}\right\rangle_{\Gamma_i}\right] \\
&= \sum_{i=1}^{2}(\nabla z_i, \nabla(u_i-u_{i,h}))_{\Omega_i} + \left\langle \left\{\frac{\partial z}{\partial n}\right\}, [\![u_h]\!]\right\rangle_{\Gamma}.
\end{aligned} \tag{55}
$$

We let $\pi_h z \in V^h$ be the Clément interpolant to z satisfying

$$
\left(\sum_{K\in\mathcal{T}_h}\left(h_K^{-4}\|z-\pi_h z\|^2_{L_2(K)} + h_K^{-3}\|z-\pi_h z\|^2_{L_2(\partial K)} + h_K^{-1}\left\|\frac{\partial(z-\pi_h z)}{\partial n_K}\right\|^2_{L_2(\partial K)}\right)\right)^{1/2} \le C\,\|z\|_{H^2(\Omega)}. \tag{56}
$$

From the consistency condition we have $a_h(u-u_h, \pi_h z) = 0$, and since $[\![z]\!] = 0$, $[\![u]\!] = 0$, on the interface, we get

$$
\begin{aligned}
0 &= a_h(u-u_h, \pi_h z) \\
&= \sum_{i=1}^{2}\left[(\nabla(u_i-u_{i,h}), \nabla\pi_h z_i)_{\Omega_i} + \gamma\sum_{E\in\mathcal{G}_h^i} h_E^{-1}\langle [\![u-u_h]\!], [\![\pi_h z]\!]\rangle_E\right] \\
&\quad -\left\langle\left\{\frac{\partial(u-u_h)}{\partial n}\right\}, [\![\pi_h z]\!]\right\rangle_\Gamma - \left\langle\left\{\frac{\partial \pi_h z}{\partial n}\right\}, [\![u-u_h]\!]\right\rangle_\Gamma \\
&= \sum_{i=1}^{2}\left[(\nabla(u_i-u_{i,h}), \nabla\pi_h z_i)_{\Omega_i} + \gamma\sum_{E\in\mathcal{G}_h^i} h_E^{-1}\langle [\![\, u_h]\!], [\![z-\pi_h z]\!]\rangle_E\right] \\
&\quad +\left\langle\left\{\frac{\partial(u-u_h)}{\partial n}\right\}, [\![z-\pi_h z]\!]\right\rangle_\Gamma + \left\langle\left\{\frac{\partial \pi_h z}{\partial n}\right\}, [\![u_h]\!]\right\rangle_\Gamma.
\end{aligned} \tag{57}
$$

Substracting (57) from (55) gives

$$
\begin{aligned}
\|u-u_h\|^2_{L_2(\Omega)} &= \sum_{i=1}^{2}\left[(\nabla(z_i-\pi_h z, \nabla(u_i-u_{i,h}))_{\Omega_i} - \gamma\sum_{E\in\mathcal{G}_h^i} h_E^{-1}\langle [\![\, u_h]\!], [\![z-\pi_h z]\!]\rangle_E\right] \\
&\quad -\left\langle\left\{\frac{\partial(u-u_h)}{\partial n}\right\}, [\![z-\pi_h z]\!]\right\rangle_\Gamma + \left\langle\left\{\frac{\partial (z-\pi_h z)}{\partial n}\right\}, [\![u_h]\!]\right\rangle_\Gamma \\
&:= R_1 + R_2 + R_3 + R_4.
\end{aligned} \tag{58}
$$

Using Schwarz inequality and the Clément estimate (56) we get

$$\begin{aligned}R_2 + R_4 &\le C \left(\sum_{K\in\mathcal{T}_h} \left(h_K^{-3} \|z - \pi_h z\|^2_{L_2(\partial K)} + h_K^{-1} \left\| \frac{\partial(z-\pi_h z)}{\partial n_K} \right\|^2_{L_2(\partial K)} \right) \right)^{1/2} \left(\sum_{E\in\mathcal{G}_h^i} h_E \|[\![u_h]\!]\|_{L_2(E)} \right)^{1/2}\\ &\le C \|z\|_{H^2(\Omega)} L(u_h) \le C \|u - u_h\|_{L_2(\Omega)} L(u_h).\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}R_1 + R_3 &= \sum_{i=1}^{2} \Big[(\nabla(z_i - \pi_h z, \nabla(u_i - u_{i,h}))_{\Omega_i}\\ &\quad - \left\langle \left\{ \frac{\partial(u-u_h)}{\partial n} \right\}, [\![z - \pi_h z]\!] \right\rangle_\Gamma\\ &= \sum_{K\in\mathcal{T}_h} \Bigg\{ (z - \pi_h z, \Delta(u_h - u))_K\\ &\quad + \left\langle \frac{\partial(u-u_h)}{\partial n_K}, z - \pi_h z \right\rangle_{\partial K} \Bigg\}\\ &\quad - \left\langle \left\{ \frac{\partial(u-u_h)}{\partial n} \right\}, [\![z - \pi_h z]\!] \right\rangle_\Gamma\\ &= \sum_{K\in\mathcal{T}_h} (z - \pi_h z, \Delta u_h + f)_K\\ &\quad + \sum_{E\in\mathcal{I}_h} \left\langle \frac{\partial(u-u_h)}{\partial n_E}, z - \pi_h z \right\rangle_E\\ &\quad + \sum_{i=1}^{2} \left\langle \frac{\partial(u_i-u_{i,h})}{\partial n_i}, z_i - \pi_h z_i \right\rangle_\Gamma\\ &\quad - \left\langle \left\{ \frac{\partial(u-u_h)}{\partial n} \right\}, [\![z - \pi_h z]\!] \right\rangle_\Gamma .\end{aligned}$$

The boundary terms in the second term above are grouped together two by two yielding jump terms in the normal derivative. This gives the estimate

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} (z - \pi_h z, \Delta u_h + f)_K + \sum_{E\in\mathcal{I}_h} \left\langle \frac{\partial(u-u_h)}{\partial n_E}, z - \pi_h z \right\rangle_E\\ &\le C \left(\sum_{K\in\mathcal{T}_h} \left(h_K^{-4} \|z - \pi_h z\|^2_{L_2(K)} + h_K^{-3} \|z - \pi_h z\|^2_{L_2(\partial K)} \right) \right)^{1/2}\\ &\quad \times \left(\sum_{K\in\mathcal{T}_h} h_K^4 \|f + \Delta u_h\|^2_{L_2(K)} + h_K^3 \left\| \left[\!\left[\frac{\partial u_h}{\partial n_K} \right]\!\right] \right\|^2_{L_2(\partial K)} \right)^{1/2}\\ &\le C \|z\|_{H^2(\Omega)} L(u_h)\\ &\le C \|u - u_h\|_{L_2(\Omega)} L(u_h).\end{aligned} \tag{59}$$

The remaining two terms are estimated as follows

$$
\begin{aligned}
&\sum_{i=1}^{2}\left\langle \frac{\partial(u_i-u_{i,h})}{\partial n_i}, z_i-\pi_h z_i\right\rangle_\Gamma - \left\langle \left\{\frac{\partial(u-u_h)}{\partial n}\right\}, [\![z-\pi_h z]\!]\right\rangle_\Gamma \\
&= \sum_{i=1}^{2}\left\langle \frac{\partial u_i}{\partial n_i}, z_i-\pi_h z_i\right\rangle_\Gamma - \sum_{i=1}^{2}\left\langle \frac{\partial u_{i,h}}{\partial n_i}, z_i-\pi_h z_i\right\rangle_\Gamma - \left\langle \frac{\partial u}{\partial n}, [\![z-\pi_h z]\!]\right\rangle_\Gamma + \left\langle \left\{\frac{\partial u_h}{\partial n}\right\}, [\![z-\pi_h z]\!]\right\rangle_\Gamma \\
&= -\sum_{i=1}^{2}\left\langle \frac{\partial u_{i,h}}{\partial n_i}, z_i-\pi_h z_i\right\rangle_\Gamma + \left\langle \left\{\frac{\partial u_h}{\partial n}\right\}, [\![z-\pi_h z]\!]\right\rangle_\Gamma \\
&= -\left\langle \frac{\partial u_{1,h}}{\partial n_1}, z_1-\pi_h z_1\right\rangle_\Gamma - \left\langle \frac{\partial u_{2,h}}{\partial n_2}, z_2-\pi_h z_2\right\rangle_\Gamma + \frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1}, z_1-\pi_h z_1\right\rangle_\Gamma - \frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1}, z_2-\pi_h z_2\right\rangle_\Gamma \\
&\quad -\frac{1}{2}\left\langle \frac{\partial u_{2,h}}{\partial n_2}, z_1-\pi_h z_1\right\rangle_\Gamma + \frac{1}{2}\left\langle \frac{\partial u_{2,h}}{\partial n_2}, z_2-\pi_h z_2\right\rangle_\Gamma \\
&= -\frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1}, z_1-\pi_h z_1\right\rangle_\Gamma - \frac{1}{2}\left\langle \frac{\partial u_{2,h}}{\partial n_2}, z_2-\pi_h z_2\right\rangle_\Gamma - \frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1}, z_2-\pi_h z_2\right\rangle_\Gamma - \frac{1}{2}\left\langle \frac{\partial u_{2,h}}{\partial n_2}, z_1-\pi_h z_1\right\rangle_\Gamma \quad (60)\\
&= -\frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1} + \frac{\partial u_{2,h}}{\partial n_2}, z_1-\pi_h z_1\right\rangle_\Gamma - \frac{1}{2}\left\langle \frac{\partial u_{1,h}}{\partial n_1} + \frac{\partial u_{2,h}}{\partial n_2}, z_2-\pi_h z_2\right\rangle_\Gamma \\
&= -\frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right], z_1-\pi_h z_1\right\rangle_\Gamma - \frac{1}{2}\left\langle \left[\!\!\left[\frac{\partial u_h}{\partial n}\right]\!\!\right], z_2-\pi_h z_2\right\rangle_\Gamma \\
&\le CL(u_h)\Big(\sum_{K\in\mathcal{T}_h} h_K^{-3}\,\|z-\pi_h z\|^2_{L_2(\partial K)}\Big)^{1/2} \\
&\le CL(u_h)\,\|z\|_{H^2(\Omega)} \le CL(u_h)\,\|u-u_h\|_{L_2(\Omega)}\,.
\end{aligned}
$$

Combining (59) and (60) gives

$$R_1 + R_3 \le CL(u_h)\,\|u-u_h\|_{L_2(\Omega)}$$

which together with (59) and (58) proves the claim. □

4. Numerical examples

4.1. **Numerical verification of the *a priori* estimates**

To verify the *a priori* estimates, we choose the model problem of a unit square with exact solution

$$u = x\,y\,(1-x)\,(1-y)$$

corresponding to a right-hand side of $f = 2(x - x^2 + y - y^2)$. The domain is divided by a vertical slit at $x = 0.7$. Two different triangulations were used: one matching and one non-matching (see Fig. 1).

In Figure 2 (left-hand side) we give the convergence in the broken energy norm. The dashed line is the non-matching grid computation. Both meshes show the same convergence with slope 0.95. which is close to the theoretical value of 1. On the right-hand side we show the convergence of the L_2-norm of the jump term (dashed line for the non-matching grid). Here we obtain a better convergence (slope 2.15) for the matching grids than for the non-matching grids (slope 1.57, close to the theoretical value of 3/2).

4.2. **Adaptive computations**

We present results of adaptive computations on the L-shaped domain

$$\Omega = (0,1)\times(0,1)\setminus(1/2,1)\times(0,1/2).$$

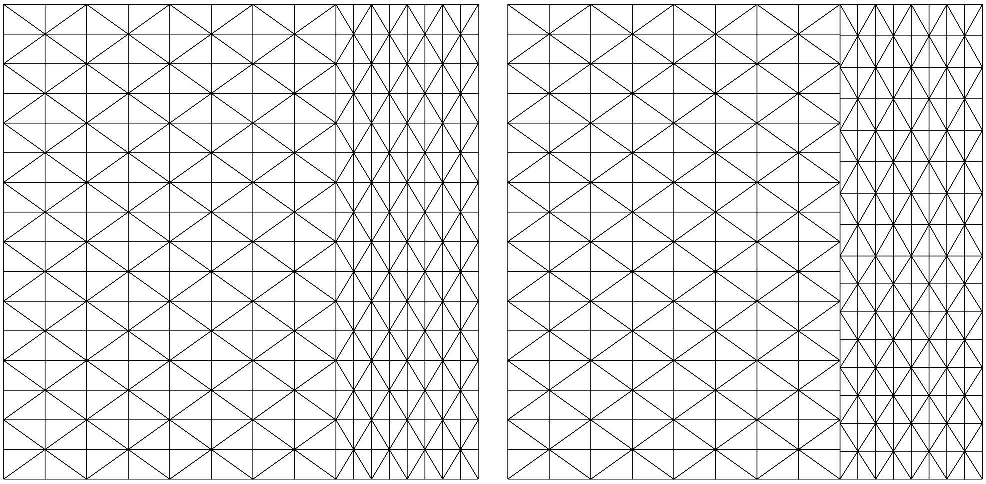

FIGURE 1. Matching and non-matching grids.

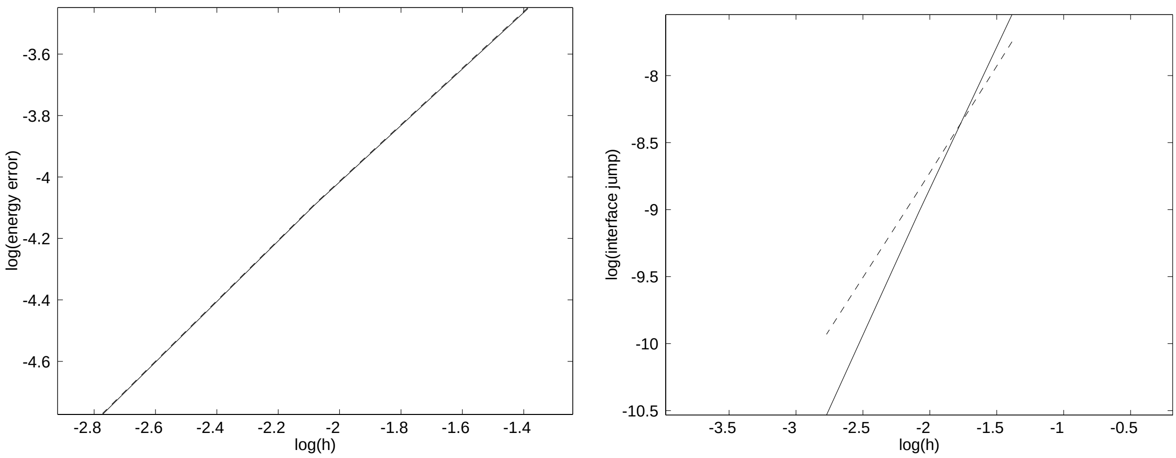

FIGURE 2. Convergence in energy and convergence of the jump term.

The problem is boundary driven ($f = 0$), with boundary data corresponding to the exact solution $u = r^{2/3}\sin(2\theta/3)$ in polar coordinates (with origin at $(1/2, 1/2)$). We let $\Omega_1 = (0, 1/2) \times (0, 1)$ and $\Omega_2 = (1/2, 1) \times (1/2, 1)$, and use a non-matching triangulation. The purpose of this example is not to obtain exact error control, but rather to show how the adaptive algorithm behaves with respect to the elements adjacent to the interface. We consider adaptive control of the L_2-norm error, but we have not made any attempt to measure the constant in the inequality (53); instead we have simply tuned the interpolation constants to approximately match the exact error.

In Figures 3 and 4 we show the first and last (adapted) meshes resulting from equilibrating the error distribution over the set of elements (for details, see [7, 14]). In Figure 5 we show the exact and estimated L_2-errors on the sequence of meshes, which show a reasonable agreement. For more exact error control, more computational effort must be invested.

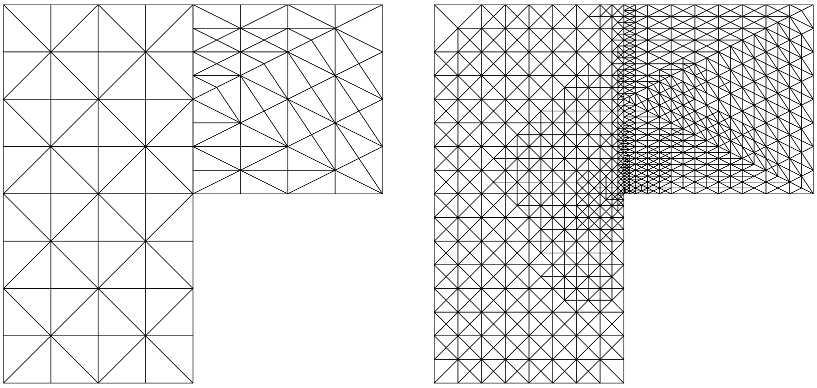

Figure 3. First mesh and final adapted mesh.

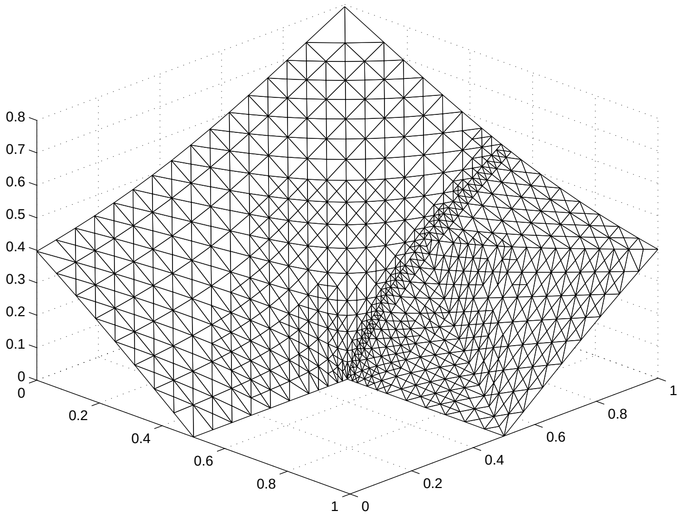

Figure 4. Elevation of the solution.

4.3. **Difference between matching and non-matching meshes**

In the previous example, the adaptive algorithm produced a slightly finer mesh on the interface. One natural question is then whether the effect of non-matching actually does lead to larger errors. In Figures 6 and 7 we show two different meshes for solving the problem described in Section 4.1, and the corresponding nodal interpolants of the errors. The interface is situated at $x = 1/2$, and it is noted that the error on the interface is markedly larger than in the interior of the domains only in the case of non-matching meshes.

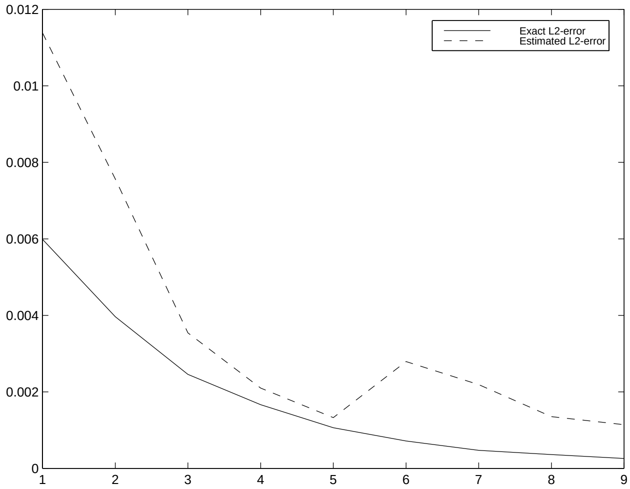

FIGURE 5. Estimated and exact errors in the $L_2(\Omega)$-norm.

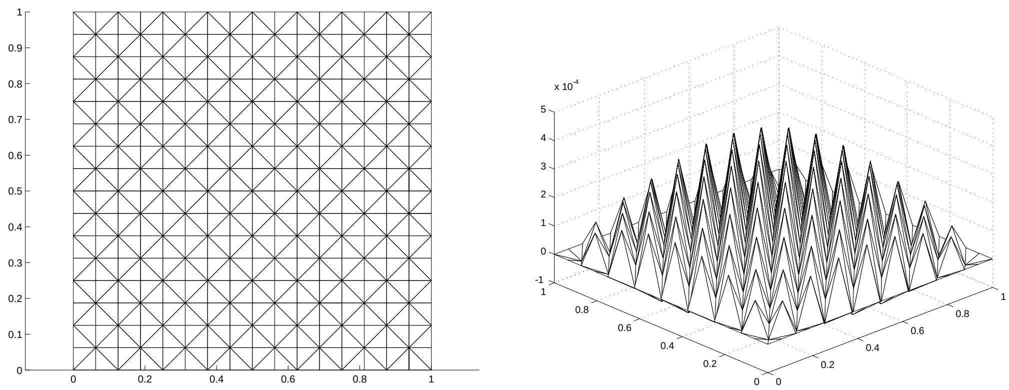

FIGURE 6. Matching meshes and nodal errors.

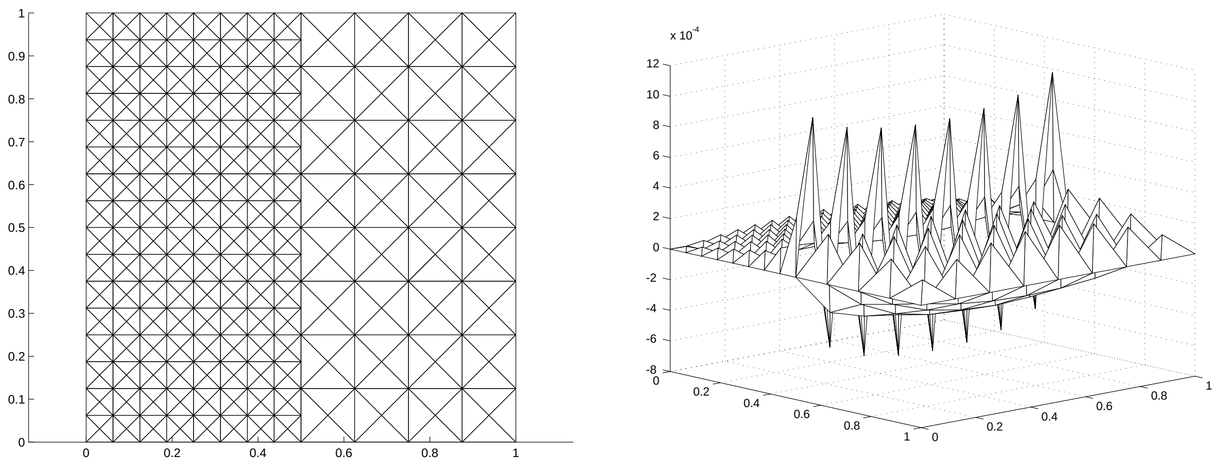

FIGURE 7. Non-matching meshes and nodal errors.

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