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SOLUTION OF DEGENERATE PARABOLIC VARIATIONAL INEQUALITIES WITH CONVECTION

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Abstract. Degenerate parabolic variational inequalities with convection are solved by means of a combined relaxation method and method of characteristics. The mathematical problem is motivated by Richard's equation, modelling the unsaturated – saturated flow in porous media. By means of the relaxation method we control the degeneracy. The dominance of the convection is controlled by the method of characteristics.

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1. Introduction

In this paper we consider the parabolic convection-diffusion variational inequality

$$(\partial_t b(u), v-u)+(\mathbf{A}\nabla u, \nabla(v-u))+(\operatorname{div}\bar{F}(x,u), v-u)+(g(t,u), v-u)_{\Gamma_2} \geq (f(t,u), v-u) \qquad \forall v \in L_2(I,K) \quad (1)$$

with $u(x,0) \in K$.

Here, $(.,.)$ is the scalar product in $L_2(\Omega)$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz continuous boundary $\partial\Omega$, $I \equiv (0,T)$, $W_2^1(\Omega)$ is the standard first order Sobolev space, $V \equiv \{v \in W_2^1 : v = 0 \text{ on } \Gamma_1\}$, K is a closed convex set in V and $u : I \to V$ is an abstract function. Let $(u,v)_{\Gamma_2} = \int_{\Gamma_2} uv \,\mathrm{d}x$; Γ_1 and $\Gamma_2 \subset \partial\Omega$ are open with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\operatorname{meas}_{N-1}\Gamma_1 + \operatorname{meas}_{N-1}\Gamma_2 = \operatorname{meas}_{N-1}\partial\Omega$.

We assume that $b(s)$ is increasing, $\mathbf{A}(x)$ is a possitive definite symmetric matrix, $\partial_s \bar{F}(x,s)$ is bounded and g, f are sublinear in u. The problem (1) includes many practical problems governed by convection-diffusion phenomena with adsorption, with unilateral conditions (in Ω or on $\partial\Omega$). If $K \equiv V$ then (1) represents the variational formulation of degenerate convection-diffusion problem, which also includes "porous media type equation" with convection, which can be dominant. It is well known that the numerical approximation (convergence analysis and implementation) of these type of problems is very difficult, since the solution can possess the finite support with sharp fronts on the boundary of the support. We present a motivating example concerning the flow in

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unsaturated-saturated porous media governed by Richard's equation

$$\partial_t \theta = \operatorname{div}(k(h)\mathbf{A}(x)\nabla(h+z)). \tag{2}$$

Here, θ is the volumetric water content, h is the pressure head, k is the hydraulic permeability. We use the Van Genuchten–Mualem model, describing the retention and permeability curves

$$\theta = \theta(h) = \theta_r + \frac{\theta_s - \theta_r}{(1+(\alpha h)^n)^m},\ \widetilde{k}(S) = S^{1/2}\left(1 - \left(1 - S^{1/m}\right)^m\right)^2,$$

where $S = \frac{\theta - \theta_r}{\theta_s - \theta_r}$ is the effective saturation and $k(h) = \widetilde{k}(\theta(h))$ for $h < 0$, $k(h) = 1$ for $h > 0$. Here, θ_r, θ_s, $1 < n$ and m, ($m = 1 - \frac{1}{n}$), are so called soil parameters. We verify that (2) is a degenerate parabolic equation where the degeneracy occurs in both elliptic and parabolic (storativity) terms. This generates two interfaces in the flow: the interface between dry and wet regions and the interface between unsaturated ($h < 0$) and saturated ($h > 0$) zones. These degeneracies can be transformed only to the parabolic term, using Kirchhoff's transformation

$$u := \beta(h) = \int_0^h k(z)\mathrm{d}z, \qquad b(u) := \theta(\beta^{-1}(u)).$$

Then, for $h < 0$ (unsaturated zone) we obtain

$$\partial_t b(u) - \operatorname{div}(\mathbf{A}(x)\nabla u) - \partial_z K(x, b(u)) = 0, \tag{3}$$

where $K(x, b(u)) = \mathbf{A}(x)\bar{e}\tilde{k}(b(u))$, $\bar{e}$ being the unit vector in direction z. The unknown u varies in the interval $(u^*, 0)$. Moreover, we can verify that $b'(u^*) = \infty$. The flow in the saturated zone $h \geq 0$ is governed by Darcy's law, which leads to the mathematical model

$$S_e \partial_t h - \operatorname{div}(\mathbf{A}(x)\nabla(h+z)) = 0, \tag{4}$$

where S_e is the specific (elasticity) storativity coefficient, $k(h) \equiv 1$ for $h \geq 0$ and $\theta = S_e h$ – see [6]. Also here we can use Kirchhoff's transformation and obtain $u = h$ for $h > 0$, $b(u) = S_e u$ and $\mathbf{K}(x, b(u)) = \mathbf{A}(x)\bar{e}$. If we prolong u and $b(u)$ for $u > 0$, then (3) describes unsaturated-saturated flow in porous media. We can shift the unknown so that u^* will be translated to 0. Then, we are looking for a nonegative solution of (1). Generally, we are not able to guarantee that the numerical approximations of such a solution will be nonnegative too. Therefore, it is natural to reformulate (3) in terms of a variational inequality and to look for the solution in a closed convex set $K = \{v \in V : v \geq 0\}$. Thus, we arrive at a variational inequality of the type (1) with degenerated $b(u)$. The degeneracy $b'(u^*) = \infty$ gives rise to sharp fronts between the wet region and the dry region ($h = -\infty$, *i.e.* $u = u^*$) in the unsaturated part of the porous media (infiltration phenomenon). The solution has a sharp front there. In this case the convective term $\mathbf{K}(x, b(u))$ is effected by the gravitation. The boundedness of $\partial_u \mathbf{K}(x, b(u))$ is required in our numerical approximation. We can verify that $\partial_u \mathbf{K}(x, b(u))$ is bounded (uniformly in $x \in \Omega$ for $u \geq u^*$) provided that $m < 1 - \frac{2}{n}$. If we accept $m = 1 - \frac{1}{n}$, then we have the requirement $n \geq 2$, which is satisfied for a large scale of porous media.

The mathematical formulation of (3) in terms of variational inequalities also allows us to consider unilateral boundary conditions which are quite natural for the unsaturated-saturated flow – see [2]. For an illustration we present the following model. Let us assume that on the part $\Gamma_2 \subset \partial\Omega$ the medium is in contact with water, or air, or with both of them. When it is in contact with water, then a positive pressure has to be prescribed. When the medium is in contact with air, then either the flux is zero and the pressure is nonpositive (not prescribed), or the flux is positive and the pressure is zero (overflow). The mathematical formulation is as follows – see [2]

(i) $h^+ = p*$ (data) on Γ_2 where $h > 0$;
(ii) $k(h)\mathbf{A}(\nabla h + \bar{e}).\nu \leq 0$ on Γ_2 where $h = 0$;
(iii) $k(h)\mathbf{A}(\nabla h + \bar{e}).\nu = 0$ on Γ_2 where $h < 0$.

Here, $h^+ := \max\{h, 0\}$. The vector ν is the outward unit normal vector to $\partial\Omega$. Both conditions (ii) and (iii) can be formulated in the form

$$k(h)\mathbf{A}(\nabla h + \bar{e}).\nu(h - w) \le 0 \quad \text{on} \quad \Gamma_2,\ \forall w : w^+ = p* .$$

We assume that $p* \in L_2(I, W_2^1)$ and $p* \ge 0$. The mathematical formulation of this type of boundary conditions leads again to a variational inequality of the type (1). Using Kirchhoff's transformation (including the shifting mentioned above) we take the convex set $L_2(I, K)$ in the form $\{v \in L_2(I, V) : v \ge 0, (v + u^*)^+ = \beta(p*) \text{ on } \Gamma_2\}$. Integrating by parts in (1) for a smooth solution u we obtain

$$\left(\mathbf{A}\left(\nabla u + \bar{e}\tilde{k}(b(u))\right).\nu, u - v\right)_{\Gamma_2} \le 0,$$

from which the conditions (ii) and (iii) follow since $\operatorname{sgn}(u - v) = \operatorname{sgn}(h - w)$.

The mathematical model (1) has been analysed with respect to existence and uniqueness of the solution in [1] and [27]. Richard's equation (3) has been solved (by finite volume approximations) in [12] under the assumption that b is Lipschitz continuous. The variational inequality (1) has been solved by the relaxation method in [3] without convective term and with a Lipschitz continuous b.

The aim of our paper is to approximate the solution to (1) by the method of semi-discretization in time and by full discretization techniques and to prove their convergence. The approximation method is based on the relaxation method (to control the degeneracy of b) – see [3, 15, 17–19, 25], and on a modification of the method of characteristics – see [4, 5, 7–11, 20, 22, 28, 29] among others. We follow the idea of regularized characteristics analysed in [20]. This allows us to include the dominance of the convective term $\bar{F}$, which, in addition, may depend on the unknown u.

In Section 2 we present the approximation scheme and we specify the assumptions on the data. The convergence of the semi-discretization scheme is proved in Section 3. The full discretization method is discussed in Section 4.

2. Variational solution and the approximation scheme

By C we denote a generic positive constant. By L_∞, L_2, W_2^1, $L_2(I, L_2)$, $L_2(I, V)$, $L_2(I, V^*)$, ($V \subset W_2^1$ is a subspace), we denote standard functional spaces (see [23]). Let $\|.\|_\infty$, $\|.\|_0$, $\|.\|$ and $\|.\|_*$ be the norms in L_∞, L_2, V and V^*, respectively.

We shall introduce the following 5 hypotheses:

- (H_1) $b'(s) \ge \gamma > 0$ and $|b(s)| \le C(1 + |s|) \quad \forall s \in \mathbb{R}$, $b(0) = 0$. There exist regularizations $b_n(s)$ with the properties:
  - (i) $b_n(s) \to b(s)$ for $n \to \infty$ locally uniformly;
  - (ii) $|b_n(s)| \le C_1 + C_2|s|$;
  - (iii) $\min\{b'(s), \mu\} \le b'_n(s) \le K_n$) $\forall s \in \mathbb{R}$, with $K_n \to \infty$ for $n \to \infty$ and with $0 < \mu < C_e$.
- (H_2) $\mathbf{A}(x)$ is symmetric and uniformly positive definite, *i.e.* $(\mathbf{A}(x)\xi, \xi) \ge C|\xi|^2 \quad \forall x \in \Omega, \forall \xi \in \mathbb{R}^N$.
- (H_3) $\|\partial_s \bar{F}(x, s)\|_\infty \le C$, $|\operatorname{div}_x \bar{F}(x, s)| \le C_1(1 + |s|)$.
- (H_4) $|f(t, x, s)| \le C(1 + |s|)$, $\quad |\partial_t g(t, x, s)| \le C(1 + |s|)$ and $|\partial_s g(t, x, s)| \le C$, $\forall (x, t) \in \Omega \times I, \forall s \in \mathbb{R}$.
- (H_5) $u_0 \in K$ (K being a closed convex set in V).

Remark 1. The regularization $b_n(s)$ of $b(s)$ can be constructed as follows. Put

$$\varrho_n(z) := \min\{b'(z), K_n\} \quad \text{and} \quad b_n(s) := \int_0^s \varrho_n(z)\mathrm{d}z,$$

where $K_n \to \infty$ for $n \to \infty$.

If $b(u)$ is not Lipschitz continuous, then we are not able to guarantee that $\partial_t b(u) \in L_2(I, V^*)$. If $\partial_t b(u) \in L_2(I, V^*)$ and $v \in L_2(I, K)$ with $\partial_t v \in L_2(I, L_2)$, then

$$\int_0^t \langle \partial_t b(u), v - u\rangle \mathrm{d}t = (b(u(t) - b(u_0), v(t) - u(t)) - \int_0^t ((b(u) - b(u_0)), \partial_t(v - u)).$$

Thus, in the general case we weaken the variational solution to (1) using integration by parts in the t-variable. We define the weak variational solution to (1) as follows

Definition 1. $u \in L_2(I, V)$, with $b(u) \in L_2(I \times \Omega)$, is a weak variational solution to (1) iff

$$- \int_0^t (b(u) - b(u_0), \partial_t(v - u)) + (b(u(t) - b(u_0), v(t) - u(t)) \\ + \int_0^t (\mathbf{A}\nabla u, \nabla(v - u))\mathrm{d}x + \int_0^t (\operatorname{div} \bar{F}(x, u), v - u)\mathrm{d}t \geq \\ \int_0^t (f(t, u), v - u)\mathrm{d}x, \quad \forall v \in L_2(I, K) \quad \text{with} \quad \partial_t v \in L_2(I, L_2).$$

Let u_i be an approximation of $u(x, t)$ at time points $t = t_i$, $t_i = i\tau$, $i = 1, \dots, n$, where $\tau = \frac{T}{n}$ is the time step.

To control the convective term generated by $\bar{F}(x, u)$ we use the method of characteristics in the following way. Let $\varphi^i(x)$ be the approximate characteristic map for the time interval (t_{i-1}, t_i) defined by

$$\varphi^i(x) := x - \tau\omega_h * \partial_u \bar{F}(x, u_{i-1}), \qquad (\omega_h * z)(x) = \int_{R^N} \omega_h(x - \xi) z(\xi)\mathrm{d}\xi,$$

where ω_h is a modifier:

$$\omega_h(x) = h^{-N}\omega_1(\frac{x}{h}),$$

with

$$\omega_1(x) = \kappa \exp\left(\frac{|x|^2}{|x|^2 - 1}\right) \text{ for } |x| \leq 1,\ \omega_1 = 0 \text{ for } |x| \geq 1,\ \int_{R^N} \omega_1(x)\mathrm{d}x = 1.$$

We shall assume that $h = \tau^\omega$ for $\omega \in (0, 1)$. The map $\varphi^i(x)$ is a regularization of the map $\bar{\varphi}(x) := x - \tau\partial_u \bar{F}(x, u_{i-1})$, which is the Euler-backwards approximation of the $\widetilde{\varphi}^i(x)$-characteristic defined by the ODE

$$\frac{\mathrm{d}X(s; t_i, x)}{\mathrm{d}s} = \bar{F}(X(s; t_i, x), u_{i-1}(X(s; t_i, x)) \quad \text{for} \quad s \in (t_{i-1}, t_i) \tag{5}$$

with the initial condition $X(t_i; t_i, x) = x$. Then $\widetilde{\varphi}^i(x) \equiv X(t_{i-1}; t_i, x)$.

To control the dominance of the convective term and the degeneracy of the parabolic term we suggest the following approximation scheme. We determine u_i at the time point t_i (assuming that u_{i-1} is known) from the elliptic variational inequality

$$\frac{1}{\tau}(\lambda_i(u_i - u_{i-1}), v - u_i) + (\mathbf{A}\nabla u_i, \nabla(v - u_i)) + (g(t, u_i), v - u_i)_{\Gamma_2} \geq \\ (H(t, u_{i-1}), v - u_i) - \frac{1}{\tau}(u_{i-1} - u_{i-1} \circ \varphi^i, v - u_i), \quad \forall v \in K, \tag{6}$$

where

$$H(t, u_{i-1}) := f(t_i, x, u_{i-1}) - \operatorname{div}_x \bar{F}(x, u_{i-1})$$

and where $\lambda_i \in L_\infty(\Omega)$ is a relaxation function which has to satisfy the "convergence condition"

$$\left\| \lambda_i - \frac{b_n(u_i) - b_n(u_{i-1})}{u_i - u_{i-1}} \right\|_\infty < \tau. \tag{7}$$

Here, the convective term is included in the source term. In fact, the scheme (6, 7) is implicit with respect to (λ_i, u_i). To determine λ_i we propose the iteration procedure

$$(\lambda_{i,k-1}(u_{i,k} - u_{i-1}), v - u_{i,k}) + \tau(\mathbf{A}\nabla u_{i,k}, \nabla(v - u_{i,k})) + \tau(g_i, v - u_{i,k})_{\Gamma_2} \geq \\ \tau(H(t_i, u_{i-1}), v - u_{i,k}) - (u_{i-1} - u_{i-1} \circ \varphi^i, v - u_{i,k}) \quad \forall v \in K, \tag{8}$$

with

$$\lambda_{i,k} := \frac{b_n(u_{i,k}) - b_n(u_{i-1})}{u_{i,k} - u_{i-1}}. \tag{9}$$

This concept of relaxation of the parabolic term has been used for variational equations in [17–19]. In fact, updating the value $\lambda_{i,k}$, we can simultaneously update the convective term, where in the place of $\partial_u \bar{F}(x, u_{i-1})$ we can take $\partial_u \bar{F}(x, u_{i,k-1})$ to evaluate $\varphi^i(x)$.

Another practical improvement can be realized by shortening the time step in the convective process and thus increasing the order of approximation. In this case we use Euler's backwards approximation in (5) with smaller time steps. Let $I_i^l = (t_{i-1}^{(l)} - t_{i-1}^{(l-1)}), l = 1, \ldots, m$, where $t_{i-1}^{(0)} = t_{i-1}$ and $t_{i-1}^{(m)} \equiv t_i$. We denote

$$v(x) := \partial_u \bar{F}(x, u_{i-1}), v^h := \omega_h * v \equiv v_0^h, v_1^h(x) := x - \left(t_{i-1}^{(1)} - t_{i-1}\right) v_0^h(x)$$

and

$$v_l^{(h)}(x) := v_{l-1}^h(x) - \left(t_{i-1}^{(l)} - t_{i-1}^{(l-1)}\right) v^h \left(v_{l-1}^h(x)\right).$$

Then we put

$$\widetilde{\varphi}^i(x) := v_{m-1}^h(x) - \left(t_i - t_{i-1}^{(m-1)}\right) v^h \left(v_{m-1}^h(x)\right). \tag{10}$$

Here, $v_l^h(x)$ represents the position of the initial point x after l smaller time steps along the approximated characteristic in the time interval $\left(t_{i-1}^{(0)}, t_{i-1}^{(l)}\right)$.

Thus, our approximation scheme (6, 7), with the map $\widetilde{\varphi}^i$ taken from (10), represents the approximation of the solution in the time interval (t_{i-1}, t_i), based on superposition of the convective part, represented by the last term of (6) (realized by shorter time steps I_i^l) and the diffusive part (realized by the shorter time step τ).

The application of the method of characteristics requires the one to one property of the map φ^i. The convergence analysis requires the Lipschitz-continuity of φ^i and its inverse. Since the velocity field represented by $v = \partial_u \bar{F}(x, u)$ depends on the unknown u, it is difficult to guarantee the one to one property for the map φ^i (resp. $\widetilde{\varphi}^i$) since it would lead to the boundedness of $\nabla \bar{v}$ and consequently to the boundedness of ∇u. But this is, generally, not true (especially when a porous media type phenomenon occurs). That's why we are regularizing $\bar{v}$ by $\bar{v}^h$. This allows us to guarantee the one to one property of φ^i (see [20]).

Lemma 1. *There exists $\tau_0, \tau_1, (\tau_1 < \tau_0)$, such that*

(i) $\frac{1}{2}|x - y| \leq |\varphi^i(x) - \varphi^i(y)| \leq 2|x - y|$ *for* $\tau \leq \tau_0$
(ii) $\frac{1}{2}|x - y| \leq |\widetilde{\varphi}^i(x) - \widetilde{\varphi}^i(y)| \leq 2|x - y|$ *for* $\tau \leq \tau_1$

for all $x, y \in \bar{\Omega}$, and $i = 1, \ldots, n$, where $\widetilde{\varphi}^i$ is from (10).

Proof. Assertion (i) is proved in [20]. To proove (ii) we use (10) and the fact that $\nabla v_l^h = \nabla v_{l-1}^h(1+\tau_l\nabla_y(v^h))$, $(\tau_l \equiv |I_l^i|, y = \nabla v_{l-1}^h(x))$. We use $|\nabla\omega_h * g| \le \frac{C}{h}\|g\|_\infty \le \frac{C}{h}$. Then we obtain

$$|\partial_{x_j}\{\widetilde{\varphi}^i(x)\}_j - 1| \le \frac{\tau}{h}C(1+\frac{\tau}{h}C + (\frac{\tau}{h})^2C^2 + \cdots + (\frac{\tau}{h})^{m-1}C^{m-1}) \le C\frac{\tau}{h}\quad\frac{1}{1-(\tau/h)C},$$
$$|\partial_{x_j}\{\widetilde{\varphi}^i(x)\}_k| \le C\frac{\tau}{h}\quad\frac{1}{1-(\tau/h)C},$$

for any j and $k = 1,\dots,N,\ j\ne k,$. Here $\{\bar{v}\}_j$ is the jth component of $\bar{v}$.
Since $h=\tau^\omega$, with $\omega\in(0,1)$, we obtain the required estimate (ii). □

The function φ^i maps Ω into $\Omega_i \subset \Omega^*$ for $i=1,\dots,n$, where $\Omega^* \supset \Omega$ is a small neighbourhood provided that $\tau \le \tau_0$. If $\varphi^i(x) \notin \Omega$ then we extend u_{i-1} from $W_2^1(\Omega)$ to $\widetilde{u}_{i-1} \in W_2^1(\Omega^*)$ so that (see [26], prolongation of Nikolskij),

$$\|\widetilde{u}_{i-1}\|_{W_2^1(\Omega^*)} \le C\|u_{i-1}\|_{W_2^1(\Omega)}. \tag{11}$$

Remark 2. The existence of the function $u_i \in K$ satysfying (6), resp. (8), is guaranteed by [24] and by the fact that $u_{i-1} \in L_2(\Omega)$ implies that $u_{i-1}\circ\varphi^i \in L_2(\Omega)$ because of Lemma 1. The convergence of the iterations (8) and (9) has been analysed in [19] for variational equalities. The analysis can be adopted for variational inequalities.

3. Convergence of the scheme (6, 7)

First, we prove some *a priori* estimates for $\{u_i\}_1^n$ and then we show the compactness of $\{\bar{u}^n\}_{n=1}^\infty$ in the corresponding functional spaces. Here, we define the Rothe's functions

$$\bar{u}^n(t) := u_i \quad\text{and } u^n(t) := u_{i-1} + \frac{t-t_{i-1}}{\tau}(u_i - u_{i-1}) \tag{12}$$

for $t\in(t_{i-1},t_i), i=1,\dots,n$.

Lemma 2. *Under the assumptions (H_1)–(H_5) it holds that*

$$\max_{1\le j\le n}\int_\Omega B_n(u_j)\mathrm{d}x \le C,\quad \sum_{i=1}^n \|u_i\|^2\tau \le C,\quad \max\|u_j\|_0 \le C,$$
$$\sum_{i=1}^j\left(\frac{b_n(u_i)-b_n(u_{i-1})}{\tau},\frac{u_i-u_{i-1}}{\tau}\right)\tau \le \varepsilon\sum_{i=1}^j\|\delta u_i\|_0^2\tau + C_\varepsilon,$$

where $\varepsilon > 0$ is any small, fixed number.

Proof. We split

$$\lambda_i(u_i-u_{i-1}) = b_n(u_i) - b_n(u_{i-1}) + \chi_i(u_i-u_{i-1})\tau, \tag{13}$$

where $\|\chi_i\|_\infty \le 1$ and next we put $v = u^*$ ($u^* \in K$ is fixed) into (6). We multiply the resulting inequality with τ and sum up for $i = 1, \dots, j$. We obtain

$$\sum_{i=1}^{j}(b_n(u_i) - b_n(u_{i-1}), u_i) + \sum_{i=1}^{j}(\mathbf{A}\nabla u_i, \nabla u_i)\tau \le \sum_{i=1}^{j}(b_n(u_i) - b_n(u_{i-1}), u^*) + \sum_{i=1}^{j}(\mathbf{A}\nabla u_i, \nabla u^*)\tau$$
$$+ \sum_{i=1}^{j}(\chi_i(u_i - u_{i-1}), u_i - u^*)\tau + \sum_{i=1}^{j}(g(t_i, u_{i-1}), u_i - u^*)_{\Gamma_2}\tau$$
$$+ \sum_{i=1}^{j}(H(t_i, u_{i-1}), u_i - u^*)\tau + \sum_{i=1}^{j}(u_{i-1} - u_{i-1} \circ \varphi^i, u_i - u^*).$$

This inequality is briefly denoted as $J_1 + J_2 \le J_3 + \cdots + J_8$. We successively estimate all terms. First we have

$$\begin{aligned} J_1 &= \sum_{i=1}^{j}(b_n(u_i) - b_n(u_{i-1}), u_i) \\ &= (b_n(u_j), u_j) - (b_n(u_0), u_0) - \sum_{i=1}^{j}(b_n(u_{i-1}), u_i - u_{i-1}) \\ &\ge (b_n(u_j), u_j) - (b_n(u_0), u_0) - \sum_{i=1}^{j}\int_\Omega\int_{u_{j-1}}^{u_j} b_n(z)\mathrm{d}z \\ &\ge \int_\Omega B_n(u_j)\mathrm{d}x - \int_\Omega B_n(u_0)\mathrm{d}x, \end{aligned} \tag{14}$$

where $B_n(s) = sb_n(s) - \int_0^s b_n(z)\mathrm{d}z$. From hypotesis (H_2) and from Young's inequality, ($ab \le \delta\frac{a^2}{2} + \frac{1}{2\delta}b^2$, $\delta > 0$ arbitrary), we can estimate

$$J_2 \ge C\sum_{i=1}^{j}\|\nabla u_i\|_0^2\tau, \quad |J_3| \le \beta\|u_j\|^2 - C_\beta,$$
$$|J_4| \le \beta\sum_{i=1}^{j}\|\nabla u_i\|_0^2 + C_\beta, \quad |J_5| \le C_1 + C_2\sum_{i=1}^{j}\|u_i\|_0^2\tau.$$

To estimate the boundary term we use (H_4), the continuous imbedding $V \hookrightarrow L_2(\partial\Omega)$ and the inequality, (see [26]),

$$\|v\|_{\partial\Omega}^2 \le \varepsilon\|\nabla v\|_0^2 + C_\varepsilon\|v\|_0^2, \quad \forall v \in W_2^1, \quad \varepsilon > 0 \text{ arbitrary}. \tag{15}$$

Then we obtain

$$|J_6| \le \varepsilon\sum_{i=1}^{j}\|\nabla u_i\|_0^2\tau + C_\varepsilon\sum_{i=1}^{j}\|u_i\|_0^2\tau + C.$$

Moreover, we readily get

$$|J_7| \le C_1 + C_2\sum_{i=1}^{j}\|u_i\|_0^2\tau.$$

The critical point is to estimate J_8. For this purpouse we use the formula

$$\frac{u_{i-1} - u_{i-1} \circ \varphi^i}{\tau} = \int_0^1 \nabla\widetilde{u}_{i-1}(x + s(\varphi^i(x) - x))\mathrm{d}s \cdot \omega_h * \partial_u\bar{F}(x, u_{i-1}). \tag{16}$$

Since $\partial_u \bar{F}(x, u_{i-1})$ is bounded, (see (H_3)), $\omega_h * \partial_u \bar{F}(x, u_{i-1})$ is bounded too. Now we use (11) and Young's inequality to obtain

$$|J_8| \le \beta \sum_{i=1}^{j} \|\nabla u_i\|_0^2 \tau + C_\beta \sum_{i=1}^{j} \|u_i\|_0^2 \tau + C.$$

The asymptotic properties of $b_n(s)$ guarantee that

$$\gamma s^2 - C_2 \le B_n(s) \le C_1 s^2 + C_2, \tag{17}$$

where γ is from hypotesis (H_1). Consequently we get from (14)

$$J_1 \ge \varepsilon \|u_j\|_0^2 - C. \tag{18}$$

Substituting the estimates for $J_1, ..., J_8$ in the inequality $J_1 + J_2 \le J_3 + ... + J_8$ and using Gronwall's argument we conclude that

$$\max_{1 \le i \le n} \|u_i\|_0 \le C, \quad \max_{1 \le i \le n} \int_\Omega B_n(u_i) \mathrm{d}x \le C, \quad \sum_{i=1}^{j} \|u_i\|^2 \tau \le C. \tag{19}$$

Next we put $v = u_{i-1}$ into (6) and sum up for $i = 1, \dots, j$. We use the decomposition (13) and (16). In addition we apply the *a priori* etimates (19). This leads to

$$\sum_{i=1}^{j} \left(\frac{b_n(u_i) - b_n(u_{i-1})}{\tau}, \frac{u_i - u_{i-1}}{\tau} \right) \tau + \sum_{i=1}^{j} (\mathbf{A} \nabla u_i, \nabla(u_i - u_{i-1})) + \sum_{i=1}^{j} (g(t, u_{i-1}), u_i - u_{i-1})_{\Gamma_2}$$
$$\le \varepsilon \sum_{i=1}^{j} \|\delta u_i\|_0^2 \tau + C_\varepsilon \sum_{i=1}^{j} \|u_i\|^2 \tau + \tau \sum_{i=1}^{j} \|\delta u_i\|_0^2 \tau. \tag{20}$$

For the 2nd term of the LHS, we use the symmetry of $\mathbf{A}$ to get

$$\sum_{i=1}^{j} (\mathbf{A} \nabla u_i, \nabla(u_i - u_{i-1})) \ge \frac{1}{2} (\mathbf{A} \nabla u_j, \nabla u_j) - \frac{1}{2} \sum_{i=1}^{j} (\mathbf{A} \nabla u_0, \nabla u_0) + \frac{1}{2} (\mathbf{A} \nabla (u_i - u_{i-1}), \nabla (u_i - u_{i-1}))$$
$$\ge C_e (\|\nabla u_j\|_0^2 + \sum_{i=1}^{j} \|\nabla (u_i - u_{i-1})\|_0^2) - C_2. \tag{21}$$

There exists $L > 0$ such that $\tilde{g}(t, x, s) := g(t, x, s) - Ls$ is decreasing in s. Then due to hypothesis (H_4) we can estimate

$$\sum_{i=1}^{j} (\tilde{g}(t, u_{i-1}), u_i - u_{i-1})_{\Gamma_2} \ge \int_{\Gamma_2} [G(t_j, u_j) - G(0, u_0)] \mathrm{d}x - C \sum_{i=1}^{j} \|u_i\|_0^2 \tau,$$

where $G(t, x, s) = \int_0^s \tilde{g}(t, x, z) \mathrm{d}z$. Moreover, we have

$$\sum_{i=1}^{j} (u_{i-1}, u_i - u_{i-1})_{\Gamma_2} = \frac{1}{2} \|u_j\|_{\Gamma_2}^2 - \frac{1}{2} \|u_0\|_{\Gamma_2}^2 - \frac{1}{2} \sum_{i=1}^{j} \|u_i - u_{i-1}\|_{\Gamma_2}^2.$$

Estimating the boundary terms we further use (15) and (19) to obtain

$$\sum_{i=1}^{j}(g(t,u_{i-1}),u_i-u_{i-1})_{\Gamma_2} \geq$$
$$\varepsilon\|\nabla u_j\|_0^2 - C_\varepsilon\|u_j\|_0^2 - C\sum_{i=1}^{j}\|u_i\|_0^2\tau - C\sum_{i=1}^{j}\|\nabla u_i\|_0^2\tau - \varepsilon\sum_{i=1}^{j}\|\nabla(u_i-u_{i-1}\|_0^2 - C_\varepsilon\tau\sum_{i=1}^{j}\|\delta u_i\|_0^2\tau - C_\varepsilon \geq$$
$$\varepsilon\|\nabla u_j\|_0^2 - C_\varepsilon\tau\sum_{i=1}^{j}\|\delta u_i\|_0^2\tau - C_\varepsilon. \quad (22)$$

We invoke (21) and (22) in (20). Then from (H_1) and from Gronwall's argument we deduce the last estimate in Lemma 2. □

A subsequence of $\{n\}$ is denoted again by $\{n\}$.

Lemma 3. *$\{\bar{u}^n\}_1^\infty$ and $\{b_n(\bar{u}^n)\}_1^\infty$ are compact in $L_2(Q_T)$. There exists $u \in L_2(I,K)$ such that $\bar{u}^n \to u$, $b_n(\bar{u}^n) \to b(u)$ in $L_2(I,L_2)$ and $\bar{u}^n \rightharpoonup u$ in $L_2(I,V)$. Moreover, $\partial_t u \in L_2(I,L_2)$ and $\partial_t u^n \rightharpoonup \partial_t u$ in $L_2(I,L_2)$.*

Proof. We define

$$\rho(s) = \min\{b'(s),\mu\} \quad \text{and} \quad W(s) = \int_0^s \rho(z)\mathrm{d}z.$$

From (H_1) we have that $b_n'(s) \geq \gamma > 0$ and

$$\left(\frac{b_n(u_i)-b_n(u_{i-1})}{\tau}, \frac{u_i-u_{i-1}}{\tau}\right) \geq \gamma\|\delta(u_i)\|_0^2.$$

Then, choosing ε sufficiently small in the last estimate in Lemma 2, we get the *a priori* estimate

$$\sum_{i=1}^{n}\|\delta u_i\|_0^2\tau \leq C.$$

This estimate and $(19)_3$ can be respectively rewritten in the form

$$\int_I \|\partial_t u^n\|_0^2\,\mathrm{d}t \leq C, \quad \int_I \|\bar{u}^n\|^2\,\mathrm{d}t \leq C, \quad (23)$$

which implies the compactness of $\{\bar{u}^n\}$ in $L_2(I,L_2)$ because of Rellich's compactness argument ($W_2^1 \hookrightarrow L_2$) – see [26]. Consequently, there exists a subsequence of $\{\bar{u}^n\}_{n=1}^\infty$ and a function $u \in L_2(I,L_2)$ such that $\bar{u}^n \to u$ in $L_2(I,L_2)$ (a subsequence of $\{\bar{u}^n\}$ is denoted again by $\{\bar{u}^n\}$). From the first estimate in (23) we deduce that

$$\int_I \|u^n-\bar{u}^n\|_0^2\,\mathrm{d}t \leq \frac{C}{n} \quad \text{and} \quad \partial_t u^n \rightharpoonup \chi \quad \text{in } L_2(I,L_2).$$

The 1st inequality implies that also $u^n \to u$ in $L_2(I,L_2)$. Thus we find that $\chi \equiv \partial_t u$. From the asymptotic properties of b_n, (see (H_1), (part (ii)) we have

$$C_2(|\bar{u}^n|-1) \leq b_n(\bar{u}^n) \leq C_1(1+|\bar{u}^n|),$$

which implies that $b_n(\bar{u}^n) \to b(u)$ in $L_2(I,L_2)$. The weak convergence $\bar{u}^n \rightharpoonup u$ in $L_2(I,V)$ is a consequence of Lemma 2. Since $L_2(I,K)$ is a closed convex subset of $L_2(I,V)$, we have that $u \in L_2(I,K)$. Thus Lemma 3 is proved. □

Now we can formulate our main result. By $\{\bar{u}^{\bar{n}}\}$ we denote a suitable subsequence of $\{\bar{u}^n\}$.

Theorem 1. *If the assumptions (H_1)–(H_5) are satisfied, then there exists a weak variational solution u to the problem (1) in the sense of Definition 1. Moreover, $\bar{u}^{\bar{n}} \to u$ in $L_2(I, L_2)$, $\bar{u}^{\bar{n}} \rightharpoonup u$ in $L_2(I, V)$, where $\{\bar{u}^n\}$ is from (6, 7) and (12). If the weak variational solution is unique, then the original sequence $\{\bar{u}^n\}$ is convergent.*

Proof. We rewrite (6) in the form

$$(\partial_t \widetilde{b}_n(\bar{u}^n), v - \bar{u}^n) + (\bar{\chi}^n(\bar{u}^n - \bar{u}^n_\tau), v - \bar{u}^n) + (\mathbf{A}\nabla\bar{u}^n, \nabla(v - \bar{u}^n)) + (\bar{g}^n(t, \bar{u}^n_\tau), v - \bar{u}^n)_{\Gamma_2} \geq$$
$$(\bar{H}^n(t, \bar{u}^n_\tau), v - \bar{u}^n) - \left(\frac{1}{\tau}(\bar{u}^n_\tau - \bar{u}^n_\tau \circ \bar{\varphi}^n), v - \bar{u}^n\right) \quad \forall v \in L_2(I, K), \quad (24)$$

where

$$\bar{u}^n_\tau := \bar{u}^n(t - \tau) \quad (\bar{u}^n(s) = u_0 \text{ for } s \in (-\tau, 0)),$$

and where

$$\tilde{b}_n(\bar{u}^n) = b_n(\bar{u}^n) + \frac{t - t_{i-1}}{\tau}(b_n(\bar{u}^n) - b_n(\bar{u}^n_\tau)) \text{ for } t \in (t_{i-1}, t_i), i = 1, \ldots, n,$$

and

$$\bar{g}^n(t, s) = g(t_i, s) \text{ for } t \in (t_{i-1}, t_i), i = 1, \ldots, n.$$

Similary, we introduce $\bar{H}^n, \bar{F}^n(t, \bar{u}^n_\tau)$ and $\bar{\varphi}^n$. We first integrate (24) over $(0, t)$. The first term leads to

$$\begin{aligned} J_n(t) &\equiv \int_0^t (\partial_t(\tilde{b}_n(\bar{u}^n) - \tilde{b}_n(u_0)), v - u^n)\mathrm{d}z \\ &= (\tilde{b}_n(\bar{u}^n(t) - b_n(u_0), v(t) - u(t)) \\ &\quad - \int_0^t (\tilde{b}_n(\bar{u}^n(z)) - b_n(u_0), \partial_t(v - u^n)\mathrm{d}z. \end{aligned}$$

From Lemma 3 we deduce that

$$J_n(t) \to (b(u(t)) - b(u(0)), v(t) - u(t)) - \int_0^t (b(u) - b(u_0), \partial_t(v - u))\mathrm{d}t. \quad (25)$$

To this end notice that

$$\begin{aligned} \left|\int_0^t (\partial_t(\tilde{b}_n(\bar{u}^n) - \tilde{b}_n(u_0), \bar{u}^n - u^n)\mathrm{d}t\right| &\leq 2\int_I \|b_n(\bar{u}^n) - b_n(\bar{u}^n_\tau)\|_0 \, \|\partial_t u^n\|_0 \,\mathrm{d}t \\ &\leq C\left(\int_I \|b_n(\bar{u}^n) - b_n(\bar{u}^n_\tau)\|_0^2 \,\mathrm{d}t\right)^{1/2} \to 0 \quad \text{for } n \to \infty, \end{aligned}$$

since $\bar{u}^n \to u$ and $\bar{u}^n_\tau \to u$ in $L_2(I, L_2)$.

Similarly we obtain for the 2nd term in (24)

$$\int_0^t (\bar{\chi}^n(\bar{u}^n - \bar{u}^n_\tau), \bar{u}^n - \bar{u}^n_\tau)\mathrm{d}t \to 0 \quad \text{for } n \to \infty. \quad (26)$$

From $\bar{u}^n \rightharpoonup u$ in $L_2(I, V)$ it follows that

$$\int_0^t (\mathbf{A}\nabla\bar{u}^n, \nabla v)\mathrm{d}t \to \int_0^t (\mathbf{A}\nabla u, \nabla v)\mathrm{d}t \quad (27)$$

and also

$$\underline{\lim} \int_0^t (\mathbf{A}\nabla \bar{u}^n, \nabla \bar{u}^n)\mathrm{d}t \geq \int_0^t (\mathbf{A}\nabla u, \nabla u)\mathrm{d}t, \tag{28}$$

since $\int_0^t (\mathbf{A}\nabla w, \nabla w)\mathrm{d}t$ is convex in w and consequently lower semicontinuous.

From $\bar{u}_\tau^n \to u$ in $L_2(I, L_2)$ and from $\bar{u}_\tau^n \rightharpoonup u$ in $L_2(I, V)$ we have that $\bar{u}_\tau^n \to u$ in $L_2(I, L_2(\partial\Omega))$ – see (14) or [21]. Then, we have for the boundary term in (24)

$$\int_0^t (\bar{g}^n(t, \bar{u}_\tau^n), v - \bar{u}^n)_{\Gamma_2}\mathrm{d}t \to \int_0^t (g(t, u), v - u)_{\Gamma_2}\mathrm{d}t. \tag{29}$$

Similarly, we obtain

$$\int_0^t (\bar{H}^n(t, \bar{u}^n), v - \bar{u}^n)\mathrm{d}t \to \int_0^t (f(t, u) - \mathrm{div}_x\ \bar{F}(x, u), v - u)\mathrm{d}t. \tag{30}$$

To take the limit $n \to \infty$ in the last term of (24) we use the formula (16) and denote

$$\nabla \bar{w}_\tau^n := \int_0^1 \nabla \widetilde{u}_\tau^n(x + s(\bar{\varphi}^n(x) - x))\mathrm{d}s.$$

Due to (11) and Lemma 2, $\{\nabla \bar{w}_\tau^n\}_0^\infty$ is bounded in $L_2(I, V)$. Thus $\nabla \bar{w}_\tau^n \rightharpoonup \psi$ in $L_2(I, L_2)$. From

$$\begin{aligned} \bar{w}_\tau^n - \bar{u}_\tau^n &= \int_0^1 (\widetilde{u}_\tau^n(x + s(\bar{\varphi}^n(x) - x) - \bar{u}_\tau^n)\mathrm{d}s \\ &= \int_0^1 \int_0^1 \nabla \widetilde{u}_\tau^n(x + sr(\bar{\varphi}^n(x) - x)\mathrm{d}s\,\mathrm{d}r.\ \omega_h * \partial_u \bar{F}^n(x, \bar{u}_\tau^n)\tau \end{aligned}$$

(for a.e. x) we deduce that

$$\int_I \|\bar{w}_\tau^n - \bar{u}_\tau^n\|_0^2\,\mathrm{d}t \leq C\|\bar{u}^n\|_{L_2(I,V)}^2\tau \leq C\tau \to 0 \quad \text{for} \quad n \to \infty$$

since $\|\omega_h * \partial_u \bar{F}(x, \bar{u}_\tau^n)\|_\infty \leq C$.

Thus, $\bar{w}_\tau^n \to u$ in $L_2(I, L_2)$. Recalling that $\nabla \bar{w}_\tau^n \rightharpoonup \psi$ in $L_2(I, L_2)$ we have $\psi \equiv \nabla u$. From Lemma 3 and hypothesis (H_3) we obtain $\partial_u \bar{F}(x, \bar{u}_\tau^n) \to \partial_u F(x, u)$ in $L_2(I, L_2)$ and, consequently,

$$\omega_h * \partial_u \bar{F}(x, \bar{u}_\tau^n) \to \partial_u \bar{F}(x, u) \quad \text{a.e in } \Omega \times I.$$

Therefore,

$$\left[\omega_h * \partial_u \bar{F}(x, \bar{u}_\tau^n)\right](v - \bar{u}^n) \to \partial_u \bar{F}(x, u)(v - u) \quad \text{in } L_2(I, L_2)$$

and

$$\int_0^t (\bar{u}_\tau^n - \bar{u}_\tau^n \circ \varphi^n, v - \bar{u}^n) \to \int_0^t (\partial_u \bar{F}(x, u) \cdot \nabla u, v - u)\mathrm{d}t. \tag{31}$$

Combining the limit results (25)–(31) in the inequality (24) we conclude that the function u from Lemma 3 satisfies the inequality

$$(b(u(t)) - b(u_0), v(t) - u(t)) - \int_0^t (b(u) - b(u_0), \partial_t(v - u))\mathrm{d}z + \int_0^t (\mathbf{A}\nabla u, \nabla(v - u)) + (\operatorname{div} \bar{F}(x, u), v - u) \geq \int_0^t (f(s, u(s)), v(s)) - u(s))\}\mathrm{d}s \quad \text{a.e. } t \in I \quad \text{and} \quad \forall v \in L_2(I, K).$$

Hence we conclude that u is a weak variational solution to (1). To show that $\bar{u}^n \to u$ in $L_2(I, V)$ we proceed in the following way. We put $v = u(t)$ in (24) and integrate over $(0, t)$. Since $\partial_t u^n \rightharpoonup \partial_t u$ and $\tilde{b}_n(\bar{u}_n) \to b(u)$ in $L_2(I, L_2)$ we find by the same arguments as before (see (25)) that $J_n \to 0$ for $n \to \infty$. Next, the elliptic part gives

$$\int_0^t (\mathbf{A}\nabla \bar{u}^n, \nabla(\bar{u}^n - u)) = \int_0^t (\mathbf{A}\nabla(\bar{u}^n - u), \nabla(\bar{u}^n - u)) - C_n \geq C_e \int_0^t \|\nabla(\bar{u}^n - u)\|_0^2 \,\mathrm{d}t - C_n,$$

with

$$C_n = \int_0^t (\mathbf{A}\nabla u, \nabla(\bar{u}^n - u)) \to 0 \quad \text{for } n \to \infty$$

since $\bar{u}^n \rightharpoonup u$ in $L_2(I, V)$.

The remaining terms in (24) converge to 0 since $\bar{u}^n, \bar{u}^n_\tau \to u$ in $L_2(I, L_2)$)and also in $L_2(I, L_2(\Gamma_2))$. Sumarizing the above arguments, we obtain $\bar{u}^n \to u$ in $L_2(I, V)$ and the proof is complete. □

A stronger variational solution can be obtained when b is Lipschitz continuous. In that case we do not regularize b by b_n and in (6) and (7) we consider $b_n \equiv b$.

Theorem 2. *Let the assumptions of Theorem 1, except of parts (i)–(iv) in hypothesis (H_1), be satisfied. If, additionally, b is Lipschitz continuous, then there exists a variational solution $u \in L_2(I, V)$, with $\partial_t b(u) \in L_2(I, L_2)$ which satisfies (1). The convergence results of Theorem 1 hold and, moreover, $\partial_t b(\bar{u}^n(t)) \rightharpoonup b(u)$ in $L_2(I, L_2)$ where $\bar{u}^n$ is from (6, 7) and (12), with $b_n(s) \equiv b(s)$. If the variational solution is unique, then the original sequences $\{\bar{u}^n\}$, $\{b_n(\bar{u}^n)\}$ are convergent.*

Proof. We obtain the same *a priori* estimates as in Lemma 2 and compactness result as in Lemma 3. From the last estimate of Lemma 2 we obtain

$$\sum_{i=1}^n \|\frac{b(u_i) - b(u_{i-1})}{\tau}\|_0^2 \tau \leq C \sum_{i=1}^n \left(\frac{b(u_i) - b(u_{i-1})}{\tau}, \frac{u_i - u_{i-1}}{\tau}\right) \tau \leq C.$$

Then, $\partial_t \widetilde{b}_n(\bar{u}^n) \rightharpoonup \chi$ in $L_2(I, L_2)$. By the same arguments as in Lemma 3 we have $b_n(\bar{u}^n) \to b(u)$, $\bar{u}^n \to u$ in $L_2(I, L_2)$. From the estimate

$$\sum_{i=1}^n \left\|\widetilde{b}_n(\bar{u}^n) - b_n(\bar{u}^n)\right\|_0^2 \tau \leq 2\tau \sum_{i=1}^n \left\|\frac{b(u_i) - b(u_{i-1})}{\tau}\right\|_0^2 \tau \to 0$$

we deduce that $\widetilde{b}_n(\bar{u}^n) \to b(u)$, which, together with the weak convergence of $\partial_t \widetilde{b}_n(\bar{u}^n)$, implies that $\chi \equiv \partial_t b(u)$. This readily leads to

$$\int_t^{t+z} \left(\partial_t \widetilde{b}_n(\bar{u}^n), v - \bar{u}^n\right) \mathrm{d}t \to \int_t^{t+z} (\partial_t b(u), v - u)\mathrm{d}t \quad \forall t \in I$$

for any $0 < z \leq z_0$. Next, we integrate (24) over $(t, t+z)$ and take the limit for $n \to \infty$ and conclude that $u \in L_2(I, K)$ satisfies (1) for a.e. $t \in I$. The rest of the proof is the same as in Theorem 1. □

Remark 3. The uniqueness results for (1) are discussed in [1] and [27] under some additional structural assumptions on $\bar{F}(x, u)$.

Remark 4. The results of Theorem 1 and Theorem 2 also hold in the case that approximated characteristics $\widetilde{\varphi}^i$ are considered. One can verify that the problem (2) for unsaturated-saturated flow, governed by Richard's equation expressed in terms of Van Genuchten-Mualem's model in the unsaturated zone and extended to (4) for the saturated zone, satisfies the assumptions made in Theorem 1.

4. Full discretization scheme

The approximation $\{\bar{u}^n\}$ of (1) have been obtained by means of u_i, $i = 1, \dots, n$, which are determined by a linear elliptic variational inequality (6). The numerical realization of (6) can be performed by projection on finite dimensional spaces, *cf.* [14].

In the case of variational equations, the space V is approximated by a finite dimensional subspace $V_\lambda \subset V$, such that $V_\lambda \to V$ in canonical sense, for $\lambda \to 0$, *e.g.* by using finite element spaces. In our variational problem we have to approximate K by a finite dimensional space K_λ. We shall assume (see [14])

(H_6) K_λ is a closed convex set in V_λ and the following conditions are satisfied:
- (i) $\forall v \in K, \exists v_\lambda \in K_\lambda$ with $v_\lambda \to v$ in V, for $\lambda \to 0$
- (ii) If $u_\lambda \in K_\lambda$ and $u_\lambda \rightharpoonup u$ in V for $\lambda \to 0$, then $u \in K$.

We can easily verify that if $K_\lambda = K \cap V_\lambda$, ($K \equiv \{v \in V : V \geq 0\}$, in our motivating problem of Section 1, then the assumption (H_6) is satisfied. Now, let $u_i^\lambda \in K_\lambda$ be the solution of the following variational inequality (see (6))

$$\frac{1}{\tau}\left(\lambda_i\left(u_i^\lambda - u_{i-1}\right), v - u_i^\lambda\right) + \left(\mathbf{A}\nabla u_i^\lambda, \nabla\left(v - u_i^\lambda\right)\right) + \left(g\left(t, u_{i-1}^\lambda\right), v - u_i^\lambda\right)_{\Gamma_2} \geq \left(H\left(t_i, u_{i-1}^\lambda\right), v - u_i^\lambda\right) - \left(u_{i-1}^\lambda - u_{i-1}^\lambda \circ \varphi_\lambda^i, v - u_i^\lambda\right), \quad \forall v_\lambda \in K_\lambda,$$

where $\varphi_\lambda^i := x - \tau\omega_h * \partial_u \bar{F}(x, u_{i-1}^\lambda)$.

By means of u_i^λ, $(i = 1, \dots, n)$, we construct the finite dimensional Rothe-function $\bar{u}_\alpha(t)$, where $\alpha = (\tau, \lambda)$. Explicitly,

$$\bar{u}_\alpha(t) = u_i^\lambda \quad \text{for } t \in (t_{i-1}, t_i), i = 1, \dots, n; \ \bar{u}_\alpha(0) = u_0^\lambda \in K_\lambda.$$

Similary we introduce $u_\alpha(t)$. Now we will prove the convergence $\bar{u}_\alpha \to u$ in the corresponding functional spaces when $\alpha \to 0$, where u is a weak variational solution to (1) (in the sense of Def. 1). Let $\{\bar{\alpha}\}$ denote a subsequence of $\{\alpha\}$, $\alpha \to 0$.

Theorem 3. *Let the assumptions (H_1)–(H_6) be satisfied and let $u_0^\lambda \to u_0$ in $L_2(\Omega)$ for $\lambda \to 0$. Then $\bar{u}_{\bar{\alpha}} \to u$ in $L_2(I, V)$ and $b_n(\bar{u}_{\bar{\alpha}}) \to b(u)$ in $L_2(I, L_2)$ for $\alpha \to 0$, where u is a weak variational solution to (1). If the variational solution is unique, then the original sequences are converging.*

Proof. We can use all *a priori* estimates for $\{\bar{u}_\alpha\}$ that we have obtained in Section 3 for $\{\bar{u}^n\}$. We have $\bar{u}_\alpha(t) \to u(t)$ for a.e. $t \in I$ and $\bar{u}_\alpha \rightharpoonup u$ in $L_2(I, V)$. We must verify that $u \in L_2(I, K)$. From the *a priori* estimates we have that $\|\bar{u}_\alpha(t)\| \leq C$ for all $t \in I$. Then, at a fixed t we can choose a subsequence $\{\bar{\alpha}\}$ of $\{\alpha\}$ so that $\bar{u}_{\bar{\alpha}}(t) \rightharpoonup w_t$ in V. On the other hand $\bar{u}_{\bar{\alpha}}(t) \to u(t)$ in L_2 for a.e. $t \in I$. Hence, $w_t \equiv u(t)$. From this it follows that the original sequence $\bar{u}_\alpha(t)$ weakly converges to $u(t)$ in V for a.e. $t \in I$. Then, $u(t) \in K$ because of (H_6), (part (ii)). Thus, $u \in L_2(I, K)$. Now, we can follow the proof of Theorem 1. For any $v \in L_2(I, K)$ we can construct a function $v_\lambda \in L_2(I, K_\lambda)$ so that $v_\alpha \to v$ in $L_2(I, V)$ – because of (H_6), (part (i)). Next, we use a test function v_α (in the place of v in (6)) and take the limit $\alpha \to 0$. We obtain that u is a weak variational solution to (1). To show the stronger convergence of $\{\bar{u}_\alpha\}$ we follow the proof of Theorem 1.

Again, in the place of $v = u$, we take the test function $v = v_\alpha \in L_2(I, K_\lambda)$ in (6) such that $v_\alpha \to u$ in $L_2(I, V)$. We find that

$$\int_I \|\nabla(\bar{u}_\alpha - v_\alpha)\|_0^2 \,\mathrm{d}t \to 0 \quad \text{for} \quad \alpha \to 0.$$

From this convergence and from the convergences $\bar{u}_\alpha \to u$ in $L_2(I, L_2)$ and $v_\alpha \to u$ in $L_2(I, V)$ for $\alpha \to 0$, we conclude that $\bar{u}_\alpha \to u$ in $L_2(I, V)$. Thus the proof is complete. □

Remark 5. In the present paper we have focused on the convergence analysis of the approximation method. The practical implementation of the presented method, which is based on the concept of regularized characteristics, may proceed similarly as for the numerical experiments in [22]. The difference is that now at every time section $t = t_i, i = 1, ..., n$ an elliptic variational inequality has to be solved by FEMs instead of an elliptic variational equality – however the transport parts remain the same. The crucial point in the implementation of the method of characteristics is the evaluation of the storativity integrals $(u_{i-1} \circ \varphi_i, v)$, which can lead to instabilities and to the violation of the mass balance. In [22] we have applied the method introduced in [7, 8]. The weak point is the preservation of mass, especially for larger time steps.

Another efficient method which is based on the method of characteristics and which is mass preserving is the ELLAM method, analysed in [11, 29]. Recently, two additional methods were developed. In the first one – see [16] – the transport part too is realized by means of splitting into directions parallel to the coordinate axes. In the second one – see [13] – "a flux-based method of characteristics" is introduced for FVMs using a general (unstructured) domain decomposition.

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