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ON A STABILIZED COLOCATED FINITE VOLUME SCHEME FOR THE STOKES PROBLEM

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Abstract. We present and analyse in this paper a novel colocated Finite Volume scheme for the solution of the Stokes problem. It has been developed following two main ideas. On one hand, the discretization of the pressure gradient term is built as the discrete transposed of the velocity divergence term, the latter being evaluated using a natural finite volume approximation; this leads to a non-standard interpolation formula for the expression of the pressure on the edges of the control volumes. On the other hand, the scheme is stabilized using a finite volume analogue to the Brezzi-Pitkäranta technique. We prove that, under usual regularity assumptions for the solution (each component of the velocity in $H^2(\Omega)$ and pressure in $H^1(\Omega)$), the scheme is first order convergent in the usual finite volume discrete H^1 norm and the L^2 norm for respectively the velocity and the pressure, provided, in particular, that the approximation of the mass balance flux is of second order. With the above-mentioned interpolation formulae, this latter condition is satisfied only for particular meshes: acute angles triangulations or rectangular structured discretizations in two dimensions, and rectangular parallelepipedic structured discretizations in three dimensions. Numerical experiments confirm this analysis and show, in addition, a second order convergence for the velocity in a discrete L^2 norm.

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1. Introduction

Finite volumes enjoy many favorable properties for the discretization of conservation equations: to cite only a few examples, (computing time) efficiency, local conservativity and the possibility to hand-build, to some extent, the discrete operators to recover properties of the continuous problem like, for instance, the positivity of the advection operator. In addition, the complexity raised by the design of actually high order schemes for multidimensional problems with finite volumes discretizations, in particular compared to finite elements methods, may often be of little concern in real-life applications, when the regularity which can be expected for the solution is rather poor. These features make finite volume attractive for industrial problems, as encountered for instance in nuclear safety, which is a part of the context of this study.

Keywords and phrases. Finite volumes, colocated discretizations, Stokes problem, incompressible flows, analysis.

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On the opposite, the difficulty to build stable pairs of discretization for the velocity and pressure in incompressible flow problems remains, to our opinion, a severe drawback of the method. In most applications, this stability is obtained by using a staggered arrangement for the velocity and pressure unknowns: the celebrated MAC scheme (see [13] for the pioneering work and [16, 17] for an analysis). Although this discretization has proved its low cost and reliability, it turns to be difficult to handle from a programming point of view: rather intricate treatment of particular boundary conditions, as inner corners for instance, difficulty to deal with general computational domains, to implement multilevel local refinement techniques, etc. For this reason, some schemes making use of discretizations where the degrees of freedom for the velocity components and pressure are seen as an approximation of the continuous solution at the same location (or as an average of the continuous solution over the same control volume) have been proposed in the last two decades [11, 18, 19, 21]. These discretizations are referred to as "colocated".

The purpose of this paper is to present and analyse a novel colocated scheme for the Stokes problem, which has the following essential features. On one hand, the discretization of the pressure gradient term is designed to be the discrete transposed operator of the velocity divergence term, the latter being evaluated using a natural finite volume approximation. On the other hand, the scheme is stabilized using a finite volume analogue of the Brezzi-Pitkäranta technique. We choose to restrict ourselves here to the linear case (*i.e.* the Stokes problem), for the sake of readability, and to specific meshes for which an optimal order of convergence can be proven. An extension of these results to the Navier-Stokes equations and to general meshes can be found in [8].

The outline of the paper is as follows: we first present the scheme under consideration, then the next section is devoted to establish consistency error estimates which will be used to prove the convergence result (Sect. 4); finally, we show some numerical experiments which substantiate the theoretical analysis.

2. The continuous problem and the discrete scheme

2.1. The Stokes problem

For the sake of simplicity, we restrict the presentation to homogeneous Dirichlet boundary conditions and regular right-hand sides. In the so-called strong or differential form, the problem under consideration reads:

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a polygonal open bounded connected subset of $\mathbb{R}^d$, $d = 2$ or 3, $\partial\Omega$ stands for its boundary, f is a function of $\mathrm{L}^2(\Omega)^d$, u and p are respectively a vector valued (*i.e.* taking values in $\mathbb{R}^d$) and a scalar (*i.e.* taking values in $\mathbb{R}$) function defined over Ω.

The weak solution of (1) (see *e.g.* [12]) (u, p) is the unique solution in $\mathrm{H}^1_0(\Omega)^d \times \mathrm{L}^2(\Omega)$ of:

$$\begin{cases} \displaystyle\int_\Omega \nabla u : \nabla v - \int_\Omega p\ \nabla \cdot v = \int_\Omega f \cdot v & \forall v \in \mathrm{H}^1_0(\Omega)^d \\ \displaystyle\int_\Omega q\ \nabla \cdot u = 0 & \forall q \in \mathrm{L}^2(\Omega) \\ \displaystyle\int_\Omega p = 0. & \end{cases} \tag{2}$$

2.2. Discretization and discrete functional spaces

2.2.1. *Admissible discretizations*

The finite volume discretizations of the polygonal domain Ω which are considered here consist in a finite family $\mathcal{M}$ of disjoint non-empty convex subdomains K of Ω (the "control volumes") such that:

– if $d = 2$, each control volume is either a rectangle or a triangle with internal angles strictly lower than $\pi/2$;
– if $d = 3$, each control volume is a rectangular parallelepiped;
– the discretization is conforming in the sense that two neighbouring control volumes share a complete $(d-1)$-dimensional side, which will be called hereafter an edge of the meshing.

To each control volume K, we associate the following point, denoted by x_K: if $d = 2$, the intersection of the perpendicular bisectors of each edge, if $d = 3$ the intersection of the lines issued from the barycenter of the edge and orthogonal to the edge. Note that, when K is a simplex of $\mathbb{R}^2$, the fact that the interior angles are lower than $\pi/2$ impose that x_K is always located inside the triangle K.

The set of edges is denoted by $\mathcal{E}$. It can be split into the set of internal edges, *i.e.* separating two control volumes, denoted by $\mathcal{E}_{\text{int}}$, and the set of external ones denoted by $\mathcal{E}_{\text{ext}}$. The set $\mathcal{E}(K)$ stands for the set of the edges of the control volume K.

An internal edge separating two control volumes K and L is denoted by $K|L$. The segment $[x_K, x_L]$ is orthogonal to $K|L$ and crosses $K|L$ at $x_{K|L}$. We denote by $d_{K|L}$ the distance between x_K and x_L and $d_{K,K|L}$ the distance between x_K and $x_{K|L}$.

Remark 2.1. The class of admissible meshings is far smaller here than usual for elliptic problems (see [7]). Additional constraints come from the need to suppose in the analysis that $x_{K|L}$ is located at the barycenter of $K|L$, to obtain a second order approximation for the mass fluxes through $K|L$.

We will adopt hereafter the following conventions. The expression:

$$\sum_{\sigma\in\mathcal{E}_{\text{int}}\ (\sigma=K|L)} F(\sigma, K|L, K, L) \qquad \text{or} \qquad \max_{\sigma\in\mathcal{E}_{\text{int}}\ (\sigma=K|L)} F(\sigma, K|L, K, L)$$

will stand for a summation or a maximum taken over the internal edges, the control volumes sharing the edge σ being denoted by K and L. Similarly, the expression:

$$\sum_{\sigma\in\mathcal{E}_{\text{ext}}\ (\sigma\in\mathcal{E}(K))} F(\sigma, K) \qquad \text{or} \qquad \max_{\sigma\in\mathcal{E}_{\text{ext}}\ (\sigma\in\mathcal{E}(K))} F(\sigma, K)$$

will stand for a summation or a maximum taken over the external edges, the unique control volume to which the edge σ belongs being denoted by K. Finally:

$$\sum_{\sigma=K|L} F(\sigma, K|L, K, L)$$

will stand, in a relation related to the control volume K, for a summation over the internal edges σ of K, the second control volume to which σ belongs being denoted by L.

We denote by h_K the diameter of each control volume and h the maximum of the values of h_K, for $K \in \mathcal{M}$. Here and throughout the paper, $m(K)$ and $m(\sigma)$ will stand for, respectively, the d-measure of the control volume K and the $(d-1)$-measure of the edge σ. The regularity of the meshing is quantified by the following set of real numbers:

$$\text{regul}(\mathcal{M}) = \left\{ \begin{array}{l} \displaystyle \max_{\sigma\in\mathcal{E}_{\text{int}}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma\ (h_K+h_L)^{d-2}}, \quad \max_{\sigma\in\mathcal{E}_{\text{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma} h_K^{d-2}}, \\ \displaystyle \max_{K\in\mathcal{M},\ L\in\mathcal{N}(K)} \frac{h_K}{h_L}, \quad \max_{K\in\mathcal{M}} \frac{h_K}{\rho_K} \end{array} \right\} \tag{3}$$

where ρ_K is the diameter of the greatest ball included in K and $\mathcal{N}(K)$ stands for the set of neighbouring control volumes of K.

Throughout the paper, the following notation:

$$F_1 \underset{\text{reg}}{\leq} F_2$$

means that there exists a positive number c, (possibly) depending on Ω and on the meshing only through $\text{regul}(\mathcal{M})$ and being a non-decreasing function of all the four elements of $\text{regul}(\mathcal{M})$, such that:

$$F_1 \leq c\ F_2.$$

2.2.2. *Discrete functional spaces*

Definition 2.2. Let $\mathcal{M}$ be a discretization as described in the preceding section. We denote by $\mathrm{H}_{\mathcal{D}}(\Omega) \subset \mathrm{L}^2(\Omega)$ the space of functions which are piecewise constant over each control volume $K \in \mathcal{M}$. For all $u \in \mathrm{H}_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by u_K the constant value of u in K. The space $\mathrm{H}_{\mathcal{D}}(\Omega)$ is equipped with the following Euclidean structure. For $(u, v) \in (\mathrm{H}_{\mathcal{D}}(\Omega))^2$, we define the following inner product (corresponding to Dirichlet boundary conditions):

$$[u, v]_{\mathcal{D}} = \sum_{\sigma \in \mathcal{E}_{\text{int}}\ (\sigma = K|L)} \frac{m(\sigma)}{d_\sigma}\ (u_L - u_K)(v_L - v_K) + \sum_{\sigma \in \mathcal{E}_{\text{ext}}\ (\sigma \in \mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ u_K\ v_K. \tag{4}$$

Thanks to the discrete Poincaré inequality (7) given below, this scalar product defines a norm on $\mathrm{H}_{\mathcal{D}}(\Omega)$:

$$\|u\|_{1,\mathcal{D}} = [u, u]_{\mathcal{D}}^{1/2}. \tag{5}$$

Definition 2.3. We also define the following different inner product (corresponding to Neumann boundary conditions), together with its associated seminorm:

$$\langle u, v\rangle_{\mathcal{D}} = \sum_{\sigma \in \mathcal{E}_{\text{int}}\ (\sigma = K|L)} \frac{m(\sigma)}{d_\sigma}\ (u_L - u_K)(v_L - v_K) \qquad |u|_{1,\mathcal{D}} = \langle u, u\rangle_{\mathcal{D}}^{1/2}. \tag{6}$$

These definitions naturally extend to vector valued functions as follows. For $u = (u^{(i)})_{i=1,\ldots,d} \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ and $v = (v^{(i)})_{i=1,\ldots,d} \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$, we define:

$$[u, v]_{\mathcal{D}} = \sum_{i=1}^{d} [u^{(i)}, v^{(i)}]_{\mathcal{D}} \qquad \|u\|_{1,\mathcal{D}} = \left(\sum_{i=1}^{d} [u^{(i)}, u^{(i)}]_{\mathcal{D}} \right)^{1/2}.$$

Proposition 2.4. *The following discrete Poincaré inequalities hold (see* [7]*):*

$$\begin{aligned} &\|u\|_{L^2(\Omega)} \leq \text{diam}(\Omega)\ \|u\|_{1,\mathcal{D}} && \forall u \in \mathrm{H}_{\mathcal{D}}(\Omega) \\ &\|u\|_{L^2(\Omega)} \leq C(\Omega)\ |u|_{1,\mathcal{D}} && \forall u \in \mathrm{H}_{\mathcal{D}}(\Omega) \text{ such that } \int_\Omega u = 0, \end{aligned} \tag{7}$$

where $C(\Omega)$ only depends on the computational domain Ω.

In addition, we define a mesh dependent seminorm $|\cdot|_h$ over $\mathrm{H}_{\mathcal{D}}(\Omega)$ by:

$$\forall u \in \mathrm{H}_{\mathcal{D}}(\Omega), \qquad |u|_h^2 = \sum_{\sigma \in \mathcal{E}_{\text{int}}\ (\sigma = K|L)} (h_K^2 + h_L^2)\ \frac{m(\sigma)}{d_\sigma}\ (u_L - u_K)^2.$$

The following inverse inequality holds:

Proposition 2.5. *Let u be a function of* $\mathrm{H}_{\mathcal{D}}(\Omega)$. *Then:*

$$|u|_h \underset{\text{reg}}{\leq} \|u\|_{L^2(\Omega)} \tag{8}$$

Proof. We have:

$$\begin{aligned} \sum_{\sigma\in\mathcal{E}_{\text{int}}\ (\sigma=K|L)} (h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}\,(u_L-u_K)^2 &\leq 2 \sum_{\sigma\in\mathcal{E}_{\text{int}}\ (\sigma=K|L)} (h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}\,(u_L^2+u_K^2) \\ &= 2\sum_{K\in\mathcal{M}} u_K^2 \sum_{\sigma=K|L} (h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}\cdot \end{aligned}$$

Owing to the fact that $\mathrm{card}(\mathcal{E}(K))$ is bounded and $\dfrac{h_L}{h_K}$, $\dfrac{m(\sigma)}{d_\sigma\ (h_K+h_L)^{d-2}}$ and $\dfrac{h_K}{\rho_K}$ are controlled by elements of $\mathrm{regul}(\mathcal{M})$, we get:

$$2\sum_{K\in\mathcal{M}} u_K^2 \sum_{\sigma=K|L} (h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma} \underset{\text{reg}}{\leq} \sum_{K\in\mathcal{M}} \rho_K^d u_K^2$$

which concludes the proof. □

2.3. The finite volume scheme

The finite volume scheme under consideration reads:

Find $u\in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ and $p\in \mathrm{H}_{\mathcal{D}}(\Omega)$ such that for each control volume K of the mesh:

$$\left|\begin{aligned} &\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}(u_L-u_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\text{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\,(-u_K) + \sum_{\sigma=K|L} m(\sigma)\frac{d_{L,\sigma}}{d_\sigma}\,(p_L-p_K)\,n_\sigma = \int_K f \\ &\sum_{\sigma=K|L} m(\sigma)\,\left(\frac{d_{L,\sigma}}{d_\sigma}\,u_K + \frac{d_{K,\sigma}}{d_\sigma}\,u_L\right)\cdot n_\sigma - \lambda \sum_{\sigma=K|L} (h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}(p_L-p_K) = 0 \end{aligned}\right. \tag{9}$$

where λ is a positive parameter and $n_{K|L}$ stands for the normal to the edge $K|L$ oriented from K to L. It is easily seen that this system of equations is singular, the vector of unknowns corresponding to a zero velocity and a constant pressure belonging to the kernel of the associated discrete operator. Consequently, we impose to the solution to fulfill the following constraint:

$$\sum_{K\in\mathcal{M}} m(K)\,p_K = 0.$$

As the functions of $\mathrm{H}_{\mathcal{D}}(\Omega)$ are constant over each control volume, this relation is an exact reformulation of the third relation of (2):

$$\int_\Omega p = 0. \tag{10}$$

On the discrete gradient expression

While the expression of the velocity divergence term is built with a natural interpolation for the velocity on the side σ, the expression of the pressure gradient is not. In fact, the latter is specially constructed to ensure that the discrete gradient is the transposed operator of the discrete divergence, *i.e.*:

$$\forall u \in (\mathrm{H}_{\mathcal{D}}(\Omega))^d, \quad \forall p \in \mathrm{H}_{\mathcal{D}}(\Omega),$$

$$\sum_{K\in\mathcal{M}} p_K \sum_{\sigma=K|L} m(\sigma)\ \left(\frac{d_{L,\sigma}}{d_\sigma}\ u_K + \frac{d_{K,\sigma}}{d_\sigma}\ u_L\right)\cdot n_\sigma = -\sum_{K\in\mathcal{M}} u_K \cdot \sum_{\sigma=K|L} m(\sigma)\frac{d_{L,\sigma}}{d_\sigma}\ (p_L - p_K)\ \ n_\sigma.$$

This property seems to be of crucial importance in the analysis of the stability of the scheme.

Another equivalent expression can be derived for the discrete gradient term. Indeed, using the fact that, for each element K, $\sum_{\sigma\in\mathcal{E}(K)} m(\sigma)\ n_\sigma = 0$, we get:

$$\begin{aligned}\sum_{\sigma=K|L} m(\sigma)\frac{d_{L,\sigma}}{d_\sigma}\ (p_L - p_K)\ n_\sigma &= \sum_{\sigma=K|L} m(\sigma)\frac{d_{L,\sigma}}{d_\sigma}\ (p_L - p_K)\ n_\sigma \\ &\qquad + p_K \left[\sum_{\sigma=K|L} m(\sigma)\ n_\sigma + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} m(\sigma)\ n_\sigma\right] \\ &= \sum_{\sigma=K|L} m(\sigma)\ (\frac{d_{L,\sigma}}{d_\sigma}\ p_L + (1-\frac{d_{L,\sigma}}{d_\sigma})\ p_K)\ n_\sigma + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} m(\sigma)\ p_K\ n_\sigma \\ &= \sum_{\sigma=K|L} m(\sigma)\ (\frac{d_{L,\sigma}}{d_\sigma}\ p_L + \frac{d_{K,\sigma}}{d_\sigma}\ p_K)\ n_\sigma + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} m(\sigma)\ p_K\ n_\sigma.\end{aligned} \tag{11}$$

This last expression can be seen as a rather natural discretization of the integral over the edge σ of the quantity $p\ n_\sigma$, although with a less natural interpolation formula to estimate the pressure on each edge. Note also that this latter formulation is conservative. Both formulations of the discrete gradient are used hereafter.

<u>On the necessity of adding a stabilization term</u>

Let us consider the particular case of a regular meshing of a square two-dimensional domain by square control volumes. It is then easy to see that, without stabilization term, the considered scheme becomes the same as the usual colocated finite volume scheme, which is known to be hardly usable. In particular, the pressure gradient term reads, with the usual notations for the computational molecule:

$$\frac{1}{2}\ h^{d-1}\ \begin{bmatrix} p_{\mathrm{W}} - p_{\mathrm{E}} \\ p_{\mathrm{N}} - p_{\mathrm{S}} \end{bmatrix}.$$

This relation enlights the risk of odd-even decoupling of the pressure (even if checkerboard pressure modes may be filtered out by boundary conditions); this phenomena has indeed been evidenced in numerical experiments.

<u>On the stabilization term</u>

In the particular case of a uniform meshing, the stabilization term is the discrete counterpart of $-2\lambda h^2\Delta p$, which explains why we consider that this stabilization falls into the Brezzi-Pitkäranta family [4]. However, it can also be seen as a sum of jump terms across each internal edge, as classically used for stabilizing finite element discretizations using a discontinuous approximation space for the pressure [14, 22].

Other stabilizations for colocated finite volumes are possible. In particular, Brezzi and Fortin proposed a few years ago the so-called "minimal stabilization procedure" [3] which consists in adding, as in the present work, a stabilization term in the continuity equation, which reads in the framework of a Galerkin numerical scheme:

$$c(p,q) = \int_\Omega \big(p - \Pi_{\bar{Q}_h} p)(q - \Pi_{\bar{Q}_h} q\big)$$

where $p \in Q_h$ is the pressure unknown, $q \in Q_h$ is a test function, $\Pi_{\bar{Q}_h}$ is a projection operator from Q_h onto $\bar{Q}_h$ and $\bar{Q}_h$ is a subset of Q_h such that the pair of discrete spaces obtained by associating the velocity discretization space to $\bar{Q}_h$ is *inf-sup* stable. Later on, Dohrmann and Bochev [6] introduced a stabilized scheme for equal degree (let say k) finite element discretizations of the Stokes problem based of the same relation, choosing for $\bar{Q}_h$ the discontinuous space of piecewise $k-1$ degree polynomials. A finite volume analogue to these procedures is to use for $\bar{Q}_h$ the space of pressures keeping a constant value over clusters of control volumes; this is the line which we follow in ongoing works [9, 10].

3. A quasi-interpolation operator and consistency error bounds

This section is devoted to state and prove the consistency error estimates useful for the analysis. It is written so as to give, as much as possible, a self-consistent presentation of the matter at hand. We thus give a proof of the whole set of relevant estimates, even if some of them can be found in already published literature (in particular [7]). These proofs are new, and rely on the Clément quasi-interpolation technique [5], which presents two advantages: first, they allow some straightforward generalizations (for instance, dealing with cases when the full regularity of the solution is not achieved); second, they make the presentation closer to the literature of some related topics, as for instance *a posteriori* error estimates.

3.1. **Two technical preliminary lemma**

The two following lemmas will be useful in the rest of the paper. The first one is a trace lemma, the proof of which can be found in the case of a simplex in ([23], Sect. 3). Applying a similar technique to parallelepipeds leads to the same relation, with the space dimension d replaced by the (lower) constant $(1+\sqrt{5})/2$.

Lemma 3.1. *Let K be an admissible control volume as defined in Section 2.2.1, h_K its diameter and σ one of its edges. Let u be a function of $\mathrm{H}^1(K)$. Then:*

$$\|u\|_{L^2(\sigma)} \leq \left(d\ \frac{m(\sigma)}{m(K)}\right)^{1/2} \ \left(\|u\|_{L^2(K)} + h_K |u|_{H^1(K)}\right).$$

For vector functions of $\mathrm{H}^1(K)^d$, applying this lemma to each component yields the following estimate:

$$\|u\|_{L^2(\sigma)} \leq \left(2d\ \frac{m(\sigma)}{m(K)}\right)^{1/2} \ \left(\|u\|_{L^2(K)} + h_K |u|_{H^1(K)}\right). \tag{12}$$

Similarly, if $u \in \mathrm{H}^2(K)$, we have:

$$|u|_{H^1(\sigma)} \leq \left(2d\ \frac{m(\sigma)}{m(K)}\right)^{1/2} \ \left(|u|_{H^1(K)} + h_K |u|_{H^2(K)}\right). \tag{13}$$

The second lemma allows to bound the $\mathrm{L}^\infty(K)$ norm of a degree one polynomial by its $\mathrm{L}^2(K)$ norm:

Lemma 3.2. *Let $\mathbb{P}_1$ be the space of linear polynomials and ϕ be an element of $\mathbb{P}_1$. Then there exists a constant $\mathrm{c}_{\infty,2}$ such that:*

$$\|\phi\|_{L^\infty(K)} \leq \mathrm{c}_{\infty,2}\ \frac{1}{m(K)^{1/2}}\ \|\phi\|_{L^2(K)}.$$

Proof. To each type of control volume under consideration, we can associate a reference control volume by an affine mapping: for instance, for two-dimensional simplices, we can choose the triangle of vertices $(0,0)$, $(1,0)$ and $(0,1)$. The vector space of degree one polynomials on the reference control volume is a finite dimensional space, on which the $\mathrm{L}^\infty(K)$ and $\mathrm{L}^2(K)$ norm are equivalent. The result then follows by a change of coordinates in the integral. □

In addition, we will repeatedly use hereafter the following inequality, which is an easy consequence of the Cauchy-Schwarz inequality:

$$\forall u \in \mathrm{L}^2(\omega) \qquad \int_\omega u \leq m(\omega)^{1/2}\ \|u\|_{L^2(\omega)} \tag{14}$$

where ω is (for the cases under consideration here) a polygonal (necessarily bounded) domain of $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$.

3.2. **Definition and properties of a quasi-interpolation operator**

Definition 3.3. Let u be a function in $\mathrm{L}^2(\Omega)$. For each control volume $K \in \mathcal{M}$, we define ω_K as the convex hull of $K \cup (\cup_{L\in\mathcal{N}(K)} L)$. Let $\mathbb{P}_1$ be the space of linear polynomials and $\phi_K \in \mathbb{P}_1$ be defined by:

$$\int_{\omega_K} (u - \phi_K)\ \psi = 0 \qquad \forall \psi \in \mathbb{P}_1. \tag{15}$$

Then we define $\Pi_\mathcal{D} u \in \mathrm{H}_\mathcal{D}(\Omega)$ by $(\Pi_\mathcal{D} u)_K = \phi_K(x_K),\ \forall K \in \mathcal{M}$.
By extension, we will keep the same notation for vector valued functions.

The following estimates hold:

Lemma 3.4. *Let $\bar{h}_K$ be the diameter of ω_K. If u is a function from ω_K to $\mathbb{R}$ and ϕ_K is defined by (15), then:*

$$\begin{aligned}
\textit{if } u \in \mathrm{H}^1(\omega_K)\ : \qquad & \|u - \phi_K\|_{L^2(\omega_K)} && \leq c_{0,1}^{\mathrm{app}}\ \bar{h}_K\ |u|_{H^1(\omega_K)} \\
& |u - \phi_K|_{H^1(\omega_K)} && \leq c_{1,1}^{\mathrm{app}}\ |u|_{H^1(\omega_K)} \\
\textit{if } u \in \mathrm{H}^2(\omega_K)\ : \qquad & \|u - \phi_K\|_{L^2(\omega_K)} && \leq c_{0,2}^{\mathrm{app}}\ \bar{h}_K^2\ |u|_{H^2(\omega_K)} \\
& |u - \phi_K|_{H^1(\omega_K)} && \leq c_{1,2}^{\mathrm{app}}\ \bar{h}_K\ |u|_{H^2(\omega_K)}
\end{aligned}$$

where $c_{0,1}^{\mathrm{app}}$, $c_{1,1}^{\mathrm{app}}$, $c_{0,2}^{\mathrm{app}}$ and $c_{1,2}^{\mathrm{app}}$ only depend on d.

The existence of these constants is due to the independence of the shape of a convex domain of the constants in Jackson's type inequalities, as stated in [24].

Remark 3.5. Let $\mathcal{M}$ be a meshing as described in Section 2.2.1. Each control volume K is intersected by a finite number of domains ω_L, $L \in \mathcal{M}$. This number is known to be bounded by a constant N_ω which can be expressed as a non-decreasing function of the parameters in regul($\mathcal{M}$) (for simplicial discretizations, the number of simplices sharing a vertex is limited by $\max\limits_{K\in\mathcal{M}}(h_K/\rho_K)$, see ([2], Sect. 2)).

In addition, it is easy to see that:

$$\forall K \in \mathcal{M}, \qquad \bar{h}_K \underset{\mathrm{reg}}{\leq} h_K.$$

The continuity of the projection operator $\Pi_\mathcal{D}$ from $\mathrm{H}^1(\Omega)$ to $\mathrm{H}_\mathcal{D}(\Omega)$ is addressed in the following proposition:

Proposition 3.6. *Let u be a function of $\mathrm{H}^1(\Omega)$. Then the following estimate holds:*

$$|\Pi_\mathcal{D} u|_{1,\mathcal{D}} \underset{\mathrm{reg}}{\leq} |u|_{H^1(\Omega)}.$$

If in addition u belongs to $\mathrm{H}^1_0(\Omega)$, then:

$$\|\Pi_\mathcal{D} u\|_{1,\mathcal{D}} \underset{\mathrm{reg}}{\leq} |u|_{H^1(\Omega)}.$$

The following result gives some insight into the way the projection of u approximates the function u itself.

Proposition 3.7. *Let u be a function of* $\mathrm{H}^1(\Omega)$*. Then the following estimate holds:*

$$\|u - \Pi_{\mathcal{D}} u\|_{L^2(\Omega)} \underset{\text{reg}}{\le} h\ |u|_{H^1(\Omega)}. \tag{16}$$

Let us now suppose that u belongs to $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$*. As a consequence, u is continuous and we can define $\bar u \in \mathrm{H}_{\mathcal{D}}(\Omega)$ by $\bar u_K = u(x_K)$, $\forall K \in \mathcal{M}$. Then we have:*

$$\|\Pi_{\mathcal{D}} u - \bar u\|_{1,\mathcal{D}} \underset{\text{reg}}{\le} h\ |u|_{H^2(\Omega)}. \tag{17}$$

The proofs of both preceding propositions are given in appendix.

3.3. Consistency error bounds

In this section, we successively provide estimates for the consistency residuals associated to the diffusive term (Lem. 3.8), the pressure gradient term (Lem. 3.9) and, finally, the velocity divergence term (Lem. 3.10).

Lemma 3.8. *Let u be a function of* $\mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$*. For any side σ of $\mathcal{E}$, we define:*

$$\begin{array}{ll} \textit{if } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L: & R_{\Delta,K|L}(u) = \dfrac{m(K|L)}{d_{K|L}}\ ((\Pi_{\mathcal{D}} u)_L - (\Pi_{\mathcal{D}} u)_K) - \displaystyle\int_{K|L} \nabla u \cdot n_{K|L} \\ \textit{if } \sigma \in \mathcal{E}_{\rm ext},\ \sigma \in \mathcal{E}(K): & R_{\Delta,\sigma}(u) = \dfrac{m(\sigma)}{d_{K,\sigma}}\ (-(\Pi_{\mathcal{D}} u)_K) - \displaystyle\int_{\sigma} \nabla u \cdot n_\sigma \end{array}$$

where n_σ is the normal vector to the edge σ oriented outward to K. Then:

if $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$:

$$|R_{\Delta,K|L}(u)| \le f(\mathrm{c}_{\infty,2}, c^{\rm app}_{0,2}, c^{\rm app}_{1,2})\ \frac{m(K|L)}{d_{K|L}\ \min\left[m(K), m(L)\right]^{1/2}}\ (\bar h_K^2\ |u|_{H^2(\omega_K)} + \bar h_L^2\ |u|_{H^2(\omega_L)});$$

if $\sigma \in \mathcal{E}_{\rm ext}$, $\sigma \in \mathcal{E}(K)$:

$$|R_{\Delta,\sigma}(u)| \le f(c^{\rm app}_{0,2}, c^{\rm app}_{1,2})\ \frac{m(\sigma)}{d_{K,\sigma}\ m(K)^{1/2}}\ \bar h_K^2\ |u|_{H^2(\omega_K)}.$$

Proof. We begin with the case of an internal side. By definition of the projection operator $\Pi_{\mathcal{D}}$, the quantity $R_{\Delta,K|L}$ reads:

$$\begin{aligned} R_{\Delta,K|L}(u) &= \frac{m(K|L)}{d_{K|L}}\ (\phi_L(x_L) - \phi_K(x_K)) - \int_{K|L} \nabla u \cdot n_{K|L} \\ &= \underbrace{\frac{m(K|L)}{d_{K|L}}\ (\phi_K(x_L) - \phi_K(x_K)) - \int_{K|L} \nabla u \cdot n_{K|L}}_{T_{1,K|L}} + \underbrace{\frac{m(K|L)}{d_{K|L}}\ (\phi_L(x_L) - \phi_K(x_L))}_{T_{2,K|L}}\ . \end{aligned} \tag{18}$$

Using the fact that ϕ_K is a linear polynomial, the first term of the right hand side of the preceding relation can be expressed as:

$$T_{1,K|L} = m(K|L)\ \nabla\phi_K \cdot \frac{\overrightarrow{x_K x_L}}{d_{K|L}}\ - \int_{K|L} \nabla u \cdot n_{K|L} = -\int_{K|L} \nabla(u - \phi_K) \cdot n_{K|L}.$$

Making use of inequality (14), with $\omega = K|L$, applying inequality (13) to $u - \phi_K$ and finally using lemma 3.4, we obtain the following estimate:

$$\begin{aligned}
|T_{1,K|L}| &\le m(K|L)^{1/2}\ |u-\phi_K|_{H^1(K|L)} \\
&\le m(K|L)^{1/2}\ \left(2d\ \frac{m(K|L)}{m(K)}\right)^{1/2}\ \big(|u-\phi_K|_{H^1(K)} + h_K|u-\phi_K|_{H^2(K)}\big) \\
&\le (2d)^{1/2}\ \frac{m(K|L)}{m(K)^{1/2}}\ \big(|u-\phi_K|_{H^1(\omega_K)} + h_K|u-\phi_K|_{H^2(\omega_K)}\big) \\
&\le (1+c^{\mathrm{app}}_{1,2})\ (2d)^{1/2}\ \frac{m(K|L)}{m(K)^{1/2}}\ \bar{h}_K\ |u|_{H^2(\omega_K)}.
\end{aligned}$$

On the other hand, using lemma 3.2, the triangular inequality, the fact that L is included in both ω_L and ω_K, then finally lemma 3.4, the second term at the right hand side of equation (18) can be estimated as follows:

$$\begin{aligned}
|T_{2,K|L}| &\le \frac{m(K|L)}{d_{K|L}}\ \|\phi_K-\phi_L\|_{L^\infty(L)} \\
&\le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{d_{K|L}\ m(L)^{1/2}}\ \|\phi_K-\phi_L\|_{L^2(L)} \\
&\le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{d_{K|L}\ m(L)^{1/2}}\ (\|\phi_K-u\|_{L^2(L)} + \|\phi_L-u\|_{L^2(L)}) \\
&\le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{d_{K|L}\ m(L)^{1/2}}\ (\|\phi_K-u\|_{L^2(\omega_K)} + \|\phi_L-u\|_{L^2(\omega_L)}) \\
&\le \mathrm{c}_{\infty,2}\ c^{\mathrm{app}}_{0,2}\ \frac{m(K|L)}{d_{K|L}\ m(L)^{1/2}}\ (\bar{h}_K^2\ |u|_{H^2(\omega_K)} + \bar{h}_L^2\ |u|_{H^2(\omega_L)}).
\end{aligned}$$

The proof is then easily completed by collecting the bounds of $T_{1,K|L}$ and $T_{2,K|L}$ and using the fact that $d_{K|L}$ is smaller than $\bar{h}_K$.

On an external side σ associated to the control volume K, we have:

$$\begin{aligned}
R_{\Delta,\sigma}(u) &= \frac{m(\sigma)}{d_{K,\sigma}}\ (-\phi_K(x_K)) - \int_\sigma \nabla u\cdot n_\sigma \\
&= \underbrace{\frac{m(\sigma)}{d_{K,\sigma}}\ (\phi_K(x_\sigma)-\phi_K(x_K)) - \int_\sigma \nabla u\cdot n_\sigma}_{T_{1,\sigma}} + \underbrace{\frac{m(\sigma)}{d_{K,\sigma}}\ (-\phi_K(x_\sigma))}_{T_{2,\sigma}}.
\end{aligned}$$

The first term $T_{1,\sigma}$ can be estimated strictly as the term T_1 of the relation (18). As the function u vanishes on the boundary of the domain, the second one reads:

$$T_{2,\sigma} = -\frac{1}{d_{K,\sigma}}\int_\sigma \phi_K = -\frac{1}{d_{K,\sigma}}\int_\sigma (\phi_K - u).$$

Making use of inequality (14), lemma 3.1 and lemma 3.4, we get:

$$\begin{aligned} |T_{2,\sigma}| &\leq \frac{m(\sigma)^{1/2}}{d_{K,\sigma}} \ \|\phi_K - u\|_{L^2(\sigma)} \\ &\leq d^{1/2} \ \frac{m(\sigma)}{d_{K,\sigma} \ m(K)^{1/2}} \ \left(\|\phi_K - u\|_{L^2(K)} + h_K \ |\phi_K - u|_{H^1(K)}\right) \\ &\leq (c^{\mathrm{app}}_{0,2} + c^{\mathrm{app}}_{1,2}) \ d^{1/2} \ \frac{m(\sigma)}{d_{K,\sigma} \ m(K)^{1/2}} \ \bar{h}^2_K |u|_{H^2(\omega_K)}. \end{aligned}$$

Once again, collecting the bounds for $T_{1,\sigma}$ and $T_{2,\sigma}$ yields the result. □

Lemma 3.9. *Let u be a function of $\mathrm{H}^1(\Omega)$. For any side σ of $\mathcal{E}$, we define the following vector valued quantity:*

$$\begin{aligned} &\text{if } \sigma \in \mathcal{E}_{\mathrm{int}}, \ \sigma = K|L: \quad R_{\mathrm{grad},K|L}(u) = m(K|L) \ \left[\frac{d_{K,K|L}}{d_{K|L}} \ (\Pi_{\mathcal{D}} u)_K + \frac{d_{L,K|L}}{d_{K|L}} \ (\Pi_{\mathcal{D}} u)_L\right] \ n_{K|L} - \int_{K|L} u \ n_{K|L}, \\ &\text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}, \ \sigma \in \mathcal{E}(K): \quad R_{\mathrm{grad},\sigma}(u) = m(\sigma) \ (\Pi_{\mathcal{D}} u)_K \ n_\sigma - \int_\sigma u \ n_\sigma, \end{aligned}$$

where n_σ is the normal vector to the edge σ oriented outward to K. Then:

$$\begin{aligned} &\text{if } \sigma \in \mathcal{E}_{\mathrm{int}}, \ \sigma = K|L: \\ &\qquad |R_{\mathrm{grad},K|L}(u)| \leq f(\mathrm{c}_{\infty,2}, c^{\mathrm{app}}_{0,1}, c^{\mathrm{app}}_{1,1}) \ \frac{m(K|L)}{\min(m(K), m(L))^{1/2}} \ \left[\bar{h}_K \ |u|_{H^1(\omega_K)} + \bar{h}_L \ |u|_{H^1(\omega_L)}\right], \\ &\text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}, \ \sigma \in \mathcal{E}(K): \\ &\qquad |R_{\mathrm{grad},\sigma}(u)| \leq f(\mathrm{c}_{\infty,2}, c^{\mathrm{app}}_{0,1}, c^{\mathrm{app}}_{1,1}) \ \frac{m(\sigma)}{m(K)^{1/2}} \ \bar{h}_K \ |u|_{H^1(\omega_K)}, \end{aligned}$$

where $|\cdot|$ stands in the preceding relation for the Euclidean norm in $\mathbb{R}^d$.

Proof. First of all, we remark that we can recast the case of an external edge into the formulation associated to an internal one by defining a fictitious external control volume L, setting $d_{L,K|L} = 0$ and giving to $(\Pi_{\mathcal{D}} u)_L$ any finite value. So the major part of the proof addresses both cases.

Using the linearity of ϕ_K, the quantity $R_{\mathrm{grad},K|L}$ can be decomposed as follows:

$$\begin{aligned} R_{\mathrm{grad},K|L}(u) &= m(K|L) \ (\frac{d_{K,K|L}}{d_{K|L}} \ \phi_K(x_K) + \frac{d_{L,K|L}}{d_{K|L}} \ \phi_L(x_L)) \ n_{K|L} - \int_{K|L} u \ n_{K|L} \\ &= m(K|L) \ \phi_K(\frac{d_{K,K|L}}{d_{K|L}} \ x_K + \frac{d_{L,K|L}}{d_{K|L}} \ x_L) \ n_{K|L} - \int_{K|L} u \ n_{K|L} \\ &\quad + m(K|L) \frac{d_{L,K|L}}{d_{K|L}} \ (\phi_L(x_L) - \phi_K(x_L)) \ n_{K|L}. \end{aligned}$$

Let x_G be the point defined by $x_G = \dfrac{d_{K,K|L}}{d_{K|L}}\, x_K + \dfrac{d_{L,K|L}}{d_{K|L}}\, x_L$. We recall that $x_{K|L}$ is defined as the midpoint of the edge $K|L$. We then have, as the midpoint integration rule is exact for the linear polynomial ϕ_K:

$$R_{\mathrm{grad},K|L}(u) = \underbrace{m(K|L)(\phi_K(x_G) - \phi_K(x_{K|L}))\; n_{K|L}}_{T_{1,K|L}} + \underbrace{\int_{K|L} (\phi_K - u)\; n_{K|L}}_{T_{2,K|L}}$$
$$+ \underbrace{m(K|L)\frac{d_{L,K|L}}{d_{K|L}}\; (\phi_L(x_L) - \phi_K(x_L))\; n_{K|L}}_{T_{3,K|L}}.$$

Next step is to bound successively the three terms $T_{1,K|L}$, $T_{2,K|L}$ and $T_{3,K|L}$.

First, we have:

$$T_{1,K|L} = m(K|L)\; (\phi_K(x_G) - \phi_K(x_{K|L}))\; n_{K|L} = m(K|L)\; (\nabla\phi_K \cdot \overrightarrow{x_{K|L}\; x_G})\; n_{K|L}.$$

As the distance between $x_{K|L}$ and x_G is smaller than $\bar{h}_K$, by Lemma 3.2 then 3.4, the following estimate holds:

$$|T_{1,K|L}| \le m(K|L)\; \bar{h}_K |\nabla\phi_K| \le \mathrm{c}_{\infty,2}\; \frac{m(K|L)}{m(K)^{1/2}}\; \bar{h}_K\; |\phi_K|_{H^1(\omega_K)} \le \mathrm{c}_{\infty,2}(1 + c^{\mathrm{app}}_{1,1})\; \frac{m(K|L)}{m(K)^{1/2}}\; \bar{h}_K\; |u|_{H^1(\omega_K)}.$$

The term $T_{2,K|L}$ is estimated as follows, using successively inequality (14), Lemmas 3.1 and 3.4:

$$\begin{aligned}
|T_{2,K|L}| &\le m(K|L)^{1/2} \|\phi_K - u\|_{L^2(K|L)} \\
&\le m(K|L)^{1/2} \left(d\; \frac{m(K|L)}{m(K)}\right)^{1/2} \big(\|\phi_K - u\|_{L^2(K)} + h_K\; |\phi_K - u|_{H^1(K)}\big) \\
&\le (c^{\mathrm{app}}_{0,1} + c^{\mathrm{app}}_{1,1})\; d^{1/2}\; \frac{m(K|L)}{m(K)^{1/2}}\; \bar{h}_K |u|_{H^1(\omega_K)}.
\end{aligned}$$

If the edge under consideration is an external one, the term $T_{3,K|L}$ is zero (since $d_{L,K|L} = 0$). Otherwise, by Lemma 3.2 then 3.4, $T_{3,K|L}$ is bounded by:

$$\begin{aligned}
|T_{3,K|L}| &\le m(K|L)\; \|\phi_L - \phi_K\|_{L^\infty(L)} \\
&\le \mathrm{c}_{\infty,2}\; \frac{m(K|L)}{m(L)^{1/2}}\; \|\phi_L - \phi_K\|_{L^2(L)} \\
&\le \mathrm{c}_{\infty,2}\; \frac{m(K|L)}{m(L)^{1/2}}\; \big(\|\phi_L - u\|_{L^2(L)} + \|\phi_K - u\|_{L^2(L)}\big) \\
&\le \mathrm{c}_{\infty,2}\; \frac{m(K|L)}{m(L)^{1/2}}\; \big(\|\phi_L - u\|_{L^2(\omega_L)} + \|\phi_K - u\|_{L^2(\omega_K)}\big) \\
&\le \mathrm{c}_{\infty,2}\; c^{\mathrm{app}}_{0,1}\; \frac{m(K|L)}{m(L)^{1/2}}\; \big(\bar{h}_L\; |u|_{H^1(\omega_L)} + \bar{h}_K\; |u|_{H^1(\omega_K)}\big).
\end{aligned}$$

Collecting the bounds of $T_{1,K|L}$, $T_{2,K|L}$ and $T_{3,K|L}$, the proof is over. $\square$

Lemma 3.10. *Let u be a function of $\mathrm{H}_0^1(\Omega)^d$ and K be a control volume of $\mathcal{M}$. For each neighbouring control volume L of K, we denote by $R_{\mathrm{div},K|L}(u)$ the following quantity:*

$$R_{\mathrm{div},K|L}(u) = m(K|L)\ (\frac{d_{L,K|L}}{d_{K|L}}\ (\Pi_{\mathcal{D}}u)_K + \frac{d_{K,K|L}}{d_{K|L}}\ (\Pi_{\mathcal{D}}u)_L) \cdot n_{K|L} - \int_{K|L} u \cdot n_{K|L}.$$

Then:

$$|R_{\mathrm{div},K|L}(u)| \leq f(\mathrm{c}_{\infty,2}, c_{0,1}^{\mathrm{app}}, c_{1,1}^{\mathrm{app}})\ \frac{m(K|L)}{\min(m(K),m(L))^{1/2}}\ (\bar{h}_K\ |u|_{H^1(\omega_K)^d} + \bar{h}_L\ |u|_{H^1(\omega_L)^d}) \tag{19}$$

and, if u belongs to $\mathrm{H}^2(\Omega)^d$:

$$|R_{\mathrm{div},K|L}(u)| \leq f(\mathrm{c}_{\infty,2}, c_{0,2}^{\mathrm{app}}, c_{1,2}^{\mathrm{app}})\ \frac{m(\sigma)}{\min(m(K),m(L))^{1/2}}\ (\bar{h}_K^2\ |u|_{H^2(\omega_K)^d} + \bar{h}_L^2\ |u|_{H^2(\omega_L)^d}). \tag{20}$$

Proof. By definition of $(\Pi_{\mathcal{D}}u)_K$ and $(\Pi_{\mathcal{D}}u)_L$, we can write $(\Pi_{\mathcal{D}}u)_K = \phi_K(x_L)$ and $(\boldsymbol{\Pi}_{\mathcal{D}}u)_K = \phi_L(x_L)$, where ϕ_K and ϕ_L are two vector valued linear polynomials. With this notation, we have:

$$\begin{aligned} R_{\mathrm{div},K|L}(u) = m(K|L)\ (&\frac{d_{L,K|L}}{d_{K|L}}\ \phi_K(x_K) + \frac{d_{K,K|L}}{d_{K|L}}\ \phi_K(x_L)) \cdot n_{K|L} - \int_{K|L} u \cdot n_{K|L} \\ &+ m(K|L)\ \frac{d_{K,K|L}}{d_{K|L}}\ (\phi_L(x_L) - \phi_K(x_L)) \cdot n_{K|L}. \end{aligned} \tag{21}$$

By linearity of ϕ_K and recognizing a one point integration rule valid up to degree one, because:

$$x_{K|L} = \frac{d_{L,K|L}}{d_{K|L}} x_K + \frac{d_{K,K|L}}{d_{K|L}}\ x_L$$

is the barycenter of the edge $\sigma = K|L$, we get for the first two terms of the preceding relation:

$$T_1 = m(K|L)\ \phi_K\left(\frac{d_{L,K|L}}{d_{K|L}}\ x_K + \frac{d_{K,K|L}}{d_{K|L}}\ x_L\right) \cdot n_{K|L} - \int_{K|L} u \cdot n_{K|L} = \int_{K|L} (u - \phi_K) \cdot n_{K|L}.$$

Inequality (14) then (12) yields:

$$\begin{aligned} |T_1| = \quad &\leq m(K|L)^{1/2} \|u - \phi_K\|_{L^2(K|L)^d} \\ &\leq m(K|L)^{1/2}\ \left(2d\ \frac{m(K|L)}{m(K)}\right)^{1/2}\ (\|u - \phi_K\|_{L^2(K)^d} + h_K |u - \phi_K|_{H^1(K)^d}) \\ &\leq (2d)^{1/2}\ \frac{m(K|L)}{m(K)^{1/2}}\ (\|u - \phi_K\|_{L^2(\omega_K)^d} + h_K |u - \phi_K|_{H^1(\omega_K)^d}). \end{aligned}$$

By the approximation lemma 3.4 and because $h_K \leq \bar{h}_K$:

$$\begin{aligned} |T_1| \quad &\leq (c_{0,1}^{\mathrm{app}} + c_{1,1}^{\mathrm{app}})\ (2d)^{1/2}\ \frac{m(K|L)}{m(K)^{1/2}}\ \bar{h}_K\ |u|_{H^1(\omega_K)^d} \qquad \text{if } u \in \mathrm{H}^1(\Omega)^d \\ |T_1| \quad &\leq (c_{0,2}^{\mathrm{app}} + c_{1,2}^{\mathrm{app}})\ (2d)^{1/2}\ \frac{m(K|L)}{m(K)^{1/2}}\ \bar{h}_K^2\ |u|_{H^2(\omega_K)^d} \qquad \text{if } u \in \mathrm{H}^2(\Omega)^d. \end{aligned} \tag{22}$$

The last part of the right hand side of equation (21) reads:

$$T_2 = m(K|L)\ \frac{d_{K,K|L}}{d_{K|L}}\ (\phi_L - \phi_K)(x_L) \cdot n_{K|L}.$$

By Lemma 3.2, we have:

$$|T_2| \le m(K|L)\ \|\phi_L - \phi_K\|_{L^\infty(L)^d} \le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{m(L)^{1/2}}\ \|\phi_L - \phi_K\|_{L^2(L)^d}.$$

Because L is included in both ω_K and ω_L, we get:

$$\begin{aligned}|T_2| &\le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{m(L)^{1/2}}\ (\|\phi_L - u\|_{L^2(L)^d} + \|\phi_K - u\|_{L^2(L)^d})\\ &\le \mathrm{c}_{\infty,2}\ \frac{m(K|L)}{m(L)^{1/2}}\ (\|\phi_L - u\|_{L^2(\omega_L)^d} + \|\phi_K - u\|_{L^2(\omega_K)^d})\end{aligned}$$

and, by Lemma 3.4:

$$\begin{aligned}|T_2| &\le \mathrm{c}_{\infty,2}\ c_{0,1}^{\mathrm{app}}\ \frac{m(K|L)}{m(L)^{1/2}}\ (\bar h_L\ |u|_{H^1(\omega_L)^d} + \bar h_K\ |u|_{H^1(\omega_K)^d}) &&\text{if } u \in \mathrm{H}^1(\Omega)^d\\ |T_2| &\le \mathrm{c}_{\infty,2}\ c_{0,2}^{\mathrm{app}}\ \frac{m(K|L)}{m(L)^{1/2}}\ (\bar h_L^2\ |u|_{H^2(\omega_L)^d} + \bar h_K^2\ |u|_{H^2(\omega_K)^d}) &&\text{if } u \in \mathrm{H}^2(\Omega)^d.\end{aligned} \tag{23}$$

Finally, both desired results follow by collecting the bounds (22) and (23). □

4. Error analysis

This section is aimed at the error analysis of the scheme under consideration. As usual in the analysis of stabilized methods for saddle point problems, the first step is to prove the stability of the scheme; this is the purpose of Proposition 4.1. Then the error bound (Thm. 4.3) follows from the consistency error estimates.

Proposition 4.1 (stability of the scheme)**.** *Let u, v and p, q be two elements of respectively $\mathrm{H}_\mathcal{D}(\Omega)^d$ and $\mathrm{H}_\mathcal{D}(\Omega)$. We define:*

$$\begin{aligned}&B(u,p;v,q) =\\ &\sum_{K\in\mathcal{M}} v_K \cdot \left[\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}\ (u_L - u_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\ (-u_K) + \sum_{\sigma=K|L} m(\sigma)\ \frac{d_{L,\sigma}}{d_\sigma}\ (p_L - p_K)\ n_\sigma\right]\\ &+ \sum_{K\in\mathcal{M}} q_K \sum_{\sigma=K|L} m(\sigma)\ (\frac{d_{L,\sigma}}{d_\sigma}\ u_K + \frac{d_{K,\sigma}}{d_\sigma}\ u_L)\cdot n_\sigma - \lambda \sum_{\sigma=K|L} (h_K^2 + h_L^2)\ \frac{m(\sigma)}{d_\sigma}(p_L - p_K).\end{aligned}$$

Then for each pair $u \in \mathrm{H}_\mathcal{D}(\Omega)^d$ and $p \in \mathrm{H}_\mathcal{D}(\Omega)$, there exists $\tilde u \in \mathrm{H}_\mathcal{D}(\Omega)^d$ and $\tilde p \in \mathrm{H}_\mathcal{D}(\Omega)$ such that:

$$\|\tilde u\|_{1,\mathcal{D}} + \|\tilde p\|_{L^2(\Omega)} \underset{\mathrm{reg}}{\le} \|u\|_{1,\mathcal{D}} + \ \|p\|_{L^2(\Omega)} \tag{24}$$

and

$$\|u\|_{1,\mathcal{D}}^2 + \|p\|_{L^2(\Omega)}^2 \underset{\mathrm{reg}}{\le} B(u,p\,;\tilde u,\tilde p). \tag{25}$$

Proof. Let u and p be given as in the proposition statement. The proof of this proposition is obtained by building explicitly $\tilde{u}$ and $\tilde{p}$ such that the relations (24) and (25) hold.

In a first step, we recall that the discrete gradient is chosen as the transpose of the divergence, *i.e.* such that:

$$\sum_{K\in\mathcal{M}} u_K \cdot \sum_{\sigma=K|L} m(\sigma)\, \frac{d_{L,\sigma}}{d_\sigma}\, (p_L - p_K)\, n_\sigma + \sum_{K\in\mathcal{M}} p_K \sum_{\sigma=K|L} m(\sigma)\, \left(\frac{d_{L,\sigma}}{d_\sigma}\, u_K + \frac{d_{K,\sigma}}{d_\sigma}\, u_L\right) \cdot n_\sigma = 0.$$

Consequently, by a standard reordering of the summations, we have:

$$B(u,p\,;u,p) = \|u\|_{1,\mathcal{D}}^2 + \lambda\, |p|_h^2.$$

In a second step, we know (see *e.g.* [15]) that, since $p \in \mathrm{L}^2(\Omega)$ and the integral of p over Ω is zero, there exists $c_{\mathrm{dr}} > 0$, which only depends on d and Ω, and $\bar{v} \in \mathrm{H}_0^1(\Omega)^d$ such that $\nabla \cdot (\bar{v}(x)) = -p(x)$ for a.e. $x \in \Omega$ and

$$|\bar{v}|_{H^1(\Omega)^d} \le c_{\mathrm{dr}}\, \|p\|_{L^2(\Omega)}. \tag{26}$$

For short, we note $\bar{v}_K$ be $(\Pi_{\mathcal{D}}\bar{v})_K$. We have:

$$\begin{aligned} B(u,p\,;\Pi_{\mathcal{D}}\bar{v},0) = \sum_{K\in\mathcal{M}} \bar{v}_K \cdot &\left[\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}\, (u_L - u_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\, (-u_K)\right] \\ &+ \sum_{K\in\mathcal{M}} \bar{v}_K \cdot \sum_{\sigma=K|L} m(\sigma)\, \frac{d_{L,\sigma}}{d_\sigma}\, (p_L - p_K)\, n_\sigma. \end{aligned}$$

The Cauchy-Schwarz inequality yields the following estimate for the first summation:

$$\sum_{K\in\mathcal{M}} \bar{v}_K \cdot \left[\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}\, (u_L - u_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\, (-u_K)\right] \ge -\, \|\Pi_{\mathcal{D}}\bar{v}\|_{1,\mathcal{D}}\, \|u\|_{1,\mathcal{D}}. \tag{27}$$

Using the fact that the discrete gradient is by construction the transpose of the discrete divergence, we obtain for the second term:

$$T_2 = \sum_{K\in\mathcal{M}} \bar{v}_K \cdot \sum_{\sigma=K|L} m(\sigma)\, \frac{d_{L,\sigma}}{d_\sigma}\, (p_L - p_K)\, n_\sigma = -\sum_{K\in\mathcal{M}} p_K \sum_{\sigma=K|L} m(\sigma)\, \left(\frac{d_{L,\sigma}}{d_\sigma}\, \bar{v}_K + \frac{d_{K,\sigma}}{d_\sigma}\, \bar{v}_L\right) \cdot n_\sigma.$$

Adding and subtracting the integral over each element of the divergence of $\bar{v}$ yields:

$$T_2 = \underbrace{-\sum_{K\in\mathcal{M}} p_K \int_K \nabla\cdot\bar{v}}_{T_3} \underbrace{-\sum_{K\in\mathcal{M}} p_K \left[\sum_{\sigma=K|L} m(\sigma)\, (\frac{d_{L,\sigma}}{d_\sigma}\, \bar{v}_K + \frac{d_{K,\sigma}}{d_\sigma}\, \bar{v}_L) \cdot n_\sigma - \int_\sigma \bar{v}\cdot n_\sigma\right]}_{T_4}.$$

The first term reads:

$$T_3 = -\sum_{K\in\mathcal{M}} \int_K p_K\, \nabla\cdot\bar{v} = \sum_{K\in\mathcal{M}} \int_K p_K^2 = \|p\|_{L^2(\Omega)}^2. \tag{28}$$

Reordering the sums in the second one, we obtain:

$$T_4 = \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \left[\left(\frac{m(\sigma)}{d_\sigma}\right)^{1/2}\, (h_K^2 + h_L^2)^{1/2}\, (p_K - p_L)\right] \left[\left(\frac{d_\sigma}{m(\sigma)}\right)^{1/2} \frac{1}{(h_K^2 + h_L^2)^{1/2}}\, R_{\mathrm{div},\sigma}(\bar{v})\right]$$

and using the Cauchy-Schwarz inequality:

$$|T_4| \leq |p|_h \left[\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{d_\sigma}{m(\sigma)} \frac{1}{h_K^2+h_L^2} \ (R_{{\rm div},\sigma}(\bar v))^2 \right]^{1/2}.$$

By Lemma 3.10, we then get:

$$\begin{aligned} |T_4| \leq\ & g({\rm c}_{\infty,2}, c_{0,1}^{\rm app}, c_{1,1}^{\rm app})\ |p|_h \\ & \left[\sum_{\sigma\in\mathcal{E}_{\rm int}(\sigma=K|L)} \frac{d_\sigma\ m(\sigma)}{\min(m(K),m(L))} \left(\frac{\bar h_L^2}{h_K^2+h_L^2}\ |\bar v|^2_{H^1(\omega_K)^d} + \frac{\bar h_K^2}{h_K^2+h_L^2}\ |\bar v|^2_{H^1(\omega_L)^d} \right) \right]^{1/2}. \end{aligned}$$

The quantity $d_\sigma\ m(\sigma)$ can be seen as the area of a volume ($d=3$) or surface ($d=2$) included in $K\cup L$. Consequently, since $\max_{K\in\mathcal{M},\ L\in\mathcal{N}(K)} h_K/h_L$ is one of the elements of regul($\mathcal{M}$), we get:

$$\frac{d_\sigma\ m(\sigma)}{\min(m(K),m(L))} \underset{\rm reg}{\leq} 1.$$

In the same way:

$$\frac{\bar h_L^2}{h_K^2+h_L^2} \underset{\rm reg}{\leq} 1 \quad , \qquad \frac{\bar h_K^2}{h_K^2+h_L^2} \underset{\rm reg}{\leq} 1.$$

Finally, as a control volume is intersected by at most N_ω domains ω_L, the integral of the H^1 seminorm of $\bar v$ over each control volume is encountered only a bounded number of times in the above summation, and we have:

$$|T_4| \underset{\rm reg}{\leq} |p|_h\ |\bar v|_{H^1(\Omega)^d}. \tag{29}$$

Finally, gathering the estimates (27),(28) and (29) yields:

$$B(u,p\,;\Pi_{\mathcal{D}}\bar v,0) \geq \|p\|^2_{L^2(\Omega)} -\ \|\bar v\|_{1,\mathcal{D}}\ \|u\|_{1,\mathcal{D}} - c_1\ |p|_h\ |\bar v|_{H^1(\Omega)^d}$$

where c_1 is a non-decreasing function of the parameters of regul($\mathcal{M}$). As, by construction, $|\bar v|_{H^1(\Omega)^d} \leq c_{\rm dr}\|p\|_{L^2(\Omega)}$ and, by continuity of the projection operator $\Pi_{\mathcal{D}}$ from $\mathrm{H}^1(\Omega)^d$ in $\mathrm{H}_{\mathcal{D}}(\Omega)^d$) (proposition 3.6), $\|\Pi_{\mathcal{D}}\bar v\|_{1,\mathcal{D}} \underset{\rm reg}{\leq} |\bar v|_{H^1(\Omega)^d}$, this inequality equivalently reads:

$$B(u,p\,;\Pi_{\mathcal{D}}\bar v,0) \geq \|p\|^2_{L^2(\Omega)} - c_2\|p\|_{L^2(\Omega)}\ \|u\|_{1,\mathcal{D}} - c_1 c_{\rm dr}|p|_h\ \|p\|_{L^2(\Omega)}$$

with c_2 being once again a non-decreasing function of the parameters of regul($\mathcal{M}$). Using Young's inequality, we obtain:

$$B(u,p\,;\Pi_{\mathcal{D}}\bar v,0) \geq \frac{1}{2}\ \|p\|^2_{L^2(\Omega)} - c_2^2\ \|u\|^2_{1,\mathcal{D}} - c_1^2 c_{\rm dr}^2\ |p|_h^2.$$

By linearity of $(v,q)\mapsto B(u,p;v,q)$, we then have, for each positive constant ξ:

$$B(u,p\,;u+\xi\ \Pi_{\mathcal{D}}\bar v,p) \geq (1-\xi\ c_2^2)\ \|u\|^2_{1,\mathcal{D}} + \frac{\xi}{2}\ \|p\|^2_{L^2(\Omega)} + (\lambda - \xi c_1^2 c_{\rm dr}^2)\ |p|_h^2.$$

Choosing a value of ξ small enough, this inequality yields an estimate of the form (25). As the relation (24) is clearly verified by the pair $(u+\xi\ \Pi_{\mathcal{D}}\bar v,p)$, this concludes the proof. □

An immediate consequence of this stability inequality is that the discrete problem is well posed:

Proposition 4.2. *The discrete system (9), completed by the constraint (10), admits a unique solution (u, p).*

Proof. We define the following finite dimensional vector space:

$$V = \left\{ (u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega) \text{ such that } \int_\Omega p = 0 \right\}.$$

Let F be the linear mapping which associates to $(u,p) \in V$ the pair $(\hat{u}, \hat{p})$ defined by the following relations, written for each control volume K of the mesh:

$$\left| \begin{aligned} m(K)\ \hat{u}_K &= \sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}(u_L - u_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\ (-u_K) + \sum_{\sigma=K|L} m(\sigma)\frac{d_{L,\sigma}}{d_\sigma}\ (p_L - p_K)\ n_\sigma \\ m(K)\ \hat{p}_K &= \sum_{\sigma=K|L} m(\sigma)\ \left(\frac{d_{L,\sigma}}{d_\sigma}\ u_K + \frac{d_{K,\sigma}}{d_\sigma}\ u_L \right) \cdot n_\sigma - \lambda \sum_{\sigma=K|L} (h_K^2 + h_L^2)\ \frac{m(\sigma)}{d_\sigma}(p_L - p_K). \end{aligned} \right.$$

By summing over each control volume of the mesh the last relation, we check that the integral of $\hat{p}$ over Ω is zero, which means that $(\hat{u}, \hat{p}) \in V$. Proposition 4.1 then implies that the kernel of F is reduced to $(0,0)$, which proves that the mapping F is one to one from V onto V. As the right hand side of (9) belongs to V, this concludes the proof. □

We are now in position to state the convergence result of the scheme:

Theorem 4.3. *We assume that the weak solution $(\bar{u}, \bar{p})$ of the Stokes problem in the sense of (2) is such that $(\bar{u}, \bar{p}) \in \mathrm{H}_0^1(\Omega)^d \cap \mathrm{H}^2(\Omega)^d \times \mathrm{H}^1(\Omega)$. Let $(u,p) \in \mathrm{H}_{\mathcal{D}}(\Omega)^d \times \mathrm{H}_{\mathcal{D}}(\Omega)$ be the solution to (9). Let $\bar{u}_{\mathcal{D}} = \Pi_{\mathcal{D}}\bar{u}$ and $\bar{p}_{\mathcal{D}} = \Pi_{\mathcal{D}}\bar{v}$ and define $e \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ and $\epsilon \in \mathrm{H}_{\mathcal{D}}(\Omega)$ by $e = u - \bar{u}_{\mathcal{D}}$ and $\epsilon = p - \bar{p}_{\mathcal{D}}$. Then:*

$$\|e\|_{1,\mathcal{D}} + \|\epsilon\|_{L^2(\Omega)} \underset{\mathrm{reg}}{\leq} h\ (|\bar{u}|_{H^2(\Omega)^d} + |\bar{p}|_{H^1(\Omega)}). \tag{30}$$

In addition, let $\hat{u}_{\mathcal{D}}$ be the function of $\mathrm{H}_{\mathcal{D}}(\Omega)^d$ defined by $\hat{u}_{\mathcal{D}}^{(i)}|_K = \bar{u}^{(i)}(x_K)$, $\forall K \in \mathcal{M}$. Then:

$$\|u - \hat{u}_{\mathcal{D}}\|_{1,\mathcal{D}} + \|\epsilon\|_{L^2(\Omega)} \underset{\mathrm{reg}}{\leq} h\ (|\bar{u}|_{H^2(\Omega)^d} + |\bar{p}|_{H^1(\Omega)}). \tag{31}$$

Finally, we also have:

$$\|u - \bar{u}\|_{L^2(\Omega)^d} + \|p - \bar{p}\|_{L^2(\Omega)} \underset{\mathrm{reg}}{\leq} h\ (|\bar{u}|_{H^1(\Omega)^d} + |\bar{u}|_{H^2(\Omega)^d} + |\bar{p}|_{H^1(\Omega)}). \tag{32}$$

Proof. Subtracting the same terms at the left and right hand side of the discrete momentum balance equation, we get, for each control volume K of $\mathcal{M}$:

$$\begin{aligned} &\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}\ (e_L - e_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} -\frac{m(\sigma)}{d_{K,\sigma}}\ (-e_K) + \sum_{\sigma=K|L} m(\sigma)\ \frac{d_{L,\sigma}}{d_\sigma}\ (\epsilon_L - \epsilon_K)\ n_\sigma = \int_K f \\ &+ \sum_{\sigma=K|L} \frac{m(\sigma)}{d_\sigma}\ ((\bar{u}_{\mathcal{D}})_L - (\bar{u}_{\mathcal{D}})_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\mathrm{ext}}} \frac{m(\sigma)}{d_{K,\sigma}}\ (-(\bar{u}_{\mathcal{D}})_K) + \sum_{\sigma=K|L} -m(\sigma)\ \frac{d_{L,\sigma}}{d_\sigma}\ ((\bar{p}_{\mathcal{D}})_L - (\bar{p}_{\mathcal{D}})_K)\ n_\sigma. \end{aligned}$$

The regularity of $\bar{u}$ and $\bar{p}$ assumed in the statement of the theorem allows to integrate the continuous partial derivative equation (1) over each element K:

$$\int_{\partial K} -\nabla\bar{u}\cdot n + \int_{\partial K} \bar{p}\ n = \int_K f.$$

Subtracting this relation to the previous one and switching, for the pressure term in the right hand side of the relation, to the equivalent formulation given by (11), we get, for each control volume K of $\mathcal{M}$:

$$\sum_{\sigma=K|L} -\frac{m(\sigma)}{d_\sigma}\,(e_L - e_K) + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}} -\frac{m(\sigma)}{d_{K,\sigma}}\,(-e_K) + \sum_{\sigma=K|L} m(\sigma)\,\left(\frac{d_{L,\sigma}}{d_\sigma}\,\epsilon_L + \frac{d_{K,\sigma}}{d_\sigma}\,\epsilon_K\right)\,n_\sigma = T_{1,K} + T_{2,K}$$

with

$$\begin{aligned}
T_{1,K} &= \sum_{\sigma=K|L}\left[\frac{m(\sigma)}{d_\sigma}\,(\bar u_{\mathcal{D}})_L - (\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right] + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}}\left[\frac{m(\sigma)}{d_{K,\sigma}}\,(-(\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right]\\
T_{2,K} &= \sum_{\sigma=K|L}\left[-m(\sigma)\,(\frac{d_{L,\sigma}}{d_\sigma}\,(\bar p_{\mathcal{D}})_L + \frac{d_{K,\sigma}}{d_\sigma}\,(\bar p_{\mathcal{D}})_K)\,n_\sigma + \int_\sigma \bar p\,n_\sigma\right] + \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}}\left[-m(\sigma)\,(\bar p_{\mathcal{D}})_K\,n_\sigma + \int_\sigma \bar p\,n_\sigma\right].
\end{aligned}$$

Repeating the same process for the mass balance equation yields, once again for each control volume K of $\mathcal{M}$:

$$\sum_{\sigma=K|L} m(\sigma)\,\left(\frac{d_{L,\sigma}}{d_\sigma}\,e_K + \frac{d_{K,\sigma}}{d_\sigma}\,e_L\right)\cdot n_\sigma - \lambda\sum_{\sigma=K|L}(h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}(\epsilon_L-\epsilon_K) = T_{3,K}+T_{4,K}$$

with

$$\begin{aligned}
T_{3,K} &= -\sum_{\sigma=K|L} m(\sigma)\,\left(\frac{d_{L,\sigma}}{d_\sigma}\,(\bar u_{\mathcal{D}})_K + \frac{d_{K,\sigma}}{d_\sigma}\,(\bar u_{\mathcal{D}})_L\right)\cdot n_\sigma - \int_\sigma \bar u\cdot n_\sigma\\
T_{4,K} &= \lambda\sum_{\sigma=K|L}(h_K^2+h_L^2)\,\frac{m(\sigma)}{d_\sigma}\,((\bar p_{\mathcal{D}})_L - (\bar p_{\mathcal{D}})_K).
\end{aligned}$$

We choose $\tilde e \in \mathrm{H}_{\mathcal{D}}(\Omega)^d$ and $\tilde\epsilon \in \mathrm{H}_{\mathcal{D}}(\Omega)$ in such a way that the stability relations (24) and (25) are satisfied with $u = e$ and $p = \epsilon$. We then obtain:

$$\|e\|_{1,\mathcal{D}}^2 + \|\epsilon\|_{L^2(\Omega)}^2 \underset{\rm reg}{\le} \underbrace{\sum_{K\in\mathcal{M}}\tilde e_K\cdot T_{1,K}}_{(T_\Delta)} + \underbrace{\sum_{K\in\mathcal{M}}\tilde e_K\cdot T_{2,K}}_{(T_{\rm grad})} + \underbrace{\sum_{K\in\mathcal{M}}\tilde\epsilon_K T_{3,K}}_{(T_{\rm div})} + \underbrace{\sum_{K\in\mathcal{M}}\tilde\epsilon_K T_{4,K}}_{(T_{\rm c})}\cdot$$

Next step consists in bounding each of these terms. Reordering the summations, we get for T_Δ:

$$\begin{aligned}
T_\Delta = \sum_{K\in\mathcal{M}}\tilde e_K\cdot\Bigg\{&\sum_{\sigma=K|L}\left[\frac{m(\sigma)}{d_\sigma}\,((\bar u_{\mathcal{D}})_L - (\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right]\\
&+ \sum_{\sigma\in\mathcal{E}(K)\cap\mathcal{E}_{\rm ext}}\left[\frac{m(\sigma)}{d_K,\sigma}\,(-(\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right]\Bigg\}\\
= &\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)}(\tilde e_K - \tilde e_L)\cdot\,\left(\frac{m(\sigma)}{d_\sigma}\,((\bar u_{\mathcal{D}})_L - (\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right)\\
&+ \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}\tilde e_K\cdot\,\left(\frac{m(\sigma)}{d_{K,\sigma}}\,(-(\bar u_{\mathcal{D}})_K) - \int_\sigma \nabla\bar u\cdot n_\sigma\right).
\end{aligned}$$

The Cauchy-Schwarz inequality yields:

$$
\begin{aligned}
|T_\Delta| \;\leq\; & \left[\sum_{\sigma\in\mathcal{E}_{\rm int}(\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\,(\tilde e_K-\tilde e_L)^2+\sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}\frac{m(\sigma)}{d_{K,\sigma}}\,\tilde e_K^2\right]^{1/2} \\
& \left[\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{d_\sigma}{m(\sigma)}\left(\frac{m(\sigma)}{d_\sigma}\,((\bar u_{\mathcal D})_L-(\bar u_{\mathcal D})_K)-\int_\sigma \nabla\bar u\cdot n_\sigma\right)^2\right. \\
& \left.\qquad + \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}\frac{d_{K,\sigma}}{m(\sigma)}\left(\frac{m(\sigma)}{d_{K,\sigma}}\,(-(\bar u_{\mathcal D})_K)-\int_\sigma\nabla\bar u\cdot n_\sigma\right)^2\right]^{1/2} \\
= \;& \|\tilde e\|_{1,\mathcal D}\left[\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)}\frac{d_\sigma}{m(\sigma)}\sum_{i=1}^d (R_{\Delta,\sigma}(\bar u^{(i)}))^2+\sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}\frac{d_{K,\sigma}}{m(\sigma)}\sum_{i=1}^d\sum_{i=1}^d (R_{\Delta,\sigma}(\bar u^{(i)}))^2\right]^{1/2}.
\end{aligned}
$$

Using Lemma 3.8 and examining the functions of the geometrical features of the mesh appearing as coefficients in the summations, we obtain:

$$
|T_\Delta| \underset{\rm reg}{\leq} h\,\|\tilde e\|_{1,\mathcal D}\left[\sum_{\sigma\in\mathcal{E}_{\rm int}(\sigma=K|L)} |\bar u|^2_{H^2(\omega_K)^d}+|\bar u|^2_{H^2(\omega_L)^d}+\sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}|\bar u|^2_{H^2(\omega_K)^d}\right]^{1/2}.
$$

And, finally, since by assumption the integral over each control volume is counted no more than a bounded number of times (depending on N_ω and the number of sides of the considered control volumes):

$$
|T_\Delta| \underset{\rm reg}{\leq} h\,\|\tilde e\|_{1,\mathcal D}\,|\bar u|_{H^2(\Omega)^d}. \tag{33}
$$

Following the same line, reordering the summations and using the Cauchy-Schwarz inequality yields for $T_{\rm grad}$:

$$
|T_{\rm grad}| \leq \|\tilde e\|_{1,\mathcal D}\left[\sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)}\frac{d_\sigma}{m(\sigma)}\,(R_{{\rm grad},\sigma}(\bar p))^2+\sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))}\frac{d_{K,\sigma}}{m(\sigma)}\,(R_{{\rm grad},\sigma}(\bar p))^2\right]^{1/2}
$$

which leads, by Lemma 3.9, to the following estimate for $T_{\rm grad}$:

$$
|T_{\rm grad}| \underset{\rm reg}{\leq} h\,\|\tilde e\|_{1,\mathcal D}\,|\bar p|_{H^1(\Omega)}. \tag{34}
$$

By the same way, we obtain for $T_{\rm div}$:

$$
|T_{\rm div}| \leq \|\tilde\epsilon\|_{L^2(\Omega)}\left[\sum_{K\in\mathcal M}\frac{1}{m(K)}\left(\sum_{\sigma=K|L}R_{{\rm div},\sigma}(\bar u)\right)^2\right]^{1/2}
$$

and, by Lemma 3.10:

$$
|T_{\rm div}| \underset{\rm reg}{\leq} h\,\|\tilde\epsilon\|_{L^2(\Omega)}\,|u|_{H^2(\Omega)^d}. \tag{35}
$$

Finally, reordering summations and using the Cauchy-Schwarz inequality, the following bound holds for $T_{\rm c}$:

$$
|T_{\rm c}| \leq 2\lambda\, h\,|\tilde\epsilon|_h\,|\bar p_{\mathcal D}|_{1,\mathcal D}.
$$

Using Proposition 3.6 and the inverse inequality (8), we finally obtain:

$$|T_{\rm c}| \underset{\rm reg}{\leq} \lambda\ h\ \|\tilde{\epsilon}\|_{L^2(\Omega)}\ |\bar{p}|_{H^1(\Omega)}. \tag{36}$$

The four terms T_Δ, $T_{\rm grad}$, $T_{\rm div}$ and $T_{\rm c}$ are now bounded respectively by the estimates (33), (34), (35) and (36), and we get:

$$\|e\|^2_{1,\mathcal{D}} + \|\epsilon\|^2_{L^2(\Omega)} \underset{\rm reg}{\leq} h\ \left[\|\tilde{e}\|_{1,\mathcal{D}} + \|\tilde{\epsilon}\|_{L^2(\Omega)}\right]\ \left[|u|_{H^2(\Omega)^d} + |\bar{p}|_{H^1(\Omega)}\right].$$

The first estimate of the theorem 30 then follows by Young's inequality and using relation (24). The second one is easily deduced from (30), using the triangular inequality and the second estimate of Proposition 3.7. The third one follows similarly from the triangular inequality, the first estimate of Proposition 3.7 and the discrete Poincaré inequality (7). □

Remark 4.4. In view of this proof (see Eq. (35)), the second order estimate for the residual associated to the divergence term seems to be necessary to obtain a first order convergence rate for the scheme. This considerably reduces the generality of the possible meshings, as it imposes for the segment $[x_K, x_L]$ to cross the edge $K|L$ at its barycenter. For more general discretizations, we prove in [8] a suboptimal $h^{1/2}$ error estimate for a scheme with an enhanced stabilization.

5. Numerical tests

The aim of this section is to check the validity of the theoretical analysis against a practical test case for which an analytic solution can be exhibited. This solution is built as follows. We choose a streamfunction and a geometrical domain such that homogeneous Dirichlet conditions hold:

$$\varphi = 1000\ \left[\ x\,(1-x)\,y\,(1-y)\ \right]^2 \quad , \qquad \Omega =]0,1[\times]0,1[\quad , \qquad \bar{u} = \begin{bmatrix} \dfrac{\partial\varphi}{\partial y} \\ -\dfrac{\partial\varphi}{\partial x} \end{bmatrix};$$

we pick an arbitrary pressure in $\mathrm{L}^2_0(\Omega)$:

$$p = 100\ \left(x^2 + y^2 - \frac{2}{3}\right)$$

and the right hand side f is computed in order that the equations of the Stokes problem (1) are satisfied.

To obtain the numerical results displayed here, the practical implementation has been performed using the software object-oriented component library PELICANS, developed at IRSN [20].

In all cases, we choose for the parameter λ the value 2×10^{-2}, which is within the range of recommended values for the stabilization parameter of the Brezzi-Pitkäranta scheme [4].

The velocity and pressure errors are defined respectively as:

$$e_K^{(i)} = u_K^{(i)} - \bar{u}^{(i)}(x_K) \quad , \qquad \epsilon = p_K - \bar{p}(x_K).$$

This pressure error definition is not the same as in the analysis. However, for a sufficiently regular pressure field, both definitions are equivalent. Indeed, let $\hat{p}$ be the function of $\mathrm{H}_\mathcal{D}$ defined by $\hat{p}_K = p(x_K)$. Due to the regularity of $\bar{p}$, the discrete Poincaré-Friedrich inequality and Proposition 3.7 yields:

$$\|\Pi_\mathcal{D} p - \hat{p}\|_{L^2(\Omega)} \underset{\rm reg}{\leq} |\Pi_\mathcal{D} p - \hat{p}|_{1,\mathcal{D}} \underset{\rm reg}{\leq} h|p|_{H^2(\Omega)}$$

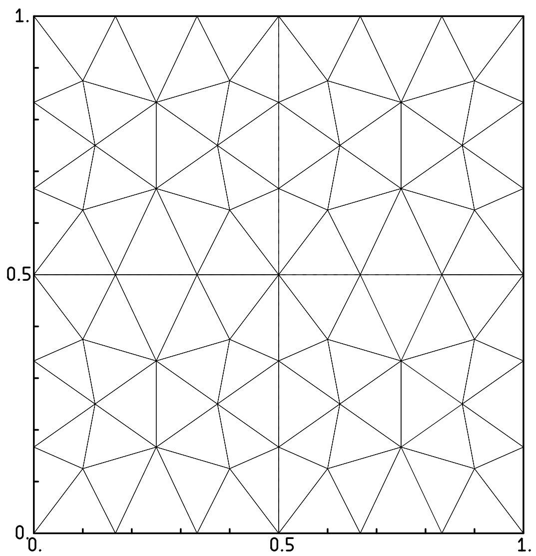

Figure 1. Coarsest meshing for simplicial control volumes.

and, by the triangular inequality, estimate (31) yields:

$$\|\epsilon\|_{L^2(\Omega)} \leq \|p - \Pi_{\mathcal{D}} p\|_{L^2(\Omega)} + \|\Pi_{\mathcal{D}} p - \hat{p}\|_{L^2(\Omega)} \underset{\text{reg}}{\leq} h\ (|p|_{H^1(\Omega)} + |p|_{H^2(\Omega)}).$$

We first solve this problem using a family of acute angles triangulations. The meshings used in this study are built by first splitting the domain into sub-squares and then cutting each sub-square into 26 triangles, all having angles of at most 80° (corresponding to Fig. 5 – bbbb in [1]). The coarsest one is displayed in Figure 1.

The obtained errors are reported in Figure 2. As foreseen by the theory, we observe a first order convergence for both the velocity in the $\|\cdot\|_{1,\mathcal{D}}$ norm and pressure in L^2 norm. In addition, a second order convergence is observed for the velocity in the discrete L^2 norm (*i.e.* $\|e\|_{L^2(\Omega)^d}$).

In a second step, we turn to regular $n \times n$ square grids. Figure 3 shows the evolution of the errors as a function of the grid parameter h. Results are better than can be expected from the theory: the convergence rate is close to $3/2$ for both the velocity in $\|\cdot\|_{1,\mathcal{D}}$ norm and the pressure in L^2 norm. In addition, as for simplicial triangulations, the velocity shows a second order convergence in the discrete L^2 norm.

In a third step, the reliability of the scheme for structured irregular grids is checked. To this purpose, we build meshings of the domain Ω generated by the same sequence of subdivisions along each direction, defined as follows: the size of the first, third, … $(2i-1)$th intervals is h while the size of the second, fourth, … $(2i)$th intervals is $h/10$. The quality of the results (not displayed here) is only slightly affected by the irregularity of this family of meshings: compared to results obtained with square control volumes, pressure and velocity errors are almost the same in the L^2 norm (the pressure approximation is even more accurate), and only twice greater for the velocity in $\|\cdot\|_{1,\mathcal{D}}$ norm.

6. Conclusion

We have presented in this paper a colocated finite volume scheme for the Stokes problem. Its stability is obtained by the addition to the continuity equation of a perturbation term which is a finite volume analogue of the well-known Brezzi-Pitkäranta stabilization term. For acute angles triangulations in 2D and for structured meshings of quadrangular (in 2D) or parallelepipedic (in 3D) control volumes, we prove a first order convergence in the natural finite volume discrete norms for both the velocity and the pressure. To the best of our knowledge,

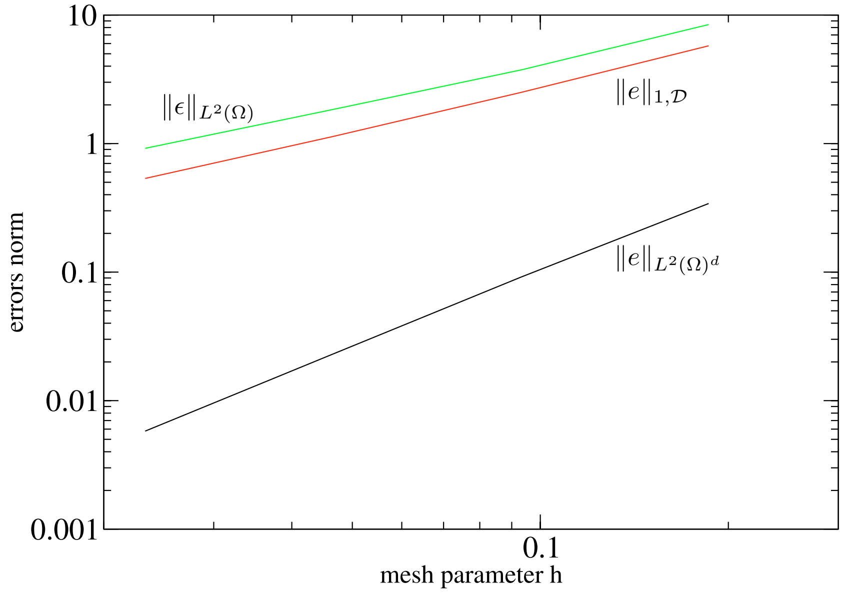

FIGURE 2. Errors for the velocity and the pressure obtained with simplicial control volumes.

this theoretical result is, for incompressible flow problems, the first of this type for colocated finite volume schemes, or even (in 2D) for finite volume schemes based on general simplicial discretizations of the equations in primitive variables. This analysis is confirmed by numerical experiments, which show, in addition, a second order convergence for the velocity in the discrete L^2 norm. A discretization for the Navier-Stokes equations following the same ideas and dealing with general meshings can be found in [8].

In view of the popularity of cheap (low degree) approximations in fluid flow applications, the considered scheme should deserve further work. In particular, a stable discretization should be derived for the more realistic Stokes problem where the divergence of the stress tensor is used instead of the velocity Laplacian, to allow to handle many practical applications involving a non-constant fluid viscosity.

Appendix A. Proof of Proposition 3.6

Proof. The proof of the first inequality of the proposition can be easily derived from the proof of the second one. Consequently, we will only address the latter here.

Let u be a function in $\mathrm{H}^1_0(\Omega)$ and the $\phi_K,\ K \in \mathcal{M}$ be defined by (15). By definition of the projection operator $\Pi_{\mathcal{D}}$, the discrete norm of $\hat{u} = \Pi_{\mathcal{D}} u$ reads:

$$\begin{aligned}
\|\hat{u}\|^2_{1,\mathcal{D}} &= \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\hat{u}_L - \hat{u}_K)^2 + \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ \hat{u}_K^2 \\
&= \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\phi_L(x_L) - \phi_K(x_K))^2 + \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ \phi_K(x_K)^2 \\
&\le 2 \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\phi_K(x_L) - \phi_K(x_K))^2 + 2 \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\phi_L(x_L) - \phi_K(x_L))^2 \\
&\quad +2 \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ (\phi_K(x_K) - \phi_K(x_\sigma))^2 + 2 \sum_{\sigma\in\mathcal{E}_{\rm ext}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ \phi_K(x_\sigma)^2.
\end{aligned} \tag{37}$$

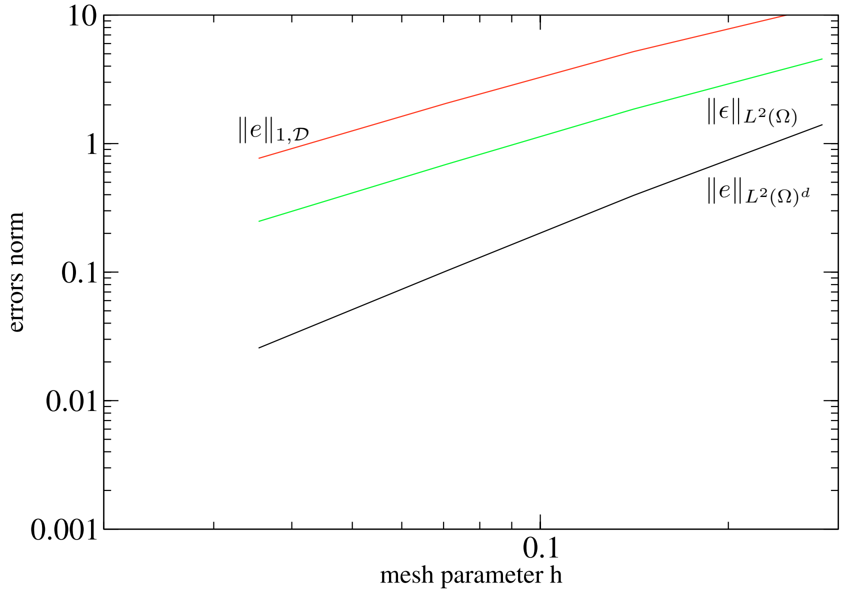

FIGURE 3. Errors for the velocity and the pressure obtained with square control volumes.

To proceed, we must now bound each term of the right hand side of this relation. Since ϕ_K is a linear polynomial, the first summation in the above relation reads:

$$T_1 = \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\phi_K(x_L)-\phi_K(x_K))^2 \ = \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\nabla\phi_K\cdot\overrightarrow{x_Kx_L})^2$$
$$= \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} m(\sigma)\ d_\sigma\ (\nabla\phi_K\cdot n_{K|L})^2.$$

The quantity $m(\sigma)\ d_\sigma$ can be seen as the measure of a domain included in $K\cup L$, and so is lower than the measure of ω_K; we then get, by lemma 3.4:

$$T_1 \le \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} |\phi_K|^2_{H^1(\omega_K)} \le (c^{\rm app}_{1,1})^2 \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} |u|^2_{H^1(\omega_K)}. \tag{38}$$

Using lemma 3.2, the second summation in (37) can be estimated as follows:

$$T_2 = \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ (\phi_L(x_L)-\phi_K(x_L))^2 \le ({\rm c}_{\infty,2})^2 \sum_{\sigma\in\mathcal{E}_{\rm int}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma\ m(L)}\ \|\phi_L-\phi_K\|^2_{L^2(L)}.$$

Then, because L is included in both ω_K and ω_L, we get by Lemma 3.4:

$$\begin{aligned} T_2 \quad & \leq 2\,(\mathrm{c}_{\infty,2})^2 \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma\ m(L)}\ \left[\|\phi_L-u\|^2_{L^2(L)}+\|\phi_K-u\|^2_{L^2(L)}\right] \\ & \leq 2\,(\mathrm{c}_{\infty,2})^2 \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma\ m(L)}\ \left[\|\phi_L-u\|^2_{L^2(\omega_L)}+\|\phi_K-u\|^2_{L^2(\omega_K)}\right] \\ & \leq 2\,(\mathrm{c}_{\infty,2}\ c^{\mathrm{app}}_{1,2})^2 \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma\ m(L)}\ \left[\bar{h}_L^2\ |u|^2_{H^1(\omega_L)}+\bar{h}_K^2\ |u|^2_{H^1(\omega_K)}\right]. \end{aligned} \tag{39}$$

Using the same arguments as for the bound of the term T_1 (the only difference is to replace d_σ by $d_{K,\sigma}$, we obtain similarly for the third term in (37):

$$T_3 \leq (c^{\mathrm{app}}_{1,1})^2 \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} |u|^2_{H^1(\omega_K)}. \tag{40}$$

Finally, using the linearity of ϕ_K and the fact that u vanishes on $\partial\Omega$, the fourth term in (37) reads:

$$T_4 = \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ \phi_K(x_\sigma)^2 = \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{1}{m(\sigma)\ d_{K,\sigma}}\ \left[\int_\sigma(\phi_K-u)\right]^2.$$

By inequality (14), Lemmas 3.1 and 3.4, we find that:

$$\begin{aligned} T_4 \quad & \leq \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{1}{d_{K,\sigma}}\|\phi_K-u\|^2_{L^2(\sigma)} \\ & \leq d \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}\ m(K)}\ \left[\|\phi_K-u\|_{L^2(K)}+h_K\ |\phi_K-u|_{H^1(K)}\right]^2 \\ & \leq d\ (c^{\mathrm{app}}_{1,2}+c^{\mathrm{app}}_{1,1})^2 \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)\ \bar{h}_K^2}{d_{K,\sigma}\ m(K)}|u|^2_{H^1(\omega_K)}. \end{aligned} \tag{41}$$

Elementary considerations show that the constants depending on the geometry in (39) and (41) are controlled by quantities which are non-decreasing functions of the parameters of the meshing gathered in regul($\mathcal{M}$). Consequently, relations (38), (39), (40) and (41) yields:

$$\|\hat{u}\|^2_{1,\mathcal{D}} \underset{\mathrm{reg}}{\leq} \sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} |u|^2_{H^1(\omega_K)}+|u|^2_{H^1(\omega_L)}+\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} |u|^2_{H^1(\omega_K)}.$$

As the integral over each element is accounted for in this summation a bounded number of times (depending on N_ω and the number of sides of the considered control volumes, see remark 3.5), this completes the proof. $\square$

Appendix B. Proof of Proposition 3.7

Proof. Proof of relation (16)

Decomposing the $L^2(\Omega)$ norm on each element and applying the triangular inequality, we get:

$$\begin{aligned}\|u-\Pi_{\mathcal{D}}u\|^2_{L^2(\Omega)} &= \sum_{K\in\mathcal{M}} \|u-\Pi_{\mathcal{D}}u\|^2_{L^2(K)} = \sum_{K\in\mathcal{M}} \|u-\phi_K(x_K)\|^2_{L^2(K)} \\ &\leq 2\sum_{K\in\mathcal{M}} \underbrace{\|u-\phi_K\|^2_{L^2(K)}}_{T_1} + \underbrace{\|\phi_K-\phi_K(x_K)\|^2_{L^2(K)}}_{T_2}.\end{aligned}$$

By Lemma 3.4, the first term is bounded by:

$$T_1 \leq \|u-\phi_K\|^2_{L^2(\omega_K)} \leq c^{\mathrm{app}}_{0,1}\ \bar{h}^2_K\ |u|^2_{H^1(\omega_K)}.$$

As ϕ_K is a linear polynomial, we have $\phi_K(x)-\phi_K(x_K) = \nabla\phi_K\cdot\overrightarrow{x_K\,x}$, $\forall x\in K$. Then, by lemma 3.4, T_2 is bounded by:

$$T_2 = \int_K (\nabla\phi_K\cdot\overrightarrow{x_K\,x})^2 \leq h^2_K \int_K |\nabla\phi_K|^2 \leq c^{\mathrm{app}}_{1,1}\ h^2_K\ |u|_{H^1(\omega_K)}.$$

These two bounds yield:

$$\|u-\Pi_{\mathcal{D}}u\|^2_{L^2(\Omega)} \underset{\mathrm{reg}}{\leq} h^2 \sum_{K\in\mathcal{M}} |u|_{H^1(\omega_K)}.$$

And consequently, by remark 3.5:

$$\|u-\Pi_{\mathcal{D}}u\|^2_{L^2(\Omega)} \underset{\mathrm{reg}}{\leq} h^2\ |u|_{H^1(\Omega)}.$$

Proof of relation (17)

By definition of the discrete $H^1(\Omega)$ norm and using the fact that u vanishes on $\partial\Omega$, we have:

$$\begin{aligned}\|\Pi_{\mathcal{D}}u-\bar{u}\|^2_{1,\mathcal{D}} = &\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \frac{m(\sigma)}{d_\sigma}\ \left[(\phi_L(x_L)-u(x_L))-(\phi_K(x_K)-u(x_K))\right]^2 \\ &+ \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \frac{m(\sigma)}{d_{K,\sigma}}\ (\phi_K(x_K)-u(x_K))^2 \\ \leq 2 &\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \underbrace{\frac{m(\sigma)}{d_\sigma}\left[(\phi_K(x_L)-\phi_K(x_K))-(u(x_L)-u(x_K))\right]^2}_{T_1} \\ +2 &\sum_{\sigma\in\mathcal{E}_{\mathrm{int}}\ (\sigma=K|L)} \underbrace{\frac{m(\sigma)}{d_\sigma}\left[\phi_L(x_L)-\phi_K(x_L)\right]^2}_{T_2} \\ +2 &\sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \underbrace{\frac{m(\sigma)}{d_{K,\sigma}}\ \left[(\phi_K(x_K)-\phi_K(x_\sigma))-(u(x_K)-u(x_\sigma))\right]^2}_{T_3} +2 \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}\ (\sigma\in\mathcal{E}(K))} \underbrace{\frac{m(\sigma)}{d_{K,\sigma}}\ (\phi_K(x_\sigma))^2}_{T_4}.\end{aligned}$$

To estimate the term T_1, a possible technique is to decompose T_1 as follows:

$$T_1 = \left[\underbrace{\frac{m(K|L)}{d_{K|L}}(\phi_K(x_L) - \phi_K(x_K)) - \int_\sigma \nabla u \cdot n_\sigma}_{T_{1,1}} + \underbrace{\int_\sigma \nabla u \cdot n_\sigma - \frac{m(K|L)}{d_{K|L}}(u(x_L) - u(x_K))}_{T_{1,2}} \right]^2 .$$

The first term is bounded in the proof of lemma 3.8 in the present paper and an estimate for the second one can be found in [7], pp. 786–790. However, for two dimensional problems, T_1 can be estimated quite simply, making use of the tools repeatedly employed in this paper. We restrict here the exposition to this case and, to this purpose, we write T_1 as:

$$T_1 = \frac{m(K|L)}{d_{K|L}} \left[\nabla\phi_K \cdot \overrightarrow{x_K\ x_L} - \int_{x_K}^{x_L} \nabla u \cdot n_{K|L} \right]^2 = \frac{m(K|L)}{d_{K|L}} \left[\int_{x_K}^{x_L} \nabla(\phi_K - u) \cdot n_{K|L} \right]^2 .$$

We denote by $D_{K|L}$ the simplex of edges $[x_K x_L]$ and a segment of first point x_K and last point located on a vertex of $K|L$. Note that $D_{K|L}$ is included in ω_K and, for the particular polygonal domains under consideration, $m(D_{K|L}) = 1/4\ m(K|L)\ d_{K|L}$. Inequalities (14) and (13) yields:

$$\begin{aligned} |T_1| &\le m(K|L)\ |\phi_K - u|^2_{H^1([x_K x_L])} \\ &\le 2d\ m(K|L)\ \frac{d_{K|L}}{m(D_{K|L})}\ \left(|\phi_K - u|_{H^1(D_{K|L})} + \mathrm{diam}(D_{K|L}) |\phi_K - u|_{H^2(D_{K|L})} \right)^2 \\ &\le 8\ (c^{\mathrm{app}}_{1,2} + 1)^2\ d\ \bar{h}^2_K\ |u|_{H^2(\omega_K)}. \end{aligned}$$

The term T_2 can be estimated using successively Lemmas 3.2 and 3.4 as follows:

$$\begin{aligned} |T_2| &= \frac{m(K|L)}{d_{K|L}} \left[(\phi_L(x_L) - \phi_K(x_L) \right]^2 \le \frac{m(K|L)}{d_{K|L}}\ \|\phi_L - \phi_K\|^2_{L^\infty(L)} \\ &\le (\mathrm{c}_{\infty,2})^2\ \frac{m(K|L)}{d_{K|L}\ m(L)} \|\phi_L - \phi_K\|^2_{L^2(L)} \\ &\le (\mathrm{c}_{\infty,2})^2\ \frac{m(K|L)}{d_{K|L}\ m(L)} (\|\phi_L - u\|^2_{L^2(L)} + \|\phi_K - u\|^2_{L^2(L)}) \\ &\le 2\ (\mathrm{c}_{\infty,2}\ c^{\mathrm{app}}_{0,2})^2\ \frac{m(K|L)}{d_{K|L}\ m(L)} (\bar{h}^4_L\ |u|^2_{H^2(\omega_L)} + \bar{h}^4_K\ |u|^2_{H^2(\omega_L)}). \end{aligned}$$

The term T_3 is estimated using the same arguments as for T_1, and we get in particular for two-dimensional problems:

$$|T_3| \le 8\ (c^{\mathrm{app}}_{1,2} + 1)^2\ d\ \bar{h}^2_K\ |u|_{H^2(\omega_K)}.$$

Finally, using the linearity of ϕ_K, the fact that x_σ is the barycenter of σ and the fact that u vanishes on $\partial\Omega$, the term T_4 reads:

$$T_4 = \frac{m(\sigma)}{d_{K,\sigma}}\ \phi_K(x_\sigma)^2 = \frac{1}{m(\sigma)\ d_{K,\sigma}}\ \left[\int_\sigma (\phi_K - u) \right]^2 .$$

By the inequality (14), lemma 3.1 and lemma 3.4, we find that:

$$
\begin{aligned}
|T_4| &\leq \frac{1}{d_{K,\sigma}} \|\phi_K - u\|^2_{L^2(\sigma)} \\
&\leq d \, \frac{m(\sigma)}{d_{K,\sigma} \; m(K)} \, \left[\|\phi_K - u\|_{L^2(K)} + h_K \; |\phi_K - u|_{H^1(K)}\right]^2 \\
&\leq (c^{\text{app}}_{0,2} + c^{\text{app}}_{1,2})^2 d \, \frac{m(\sigma) \; \bar{h}_K^4}{d_{K,\sigma} \; m(K)} |u|^2_{H^2(K)}.
\end{aligned}
$$

The proof is then completed by collecting the bounds, checking that the geometrical coefficients can be bounded by non-decreasing functions of of the parameters gathered in regul($\mathcal{M}$) and using the remark 3.5. □

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