LIMIT THEOREMS FOR MEASURE-VALUED PROCESSES OF THE LEVEL-EXCEEDANCE TYPE

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Abstract. Let, for each $t \in T, \psi(t, \cdot)$ be a random measure on the Borel σ-algebra in $\mathbb{R}^d$ such that $\mathsf{E}\psi(t, \mathbb{R}^d)^k < \infty$ for all k and let $\widehat{\psi}(t, \cdot)$ be its characteristic function. We call the function $\widetilde{\psi}^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = \mathsf{E} \prod_{j=1}^{l} \widehat{\psi}(t_j, z_j)$ of arguments $l \in \mathbb{N}, t_1, t_2 \ldots \in T, z_1, z_2 \ldots \in \mathbb{R}^d$ the *covaristic* of the measure-valued random function (MVRF) $\psi(\cdot, \cdot)$. A general limit theorem for MVRF's in terms of covaristics is proved and applied to functions of the kind $\psi_n(t, B) = \mu\{x : \xi_n(t, x) \in B\}$, where μ is a nonrandom finite measure and, for each n, ξ_n is a time-dependent random field.

INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $(X, \mathcal{X})$ and $(S, \mathcal{S})$ be measurable spaces. An $\mathcal{X} \otimes \mathcal{F}|\mathcal{S}$-measurable function $\xi : X \times \Omega \to S$ will be called an S-valued *random relief* on X. The argument $\omega \in \Omega$ of this and other functions will be usually omitted. If a measure μ on $\mathcal{X}$ is given, then a random relief ξ induces the random measure

$$\mu^\xi(B) \stackrel{\text{def}}{=} \mu\{x : \xi(x) \in B\}$$

on $\mathcal{S}$. In case $S = \mathbb{R}$ it is called the *level-exceedance measure* of the random field (in our terminology – relief) ξ. In case S is arbitrary we will say that μ^ξ is a random measure of the level-exceedance type.

A random relief depending additionally on a parameter (which may and will perform as a variable) will be called *varying*. Such a relief induces, together with μ, a measure-valued random function (MVRF).

The goals of the article are to provide a tool for proving functional limit theorems for MVRF's (this is accomplished in Sect. 2) and to apply it to measure-valued processes of the level-exceedance type arising in some model related to stochastic geometry (Sect. 3). Proofs of the theorems of Section 3 are placed to Section 4. Section 1 contains some general theorems about random measures and measure-valued functions.

The theory of random measures and measure-valued processes is well developed [1–3,8,16]. But the existing methods of this theory do not provide sufficiently general `functional` limit theorems for the processes we are going to consider. Our approach is based on the use of the Fourier-Stieltjes transform of moment measure (more precisely, some generalization of the latter). In these terms, we shall prove two functional analogues of Lévy's continuity theorem (see Thms. 2.3 and 2.4 below) and demonstrate their application.

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1. Universal constructions of measure-valued random functions

In what follows, $\mathcal{B}(X)$ signifies the σ-algebra of Borel sets in a topological space X; $\mathcal{B}$ means $\mathcal{B}(\mathbb{R})$ (so that $\mathcal{B}(\mathbb{R}^d) = \mathcal{B}^{\otimes d}$), $\mathcal{B}_+$ stands for $\mathcal{B}(\mathbb{R}_+)$; $C_b(X)$ is the set of all bounded continuous functions on X.

If $(S, \mathcal{S})$ is a measurable space, then $\mathbb{M}_+(S), \mathbb{M}(S)$ and $\mathbb{P}(S)$ signify the collections of: all (finite) measures on $\mathcal{S}$, all signed measures on $\mathcal{S}$ and all probability measures on $\mathcal{S}$, respectively. The σ-algebra in $\mathbb{M}_+(S)$ generated by the sets

$$H_B^A = \{m \in \mathbb{M}_+(S): \ m(A) \in B\}, \ A \in \mathcal{S}, \ B \in \mathcal{B}_+,$$

will be denoted $\mathfrak{M}(\mathcal{S})$. In case S is a topological space we equip $\mathbb{M}(S)$ with the topology generated by the sets $\left\{ q \in \mathbb{M}(S): \ \int f \mathrm{d}q \in U \right\}$, where f and U range over $C_b(S)$ and the collection of open sets in $\mathbb{R}$, respectively. In functional analysis, this topology is called $*weak$; in probability theory, its restriction to $\mathbb{M}_+(S)$ is called, somewhat loosely, the *weak topology*. The latter unlike the former is, under rather general assumptions, metrisable ([17], Proposition 1.3.11; this result is attributed in [17] to Prokhorov, but what he did prove in [11] is metrisability of weak `convergence`). We call a topological space S *concordant* if $\mathfrak{M}(\mathcal{S}) = \mathcal{B}(\mathbb{M}_+(S))$, where $\mathcal{S}$ is the σ-algebra of Baire sets in S.

The following statement asserts that a random measure of the level-exceedance type is an $\mathbb{M}_+$-valued random element.

Lemma 1.1. *Let Y be a concordant topological space, $(\Omega, \mathcal{F})$ be a measurable space and υ be a function on $\Omega \times \mathcal{B}(Y)$ such that for any $\omega \in \Omega$ $\upsilon(\omega, \cdot) \in \mathbb{M}_+(Y)$ and for any $A \in \mathcal{B}(Y)$ $\upsilon(\cdot, A)$ is $\mathcal{F}|\mathcal{B}$-measurable. Then $\{\omega : \upsilon(\omega, \cdot) \in D\} \in \mathcal{F}$ for all $D \in \mathcal{B}(\mathbb{M}_+(Y))$.*

Proof. By assumption this is true for all $D = H_B^A$ $(A \in \mathcal{B}(Y), B \in \mathcal{B})$ and therefore, by the definition of $\mathfrak{M}(Y)$, for all its elements. $\qquad\square$

Remark 1.1. The class of concordant topological spaces is vast. Theorem A.2.6 in [2] asserts that it contains all locally compact Polish spaces, in particular $\mathbb{R}^d$. One can prove that it contains even all metric spaces. We shall not use this fact.

Henceforth $\mathcal{X}$ and $\mathcal{Y}$ stand for $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, respectively. We remind that a measure is called *diffuse* if it equals zero at each singleton.

Theorem 1.1. *Let Y be a concordant separable complete metric space, $(\Omega, \mathcal{F})$ be a measurable space and Ψ be an $\mathcal{F}|\mathfrak{M}(\mathcal{Y})$-measurable map from Ω to $\mathbb{P}(Y)$. Then for any separable complete metric space X and any diffuse probability measure μ on $\mathcal{X}$ there exists an $\mathcal{X} \otimes \mathcal{F}|\mathcal{Y}$-measurable map $\xi: \ X \times \Omega \to Y$ such that $\Psi = \mu^\xi$.*

We prove first a particular case of the theorem, formulating it in the autonomous notation.

Lemma 1.2. *Let ν be a probability measure on the Borel σ-algebra of a separable complete metric space Z. Then for any separable complete metric space X and any diffuse probability measure μ on $\mathcal{X}$ there exists a Borel function $f: X \to Z$ such that $\nu = \mu \circ f^{-1}$.*

Proof. By Lemma 4.1.1 in [4] there exists a Borel function $g: X \to [0,1]$ such that $g(x_1) \neq g(x_2)$ as $x_1 \neq x_2$. Denote $m = \mu \circ g^{-1}$, $F(u) = m([0,u])$ $(= 0$ as $u < 0)$, $\widetilde{F}(v) = \sup\{u \in [0,1] : F(u) \leq v\}$. For any $u \in \mathbb{R}$, $\mu(g^{-1}\{u\}) = 0$, because, by the choice of g, the set $g^{-1}\{u\}$ contains either one element or none and the measure μ is, by assumption, diffuse. So F is continuous and therefore $F\left(\widetilde{F}(v)\right) = v$. But $F\left(\widetilde{F}(v)\right) = m\left(F^{-1}[0,v]\right)$. Consequently, $m \circ F^{-1} = \lambda$, where λ is the Lebesgue measure on $\mathcal{B}[0,1]$. Juxtaposing this with the definition of m, we see that

$$\mu \circ (F \circ g)^{-1} = \lambda. \tag{1.1}$$

Skorokhod's principle of a common probability space [13,14] asserts, in particular, existence of a Borel function $h: [0,1] \to Z$ such that $\nu = \lambda \circ h^{-1}$. Comparing this with (1.1), we see that $\nu = \mu \circ (h \circ F \circ g)^{-1}$. $\qquad\square$

Proof of Theorem 1.1. For $X = [0,1]$ and $\mu = \lambda$ this is Theorem 1.2.1 in [15]. Consequently, there exists a $\mathcal{B}[0,1] \otimes \mathcal{F}|\mathcal{Y}$-measurable map $\zeta : [0,1] \times \Omega \to Y$ such that for any $B \in \mathcal{Y}$

$$\psi(B) = \lambda\{z : \zeta(z) \in B\}. \tag{1.2}$$

Evidently, for any $B \subset Y$ and $q \in [0,1]^X$

$$q^{-1}\{z : \zeta(z) \in B\} = \{x : \zeta(q(x)) \in B\}. \tag{1.3}$$

Lemma 1.2 asserts existence of a Borel function $f : X \to [0,1]$ such that $\mu \circ f^{-1} = \lambda$. So, putting in (1.3) $q = f$, we get for any $B \in \mathcal{Y}$

$$\lambda\{z : \zeta(z) \in B\} = \mu\{x : \zeta(f(x)) \in B\}.$$

It remains, in view of (1.2), to set $\xi = \zeta \circ f$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Wishing to discern a measure valued-function and its value on a set, we use throughout the notation $\Psi(t) = \psi(t, \cdot)$, $\Upsilon(t) = \upsilon(t, \cdot)$ which will not be explained further.

Corollary 1.1. *Let Y be a concordant separable complete metric space and Ψ be a $\mathbb{P}(Y)$-valued random function on some set T given on some probability space. Then for any separable complete metric space X and any diffuse probability measure μ on $\mathcal{X}$ there exists a Y-valued varying random relief ξ on X given on the same probability space and such that for all $t \in T$ and $B \in \mathcal{Y}$*

$$\psi(t, B) = \mu\{x : \xi(t, x) \in B\}.$$

Let $(S, \mathcal{S})$ be an arbitrary measurable space and Ψ be an $\mathbb{M}_+(S)$-valued random function on some infinite set T. We denote

$$\overline{\psi}^{\,l}(t_1, \ldots, t_l; A_1, \ldots, A_l) = \mathsf{E}\prod_{j=1}^{l} \psi(t_j, A_j)$$

and call the $\mathbb{R}_+ \cup \{\infty\}$-valued function $\overline{\psi}$ of variables $l \in \mathbb{N}$, $t_1, t_2 \ldots \in T$ ($t_i \neq t_j$ as $i \neq j$), $A_1, A_2 \ldots \in \mathcal{S}$ the *multiplex* of Ψ. The likewise defined and denoted function on $\mathbb{N} \times T^{\mathbb{N}} \times \mathcal{S}^{\mathbb{N}}$ (the equality $t_i = t_j$ is allowed and T may be finite) will be called the *extended multiplex* of Ψ. The multiplex (in both variants) can be identified with the sequence $(\overline{\psi}^{\,l}, l \in \mathbb{N})$ whose lth member is a function on $\mathbf{T} \times \mathcal{S}^l, \mathbf{T} \subset T^l$. The short notation of this sequence will be $\overline{\psi}$.

Theorem 1.2. *Let a separable complete metric space Y, an infinite set T and, for each $l \in \mathbb{N}$, an $\mathbb{R}$-valued function Λ^l of variables $t_1 \in T, t_2 \in T \setminus \{t_1\}, \ldots, t_l \in T \setminus \{t_1, \ldots, t_{l-1}\}$, $A_1 \in \mathcal{Y}, \ldots, A_l \in \mathcal{Y}$ be given. Suppose that these functions have the properties: (a) $\Lambda^l(t_1, \ldots, t_l; A_1, \ldots, A_{l-1}, \cdot) \in \mathbb{M}_+(Y)$;*
(b) for any permutation $\begin{pmatrix} 1 & \ldots & l \\ i_1 & \ldots & i_l \end{pmatrix}$

$$\Lambda^l(t_{i_1}, \ldots, t_{i_l}; A_{i_1}, \ldots, A_{i_l}) = \Lambda^l(t_1, \ldots, t_l; A_1, \ldots, A_l);$$

(c) $\Lambda^l(t_1, \ldots, t_l; A_1, \ldots, A_{l-1}, Y) = \Lambda^{l-1}(t_1, \ldots, t_{l-1}; A_1, \ldots, A_{l-1})$ $(l > 1)$; (d) $\Lambda^1(t; Y) = 1$.
Then there exists a $\mathbb{P}(Y)$-valued random function Υ on T such that Λ is its multiplex.

Proof. Properties (a)–(d) together with separability and completeness of Y imply, by Kolmogorov's theorem, existence of a Y-valued random function ζ on T such that for any $l \in \mathbb{N}, t_1 \in T, t_2 \in T \setminus \{t_1\}, \ldots, t_l \in T \setminus \{t_1, \ldots, t_{l-1}\}$ $A_1 \in \mathcal{Y}, \ldots, A_l \in \mathcal{Y}$

$$\mathsf{P}\{\zeta(t_1) \in A_1, \ldots, \zeta(t_l) \in A_l\} = \Lambda^l(t_1, \ldots, t_l; A_1, \ldots, A_l). \tag{1.4}$$

Put

$$v(t, A) = \begin{cases} 1 & \text{if } \zeta(t) \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Then $v(t, \cdot)$ is a random probability measure on $\mathcal{Y}$ and

$$\overline{v}^l(t_1, \ldots, t_l; A_1, \ldots, A_l) = \mathsf{P}\{\zeta(t_1) \in A_1, \ldots, \zeta(t_l) \in A_l\}. \tag{1.6}$$
$\square$

Theorem 1.3. *Let Y be a separable complete metric space, T be an infinite set and, for each $l \in \mathbb{N}, \Lambda^l$ be an $\mathbb{R}$-valued function on $T^l \times \mathcal{Y}^l$. Suppose that the functions Λ^l have properties (a)–(d) and one more property*

$$\Lambda^l(t_1, \ldots, t_{l-1}, t_{l-1}; A_1, \ldots, A_l) = \Lambda^{l-1}(t_1, \ldots, t_{l-1}; A_1, \ldots, A_{l-2}, A_{l-1} \cap A_l). \tag{1.7}$$

Then there exists a $\mathbb{P}(Y)$-valued random function Υ on T such that Λ is its extended multiplex.

Proof. Let us construct ζ and Υ as in the proof of Theorem 1.2. Then equalities (1.4) and (1.6) hold for all $A_1, \ldots, A_l \in \mathcal{Y}$ and pairwise different $t_1, \ldots, t_l \in T$. Besides that, by the construction of v

$$\overline{v}^l(t_1, \ldots, t_{l-1}, t_{l-1}; A_1, \ldots, A_l) = \overline{v}^{l-1}(t_1, \ldots, t_{l-1}; A_1, \ldots, A_{l-2}, A_{l-1} \cap A_l).$$

This together with condition (1.7) shows that equalities (1.4) and (1.6) remain valid in case some of the points $t_1, \ldots, t_l$ coincide. $\square$

Remark 1.2. The analogue of Theorems 1.2 and 1.3 for the case when T is a singleton (so that MVRF becomes simply random measure and multiplex becomes moment function [2,8,16,26]) was proved in [22]. It contains one more condition and its proof based on convex analysis is by far more difficult.

Let $(S, \mathcal{S})$ be an arbitrary measurable space. We call an $\mathbb{M}_+(S)$-valued random function Ψ on T *momentable* if its extended multiplex assumes only finite values and determines the finite-dimensional distributions of Ψ. Recalling the definition of the σ-algebra $\mathfrak{M}(\mathcal{S})$, we see that the second demand is tantamount to the property that the extended multiplex determines the probabilities $\mathsf{P}\left\{\Psi(t_1) \in H_{B_1}^{A_1}, \ldots, \Psi(t_l) \in H_{B_l}^{A_l}\right\}$ for all possible $l \in \mathbb{N}, A_1, \ldots, A_l \in \mathcal{S}$ and pairwise different $t_1, \ldots, t_l \in T$. But the relation $\Psi(t) \in H_B^A$ means that $\psi(t, A) \in B$. So Ψ is momentable if and only if its extended multiplex is finite and determines the distributions of all the vectors $(\psi(t_1, A_1), \ldots, \psi(t_l, A_l))$, $l \in \mathbb{N}, t_j \in T, A_j \in \mathcal{S}$ (it may seem that we must add "$t_i \neq t_j$ as $i \neq j$", but, obviously, this restriction in this context is redundant).

Denote $\Xi(t) = \psi(t, S)$.

Lemma 1.3. *Let Ψ be an $\mathbb{M}_+(S)$-valued random function on T such that for all $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in T$*

$$\sum_{l=1}^{\infty} \left(\sum_{i=1}^{k} \mathsf{E}\,\Xi(t_i)^l\right)^{-\frac{1}{2l}} = \infty. \tag{1.8}$$

Then Ψ is momentable.

Proof. Let us take arbitrary positive numbers $a_1, \ldots, a_k$ and denote $\beta_i = \psi(t_i, A_i)$, $\alpha = \sum_{i=1}^{k} a_i \beta_i$. By Hölder's inequality

$$\alpha^l \leq \left(\sum_{i=1}^{k} a_i^{l/(l-1)}\right)^{l-1} \sum_{i=1}^{k} \beta_i^l,$$

whence

$$\mathsf{E}\alpha^l \leq k^{l-1} \max_i a_i{}^l \sum_{i=1}^{k} \mathsf{E}\,\Xi(t_i)^l.$$

Then (1.8) yields $\sum (\mathsf{E}\alpha^l)^{-1/2l} = \infty$. This means, by Carleman's theorem ([12], Sect. II.12) that the distribution of the random variable α (and, consequently, the value of $\mathsf{E}e^{-\alpha}$) is determined by its moments. And they, in turn, are determined by the extended multiplex of Ψ. $\qquad\square$

Corollary 1.2. *Any probability-valued random function is momentable.*

Corollary 1.3. *Let Y be a separable complete metric space and Ψ be a $\mathbb{P}(Y)$-valued random function on some infinite set T. Then there exist a Markov kernel $P = P(y, A)$ ($y \in Y, A \in \mathcal{Y}$) and a Y-valued random function ζ on T such that Ψ has the same finite-dimensional distributions as Υ, where*

$$v(t, A) = P(\zeta(t), A). \tag{1.9}$$

Proof. Setting in Theorem 1.3 $\Lambda = \overline{\psi}$, where $\overline{\psi}$ is the extended multiplex of Ψ, we conclude from its proof that $\overline{\psi} = \overline{v}$, where $v(t, A)$ is defined by equality (1.5). And the last is a particular case ($P(y, A) = I_A(y)$) of (1.9). It remains to refer to Corollary 1.2. $\qquad\square$

Remark 1.3. Corollary 1.3 sustains if the set T is finite. For the case when T is a singleton it was proved in [22]. In this case (as well as for any finite T) the representation (1.9) need not perform in its simplest form (1.5).

2. THE TOOL

We shall denote the characteristic function of a measure on $\mathcal{B}^{\otimes d}$ in the following manner:

$$\widehat{m}(z) = \int e^{izy} m(\mathrm{d}y).$$

Here $\int = \int_{\mathbb{R}^d}$, $z \in \mathbb{R}^d$ is a row vector (and other vectors are meant as columns). Henceforth "measure-valued" means "$\mathbb{M}_+(\mathbb{R}^d)$-valued" and "probability-valued" means "$\mathbb{P}(\mathbb{R}^d)$-valued".

We call the function

$$\widetilde{v}^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = \mathsf{E} \prod_{j=1}^{l} \widehat{v}(t_j, z_j) \tag{2.1}$$

of variables $l \in \mathbb{N}, t_1, t_2 \ldots \in T, z_1, z_2 \ldots \in \mathbb{R}^d$ the *covaristic* (the short term for *covariance-characteristic function*) of a momentable MVRF Υ on T. This notion was introduced first for independent of t random probability measures in [18,20], then for probability-valued random functions in [19] and at last, in the present form, in [23] where the following statement was proved.

Theorem 2.1. *The covaristic of a momentable MVRF determines its finite-dimensional distributions.*

Let Υ be a momentable MVRF. Then, for each $l, t_1, \ldots, t_l$, the function $\overline{v}^l(t_1, \ldots, t_l; \cdot, \ldots, \cdot)$ induces a measure on $\mathcal{B}^{\otimes dl}$. This together with the definition of multiplex and Fubini's theorem yields an equivalent form of equality (2.1):

$$\widetilde{v}^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = \widehat{\overline{v}^l}(t_1, \ldots, t_l; z_1, \ldots, z_l) \equiv \int \ldots \int \exp\left(i \sum_{j=1}^{l} z_j y_j\right) \overline{v}^l(t_1, \ldots, t_l; \mathrm{d}y_1, \ldots, \mathrm{d}y_l). \tag{2.2}$$

We shall identify the covaristic of a momentable MVRF Υ with the sequence $(\widetilde{v}^l)$ whose lth member is defined by (2.1).

Theorem 2.2. *Let T be an infinite set and, for $l = 1, 2, \ldots,$ Φ^l be a $\mathbb{C}$-valued function on $T^l \times \mathbb{R}^{dl}$ such that:* $\Phi^l(t_1, \ldots, t_l; \cdot, \ldots, \cdot)$ *is positive definite,*

$$\lim_{z_1 \to 0, \ldots, z_l \to 0} \Phi^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = 1$$

and for any permutation $\begin{pmatrix} 1 & \ldots & l \\ i_1 & \ldots & i_l \end{pmatrix}$

$$\Phi^l(t_{i_1}, \ldots, t_{i_l}; z_{i_1}, \ldots, z_{i_l}) = \Phi^l(t_1, \ldots, t_l; z_1, \ldots, z_l). \tag{2.3}$$

Suppose also that

$$\Phi^1(t; 0) = 1 \tag{2.4}$$

and for any $l \geq 2, t_1, \ldots, t_l, z_1, \ldots, z_l$

$$\Phi^l(t_1, \ldots, t_l; z_1, \ldots, z_{l-1}, 0) = \Phi^{l-1}(t_1, \ldots, t_{l-1}; z_1, \ldots, z_{l-1}). \tag{2.5}$$

Then there exists a probability-valued random function Υ on T such that

$$\widetilde{v}^{\,l}(t_1, \ldots, t_l; z_1, \ldots, z_l) = \Phi^l(t_1, \ldots, t_l; z_1, \ldots, z_l) \tag{2.6}$$

for all $l \in \mathbb{N}, t_1 \in T, t_2 \in T \setminus \{t_1\}, \ldots, t_l \in T \setminus \{t_1, \ldots, t_{l-1}\}, z_1, \ldots, z_l \in \mathbb{R}^d$.

Proof. Denote $\mathbf{t} = (t_1, \ldots, t_l)$, $\mathbf{z} = (z_1, \ldots, z_l)$. The first two assumptions imply, by the Bochner-Khinchin theorem, existence of a probability measure $P^l(\mathbf{t}; \cdot)$ on $\mathcal{B}^{\otimes dl}$ such that

$$\Phi^l(\mathbf{t}; \mathbf{z}) = \int_{\mathbb{R}^{dl}} e^{i\mathbf{z}\mathbf{y}} P^l(\mathbf{t}; d\mathbf{y}). \tag{2.7}$$

Denote

$$\Lambda^l(t_1, \ldots, t_l; A_1, \ldots, A_l) = P^l(\mathbf{t}; A_1 \times \ldots \times A_l). \tag{2.8}$$

Obviously, this function has property (a) from Theorem 1.2. Equality (2.3) shows that the sequence (Λ^l) has property (b). Equalities (2.5), (2.7) and (2.8) yield property (c). Finally, (d) is immediate from (2.4), (2.7) and (2.8). Then Theorem 1.2 asserts existence of a probability-valued random function Υ on T such that

$$P^l(\mathbf{t}; A_1 \times \ldots \times A_l) = \overline{v}^{\,l}(\mathbf{t}; A_1, \ldots, A_l).$$

Substituting this to (2.7) and taking to account (2.2), we arrive at (2.6). □

Remark 2.1. The analogue of Theorem 2.2 for the case when T is a singleton (so that MVRF becomes simply random measure) was proved in [22]. It contains an extra condition, and its proof is more complicated.

Corollary 2.1. *Let the conditions of Theorem 2.2 be fulfilled. Then for any separable complete metric space X and any diffuse probability measure μ on $\mathcal{X}$ there exists an $\mathbb{R}^d$-valued varying random relief Q on X such that*

$$\Phi^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = \mathsf{E} \prod_{j=1}^{l} \int e^{iz_j Q(t_j, x)} \mu(dx). \tag{2.9}$$

Proof. Corollary 1.1 asserts existence of a varying random relief Q such that $v(t, B) = \mu\{x : Q(t, x) \in B\}$. Hence $\widehat{v}(t, z) = \int e^{izQ(t,x)} \mu(dx)$, which together with (2.1) and (2.6) yields (2.9). □

Let Y be a metric space. We denote

$$\mathfrak{F} = \left\{ \varphi \in \mathbb{R}^Y : \forall\, y, y' \in Y \;\; |\varphi(y)| \le 1 \;\&\; |\varphi(y) - \varphi(y')| \le \operatorname{dist}(y, y') \wedge 1 \right\}$$

and equip $\mathbb{M}(Y)$ with the *bounded Lipschitz seminorm* $\|q\|_{BL} = \sup\limits_{\varphi \in \mathfrak{F}} \left| \int \varphi \mathrm{d}q \right|$. The respective pseudometric will be denoted d_{BL}.

Lemma 2.1. *Let Y be a separable complete metric space. Then:* (1) d_{BL} *is a metric in* $\mathbb{M}_+(Y)$; (2) *the metric space* $(\mathbb{M}_+(Y), d_{BL})$ *is complete;* (3) *the convergence in this metric space coincides with the weak one.*

Proof. Since any finite measure is proportional to a probability measure, all the three statements are easy consequences of the well-known similar properties of $\mathbb{P}_+(Y)$ (see, *e.g.*, Thms. 2.3.8 and 2.3.9, Lem. 2.4.1 and Cor. 2.4.3 in [6]). $\qquad\square$

Remark 2.2. The second and the third statement of Lemma 2.1 do not carry over to the space $\mathbb{M}(Y)$!

We use the following notation of convergences: $\overset{\mathrm{d}}{\longrightarrow}$ means weak convergence of finite-dimensional distributions of $\mathbb{R}^k$-valued or $\mathbb{C}^k$-valued random functions (whose argument may take values in a finite set as well as in an infinite one); $\overset{d}{\longrightarrow}$ means the same for measure-valued functions. If $\alpha_1, \alpha_2, \ldots, \alpha$ are $\mathbb{R}^k$- or $\mathbb{C}^k$-valued càdlàg random processes such that α is continuous and $\phi(\alpha_n) \overset{\mathrm{d}}{\longrightarrow} \phi(\alpha)$ for any continuous in the locally uniform topology functional ϕ on the Skorokhod space D, then we write $\alpha_n \overset{\mathrm{C}}{\longrightarrow} \alpha$. We say that a sequence (α_n) is *relatively compact* (r.c.) in C if each its subsequence contains, in turn, a subsequence convergent in the above sense. This terminology carries over to measure-valued processes (MVP) with $\mathcal{C}$ instead of C in notation.

We also denote $A_L = \{ x \in \mathbb{R}^d : |x| > L \}$. The choice of norm in $\mathbb{R}^d$ is, as will be seen from the conditions where A_L performs, inessential.

Theorem 2.3. *Let (Ψ_n) be a sequence of MVP's such that: for any $l \in \mathbb{N}, t_1, \ldots, t_l \in \mathbb{R}_+, z_1, \ldots, z_l \in \mathbb{R}^d$*

$$\widetilde{\psi}_n^l(t_1, \ldots, t_l; z_1, \ldots, z_l) \to U^l(t_1, \ldots, t_l; z_1, \ldots, z_l); \tag{2.10}$$

for any $k \in \mathbb{N}$, $s_1, \ldots, s_k, t \in \mathbb{R}_+$

$$\sum_{l=1}^{\infty} \left(\sum_{j=1}^{k} U^l(s_j, \ldots, s_j; 0, \ldots, 0) \right)^{-\frac{1}{2l}} = \infty, \tag{2.11}$$

$$\sup_n \mathsf{E}\, \Xi_n(t)^k < \infty; \tag{2.12}$$

for any $t \in \mathbb{R}_+$

$$\lim_{z \to 0} U^1(t; z) = U^1(t; 0); \tag{2.13}$$

for any $t > 0, \varepsilon > 0$ and $z \in \mathbb{R}^d$

$$\lim_{L \to \infty} \overline{\lim_{n \to \infty}} \sup_{s \le t} \mathsf{P}\{\Xi_n(s) > L\} = 0, \tag{2.14}$$

$$\lim_{L \to \infty} \overline{\lim_{n \to \infty}} \sup_{s \le t} \mathsf{P}\{\psi_n(s, A_L) > \varepsilon\} = 0, \tag{2.15}$$

$$\lim_{c \to \infty} \overline{\lim_{n \to \infty}} \sup_{(v-c)_+ \le u < v \le t} \mathsf{P}\left\{ \left| \widehat{\psi}_n(u, z) - \widehat{\psi}_n(v, z) \right| > \varepsilon \right\} = 0. \tag{2.16}$$

Then U is the covaristic of some momentable MVP Ψ and $\Psi_n \overset{d}{\longrightarrow} \Psi$.

Proof. Let T be an arbitrary countable dense set in $\mathbb{R}_+$. Theorem 2 [23] asserts that under conditions (2.10)–(2.13) where $t_1, \ldots, t_l, s_1, \ldots, s_k$ and t are taken from T there exists an MVRF Ψ on T such that the conclusion of the theorem holds on T. Consequently, for any $k \in \mathbb{N}, t_1, \ldots, t_k \in T$ and bounded continuous functional ϕ on $(\mathbb{M}_+(\mathbb{R}^d))^k$

$$\mathsf{E}\phi(\Psi_n(t_1), \ldots, \Psi_n(t_k)) \to \mathsf{E}\phi(\Psi(t_1), \ldots, \Psi(t_k)). \tag{2.17}$$

To assert the same for arbitrary $t_1, \ldots, t_k \in \mathbb{R}_+$ it suffices, in the light of Lemma 2.1, to show that for all $t > 0, \varepsilon > 0$

$$\lim_{c \to \infty} \overline{\lim_{n \to \infty}} \sup_{(v-c)_+ \leq u < v \leq t} \mathsf{P}\left\{ \|\Psi_n(u) - \Psi_n(v)\|_{BL} > \varepsilon \right\} = 0. \tag{2.18}$$

Indeed, relation (2.18) ensures uniform in any interval stochastic continuity of Ψ and thus allows to uniquely extend Ψ to a stochastically continuous MVP on $\mathbb{R}_+$, thereupon (2.17) carries over, again due to (2.18), from T to $\mathbb{R}_+$.

Let us equip $\mathfrak{F}$ (defined for $Y = \mathbb{R}^d$) with the metric

$$\delta(f, g) = \sum_{L=1}^{\infty} 2^{-L} \max_{|x| \leq L} |f(x) - g(x)|.$$

Functions from $\mathfrak{F}$ being uniformly bounded and equicontinuous, each sequence of its members contains a subsequence converging uniformly on each compact set. So for any $a > 0$ there exists a finite a-mesh in $\mathfrak{F}$.

Let us fix $f \in \mathfrak{F}, L \in \mathbb{N}$ and a-mesh $\{g_1, \ldots, g_r\}$. We choose $g \in \{g_1, \ldots, g_r\}$ such that $\delta(f, g) < a$ and, for each $j \in \{1, \ldots, r\}$, take a trigonometric polynomial p_j such that $\sup_{|x| \leq L} |g_j(x) - p_j(x)| < a$ (the p_j corresponding to g will be written without subscript). Denote $C = \|g - p\|_\infty$. Then for any finite measure m

$$\int |f - g| \mathrm{d}m \leq 2^L am\left(\mathbb{R}^d\right) + 2m(A_L), \quad \int |g - p| \mathrm{d}m \leq am\left(\mathbb{R}^d\right) + Cm(A_L).$$

These inequalities together with the identity

$$\int f \mathrm{d}m_1 - \int f \mathrm{d}m_2 = \int (f - g)\mathrm{d}m_1 - \int (f - g)\mathrm{d}m_2 + \int (g - p)\mathrm{d}m_1 - \int (g - p)\mathrm{d}m_2 + \int p \mathrm{d}m_1 - \int p \mathrm{d}m_2$$

imply that

$$\|m_1 - m_2\| \leq \sum_{i=1}^{2} \left(\left(2^L + 1\right) am_i\left(\mathbb{R}^d\right) + (C+2)m_i(A_L) \right) + \max_{1 \leq j \leq r} \left| \int p_j \mathrm{d}m_1 - \int p_j \mathrm{d}m_2 \right|.$$

Setting $m_1 = \Psi_n(u)$, $m_2 = \Psi_n(v)$, we deduce (2.18) from (2.14)–(2.16). $\square$

Corollary 2.2. *Let the conditions of Theorem 2.3 be fulfilled. Then for all $l \in \mathbb{N}$ and $z_1, \ldots, z_l \in \mathbb{R}^d$*

$$\left(\widehat{\psi}_n(\cdot, z_1), \ldots, \widehat{\psi}_n(\cdot, z_l) \right) \xrightarrow{\mathrm{d}} \left(\widehat{\psi}(\cdot, z_1), \ldots, \widehat{\psi}(\cdot, z_l) \right).$$

Corollary 2.3. *Let a sequence (Ψ_n) of probability-valued processes (PVP) satisfy conditions (2.10), (2.15), (2.16) and let for any $t \in \mathbb{R}_+$*

$$\lim_{z \to 0} U^1(t; z) = 1. \tag{2.19}$$

Then U is the covaristic of some PVP Ψ and $\Psi_n \xrightarrow{d} \Psi$.

We denote $\Xi_n^*(t) = \sup_{u \leq t} \Xi_n(u)$ and introduce the conditions:

RC. For any z the sequence $\left(\widehat{\psi}_n(\cdot, z) \right)$ is r.c. in C.

UI. For any t the sequence $(\Xi_n^*(t))$ is uniformly integrable.

Taking to account that condition **RC** entails (2.16) and that relative compactness in C of a sequence of vector processes is a component-wise property ([7], VI.3.33), we can refine Corollary 2.2 as follows:

Corollary 2.4. *Let conditions* (2.10)–(2.15) *and* **RC** *be fulfilled. Then the conclusion of Theorem* 2.3 *holds and for all* $l \in \mathbb{N}$, $z_1, \ldots, z_l \in \mathbb{R}^d$

$$\left(\widehat{\psi}_n(\cdot, z_1), \ldots, \widehat{\psi}_n(\cdot, z_l) \right) \xrightarrow{\text{C}} \left(\widehat{\psi}(\cdot, z_1), \ldots, \widehat{\psi}(\cdot, z_l) \right).$$

Below, l.i.p. signifies the limit in probability.

Theorem 2.4. *Let a sequence* (Ψ_n) *of MVP's satisfy conditions* (2.10)–(2.15), **RC** *and* **UI***. Suppose that the characteristic function of the existing by Corollary* 2.4 *MVP* Ψ *such that* $\Psi_n \xrightarrow{d} \Psi$ *and* $\widetilde{\psi} = U$ *satisfies, for any* t, *the relation*

$$\text{l.i.p.} \sup_{z \to 0} \operatorname{Re} \left(\widehat{\psi}(u, 0) - \widehat{\psi}(u, z) \right) = 0. \tag{2.20}$$

Then $\Psi_n \xrightarrow{\mathcal{C}} \Psi$.

Proof. In view of Corollary 2.4 it suffices to show that the sequence (Ψ_n) is r.c. in $\mathcal{C}$.

We fix t and denote $S_n(u, z) = \operatorname{Re} \left(\widehat{\psi}_n(u, 0) - \widehat{\psi}_n(u, z) \right)$, $S_n^*(z) = \sup_{u \le t} S_n(u, z)$, $K_n^L = \sup_{u \le t} \psi_n(u, A_L)$, $B_R = \{x : |x| < R\}$ (in this proof, $|\cdot|$ is the sup-norm), $f_R(y) = \int_{B_R} (1 - e^{izy}) dz$. The jth component of $y \in \mathbb{R}^d$ will be denoted y^j.

Obviously,

$$f_R(y) = (2R)^d \left(1 - \prod_{j=1}^{d} \frac{\sin Ry^j}{Ry^j} \right),$$

so that f_R assumes only real nonnegative values and

$$f_R(y) > 2^{d-1} R^d \quad \text{as} \quad |y| > 2/R. \tag{2.21}$$

By Fubini's theorem and due to nonnegativeness of f_R

$$\int_{B_R} \left(\widehat{\psi}_n(u, 0) - \widehat{\psi}_n(u, z) \right) dz = \int f_R(y) \psi_n(u, dy) \ge \int_{A_L} f_R(y) \psi_n(u, dy),$$

whence, setting $R = 2/L$, we get with account of (2.21)

$$\int_{B_{2/L}} S_n(u, z) dz \ge 2^{2d-1} L^{-d} \psi_n(u, A_L).$$

Consequently,

$$K_n^L \le 2^{1-2d} L^d \int_{B_{2/L}} S_n^*(z) dz. \tag{2.22}$$

The evident inequalities

$$0 \le S_n^*(z) \le \Xi_n^*(t) \tag{2.23}$$

together with condition **UI** allow to apply to (2.22) consecutively Fubini's and Fatou's theorems, which results in

$$\overline{\lim_{n \to \infty}} \, \mathsf{E} K_n^L \le 2^{1-2d} L^d \int_{B_{2/L}} \overline{\lim_{n \to \infty}} \, \mathsf{E} S_n^*(z) dz. \tag{2.24}$$

Obviously, the functional $\phi(f) = \sup_{u \leq t} \operatorname{Re} f(u)$ is continuous in the locally uniform topology and $S_n^*(z) = \phi\left(\widehat{\psi}_n(\cdot, 0) - \widehat{\psi}_n(\cdot, z)\right)$, $\Xi_n^*(t) = \phi\left(\widehat{\psi}_n(\cdot, 0)\right)$. Then it follows from Corollary 2.4 that

$$(S_n^*(z), \Xi_n^*(t)) \xrightarrow{\mathrm{d}} (S^*(z), \Xi^*(t)) \equiv \left(\phi\left(\widehat{\psi}(\cdot, 0) - \widehat{\psi}(\cdot, z)\right), \phi\left(\widehat{\psi}(\cdot, 0)\right)\right),$$

whence on the strength of (2.23) and **UI**

$$S^*(z) \leq \Xi^*(t), \ \mathsf{E}\,\Xi^*(t) < \infty \tag{2.25}$$

and $\varlimsup_{n \to \infty} \mathsf{E} S_n^*(z) = \mathsf{E} S^*(z)$. The last equality together with (2.24) yields

$$\varlimsup_{n \to \infty} \mathsf{E} K_n^L \leq 2 \sup_{z \in B_{2/L}} \mathsf{E} S^*(z).$$

From (2.20) and (2.25) we deduce, by the dominated convergence theorem, that the right-hand side of the last inequality tends to zero as $L \to \infty$. So

$$\lim_{L \to \infty} \varlimsup_{n \to \infty} \mathsf{P}\left\{\sup_{u \leq t} \psi_n(u, A_L) > \varepsilon\right\} = 0$$

for any $\varepsilon > 0$. Now, relative compactness of (Ψ_n) in $\mathcal{C}$ follows from **RC** by Corollary 1 in [24]. $\qquad\square$

Corollary 2.5. *Let a sequence (Ψ_n) of PVP's satisfy conditions* (2.10), (2.15), (2.19) *and* **RC** *and let the characteristic function of the existing by Corollary* 2.3 *PVP Ψ such that $\Psi_n \xrightarrow{d} \Psi$ satisfy condition* (2.20). *Then $\Psi_n \xrightarrow{\mathcal{C}} \Psi$.*

Now, we are ready to apply the covaristics method. Another application – to empirical processes – was given in [23].

3. A GEOMETRIC MODEL

We will study from this time on sequences of MVP's of the level-exceedance type: $\Psi_n = \mu^{\xi_n}$, or, minutely,

$$\psi_n(t, B) = \mu\{x : \xi_n(t, x) \in B\}, \ B \in \mathcal{B}^{\otimes d}, \tag{3.1}$$

where ξ_n is an $\mathbb{R}^d$-valued varying random relief on X and μ is a nonrandom finite measure on $\mathcal{X}$. Such processes are usual in stochastic geometry, to mention especially the coverage problem [5] studying geometric characteristics of k-multiple $(k \in \mathbb{Z}_+)$ intersections of random sets. If this characteristic is measure, then we come to (3.1), where ξ_n is a sum of indicators of arguments t, x and ω. If we ascribe, for physical or other reasons, to each covering set its thickness, then ξ_n will be a finite linear combination of indicators. We may go farther and assume that ξ_n is a sum of random "humps" and "hollows" which appear at random points of space and time, changing afterwards their location, size and, perhaps, shape. We will consider a model where shapes are fixed. It is described by the following assumptions.

A1. $X = \mathbb{R}^p$.

A2. $\xi_n(t, x) = \sum_{k:\tau_{nk} \leq t} F\left(\frac{n^{1/p}(x - \zeta_{nk} - \eta_{nk}(t))}{\rho_{nk}(t)}\right)$,

where $F : X \to \mathbb{R}^d$ is a nonrandom Borel function; $(\tau_{nk}, k \in \mathbb{N})$ is a strictly increasing and a.s. tending to infinity sequence of positive random variables; $\zeta_{nk}, \eta_{nk}(t)$ and $\rho_{nk}(t)$ are random variables with values in X, X and $\mathbb{R}_+$, respectively (if $\rho_{nk}(t) = 0$, then, as the subsequent assumptions **A5** and (3.3) will show, the respective summand equals zero for almost all x). All these random variables are given on a common probability space $(\Omega_n, \mathcal{F}_n, \mathsf{P}_n)$ (but in formulae we write P and E without subscript).

A3. There exists a σ-algebra $\mathcal{F}_n^0 \subset \mathcal{F}_n$ such that: (i) for all $k \in \mathbb{N}$ and $t > 0$ the random variables τ_{nk} and $\eta_{nk}(t)$ are $\mathcal{F}_n^0$-measurable; (ii) for any $l \in \mathbb{N}$ and positive numbers $t_1, \ldots, t_l$ the random vectors $(\zeta_{nk}, \rho_{nk}(t_1), \ldots, \rho_{nk}(t_l))$, $k \in \mathbb{N}$, are conditionally w.r.t. $\mathcal{F}_n^0$ totally independent; (iii) for any $k \in \mathbb{N}$ and $t > 0$ ζ_{nk} and $\rho_{nk}(t)$ are conditionally w.r.t. $\mathcal{F}_n^0$ independent.

A4. For each k ζ_{nk} has a conditional w.r.t. $\mathcal{F}_n^0$ distribution density h_{nk}.

A5. The measure μ has a bounded density w.r.t. the Lebesgue measure.

Remark 3.1. Similar models were studied in [19,21]. They feature with milder probabilistic assumptions but stricter geometric ones. In those models locations and sizes of "elementary reliefs" (the summands in **A2**) do not depend on time: $\eta_{nk} = 0$, $\rho_{nk}(t) = \rho_{nk}$. The refusal from this assumption cardinally changes the emphasis and drastically complicates the rationale.

We introduce the notation: $\mathsf{E}^0 = \mathsf{E}[\cdots | \mathcal{F}_n^0]$ $R_{nk}(t, A) = \mathsf{P}\{\rho_{nk}(t) \in A | \mathcal{F}_n^0\}$ (the regular version),

$$b_{nk}(t) = \mathsf{E}^0 \rho_{nk}(t)^p \equiv \int_0^\infty r^p R_{nk}(t, \mathrm{d}r), \quad N_n(t) = \#\{k : \tau_{nk} \le t\}, \quad Q_n(t, x) = \frac{1}{n} \sum_{k=1}^{N_n(t)} b_{nk}(t) h_{nk}(x - \eta_{nk}(t)),$$

$Q_n(t) = Q_n(t, \cdot)$, $\xi_n(t) = \xi_n(t, \cdot)$, $\mathbb{C}_- = \{\mathfrak{z} \in \mathbb{C} : \mathrm{Re}\,\mathfrak{z} \le 0\}$, $\int = \int_{\mathbb{R}^m}$ (m will be determined by the context), $\mathsf{M}^l = \int_{X^l} \cdots \mathrm{d}\mu^{\otimes l} \equiv \int \cdots \int \cdots \mu(\mathrm{d}x_1) \ldots \mu(\mathrm{d}x_l)$, $\mathsf{M} = \mathsf{M}^1$. Then (3.1) yields

$$\widehat{\psi}_n(t, z) = \mathsf{M}\mathrm{e}^{\mathrm{i}z\xi_n(t)}. \tag{3.2}$$

This formula, otherwise written, was already used in the proof of Corollary 2.1.

Theorem 3.1. *Let, for each $n \in \mathbb{N}$, an MVP Ψ_n be defined by equality* (3.1) *and assumptions* **A1**–**A5**. *Suppose also that:*

$$\int |F(y)| \mathrm{d}y < \infty; \tag{3.3}$$

for each $k \in \mathbb{N}$ there exists an $\mathcal{F}_n^0$-measurable random variable ϑ_{nk} such that

$$\sup_{x \in X} h_{nk}(x) \le \vartheta_{nk}; \tag{3.4}$$

for each k there exists a $\mathcal{B}_+ \otimes \mathcal{F}_n^0$-measurable in $(r, \omega) \in \mathbb{R}_+ \times \Omega_n$ random function D_{nk} on $\mathbb{R}_+$ such that for all $x', x'' \in X$

$$|h_{nk}(x') - h_{nk}(x'')| \le D_{nk}(|x' - x''|); \tag{3.5}$$

for all $t > 0, c > 0, \varepsilon > 0$ and $z \in \mathbb{R}^d$

$$\sup_{s \le t} \mathsf{P}\left\{ \frac{1}{n} \sum_{k=1}^{N_n(s)} \vartheta_{nk} \int_0^\infty r^p R_{nk}(s, \mathrm{d}r) \int_{|y| > cn^{1/p}/r} |\sin zF(y)| \mathrm{d}y > \varepsilon \right\} \to 0, \tag{3.6}$$

$$\sup_{s \le t} \mathsf{P}\left\{ \frac{1}{n} \sum_{k=1}^{N_n(s)} \int_0^\infty r^p R_{nk}(s, \mathrm{d}r) \int |\sin zF(y)| D_{nk}\left(\frac{r|y|}{n^{1/p}} \right) \mathrm{d}y > \varepsilon \right\} \to 0, \tag{3.7}$$

$$\lim_{L \to \infty} \overline{\lim_{n \to \infty}} \sup_{s \le t} \mathsf{P}\left\{ \sum_{k=1}^{N_n(s)} \vartheta_{nk} b_{nk}(s) > nL \right\} = 0; \tag{3.8}$$

for all $l \in \mathbb{N}, t_1, \ldots, t_l \in \mathbb{R}_+$ and $s_1, \ldots, s_l \in \mathbb{R}$

$$\mathsf{E} \prod_{j=1}^{l} \mathsf{M} \mathrm{e}^{\mathrm{i}s_j Q_n(t_j)} \to Z^l(t_1, \ldots, t_l; \mathrm{i}s_1, \ldots, \mathrm{i}s_l), \tag{3.9}$$

$$\lim_{\substack{c_1, \ldots, c_l \in \mathbb{R}, \\ c_1 \to 0, \ldots, c_l \to 0}} Z^l(t_1, \ldots, t_l; \mathrm{i}c_1, \ldots, \mathrm{i}c_l) = \mu(X)^l; \tag{3.10}$$

for any $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_-$ and $t > 0$

$$\lim_{n \to \infty} \sup_{0 \le t_1 \le t_2 \le t} \left| \mathsf{E} \prod_{j=1}^{2} \mathsf{M} \mathrm{e}^{\mathfrak{z}_j Q_n(t_j)} - Z^2(t_1, t_2; \mathfrak{z}_1, \mathfrak{z}_2) \right| = 0 \tag{3.11}$$

and the function $Z^2(\cdot, \cdot; \mathfrak{z}_1, \mathfrak{z}_2)$ is continuous. Then:

(1) there exists an $\mathbb{R}_+$-valued random varying relief Q on X such that for all $l \in \mathbb{N}, t_1, \ldots, t_l \in \mathbb{R}_+$ and $\mathfrak{z}_1, \ldots, \mathfrak{z}_l \in \mathbb{C}_-$

$$Z^l(t_1, \ldots, t_l; \mathfrak{z}_1, \ldots, \mathfrak{z}_l) = \mathsf{E} \prod_{j=1}^{l} \mathsf{M} \mathrm{e}^{\mathfrak{z}_j Q(t_j)}; \tag{3.12}$$

(2) $\Psi_n \overset{d}{\longrightarrow} \Psi$, where

$$\widehat{\psi}(t, z) = \mathsf{M} \exp \left\{ Q(t) \int \left(\mathrm{e}^{\mathrm{i}z F(y)} - 1 \right) \mathrm{d}y \right\}. \tag{3.13}$$

Theorem 3.2. *Let, for each n, an MVP Ψ_n be defined by equality (3.1) and assumptions **A1**–**A5**. Suppose that conditions (3.3)–(3.10) and **RC** are fulfilled. Then $\Psi_n \overset{\mathcal{C}}{\longrightarrow} \Psi$, where Ψ and Q are the same as in Theorem 3.1.*

The comparison of conditions (3.9) and (3.10) with similar conditions (2.10) and (2.13) of Theorem 2.3 and Corollaries 2.2–2.5 reveals the gist of Theorems 3.1 and 3.2: they reduce the problem we solve for the MVP's $\Psi_n \equiv \mu^{\xi_n}$ to a similar problem for the MVP's μ^{Q_n}. The latter is much simpler, because so are the reliefs Q_n in comparison with the ξ_n's and, besides, we do not impose on (Q_n) the analogues of conditions (2.15) and (2.16). Herein the sequences (ξ_n) and (Q_n) are not asymptotically close in any customary sense. Indeed, the discontinuities of $\xi_n(\cdot, x)$ need not be small, whereas the processes $Q_n(\cdot, x)$ are, under rather general assumptions, asymptotically continuous. It will be seen from the proof of Theorem 3.1 that $Q_n(t, x)$ is an asymptotic equivalent (w.r.t. the proximity in probability) of $\mathsf{E}^0 \xi_n(t, x)$.

The proofs of both theorems will be adduced in Section 4. And our present goal is to prove three ancillary statements facilitating the verification of their conditions.

Lemma 3.1. *Let $\{h_n\}$ be a set of nonnegative Borel functions on $\mathbb{R}^p$ such that $\sup_n \int h_n(x) \mathrm{d}x < \infty$ and*

$$\sup_n \sup_{|x-y| \le r} |h_n(x) - h_n(y)| < \infty$$

for some $r > 0$. Then there exists a constant C such that for all n and x

$$h_n(x) \le C. \tag{3.14}$$

Proof. $h_n(x) = \left(\int_{|y-x|<r} (h_n(x) - h_n(y)) \mathrm{d}y + \int_{|y-x|<r} h_n(y) \mathrm{d}y \right) \Big/ \int_{|y|<r} \mathrm{d}y.$ $\qquad \square$

Lemma 3.2. *Let, for each n, N_n be a counting process with compensator Λ_n, and let the sequence (Λ_n/n) be r.c. in $\mathbb{C}$. Then so is (N_n/n).*

Proof. Denote $\breve{N}_n = N_n - \Lambda_n$. This is a local square integrable martingale such that for all $v \geq u \geq 0$

$$\left\langle \breve{N}_n \right\rangle(v) - \left\langle \breve{N}_n \right\rangle(u) \leq \Lambda_n(v) - \Lambda_n(u) \tag{3.15}$$

(equality is attained if Λ_n is continuous). This inequality and relative compactness of (Λ_n/n) in C imply, by Propositions VI.3.26 and VI.3.9 in [7], that the sequence $\left(\left\langle \breve{N}_n/n \right\rangle\right)$ is r.c. in C, too. Then by Rebolledo's theorem ([7], VI.4.13) the sequence $\left(\breve{N}_n/n\right)$ is r.c. in D. Hence, writing $N_n = \breve{N}_n + \Lambda_n$, we conclude by Lemma VI.3.31 [7] that the sequence (N_n/n) is also r.c. in D. But all jumps of N_n/n are of size $1/n$, so Proposition VI.3.26 in [7] asserts that this sequence is r.c. in C, as well. $\qquad\square$

To apply Theorem 3.2 one must be able to check condition **RC**. Since $\left|\widehat{\psi}_n(t, z)\right| = 1$, it is equivalent, by Proposition VI.3.26 [7], to the relation

$$\lim_{c \to 0} \overline{\lim_{n \to \infty}} \, \mathsf{P} \left\{ \sup_{(v-c)_+ < u < v \leq t} \left|\widehat{\psi}_n(v, z) - \widehat{\psi}_n(u, z)\right| > \varepsilon \right\} = 0, \tag{3.16}$$

where t and ε are arbitrary positive numbers and z is an arbitrary row vector in $\mathbb{R}^d$. The next statement reduces it to simpler ones.

Henceforth, a stands for $1/p$. Also, we denote $F_{nk}(t, x) = F\left(\frac{n^a(x - \zeta_{nk} - \eta_{nk}(t))}{\rho_{nk}(t)}\right)$, $F_{nk}(t) = F_{nk}(t, \cdot), \Phi_n(u, t) = \sum_{\tau_{nk} \leq u} F_{nk}(t)$ (so that $\xi_n(t) = \Phi_n(t, t)$).

Lemma 3.3. *Let, for each $n \in \mathbb{N}$, an MVP Ψ_n be defined by equality* (3.1) *and assumptions* **A1**, **A2** *and* **A5**. *Suppose also that for some $z \in \mathbb{R}^d$, $t > 0$ and any $\varepsilon > 0$*

$$\int \left|\sin \frac{zF(y)}{2}\right| dy < \infty, \tag{3.17}$$

$$\lim_{c \to 0} \overline{\lim_{n \to \infty}} \, \mathsf{P} \left\{ \sup_{(v-c)_+ < u < v \leq t} \sum_{u < \tau_{nk} \leq v} \rho_{nk}(v)^p > n\varepsilon \right\} = 0, \tag{3.18}$$

$$\lim_{c \to 0} \overline{\lim_{n \to \infty}} \, \mathsf{P} \left\{ \sup_{(v-c)_+ < u < v \leq t} \left|\mathsf{Me}^{\mathrm{i}z\Phi_n(u,v)} - \mathsf{Me}^{\mathrm{i}z\Phi_n(u,u)}\right| > \varepsilon \right\} = 0. \tag{3.19}$$

Then relation (3.16) *holds for these z, t and all ε.*

Proof. The identity

$$\mathrm{e}^{\mathrm{i}z\xi_n(v)} - \mathrm{e}^{\mathrm{i}z\xi_n(u)} = \mathrm{e}^{\mathrm{i}z\Phi_n(u,v)} \left(\exp\left\{\mathrm{i}z \sum_{u < \tau_{nk} \leq v} F_{nk}(v)\right\} - 1\right) + \mathrm{e}^{\mathrm{i}z\Phi_n(u,v)} - \mathrm{e}^{\mathrm{i}z\Phi_n(u,u)}$$

together with (3.2) yields

$$\left|\widehat{\psi}_n(v, z) - \widehat{\psi}_n(u, z)\right| \leq \sum_{u < \tau_{nk} \leq v} \mathsf{M} \left|\mathrm{e}^{\mathrm{i}zF_{nk}(v)} - 1\right| + \left|\mathsf{Me}^{\mathrm{i}z\Phi_n(u,v)} - \mathsf{Me}^{\mathrm{i}z\Phi_n(u,u)}\right|. \tag{3.20}$$

Let f denote the existing by assumption **A5** Lebesgue density of μ. By the same assumption f is bounded. Denote also $\sigma_{nk}(t) = \zeta_{nk} + \eta_{nk}(t)$. Then

$$
\begin{aligned}
\mathsf{M}\left|\mathrm{e}^{\mathrm{i}zF_{nk}(v)} - 1\right| &= 2\mathsf{M}\int\left|\sin\left(\frac{z}{2}F\left(\frac{n^a(x - \sigma_{nk}(v))}{\rho_{nk}(v)}\right)\right)\right|f(x)\mathrm{d}x \\
&= \frac{2\rho_{nk}(v)^p}{n}\int\left|\sin\frac{zF(y)}{2}\right|f\left(\sigma_{nk}(v) + \frac{\rho_{nk}(v)y}{n^a}\right)\mathrm{d}y,
\end{aligned}
$$

which jointly with (3.20)–(3.17) and boundedness of f implies (3.16).   □

The first of three examples illustrates Theorem 3.1. Assumptions **A1**–**A5** enter (maybe, in a sharpened form stated explicitly) all of them. We consider also, for explanatory purposes, that $\mu(X) = 1$.

Example 3.1. Let $R_{nk}(t, A) = R(A)$, $h_{nk}(x) = h_n(x)$, where R is a nonrandom measure and h_n is a deterministic function. Assume so far (further we shall demand more) that

$$
\lim_{L\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\{N_n(t) > nL\} = 0 \tag{3.21}
$$

and the functions h_n are equicontinuous:

$$
\lim_{r\to 0}\sup_n\sup_{|x-y|\leq r}|h_n(x) - h_n(y)| = 0. \tag{3.22}
$$

Then by Lemma 3.1 they are uniformly bounded. So, to verify condition (3.6) it suffices to show that for any $c > 0$

$$
\int_0^\infty r^p R(\mathrm{d}r)\int_{|y|>cn^a/r}|F(y)|\mathrm{d}y \to 0.
$$

And this will follow from (3.3) and Lebesgue's dominated convergence theorem applied twice if we demand that

$$
b \equiv \int_0^\infty r^p R(\mathrm{d}r) < \infty. \tag{3.23}
$$

Relations (3.14) (a consequence of (3.22)), (3.21) and (3.23) imply (3.8) as well.

Denote $D_n(r) = \sup_{|x-y|\leq r}|h_n(x) - h_n(y)|$ (so that relation (3.5) holds automatically), $\delta_n(r) = \int|F(y)|D_n(n^{-a}r|y|)\mathrm{d}y$. Due to (3.22) (and hereon (3.14)) $D_n(n^{-a}r) \to 0$ for any r and the sequence (D_n) is uniformly bounded, which together with (3.3), (3.23) and Lebesgue's theorem applied twice yields

$$
\int_0^\infty r^p\delta_n(r)R(\mathrm{d}r) \to 0.
$$

Hence in view of (3.21) relation (3.7) follows.

To check (3.9)–(3.11) we assume that

$$
\eta_{nk}(t) = g(\tau_{nk}, (t - \tau_{nk})_+), \tag{3.24}
$$

where g is a nonrandom Borel function. Then

$$
Q_n(t) = \frac{b}{n}\int_0^t q_n(\tau, t - \tau)\mathrm{d}N_n(\tau),
$$

where $q_n(\tau, u) = q_n(\tau, u, \cdot)$, $q_n(\tau, u, x) = h_n(x - g(\tau, u_+))$.

We take an arbitrary bounded sequence $(u_n) \in \mathbb{R}_+^{\mathbb{N}}$ and introduce (partly replicate) the notation: Λ_n – the compensator of N_n, $\check{N}_n = N_n - \Lambda_n$,

$$\widetilde{Q}_n(t) = \frac{b}{n} \int_0^t q_n(\tau, t - \tau) \mathrm{d}\Lambda_n(\tau), \ \alpha_n(t) = \frac{1}{n} \int_0^t q_n(\tau, u_n - \tau) \mathrm{d}\check{N}_n(\tau).$$

The last process is a local square integrable martingale with quadratic characteristic

$$\langle \alpha_n \rangle(t) = \frac{1}{n^2} \int_0^t q_n(\tau, u_n - \tau)^2 \mathrm{d}\langle \check{N}_n \rangle(\tau) \leq n^{-2} C^2 \Lambda_n(t) \tag{3.25}$$

(the inequality relies on (3.14) and (3.15)). So, if we demand that for any t

$$\Lambda_n(t)/n^2 \xrightarrow{\mathsf{P}} 0, \tag{3.26}$$

then Lenglart's inequality ([7], I.3.30)

$$\mathsf{P}\left\{ \sup_{v \leq t} |\alpha_n(v)| \geq \varepsilon \right\} \leq c\varepsilon^{-2} + \mathsf{P}\{\langle \alpha_n \rangle(t) \geq c\}$$

and (3.25) will imply that, for any bounded sequence (v_n), $\alpha_n(v_n) \xrightarrow{\mathsf{P}} 0$. Setting $v_n = u_n$, we get $Q_n(u_n) - \widetilde{Q}_n(u_n) \xrightarrow{\mathsf{P}} 0$, whence by Lebesgue's theorem

$$\mathsf{ME} \left| \mathrm{e}^{\mathfrak{z}Q_n(u_n)} - \mathrm{e}^{\mathfrak{z}\widetilde{Q}_n(u_n)} \right| \to 0$$

for all $\mathfrak{z} \in \mathbb{C}_-$. This together with the evident inequality

$$\left| \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathfrak{z}_j Q_n(t_j)} - \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathfrak{z}_j \widetilde{Q}_n(t_j)} \right| \leq \sum_{j=1}^l \mathsf{M} \left| \mathrm{e}^{\mathfrak{z}_j Q_n(t_j)} - \mathrm{e}^{\mathfrak{z}_j \widetilde{Q}_n(t_j)} \right|,$$

where $\mathfrak{z}_j \in \mathbb{C}_-$, shows that

$$\sup_{t_1 \leq \ldots \leq t_l \leq t} \left| \mathsf{E} \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathfrak{z}_j Q_n(t_j)} - \mathsf{E} \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathfrak{z}_j \widetilde{Q}_n(t_j)} \right| \to 0.$$

So, demanding that for any $l \in \mathbb{N}$, $s_1, \ldots, s_l \in \mathbb{R}$, $t \in \mathbb{R}_+$ and $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}_-$

$$\mathsf{E} \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathrm{i}s_j \widetilde{Q}_n(t_j)} \to \mathsf{E} \prod_{j=1}^l \mathsf{M}\mathrm{e}^{\mathrm{i}s_j Q(t_j)} \tag{3.27}$$

and

$$\lim_{n \to \infty} \sup_{t_1 \leq t_2 \leq t} \left| \mathsf{E} \prod_{j=1}^2 \mathsf{M}\mathrm{e}^{\mathfrak{z}_j \widetilde{Q}_n(t_j)} - \mathsf{E} \prod_{j=1}^2 \mathsf{M}\mathrm{e}^{\mathfrak{z}_j Q(t_j)} \right| = 0, \tag{3.28}$$

where Q is some varying random relief, we guarantee the fulfilment of conditions (3.9)–(3.11). This ends the theoretic part of the example which can be summarized as follows: if $R_{nk}(t, \cdot) = R(\cdot)$, $h_{nk} = h_n$, R and h_n are nonrandom, conditions (3.3), (3.21)–(3.24) (with nonrandom g) and (3.26)–(3.28) are fulfilled and, for each

$\mathfrak{z} \in \mathbb{C}_-$, the random process $\mathsf{Me}^{\mathfrak{z}Q(\cdot)}$ is stochastically continuous, then the second statement of Theorem 3.1 holds (whereas the first one is incorporated into the assumptions). It remains to illustrate the fulfilment of conditions (3.27) and (3.28).

Assume that g is continuous and there exists a random process K such that

$$n^{-1}\Lambda_n \xrightarrow{\ \mathrm{C}\ } K. \tag{3.29}$$

Denote

$$\widetilde{B}_n(t,x) = \int_0^t h_n(x - g(\tau, t-\tau))\mathrm{d}\Lambda_n(\tau)/n, \ \ B_n(t,x) = \int_0^t h_n(x - g(\tau, t-\tau))\mathrm{d}K(\tau),$$

$B_n(t) = B_n(t, \cdot)$ (likewise with $\tilde{\ }$). It easily follows from (3.14), (3.29) and Theorem III.8.3 [12] that for any t

$$\sup_{u \le t} \mathsf{M} \left| \widetilde{B}_n(u) - B_n(u) \right| \xrightarrow{\ \mathrm{P}\ } 0.$$

Hence, noting that: (1) $\widetilde{Q}_n = b\widetilde{B}_n$; (2) for any $\mathfrak{z} \in \mathbb{C}_-$ and $t > 0$ the functions $\mathsf{Me}^{\mathfrak{z}B_n}, n \in \mathbb{N}$, of argument $u \in [0, t]$ are equicontinuous, we conclude that relations (3.27) and (3.28) will hold if for all $t \in \mathbb{R}_+$ and $s \in \mathbb{R}$ the sequence $\left(\mathsf{Me}^{\mathrm{i}sB_n(t)} \right)$ converges in probability to some limit $\mathfrak{f}(s,t)$ and l.i.p. $\mathfrak{f}(c,t) = 1$. Lévy's continuity theorem allows to re-formulate this assumption in the more visual form:

$$\mathsf{Me}^{\mathrm{i}sB_n(t)} \xrightarrow{\ \mathrm{P}\ } \mathsf{Me}^{\mathrm{i}sB(t)}, \tag{3.30}$$

where, for each t, $B(t) \equiv B(t, \cdot)$ is a random function on X. This entails, together with (3.22), (3.29) and continuity of g, relations (3.27) and (3.28) with $Q(t) = bB(t)$.

Conditions (3.21) and (3.26) can be waived, once we have imposed (3.29). And this condition can, in turn, be substituted by the demand that for any t $n^{-1}\Lambda_n(t) \xrightarrow{\ \mathrm{P}\ } K(t)$, where K is a continuous process. Indeed, in this case, as was shown in [10], $\sup_{u \le t} \left| n^{-1}\Lambda_n(u) - K(u) \right| \xrightarrow{\ \mathrm{P}\ } 0$ for all t, whence (3.29) follows.

Now, we will exemplify Theorem 3.2.

Example 3.2. Let assumptions **A1** and **A2** hold with

$$p = 1, \quad F = \nu I_{[0,\ 1]}, \quad \rho_{nk}(t) = 1, \tag{3.31}$$

where ν is a fixed d-dimensional vector. Then $F_{nk}(t,x) = \nu I_{[\sigma_{nk}(t),\ \sigma_{nk}(t)+1/n]}(x)$ and, consequently, for any $s \ge u \ge 0$

$$\mathsf{Me}^{\mathrm{i}z\Phi_n(u,s)} = 1 + \sum_{m=1}^{N_n(u)} \left(\mathrm{e}^{\mathrm{i}z\nu} - 1 \right)^m \sum_{1 \le k_1 < \dots < k_m \le N_n(u)} \varphi_{\mathbf{k}_m}^n(s), \tag{3.32}$$

where $\mathbf{k}_m = (k_1, \dots, k_m)$,

$$\varphi_{\mathbf{k}_m}^n(s) = \mu \left(\bigcap_{j=1}^m \left[\sigma_{nk_j}(s),\ \sigma_{nk_j}(s) + 1/n \right] \right). \tag{3.33}$$

Denote $\alpha_{\mathbf{k}_m}^n(s) = \max_{1 \le j \le m} \sigma_{nk_j}(s)$, $\beta_{\mathbf{k}_m}^n(s) = \min_{1 \le j \le m} \sigma_{nk_j}(s) + 1/n$,

$I_{\mathbf{k}_m}^n(s) = I \left\{ \alpha_{\mathbf{k}_m}^n(s) < \beta_{\mathbf{k}_m}^n(s) \right\}$, $\chi_{kk'}^n(s) = I \left\{ |\sigma_{nk}(s) - \sigma_{nk'}(s)| < 1/n \right\}$. Obviously,

$$I_{\mathbf{k}_m}^n(s) \le \prod_{1 \le i < j \le m} \chi_{k_i k_j}^n(s) \tag{3.34}$$

and the intersection of $\left[\sigma_{nk_j}(s),\ \sigma_{nk_j}(s) + 1/n\right]$, $j = 1, \dots, m$, equals either $\left[\alpha^n_{\mathbf{k}_m}(s),\ \beta^n_{\mathbf{k}_m}(s)\right]$ (if this interval is nonvoid) or $\emptyset$ (otherwise). So, assuming **A5**, we can rewrite (3.33) in the form

$$\varphi^n_{\mathbf{k}_m}(s) = I^n_{\mathbf{k}_m}(s) \int_{\alpha^n_{\mathbf{k}_m}(s)}^{\beta^n_{\mathbf{k}_m}(s)} f(x)\mathrm{d}x,$$

whence

$$\dot{\varphi}^n_{\mathbf{k}_m}(s) = I^n_{\mathbf{k}_m}(s)\left(f\left(\beta^n_{\mathbf{k}_m}(s)\right)\dot{\beta}^n_{\mathbf{k}_m}(s) - f\left(\alpha^n_{\mathbf{k}_m}(s)\right)\dot{\alpha}^n_{\mathbf{k}_m}(s)\right) \tag{3.35}$$

provided the derivatives on the right-hand side exist.

Suppose further that

$$\eta_{nk}(t) = t - \tau_{nk} \tag{3.36}$$

and f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq C_0|x - y| \tag{3.37}$$

with some constant C_0. Then $\sigma_{nk_j}(s) - \sigma_{nk_1}(s) = \zeta_{nk_j} - \zeta_{nk_1} - \tau_{nk_j} + \tau_{nk_1}, \alpha^n_{\mathbf{k}_m}(s) = s + \max_{1 \leq j \leq m}\left(\zeta_{nk_j} - \tau_{nk_j}\right)$, $\beta^n_{\mathbf{k}_m}(s) = s + \min_{1 \leq j \leq m}\left(\zeta_{nk_j} - \tau_{nk_j}\right) + 1/n$ as $s > \tau_{nk_m}$. These equalities together with (3.35), (3.34) and (3.37) yield for $v \geq u$

$$\left|\varphi^n_{\mathbf{k}_m}(v) - \varphi^n_{\mathbf{k}_m}(u)\right| \leq \frac{C_0(v - u)}{n} \prod_{j=2}^m I\left\{\left|\zeta_{nk_j} - \zeta_{nk_1} - \tau_{nk_j} + \tau_{nk_1}\right| < \frac{1}{n}\right\}. \tag{3.38}$$

Assume the following:

A6. The random variables ζ_{nk}, $k \in \mathbb{N}$, are independent of τ_{nk}, $k \in \mathbb{N}$, and of each other.

A7. Each ζ_{nk} has a distribution density h_n, the same for all k.

Due to (3.36) and nonrandomness of ρ_{nk} these assumptions entail **A3** and **A4** (one can take for $\mathcal{F}^0_n$ the σ-algebra generated by τ_{nk}, $k \in \mathbb{N}$).

Let $\mathcal{G}_{nl}$ denote the σ-algebra generated by ζ_{nl} and τ_{nk}, $k \in \mathbb{N}$. Then: assumption **A6** implies that for any $k_1 < \dots < k_m$ and $\mathcal{G}_{nk_1}$-measurable random variables $\varpi_2, \dots \varpi_m$

$$\mathsf{E}\left(\prod_{j=2}^m I\left\{\left|\zeta_{nk_j} - \varpi_j\right| < \frac{1}{n}\right\}\,\middle|\,\mathcal{G}_{nk_1}\right) = \prod_{j=2}^m \mathsf{P}\left\{\left|\zeta_{nk_j} - \varpi_j\right| < \frac{1}{n}\,\middle|\,\mathcal{G}_{nk_1}\right\};$$

assumptions **A6** and **A7** yield

$$\mathsf{P}\left\{\left|\zeta_{nk_j} - \varpi_j\right| < 1/n\,\middle|\,\mathcal{G}_{nk_1}\right\} = \int_{\varpi_j - 1/n}^{\varpi_j + 1/n} h_n(x)\mathrm{d}x.$$

Supposing furthermore (3.14), we get from (3.38) and the last two equalities

$$\mathsf{E}^0 \sup_{(v-c)_+ < u < v \leq t}\left|\varphi^n_{\mathbf{k}_m}(v) - \varphi^n_{\mathbf{k}_m}(u)\right| \leq C_0 c(2C)^{m-1}n^{-m}.$$

And this together with (3.32) implies that

$$\mathsf{E}^0 \sup_{(v-c)_+ < u < v \leq t}\left|\mathsf{M}\mathrm{e}^{iz\Phi_n(u,v)} - \mathsf{M}\mathrm{e}^{iz\Phi_n(u,u)}\right| \leq \frac{C_0 c}{2C} \sum_{m=1}^{N_n(t)} \binom{N_n(t)}{m}\left(\frac{4C}{n}\right)^m = \frac{C_0 c}{2C}\left(\left(1 + \frac{4C}{n}\right)^{N_n(t)} - 1\right).$$

So, postulating additionally (3.21), we provide (3.19). Condition (3.17) is in our case trivial, and condition (3.18) for $\rho_{nk} = 1$ combined with (3.21) is tantamount to relative compactness of (N_n/n) in C.

Now, taking to account that this example is a particular case of the previous one, we write down the complete list of assumptions: **A1**, **A2**, **A5**–**A7**, (3.22) ($\Longrightarrow$(3.14)), (3.29) ($\Longrightarrow$(3.21)), (3.30) with

$$B_n(t, x) = \int_0^t h_n(x - t + \tau)\mathrm{d}K(\tau),$$

(3.31), (3.36), (3.37). If these assumptions are valid, then all the conditions of Lemmas 3.1–3.3 and Theorem 3.2 are fulfilled and therefore the conclusion of the theorem holds true. Since in our case $\mathrm{e}^{\mathrm{i}z F(y)} - 1 = \left(\mathrm{e}^{\mathrm{i}z\nu} - 1\right) I_{[0,1]}(y)$, equality (3.13) acquires the more concrete form

$$\widehat{\psi}(t, z) = \int \exp\left\{bB(t, x)\left(\mathrm{e}^{\mathrm{i}z\nu} - 1\right)\right\}\mu(\mathrm{d}x). \tag{3.39}$$

So $\Psi(t)$ is a mixture (averaging in the spatial variable x) of Poisson distributions on the lattice $\nu\mathbb{Z}_+$.

Example 3.3. Let $\eta_{nk}(t) = 0$ and

$$\rho_{nk}(t) = \rho_{nk}^0 T\left(t - \tau_{nk}\right), \tag{3.40}$$

where the ρ_{nk}^0's are positive random variables and T is a nonrandom positive increasing continuous function. Then denoting $\gamma_{nk}(x) = n^a(x - \zeta_{nk})/\rho_{nk}^0$, we can write for $v \geq u \geq 0$ and $k \leq N_n(u)$

$$\begin{aligned}
\mathsf{M}\left|\mathrm{e}^{2\mathrm{i}z F_{nk}(v)} - \mathrm{e}^{2\mathrm{i}z F_{nk}(u)}\right| &= 2\int\left|\sin z\left(F\left(\frac{\gamma_{nk}(x)}{T(v - \tau_{nk})}\right) - F\left(\frac{\gamma_{nk}(x)}{T(u - \tau_{nk})}\right)\right)\right| f(x)\mathrm{d}x \\
&= \frac{2}{n}\left(\rho_{nk}^0\right)^p\int\left|\sin z\left(F\left(\frac{y}{T(v - \tau_{nk})}\right) - F\left(\frac{y}{T(u - \tau_{nk})}\right)\right)\right| f\left(\zeta_{nk} + \frac{\rho_{nk}^0 y}{n^a}\right)\mathrm{d}y,
\end{aligned}$$

whence, since f is bounded,

$$\mathsf{M}\left|\mathrm{e}^{2\mathrm{i}z\Phi_n(u,v)} - \mathrm{e}^{2\mathrm{i}z\Phi_n(u,u)}\right| \leq \frac{C}{n}\max_{k \leq N_n(u)}\int\left|\sin z\left(F\left(\frac{y}{T(v - \tau_{nk})}\right) - F\left(\frac{y}{T(u - \tau_{nk})}\right)\right)\right|\mathrm{d}y\sum_{k \leq N_n(u)}\left(\rho_{nk}^0\right)^p$$

with some constant C. So, demanding that for any $t > 0$ and $z \in \mathbb{R}^d$

$$\lim_{L \to \infty}\overline{\lim_{n \to \infty}}\,\mathsf{P}\left\{\sum_{k=1}^{N_n(t)}\left(\rho_{nk}^0\right)^p > nL\right\} = 0 \tag{3.41}$$

and

$$\lim_{c \to 0}\sup_{(v-c)_+ < u < v \leq t}\int\left|\sin z\left(F\left(\frac{y}{T(v)}\right) - F\left(\frac{y}{T(u)}\right)\right)\right|\mathrm{d}y = 0, \tag{3.42}$$

we guarantee the fulfilment of condition (3.19) for all t, ε and z. Condition (3.18) will be fulfilled, too, as equality (3.40) shows, if we demand additionally that the sequence (N_n/n) be r.c. in C. And this is, by Lemma 3.2, a consequence of (3.29). So conditions (3.3), (3.29), (3.41) and (3.42) imply (3.17) – (3.19) and therefore, by Lemma 3.3, **RC**.

Assume also **A7** and a sharpened form of both **A3** and **A6**:

A8. The random variables $\zeta_{nk}, \rho^0_{nj}, \ k, j \in \mathbb{N}$, are independent of $\tau_{nk}, \ k \in \mathbb{N}$, and of each other.

Suppose that the distribution of ρ^0_{nk} does not depend on n and k. Then

$$R_{nk}(t, A) = R^0(A/T(t - \tau_{nk})), \tag{3.43}$$

$b_{nk}(t) = b^0 \ T(t - \tau_{nk})$ and

$$Q_n(t, x) = b^0 \ h_n(x) \int_0^t T(t - \tau) \mathrm{d}N_n(\tau)/n, \tag{3.44}$$

where $R^0(A) = \mathsf{P}\left\{\rho^0_{nk} \in A\right\}, \ b^0 = \mathsf{E}\left(\rho^0_{nk}\right)^p$. So, it is natural to demand instead of (3.23) that

$$b^0 \equiv \int_0^\infty r^p R^0(\mathrm{d}r) < \infty. \tag{3.45}$$

The last condition together with (3.14) and (3.21) implies, obviously, (3.8).

For all $t \geq s \geq 0$ and $k \leq N_n(s)$ we have, since T increases, $T(s - \tau_{nk}) \leq T(t)$. Consequently, the left-hand side of (3.6) does not exceed

$$\mathsf{P}\left\{T(t)^p N_n(t) \int_0^\infty r^p R^0(\mathrm{d}r) \int |F(y)| I\left\{|y| > \frac{cn^a}{T(t)r}\right\} \mathrm{d}y > n\varepsilon\right\},$$

which together with (3.3), (3.14), (3.21) and (3.45) yields, in the same way as in Example 3.1, relation (3.6). Likewise (3.7) follows from these conditions and the assumption that T increases.

Finally, assume (3.29) and

A9. The functions h_n regarded as random variables on the probability space $(X, \mathcal{X}, \mu)$ converge in distribution to some function h.

Then equality (3.44) where the function T is, by assumption, continuous shows that relations (3.9), (3.10) and (3.12) with

$$Q(t, x) = b^0 \ h(x) \int_0^t T(t - \tau) \mathrm{d}K(\tau) \tag{3.46}$$

hold for all values of the free variables.

As was noted above, condition (3.29) entails (3.21). This allows us to replace condition (3.41) with the following one:

$$\forall m \in \mathbb{N} \qquad \lim_{L \to \infty} \varlimsup_{n \to \infty} \mathsf{P}\left\{\sum_{k=1}^{mn} \left(\rho^0_{nk}\right)^p > nL\right\} = 0. \tag{3.47}$$

The ultimate list of assumptions is: **A1**, **A2**, **A5**, **A7**–**A9**, $\eta_{nk}(t) = 0$, (3.3), (3.22), (3.29), (3.40) (with the subsequent explanations), (3.42), (3.43) and (3.47). If they are valid, then the conclusion of Theorem 3.2 holds true with Q given by (3.46).

Remark 3.2. The model considered in Example 3.3 is a slight modification of the crystallization model studied by Kolmogorov [9]. In the latter, the rate of growth is at any instant the same for all crystals, whereas in the former it depends on the age of a crystal. Both cases are physically possible [25].

Remark 3.3. Rather general sufficient conditions for **RC** in a simpler but still meaningful model where $\eta_{nk} = 0$ and ρ_{nk} does not depend on time were obtained in [19,21,24].

4. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. We will consider that $\mu(X) = 1$ and $t_1 \leq \ldots \leq t_l$, which, of course, does not diminish generality.

1°. Denote $\Phi^l(t_1, \ldots, t_l; s_1, \ldots, s_l) = Z^l(t_1, \ldots, t_l; is_1, \ldots, is_l)$. Conditions (3.9) and (3.10) imply that the function Φ of variables $l \in \mathbb{N}$, $t_1 \in \mathbb{R}_+, t_2 \in \mathbb{R}_+ \setminus \{t_1\}, \ldots, t_l \in \mathbb{R}_+ \setminus \{t_1, \ldots, t_{l-1}\}$, $s_1 \in \mathbb{R}, \ldots, s_l \in \mathbb{R}$ satisfies all the conditions of Theorem 2.2 which therefore asserts existence of a $\mathbb{P}(\mathbb{R})$-valued process Υ such that equality (2.6) holds for all values of the variables. By assumption **A5** the measure μ is diffuse. Then Theorem 1.1 asserts existence of an $\mathbb{R}$-valued varying random relief Q on X such that, for all $B \in \mathcal{B}$,

$$\upsilon(t, B) = \mu\{x : Q(t, x) \in B\},$$

or, the same, $\widehat{\upsilon}(t, s) = \mathsf{M} \mathrm{e}^{isQ(t)}$ for any $s \in \mathbb{R}$. Hence and from the definition of covaristic (formula (2.1)) we obtain

$$\widetilde{\upsilon}^l(t_1, \ldots, t_l; s_1, \ldots, s_l) = \mathsf{E} \prod_{j=1}^l \mathsf{M} \mathrm{e}^{is_j Q(t_j)}.$$

This together with (2.6) proves the first statement of the theorem.

2°. Regarding $Q_n(\cdot)$ (here the dot marks the place for t) as a random process given on the probability space $(X^{\mathbb{N}} \otimes \Omega_n, \mathcal{X}^{\otimes \mathbb{N}} \otimes \mathcal{F}_n, \mu^{\otimes \mathbb{N}} \otimes \mathsf{P}_n)$, we see that relation (3.9) with right-hand side given by (3.12) asserts weak convergence of the finite-dimensional distributions of these processes to those of $Q(\cdot)$. Hence, taking to account nonnegativeness of all the Q_n's (and therefore Q), we deduce that for all $l \in \mathbb{N}, t_1, \ldots, t_l \in \mathbb{R}_+, \mathfrak{z}_1, \ldots, \mathfrak{z}_l \in \mathbb{C}_-$

$$\mathsf{E} \prod_{j=1}^l \mathsf{M} \mathrm{e}^{\mathfrak{z}_j Q_n(t_j)} \to \mathsf{E} \prod_{j=1}^l \mathsf{M} \mathrm{e}^{\mathfrak{z}_j Q(t_j)}. \tag{4.1}$$

3°. Denote $G(y, z) = \mathrm{e}^{izF(y)} - 1$, $\mathfrak{g}(z) = \int G(y, z) \mathrm{d}y$ (the integral converges due to (3.3), and its value belongs to $\mathbb{C}_-$, because $\cos zF(y) - 1 \leq 0$). Then equality (3.12) shows that the covaristic of the MVP Ψ with characteristic function (3.13) is given by the formula

$$\widetilde{\psi}^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = Z^l(t_1, \ldots, t_l; \mathfrak{g}(z_1), \ldots, \mathfrak{g}(z_l)).$$

Herein $\lim_{z \to 0} Z^1(t; \mathfrak{g}(z)) = 1$ because of (3.12) and nonnegativeness of Q. So, to deduce the second statement from Corollary 2.3 and Theorem 2.1 it suffices to check the three conditions: (2.10) with

$$U^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = Z^l(t_1, \ldots, t_l; \mathfrak{g}(z_1), \ldots, \mathfrak{g}(z_l)),$$

(2.15) and (2.16). In view of (4.1) and (3.12) condition (2.10) with this U is tantamount to the relation

$$\widetilde{\psi}_n^l(t_1, \ldots, t_l; z_1, \ldots, z_l) - \mathsf{E} \prod_{j=1}^l \mathsf{M} \mathrm{e}^{\mathfrak{g}(z_j) Q_n(t_j)} \to 0.$$

Having in mind verification of the other conditions, too, we will prove in items 4°−11° even more:

$$\sup_{t_1 \leq \ldots \leq t_l \leq t} \left| \widetilde{\psi}_n^l(t_1, \ldots, t_l; z_1, \ldots, z_l) - \mathsf{E} \prod_{j=1}^l \mathsf{M} \mathrm{e}^{\mathfrak{g}(z_j) Q_n(t_j)} \right| \to 0. \tag{4.2}$$

The notation $\chi_n \overset{\mathsf{M}^l\mathsf{P}}{\rightrightarrows} 0$, where $\chi_1, \chi_2 \ldots$ are random functions on $[0, t]^l \times X^l$, will mean that, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \sup_{t_1 \leq \ldots \leq t_l \leq t} \mathsf{M}^l\mathsf{P}\{|\chi_n| > \varepsilon\} = 0.$$

In case the χ_n's do not depend on $x_1, \ldots, x_l$ we write $\overset{\mathsf{P}}{\rightrightarrows}$.

$4°$. Let us fix $l, z_1, \ldots, z_l$ and denote

$$S_n = \exp\left(\sum_{j=1}^{l} \mathfrak{g}(z_j) Q_n(t_j, x_j)\right), \quad \xi_{nj} = \xi_n(t_j, x_j),$$

$$H_n = \mathsf{E}^0 \prod_{j=1}^{l} e^{iz_j \xi_{nj}}. \tag{4.3}$$

By construction

$$|S_n| \leq 1, \ |H_n| \leq 1. \tag{4.4}$$

Because of (3.2)

$$\prod_{j=1}^{l} \widehat{\psi}_n(t_j, z_j) = \mathsf{M}^l \prod_{j=1}^{l} e^{iz_j \xi_{nj}},$$

which together with the definition of the covaristic and formula (4.3) yields

$$\widetilde{\psi}_n^l(t_1, \ldots, t_l; z_1, \ldots, z_l) = \mathsf{M}^l \mathsf{E} H_n.$$

And this jointly with (4.4) shows that relation (4.2) is equivalent to

$$|H_n - S_n| \overset{\mathsf{M}^l\mathsf{P}}{\rightrightarrows} 0. \tag{4.5}$$

$5°$. We introduce the notation: $\rho_{nkj} = \rho_{nk}(t_j)$ (likewise $\eta_{nkj}, b_{nkj}, \ R_{nkj}(\cdot)$), $F_{nkj} = F\left(\frac{n^a(x_j - \zeta_{nk} - \eta_{nkj})}{\rho_{nkj}}\right) \equiv$

$F_{nk}(t_j, x_j)$ (recall that $a = 1/p$), $\prod_k = \prod_{k=1}^{N_n(t_l)}$ (the same for sums), $J_{nk} = \{j \in \{1, \ldots, l\} : t_j \geq \tau_{nk}\} \equiv \{j \in$

$\{1, \ldots, l\} : k \leq N_n(t_j)\}$, $\#J$ – the number of members in a finite set J, $\displaystyle\sum_{J(2)} = \sum_{J \subset J_{nk}:\#J \geq 2}$ (the same for max),

$$G_{nkj} = e^{iz_j F_{nkj}} - 1 \equiv G\left(\frac{n^a(x_j - \zeta_{nk} - \eta_{nkj})}{\rho_{nkj}}, z_j\right),$$

$w_{nk} = \sum_{j \in J_{nk}} \mathsf{E}^0 G_{nkj}, \ \Gamma_{nk}^J = \mathsf{E}^0 \prod_{j \in J} |G_{nkj}|$. Obviously, for any summands

$$\sum_k \sum_{j \in J_{nk}} \cdots = \sum_{j=1}^{l} \sum_{k=1}^{N_n(t_j)} \cdots \tag{4.6}$$

By assumption **A2** $\quad \xi_{nj} = \sum_{k=1}^{N_n(t_j)} F_{nkj}$, whence

$$\prod_{j=1}^{l} e^{iz_j \xi_{nj}} = \prod_k \prod_{j \in J_{nk}} (1 + G_{nkj}).$$

This together with **A3** (i, ii) yields another form of equality (4.3):

$$H_n = \prod_k \left(1 + w_{nk} + \sum_{J(2)} \mathsf{E}^0 \prod_{j \in J} G_{nkj} \right). \tag{4.7}$$

Obviously, for an arbitrary natural number m and any complex numbers $c_1, \ldots, c_m, d_1, \ldots, d_m$ such that $|c_1| \le 1, \ldots, |c_m| \le 1$ the inequality

$$\left| \prod_{k=1}^m c_k - \prod_{k=1}^m d_k \right| \le \sum_{k=1}^m \left(|c_k - d_k| \prod_{i=k+1}^m |d_i| \right) \tag{4.8}$$

holds. Setting $m = N_n(t_l)$, $c_k = \mathsf{E}^0 \prod_{j \in J_{nk}} \mathrm{e}^{\mathrm{i} z_j F_{nkj}}$ (this is the kth factor in (4.7)), $d_k = 1 + w_{nk}$, we get

$$\left| H_n - \prod_k (1 + w_{nk}) \right| \le V_n W_n,$$

where $V_n = \sum_k \sum_{J(2)} \Gamma_{nk}^J$ and

$$W_n = \prod_k \mathrm{e}^{|w_{nk}|}. \tag{4.9}$$

Now, our goal is to establish the relation

$$\left| H_n - \prod_k (1 + w_{nk}) \right| \overset{\mathsf{M}^l \mathsf{P}}{\rightrightarrows} 0. \tag{4.10}$$

It will be achieved if we show that

$$V_n \overset{\mathsf{M}^l \mathsf{P}}{\rightrightarrows} 0 \tag{4.11}$$

and

$$\lim_{L \to \infty} \varlimsup_{n \to \infty} \sup_{t_1 \le \ldots \le t_l \le t} \mathsf{M}^l \mathsf{P} \left\{ W_n > \mathrm{e}^L \right\} = 0. \tag{4.12}$$

The proof of (4.11) is the subject of the next four items.

6°. Let us fix $c > 0$, denote

$$\gamma_{nkj}^c = |G_{nkj}| I\{|x_j - \eta_{nkj} - \zeta_{nk}| > c\}, \quad I_{nk}^{Jc} = I \left\{ \max_{i,j \in J} |x_i - \eta_{nki} - (x_j - \eta_{nkj})| > 2c \right\},$$

$$V_n^c = \sum_n \sum_{J(2)} I_{nk}^{Jc} \, \Gamma_{nk}^J \tag{4.13}$$

and show that

$$\sup_{t_1 \le \ldots \le t_l \le t} \mathsf{M}^l V_n^c \overset{\mathsf{P}}{\longrightarrow} 0. \tag{4.14}$$

Obviously, for any $y, y_1, \ldots, y_l \in X$ and nonvoid set $J \subset \{1, \ldots, l\}$ the inequality

$$I \left\{ \max_{i,j \in J} |y_i - y_j| > 2c \right\} \le \sum_{j \in J} I\{|y_j - y| > c\}$$

holds. Setting $y_j = x_j - \eta_{nkj}$, $y = \zeta_{nk}$ and taking to account that

$$|G_{nkj}| \leq 2, \tag{4.15}$$

we get, for $J \subset J_{nk}$ such that $\#J \geq 2$,

$$\mathsf{M}^l I_{nk}^{Jc} \, \Gamma_{nk}^J \leq 2^{\#J-1} \sum_{j \in J} \mathsf{M}\mathsf{E}^0 \gamma_{nkj}^c. \tag{4.16}$$

By assumptions **A3** (i, iii) and **A4** for any $k \in \mathbb{N}, s \geq \tau_{nk}, x \in X$ and bounded Borel function $\varphi : X \times \mathbb{R}_+ \to \mathbb{C}$

$$\mathsf{E}^0 \varphi(x - \eta_{nk}(s) - \zeta_{nk}, \rho_{nk}(s)) = \int_0^\infty R_{nk}(s, \mathrm{d}r) \int \varphi(x - \eta_{nk}(s) - y, r) h_{nk}(y) \mathrm{d}y.$$

Hence, taking $\varphi(x, r) = g(\theta x / r) I\{|x| > c\}$, where $g : X \to \mathbb{C}$ is a bounded Borel function and θ is a positive parameter, and using the identity

$$\int_{|y_0 - y| > c} g\left(\frac{\theta(y_0 - y)}{r}\right) f(y) \mathrm{d}y = \left(\frac{r}{\theta}\right)^p \int_{|y| > c\theta/r} g(y) f\left(y_0 - \frac{ry}{\theta}\right) \mathrm{d}y,$$

we get

$$\mathsf{E}^0 g\left(\frac{\theta(x - \eta_{nk}(s) - \zeta_{nk})}{\rho_{nk}(s)}\right) I\{|x - \eta_{nk}(s) - \zeta_{nk}| > c\} =$$
$$\theta^{-p} \int_0^\infty r^p R_{nk}(s, \mathrm{d}r) \int_{|y| > c\theta/r} g(y) h_{nk}\left(x - \eta_{nk}(s) - \frac{ry}{n^a}\right) \mathrm{d}y. \tag{4.17}$$

Setting here $\theta = n^a, g = |G(\cdot, z_j)|, x = x_j, s = t_j \ (j \in J_{nk})$, we obtain with account of (3.4)

$$\mathsf{E}^0 \gamma_{nkj}^c \leq 2\alpha_{nkj}^c , \tag{4.18}$$

where

$$\alpha_{nkj}^c = \frac{\vartheta_{nk}}{n} \int_0^\infty r^p R_{nkj}(\mathrm{d}r) \int_{|y| > cn^a/r} \left|\sin \frac{z_j F(y)}{2}\right| \mathrm{d}y. \tag{4.19}$$

Noting that α_{nkj}^c does not depend on $x_1, \ldots, x_l$, we get from (4.13), (4.16) and (4.18)

$$\mathsf{M}^l V_n^c \leq \sum_k \sum_{J \subset J_{nk}} 2^{\#J} \alpha_{nkj}^c. \tag{4.20}$$

Obviously, for any finite set M and numbers $u_j, j \in M$, the equality

$$\sum_{J \subset M} \sum_{j \in J} u_j = 2^{\#M-1} \sum_{j \in M} u_j$$

holds. Consequently, for any nonnegative numbers a_J, where J ranges over the subsets of $\{1, \ldots, l\}$,

$$\sum_k \sum_{J \subset J_{nk}} a_J \sum_{j \in J} \alpha_{nkj}^c \leq 2^{l-1} \max_J a_J \sum_k \sum_{j \in J_{nk}} \alpha_{nkj}^c \stackrel{(4.6)}{=} 2^{l-1} \max_J a_J \sum_{j=1}^l \sum_{k=1}^{N_n(t_j)} \alpha_{nkj}^c, \tag{4.21}$$

which together with (4.20), (4.19) and (3.6) proves (4.14).

7°. Let us denote $\iota_{nk}^c = \max\limits_{J(2)} \left(1 - I_{nk}^{Jc}\right)$, $E_n = \{k : \#J_{nk} \geq 2\}$, $\overline{V}_n^c = \sum\limits_{k} \sum\limits_{J(2)} \left(1 - I_{nk}^{Jc}\right) \Gamma_{nk}^J$, so that

$$V_n = V_n^c + \overline{V}_n^c, \tag{4.22}$$

and show that

$$\mathsf{M}^l \overline{V}_n^c \leq 2^{2l-1} \max\limits_{k \in E_n} \mathsf{M}^l \iota_{nk}^c \cdot \sum_{j=1}^{l} \sum_{k=1}^{N_n(t_j)} \alpha_{nkj}^0. \tag{4.23}$$

Obviously,

$$\overline{V}_n^c \leq \sum_{k} \iota_{nk}^c \sum_{\emptyset \neq J \subset J_{nk}} \Gamma_{nk}^J \tag{4.24}$$

and $|G_{nkj}| = \gamma_{nkj}^0$. From the last equality, (4.15) and (4.18) we have $\Gamma_{nk}^J \leq 2^{\#J} \alpha_{nkj}^0$ for an arbitrary $j \in J$. Consequently,

$$\Gamma_{nk}^J \leq 2^l \sum_{j \in J} \alpha_{nkj}^0 \quad \text{for } J \neq \emptyset.$$

The variables $x_1, \ldots, x_l$ do not enter, as is seen from (4.19), the right-hand side of this inequality. So, applying M^l to both sides of (4.24), we get

$$\mathsf{M}^l \overline{V}_n^c \leq 2^l \max\limits_{k \in E_n} \mathsf{M}^l \iota_{nk}^c \cdot \sum_{k} \sum_{\emptyset \neq J \subset J_{nk}} \sum_{j \in J} \alpha_{nkj}^0.$$

Now, (4.23) follows from (4.21).

8°. Let f denote the existing by assumption **A5** density of μ. By the same assumption

$$f(x) \leq C \tag{4.25}$$

for some constant C. Let us show that for any $k \in E_n$

$$\mathsf{M}^l \iota_{nk}^c \leq 2^{p-1} c^p l(l-1) C \int_{|y|<1} \mathrm{d}y. \tag{4.26}$$

Obviously,

$$\iota_{nk}^c = I\left\{\min\limits_{i,j \in J_{nk}, i \neq j} |x_i - \eta_{nki} - (x_j - \eta_{nkj})| \leq 2c\right\}.$$

So

$$\mathsf{M}^l \iota_{nk}^c \leq \sum_{1 \leq i < j \leq l} \mu^{\otimes l}\left(A_{ij}^{(c)}(\eta_{nki}, \eta_{nkj})\right), \tag{4.27}$$

where $A_{ij}^{(c)}(y', y'') = \{(x_1, \ldots, x_l) : |x_i - y' - (x_j - y'')| \leq 2c\}$. The evident equalities

$$\mu^{\otimes l}\left(A_{ij}^{(c)}(y', y'')\right) = \iint\limits_{|x_i - x_j| < 2c} f(x_i + y')f(x_j + y'')\mathrm{d}x_i\mathrm{d}x_j = 2^p \iint\limits_{|y|<c} f(x + y + y')f(x - y + y'')\mathrm{d}x\mathrm{d}y$$

and inequality (4.25) show that

$$\mu^{\otimes l}\left(A_{ij}^{(c)}(y',y'')\right) \le 2^p C \int_{|y|<c} \mathrm{d}y \int f(x+y+y')\mathrm{d}x \equiv (2c)^p C \int_{|y|<1} \mathrm{d}y,$$

which together with (4.27) proves (4.26).

9°. From (4.19) we have

$$\sum_{k=1}^{N_n(t_j)} \alpha_{nkj}^0 \le \frac{|z_j|}{n} \int |F(y)|\mathrm{d}y \sum_{k=1}^{N_n(t_j)} \vartheta_{nk} b_{nk}(t_j),$$

whence because of (4.23) and (4.26)

$$\lim_{c\to 0}\overline{\lim_{n\to\infty}}\sup_{t_1\le\ldots\le t_l\le t} \mathsf{P}\left\{\mathsf{M}^l\overline{V}_n^c > \varepsilon\right\} = 0$$

for any $\varepsilon > 0$. This relation together with (4.14) and (4.22) implies that

$$\mathsf{M}^l V_n \overset{\mathsf{P}}{\rightrightarrows} 0 \tag{4.28}$$

and, consequently, for any $r > 0$

$$\sup_{t_1\le\ldots\le t_l\le t} \mathsf{E}\left(r \wedge \mathsf{M}^l V_n\right) \to 0. \tag{4.29}$$

Let ϱ be an arbitrary nonnegative measurable random function on X^l; ε and r be arbitrary positive numbers. Then the evident inequalities

$$I\left\{\mathsf{M}^l\varrho > r\right\}\mathsf{M}^l I\{\varrho > \varepsilon\} \le I\left\{\mathsf{M}^l\varrho > r\right\},$$
$$I\left\{\mathsf{M}^l\varrho \le r\right\}\mathsf{M}^l I\{\varrho > \varepsilon\} \le \varepsilon^{-1} I\left\{\mathsf{M}^l\varrho \le r\right\}\mathsf{M}^l\varrho$$

show that

$$\mathsf{M}^l\mathsf{P}\{\varrho > \varepsilon\} \le \mathsf{P}\left\{\mathsf{M}^l\varrho > r\right\} + \varepsilon^{-1}\mathsf{E}\left(r \wedge \mathsf{M}^l\varrho\right). \tag{4.30}$$

Putting $\varrho = V_n$, we deduce (4.11) from (4.28) and (4.29).

10°. Let us prove (4.12), where W_n is defined by (4.9). Recall that $w_{nk} = \sum_{j\in J_{nk}} \mathsf{E}^0 G_{nkj}$.

Denote

$$\varkappa_{nk} = \frac{1}{n}\sum_{j\in J_{nk}} \mathfrak{g}(z_j)b_{nkj}h_{nk}(x_j - \eta_{nkj})$$

($\mathfrak{g}$ was defined in item 3°). Setting in (4.17) $\theta = n^a, g = G(\cdot, z_j), x = x_j, s = t_j$ $(j \in J_{nk}), c = 0$, we get

$$\mathsf{E}^0 G_{nkj} = \frac{1}{n}\int_0^\infty r^p R_{nkj}(\mathrm{d}r) \int G(y,z_j)h_{nk}\left(x_j - \eta_{nkj} - \frac{ry}{n^a}\right)\mathrm{d}y,$$

which together with (3.5) and the definition of G (item 3°) yields

$$|w_{nk} - \varkappa_{nk}| \le \frac{1}{n}\sum_{j\in J_{nk}} \beta_{nk}(t_j, z_j), \tag{4.31}$$

where

$$\beta_{nk}(s,z) = 2\int_0^\infty r^p R_{nk}(s, \mathrm{d}r)\int \left|\sin\frac{zF(y)}{2}\right| D_{nk}\left(\frac{r|y|}{n^a}\right)\mathrm{d}y. \tag{4.32}$$

By the definition of $\varkappa_{nk}$ and condition (3.4)

$$|\varkappa_{nk}| \le \frac{C\vartheta_{nk}}{n}\sum_{j\in J_{nk}} b_{nkj}, \tag{4.33}$$

where $C = \max_{j \le l} |\mathfrak{g}(z_j)|$ ($< \infty$ by condition (3.3)). This jointly with (4.6) and (4.31) implies that $\sum_k |w_{nk}| \le (C+1)T_n/n$, where

$$T_n = \sum_{j=1}^{l} \sum_{k=1}^{N_n(t_j)} (\vartheta_{nk}b_{nk}(t_j) + \beta_{nk}(t_j, z_j)). \tag{4.34}$$

This quantity does not depend on $x_1, \ldots, x_l$, so $\mathsf{M}^l \sum_k |w_{nk}| \le (C+1)T_n/n$, too. Then from (4.9) and (4.30) we have for any $L, r > 0$

$$\sup_{t_1 \le \ldots \le t_l \le t} \mathsf{M}^l \mathsf{P}\left\{W_n > \mathrm{e}^L\right\} = \sup_{t_1 \le \ldots \le t_l \le t} \mathsf{M}^l \mathsf{P}\left\{\sum_k |w_{nk}| > L\right\} \le \sup_{t_1 \le \ldots \le t_l \le t} \mathsf{P}\left\{T_n > \frac{rn}{C+1}\right\} + \frac{r}{L}. \tag{4.35}$$

Equality (4.34) shows us that for any $\sigma > 0$

$$\sup_{t_1 \le \ldots \le t_l \le t} \mathsf{P}\{T_n > \sigma\} \le \sum_{j=1}^{l} \left[\sup_{t_j \le t} \mathsf{P}\left\{\sum_{k=1}^{N_n(t_j)} \vartheta_{nk}b_{nk}(t_j) > \frac{\sigma}{2l}\right\} + \sup_{t_j \le t} \mathsf{P}\left\{\sum_{k=1}^{N_n(t_j)} \beta_{nk}(t_j, z_j) > \frac{\sigma}{2l}\right\}\right],$$

which together with (4.35), (4.32) and (3.7) yields

$$\lim_{L \to \infty} \overline{\lim_{n \to \infty}} \sup_{t_1 \le \ldots \le t_l \le t} \mathsf{M}^l \mathsf{P}\left\{W_n > \mathrm{e}^L\right\} \le \sum_{j=1}^{l} \lim_{L \to \infty} \overline{\lim_{n \to \infty}} \sup_{t_j \le t} \mathsf{P}\left\{\sum_{k=1}^{N_n(t_j)} \vartheta_{nk}b_{nk}(t_j) > \frac{n\mathrm{e}^L}{2l(C+1)}\right\}.$$

Now, (4.12) ensues from (3.8).

Once relations (4.11) and (4.12) together implying (4.10) have been proved, it remains (recall that we are proving (4.5)) to show that

$$\prod_k (1 + w_{nk}) - S_n \overset{\mathsf{M}^l \mathsf{P}}{\rightrightarrows} 0. \tag{4.36}$$

11°. Writing, for arbitrary positive ε and L,

$$\mathsf{M}^l \mathsf{P}\left\{W_n \sum_k |w_{nk} - \varkappa_{nk}| > \varepsilon\right\} \le \mathsf{M}^l \mathsf{P}\left\{W_n > \mathrm{e}^L\right\} + \mathsf{M}^l \mathsf{P}\left\{\sum_k |w_{nk} - \varkappa_{nk}| > \mathrm{e}^{-L}\varepsilon\right\},$$

we deduce from (4.12), (4.31), (4.32) and (3.7) (the details are the same as in the previous item) that

$$W_n \sum_k |w_{nk} - \varkappa_{nk}| \overset{\mathsf{M}^l \mathsf{P}}{\rightrightarrows} 0. \tag{4.37}$$

Obviously, $S_n = \prod_k \mathrm{e}^{\varkappa_{nk}}$. Noting that $\mathrm{Re}\,\mathfrak{g}(z_j) \le 0$, $b_{nkj} \ge 0$, $h_{nk}(x) \ge 0$, we see that $\mathrm{Re}\,\varkappa_{nk} \le 0$. Then, putting in (4.8) $m = N_n(t_l)$, $c_k = \mathrm{e}^{\varkappa_{nk}}$, $d_k = 1 + w_{nk}$ and taking to account (4.9), we get

$$\left|S_n - \prod_k (1 + w_{nk})\right| \le W_n \sum_k |\mathrm{e}^{\varkappa_{nk}} - 1 - w_{nk}|. \tag{4.38}$$

According to the maximum principle for analytic functions

$$\sup_{\mathfrak{z} \in \mathbb{C}_-} \left|\frac{\mathrm{e}^{\mathfrak{z}} - 1 - \mathfrak{z}}{\mathfrak{z}^2}\right| = \sup_{s \in \mathbb{R}} \left|\frac{\mathrm{e}^{\mathrm{i}s} - 1 - is}{s^2}\right|.$$

Hence, writing by Euler's and Taylor's formulae

$$e^{is} = 1 + is - (\cos\theta_1 s + i\sin\theta_2 s)s^2/2,$$

we get $|e^{\varkappa_{nk}} - 1 - \varkappa_{nk}| \le |\varkappa_{nk}|^2$, which together with (4.33) and (4.6) yields

$$\sum_k |e^{\varkappa_{nk}} - 1 - \varkappa_{nk}| \le \frac{C^2}{n^2}\left(\sum_{j=1}^{l}\sum_{k=1}^{N_n(t_j)}\vartheta_{nk}b_{nk}(t_j)\right)^2.$$

Noting that the right-hand side of this inequality does not exceed

$$\frac{C^2 l}{n^2}\sum_{j=1}^{l}\left(\sum_{k=1}^{N_n(t_j)}\vartheta_{nk}b_{nk}(t_j)\right)^2,$$

we deduce (4.36) from (4.38), (4.37) and (3.8).

$12°$. Let us check condition (2.15) (the set A_L was defined before Theorem 2.3).

Denote $\kappa_n(s) = |\xi_n(s,\cdot)|$, $\kappa_n^0(s) = \mathsf{E}^0\kappa_n(s)$. Then

$$\psi_n(s, A_L) = \mathsf{M}I\{\kappa_n(s) > L\}, \ \ \mathsf{E}\psi_n(s, A_L) = \mathsf{M}\mathsf{P}\{\kappa_n(s) > L\},$$

so (2.15) will follow from Chebyshev's inequality if we show that

$$\lim_{r\to\infty}\overline{\lim}_{n\to\infty}\sup_{s\le t}\mathsf{M}\mathsf{P}\{\kappa_n(s) > r\} = 0.$$

Writing $\mathsf{P}\{\kappa > r\} \le \mathsf{E}\left(I\left\{\kappa_n^0 \le C\right\}\mathsf{E}^0 I\{\kappa_n > r\} + I\left\{\kappa_n^0 > C\right\}\right) \le r^{-1}\mathsf{E}\kappa_n^0 I\left\{\kappa_n^0 \le C\right\} + \mathsf{P}\left\{\kappa_n^0 > C\right\} \le C/r + \mathsf{P}\left\{\kappa_n^0 > C\right\}$ and then, on the basis of (4.30),

$$\mathsf{M}\mathsf{P}\{\kappa_n^0 > C\} \le \mathsf{P}\{\mathsf{M}\kappa_n^0 > L\} + L/C,$$

we reduce the task to the proof of the relation

$$\lim_{L\to\infty}\overline{\lim}_{n\to\infty}\sup_{s\le t}\mathsf{P}\left\{\mathsf{M}\kappa_n^0(s) > L\right\} = 0. \tag{4.39}$$

By assumptions **A1** and **A3** (i)

$$\kappa_n^0(s) \le \sum_{k=1}^{N_n(s)}\epsilon_{nk}(s), \tag{4.40}$$

where $\epsilon_{nk}(s) = \epsilon_{nk}(s,\cdot)$, $\epsilon_{nk}(s,x) = \mathsf{E}^0\left|F\left(\frac{n^a(x-\zeta_{nk}-\eta_{nk}(s))}{\rho_{nk}(s)}\right)\right|$. Taking in (4.17) $g = |F|$, $\theta = n^a, c = 0$, we get

$$\epsilon_{nk}(s,x) = \frac{1}{n}\int_0^\infty r^p R_{nk}(s,\mathrm{d}r)\int |F(y)|h_{nk}\left(x - \eta_{nk}(s) - \frac{ry}{n^a}\right)\mathrm{d}y,$$

which together with (4.40) and (3.4) yields

$$\mathsf{M}\kappa_n^0(s) \le \frac{1}{n}\sum_{k=1}^{N_n(s)}\vartheta_{nk}b_{nk}(s)\int |F(y)|\mathrm{d}y.$$

Now, (4.39) emerges from (3.3) and (3.8).

13°. Let us check condition (2.16).

Denote $\delta_n(u, v) = \widehat{\psi}_n(u, z) - \widehat{\psi}_n(v, z)$, $\mathbf{t} = (t_1, t_2)$. Writing, for $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathbb{C}$,

$$|\mathfrak{z}_1 - \mathfrak{z}_2|^2 = \mathfrak{z}_1 \overline{\mathfrak{z}}_1 - \mathfrak{z}_1 \overline{\mathfrak{z}}_2 - \mathfrak{z}_2 \overline{\mathfrak{z}}_1 + \mathfrak{z}_2 \overline{\mathfrak{z}}_2,$$

setting $\mathfrak{z}_1 = \widehat{\psi}_n(v, z)$ (so that $\overline{\mathfrak{z}}_1 = \widehat{\psi}_n(v, -z)$), $\mathfrak{z}_2 = \widehat{\psi}_n(u, z)$ and recalling the definition of covaristic, we become convinced that

$$\mathsf{E}|\delta_n(u, v)|^2 = \widetilde{\psi}_n^2(v, v; z, -z) - \widetilde{\psi}_n^2(v, u; z, -z) - \left(\widetilde{\psi}_n^2(u, v; z, -z) - \widetilde{\psi}_n^2(u, u; z, -z) \right).$$

So it suffices to show that for any $t > 0, z_1, z_2 \in \mathbb{R}^d$

$$\lim_{c \to 0} \overline{\lim_{\substack{\mathbf{t}', \mathbf{t}'' \in [0,t]^2, \\ |\mathbf{t}' - \mathbf{t}''| < c}}} \left| \widetilde{\psi}_n^2(\mathbf{t}'; z_1, z_2) - \widetilde{\psi}_n^2(\mathbf{t}''; z_1, z_2) \right| = 0. \tag{4.41}$$

Writing on the basis of (4.2) and (3.11)

$$\sup_{\mathbf{t} \in [0,t]^2} \left| \widetilde{\psi}_n^2(t_1, t_2; z_1, z_2) - Z^2(t_1, t_2; \mathfrak{g}(z_1), \mathfrak{g}(z_2)) \right| \to 0,$$

we deduce (4.41) from the assumed continuity of $Z^2(\cdot, \cdot; \mathfrak{z}_1, \mathfrak{z}_2)$ and Cantor's theorem. $\qquad \square$

Proof of Theorem 3.2. The only distinction of the assumptions of this theorem from those of Theorem 3.1 is that condition (3.11) is substituted by **RC**. Noting that: firstly, (3.11) was used in the previous proof only for verification of condition (2.16); secondly, relation (2.19) with $U^1 = \mathsf{E}\widehat{\psi}$ and relation (2.20) are immediate from (3.13) and (3.3), we deduce the desired conclusion from Corollary 2.5. $\qquad \square$

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