

EXPONENTIAL DEFICIENCY OF CONVOLUTIONS OF DENSITIES *

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Abstract. If a probability density $p(\mathbf{x})$ ($\mathbf{x} \in \mathbb{R}^k$) is bounded and $R(t) := \int \mathbf{e}^{(\mathbf{x}, t\mathbf{u})} p(\mathbf{x}) \, d\mathbf{x} < \infty$ for some linear functional \mathbf{u} and all $t \in (0, 1)$, then, for each $t \in (0, 1)$ and all large enough n , the n -fold convolution of the t -tilted density $\tilde{p}_t(\mathbf{x}) := \mathbf{e}^{(\mathbf{x}, t\mathbf{u})} p(\mathbf{x}) / R(t)$ is bounded. This is a corollary of a general, “non-i.i.d.” result, which is also shown to enjoy a certain optimality property. Such results and their corollaries stated in terms of the absolute integrability of the corresponding characteristic functions are useful for saddle-point approximations.

Mathematics Subject Classification. 60E05, 60E10, 60F10, 62E20, 60E15.

Received June 17, 2009. Revised February 17, 2010 and March 9, 2010.

1. INTRODUCTION

Let \mathbf{X} be a random vector in \mathbb{R}^k such that

$$M := \mathbb{E} e^{\lambda \mathbf{e}\mathbf{X}} < \infty \tag{1.1}$$

for some unit vector $\mathbf{e} \in \mathbb{R}^k$ and some $\lambda \in (0, \infty)$; here the juxtaposition $\mathbf{e}\mathbf{x}$ denotes the Euclidean scalar product of vectors \mathbf{e} and \mathbf{x} in \mathbb{R}^k . By Chebyshev’s inequality, the exponential integrability condition (1.1) implies the tail estimate

$$\mathbb{P}(\mathbf{e}\mathbf{X} \geq x) \leq M e^{-\lambda x} \quad \text{for all } x \in \mathbb{R}. \tag{1.2}$$

Vice versa, for any given $\lambda_0 \in (0, \infty]$ one has the following: if (1.2) holds for each $\lambda \in [0, \lambda_0)$ and some $M = M(\lambda) \in (0, \infty)$, then $\mathbb{E} e^{\lambda \mathbf{e}\mathbf{X}} < \infty$ for each $\lambda \in [0, \lambda_0)$.

Suppose also that (the distribution of) \mathbf{X} has a density p (relative to the Lebesgue measure) such that, for some $\mu \in [0, \lambda)$ and some $C \in (0, \infty)$,

$$p(\mathbf{x}) \leq C e^{-\mu \mathbf{e}\mathbf{x}} \quad \text{for all } \mathbf{x} \in \mathbb{R}^k. \tag{1.3}$$

Note that, if $\mu = 0$, then condition (1.3) simply means that the density p is bounded.

Keywords and phrases. Probability density, saddle-point approximation, sums of independent random variables/vectors, convolution, exponential integrability, boundedness, exponential tilting, exponential families, absolute integrability, characteristic functions.

* Supported by NSF grant DMS-0805946.

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If p is varying regularly enough in an appropriate sense then, given the condition (1.1), one will have (1.3) for $\mu = \lambda$; that is, one will have an exact “local” counterpart to the “integral” upper bound (1.2). The difference

$$\varepsilon := \lambda - \mu$$

(between the largest possible λ and μ for which (1.1) and (1.3) will still hold) may therefore be referred to as the (exponential) “deficiency” of the density p , which is a measure of its irregularity.

The main result of this paper implies that the deficiency decreases fast under convolution: starting with condition (1.3) for p with $\mu = \lambda - \varepsilon$, one has this condition for the n -fold convolution p^{*n} (in place of p) with $\mu = \lambda - \varepsilon/n$; that is, for the n -fold convolution, the deficiency is n times as small as the original one. More generally, it is proved that, for any probability densities p_1, \dots, p_n on \mathbb{R}^k satisfying the exponential integrability condition with the same λ and with respective deficiencies $\varepsilon_1, \dots, \varepsilon_n$, the deficiency of the convolution $p_1 * \dots * p_n$ is no greater than ε^\sharp/n , where ε^\sharp stands for the harmonic mean of the original deficiencies $\varepsilon_1, \dots, \varepsilon_n$. Moreover, it is shown that this bound, ε^\sharp/n , cannot be improved.

2. STATEMENTS OF THE RESULTS

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be any independent random vectors in \mathbb{R}^k , with densities p_1, \dots, p_n . Take any unit vector \mathbf{e} in \mathbb{R}^k .

Assume the following conditions: for some $\lambda \in (0, \infty)$

$$M_i := \mathbf{E} e^{\lambda \mathbf{e} \mathbf{X}_i} = \int_{\mathbb{R}^k} e^{\lambda \mathbf{e} \mathbf{x}} p_i(\mathbf{x}) \, d\mathbf{x} < \infty \quad (2.1)$$

and

$$p_i(\mathbf{x}) \leq C_i e^{-\mu_i \mathbf{e} \mathbf{x}} \quad (2.2)$$

for some C_i 's in $(0, \infty)$, some μ_i 's in $[0, \lambda)$, all $i \in \{1, \dots, n\}$, and all $\mathbf{x} \in \mathbb{R}^k$. Consider the convolution

$$p^{(n)} := p_1 * \dots * p_n, \quad (2.3)$$

which is the density of the sum $\mathbf{X}_1 + \dots + \mathbf{X}_n$.

Theorem 2.1. *There exists a finite constant K_n , which depends only on the numbers n , λ , μ_i , M_i , and C_i , such that*

$$p^{(n)}(\mathbf{x}) \leq K_n e^{-(\lambda - \varepsilon^{(n)}) \mathbf{e} \mathbf{x}} \quad \text{for all } \mathbf{x} \in \mathbb{R}^k, \quad (2.4)$$

where

$$\varepsilon^{(n)} := \frac{1}{\frac{1}{\varepsilon_1} + \dots + \frac{1}{\varepsilon_n}} \quad \text{and} \quad \varepsilon_i := \lambda - \mu_i > 0. \quad (2.5)$$

An upper bound on the constant K_n is given by (4.10).

The necessary proofs will be presented in Section 4.

Note that $\varepsilon^{(n)} = \varepsilon^\sharp/n$, where ε^\sharp denotes the harmonic mean of $\varepsilon_1, \dots, \varepsilon_n$. One may also note that $\varepsilon^{(n)} < \min(\varepsilon_1, \dots, \varepsilon_n)$.

It turns out that the coefficient $\lambda - \varepsilon^{(n)}$ in the exponent in the bound (2.4) is the best possible:

Proposition 2.2. *For any natural k and n , any $\lambda \in (0, \infty)$, and any μ_i 's in $[0, \lambda)$, the estimate (2.4) will fail to hold if the number $\varepsilon^{(n)}$ given by (2.5) is replaced by any smaller number.*

From Theorem 2.1, one immediately obtains the particular “i.i.d.” case:

Corollary 2.3. *If conditions (1.1) and (1.3) hold, then for each natural n there exists a constant K_n , which depends only on the numbers n , λ , μ , M , and C , such that*

$$p^{*n}(\mathbf{x}) \leq K_n e^{-(\lambda-\varepsilon/n)\mathbf{e}\mathbf{x}} \quad \text{for all } \mathbf{x} \in \mathbb{R}^k, \quad (2.6)$$

where $\varepsilon := \lambda - \mu$. An upper bound on the constant K_n in (2.6) is given by (4.12).

It follows from Proposition 2.2 that the coefficient $\lambda - \varepsilon/n$ in the exponent in the bound (2.6) is the best possible.

In turn, Corollary 2.3 yields

Corollary 2.4. *If conditions (1.1) and (1.3) hold, then for each $t \in (0, \lambda)$ there exists a natural number n_t such that for all natural $n \geq n_t$ the n -fold convolution \tilde{p}_t^{*n} of the t -tilted density*

$$\tilde{p}_t(\mathbf{x}) := \frac{e^{t\mathbf{e}\mathbf{x}}p(\mathbf{x})}{\mathbb{E}e^{t\mathbf{e}\mathbf{X}}} \quad (\mathbf{x} \in \mathbb{R}^k) \quad (2.7)$$

is bounded.

In fact, in Corollary 2.4 one may take $n_t = \lceil \frac{\lambda-\mu}{\lambda-t} \rceil$.

Corollary 2.4 can be rewritten as

Corollary 2.5. *If conditions (1.1) and (1.3) hold, then for each $t \in (0, \lambda)$ there exists some $\gamma_t \in (0, \infty)$ such that for all $\gamma \geq \gamma_t$*

$$\int_{\mathbb{R}^k} |\tilde{f}_t(\mathbf{s})|^\gamma d\mathbf{s} < \infty, \quad (2.8)$$

where $\tilde{f}_t(\mathbf{s}) := \int_{\mathbb{R}^k} e^{i\mathbf{s}\mathbf{x}} \tilde{p}_t(\mathbf{x}) d\mathbf{x}$, the characteristic function of the t -tilted density \tilde{p}_t ; here, of course, i stands for the imaginary unit.

Remark 2.6. In applications, one may of course assume the “grouping”: $\mathbf{X}_j = \mathbf{Y}_{m_{j-1}+1} + \dots + \mathbf{Y}_{m_j}$ for $j = 1, \dots, n$, where $0 = m_0 < m_1 < \dots$ and the \mathbf{Y} ’s are independent random vectors, whose distributions may themselves not have a density. Then the densities p_1, \dots, p_n as in Theorem 2.1 will be the densities of the convolutions of the distributions of the corresponding \mathbf{Y} ’s.

3. DISCUSSION AND APPLICATION

The condition that the convolution p^{*n_0} of the underlying population density p be bounded for some natural n_0 is quite commonly required to derive an Edgeworth expansion, as *e.g.* is done in [1] or, in an equivalent “Fourier” form as in Corollary 2.5, in [5]; then, of course, p^{*n} will be bounded (by the same constant) for all $n \geq n_0$. In many existing accounts in the literature, n_0 is taken by the authors to be simply 1.

However, when an Edgeworth expansion is used to derive a saddle-point approximation, one needs the boundedness of a convolution $\tilde{p}_t^{*n_0}$ of the *tilted* density \tilde{p}_t . Such a condition appears to be usually imposed outright; see *e.g.* Barndorff-Nielsen and Cox ([1], p. 298, condition c); Lugannani and Rice ([7], p. 481, condition (ii)) impose an even stronger condition, requiring (for $k = 1$) that $|\tilde{f}_t(s)| = O((1 + |s|)^{-\gamma})$ for some $\gamma > 0$ and all $s \in \mathbb{R}$.

On the other hand, such results as Corollaries 2.4 and 2.5 together with Remark 2.6 show that one need *a priori* require the boundedness of $\tilde{p}_t^{*n_0}$ only for $t = 0$ and some natural n_0 , that is, only for some convolution p^{*n_0} of the original, non-tilted density p .

Let us state the corresponding result, which extends the mentioned result in [1]. (Surveys of literature on saddle-point approximations are given *e.g.* in [3,9]; for more recent work see *e.g.* [5,10].) Let $\mathbf{W}, \mathbf{W}_1, \mathbf{W}_2, \dots$ be independent identically distributed zero-mean random vectors in \mathbb{R}^k each with a density function p such

that the convolution p^{*n_0} is bounded for some natural n_0 . More generally, one could require here only that the n_0 -fold convolution of the distribution of \mathbf{W} have a bounded density, say p_{n_0} , and then use p_{n_0} in place of p^{*n_0} .

Assume also that

$$\beta(\boldsymbol{\theta}) := \ln \mathbf{E} e^{\boldsymbol{\theta} \mathbf{W}} < \infty \quad (3.1)$$

for some $\delta_0 > 0$ and all $\boldsymbol{\theta} \in B(\delta_0)$, where $B(\delta_0) := \{\boldsymbol{\theta} \in \mathbb{R}^k : \|\boldsymbol{\theta}\| \leq \delta_0\}$ and $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^k . The crucial point here is that, by Corollary 2.3, for any $\delta \in (0, \delta_0)$ there exists some natural n_δ such that the n_δ -fold convolution $p_\theta^{*n_\delta}$ of the tilted density

$$p_\theta(\mathbf{w}) := e^{\boldsymbol{\theta} \mathbf{w}} p(\mathbf{w}) e^{-\beta(\boldsymbol{\theta})} \quad (3.2)$$

is bounded uniformly in all $\boldsymbol{\theta} \in B(\delta)$.

Indeed, by our assumptions, $C := \sup\{p^{*n_0}(\mathbf{w}) : \mathbf{w} \in \mathbb{R}^k\} < \infty$ and $M := \max\{\int_{\mathbb{R}^k} e^{\boldsymbol{\theta} \mathbf{w}} p^{*n_0}(\mathbf{w}) d\mathbf{w} : \boldsymbol{\theta} \in B(\delta_0)\} < \infty$, the latter relation taking place because $\int_{\mathbb{R}^k} e^{\boldsymbol{\theta} \mathbf{w}} p^{*n_0}(\mathbf{w}) d\mathbf{w} = e^{n_0 \beta(\boldsymbol{\theta})}$ and at that $\beta(\boldsymbol{\theta})$ is finite and hence continuous and bounded in $\boldsymbol{\theta} \in B(\delta_0)$. So, for any natural m , all $\boldsymbol{\tau} \in B(\delta)$, and all $\mathbf{w} \in \mathbb{R}^k$

$$p^{*n_0 m}(\mathbf{w}) \leq K_m e^{-(1-1/m)\boldsymbol{\tau} \mathbf{w}}, \quad (3.3)$$

where the constant K_m can, for instance, be taken to be $(m-1)! C M^{m-1} e$, in accordance with (4.13); inequality (3.3) follows by Corollary 2.3, with $\mathbf{e} = \boldsymbol{\tau}/\|\boldsymbol{\tau}\|$, $\lambda = \|\boldsymbol{\tau}\|$, $\mu = 0$, m and p^{*n_0} in place of n and p , and M and C as defined above in this paragraph.

Take now any $\delta \in (0, \delta_0)$ and then take m in (3.3) to be $m_\delta := \lceil \frac{\delta_0}{\delta_0 - \delta} \rceil$, so that the image of $B(\delta_0)$ under the map $\boldsymbol{\tau} \mapsto (1-1/m)\boldsymbol{\tau}$ contain $B(\delta)$; then there exists a natural number n_δ such that the tilted convolution $p_\theta^{*n_\delta}$ is bounded uniformly in $\boldsymbol{\theta} \in B(\delta)$ – it is enough to take $n_\delta = n_0 m_\delta$:

$$p_\theta^{*n_\delta}(\mathbf{w}) = e^{\boldsymbol{\theta} \mathbf{w}} p^{*n_0 m_\delta}(\mathbf{w}) e^{-n_\delta \beta(\boldsymbol{\theta})} \leq K_{m_\delta}, \quad (3.4)$$

because $\beta(\boldsymbol{\theta}) \geq 0$ for all $\boldsymbol{\theta}$, by Jensen's inequality. Thus, the mentioned condition ([1], p. 298, condition c), will be satisfied, even uniformly in $\boldsymbol{\theta} \in B(\delta)$.

Therefore (cf. [1], (A.3)), one will have the Edgeworth expansion

$$n^{k/2} p_\theta^{*n}(\mathbf{t}) = g_d(\mathbf{z}, \Sigma_\theta) \left(1 + \sum_{j=1}^r Q_j(\mathbf{z}, \boldsymbol{\theta}) n^{-j/2}\right) + O(n^{-(r+1)/2}) \quad (3.5)$$

for each $\delta \in (0, \delta_0)$ and each $r = 0, 1, \dots$ uniformly in $\mathbf{t} \in \mathbb{R}^k$ and $\boldsymbol{\theta} \in B(\delta)$, where $\mathbf{z} := n^{-1/2}(\mathbf{t} - n \mathbf{E}_\theta \mathbf{W})$, $\Sigma_\theta := \text{Cov}_\theta \mathbf{W}$ is nonsingular, \mathbf{E}_θ and Cov_θ are the expectation and covariance operators with respect to the distribution with the tilted density p_θ , $g_d(\cdot, \Sigma_\theta)$ is the density of the centered normal distribution in \mathbb{R}^k with covariance matrix Σ_θ ; for each j , $Q_j(\mathbf{z}, \boldsymbol{\theta})$ is a polynomial in \mathbf{z} defined by the identity

$$Q_j(\mathbf{z}, \boldsymbol{\theta}) \phi(\Sigma_\theta^{-1/2} \mathbf{z}) = (2\pi)^{-k/2} \int_{\mathbb{R}^k} e^{i\boldsymbol{\tau} \Sigma_\theta^{-1/2} \mathbf{z}} P_j(\boldsymbol{\tau}, \boldsymbol{\theta}) \phi(\boldsymbol{\tau}) d\boldsymbol{\tau},$$

ϕ is the standard normal density in \mathbb{R}^k , and the $P_j(\boldsymbol{\tau}, \boldsymbol{\theta})$'s are polynomials in $\boldsymbol{\tau}$ defined by the identity

$$\mathbf{E}_\theta \exp \left\{ i\boldsymbol{\tau} (n \Sigma_\theta)^{-1/2} \sum_{j=1}^n (\mathbf{W}_j - \mathbf{E}_\theta \mathbf{W}) \right\} = e^{-\boldsymbol{\tau} \boldsymbol{\tau} / 2} \left(1 + \sum_{j=1}^\infty P_j(\boldsymbol{\tau}, \boldsymbol{\theta}) n^{-j/2}\right)$$

for all $\boldsymbol{\tau}$ in a neighborhood of $\mathbf{0}$; of course, i in the latter two displays stands for the imaginary unit (note that the needed factor $n^{k/2}$ on the left-hand side of (3.5) is missing in ([1], (A.3))). By the inverse function theorem, for any $\mathbf{t} \in \mathbb{R}^k$ such that \mathbf{t}/n lies in a small enough neighborhood, say \mathcal{V} , of $\mathbf{0}$ there exists a unique root $\boldsymbol{\theta} = \boldsymbol{\theta}_\mathbf{t}$ of the equation $n \mathbf{E}_\theta \mathbf{W} = \mathbf{t}$, so that the corresponding $\mathbf{z} = n^{-1/2}(\mathbf{t} - n \mathbf{E}_\theta \mathbf{W})$ in (3.5) is $\mathbf{0}$; thus, for such \mathbf{t} and any $m = 0, 1, \dots$ one obtains the saddlepoint approximation

$$p^{*n}(\mathbf{t}) = (2\pi n)^{-k/2} (\det \Sigma_{\boldsymbol{\theta}_\mathbf{t}})^{-1/2} \exp \{ n\beta(\boldsymbol{\theta}_\mathbf{t}) - \boldsymbol{\theta}_\mathbf{t} \mathbf{t} \} \left[1 + \sum_{k=1}^m Q_{2k}(\mathbf{0}, \boldsymbol{\theta}_\mathbf{t}) n^{-k} + O(n^{-(m+1)})\right], \quad (3.6)$$

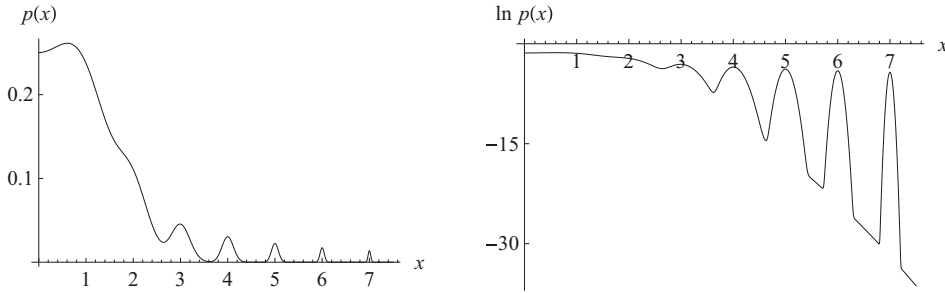


FIGURE 1. Graphs $\{(x, p(x)): 0 < x < 7.5\}$ and $\{(x, \ln p(x)): 0 < x < 7.5\}$ for $p = \tilde{p}_{\lambda, \varepsilon, \kappa, \alpha}$ with $\lambda = 0.55$, $\varepsilon = 0.50$, $\kappa = 0.9$, and $\alpha = 0.6$.

since the polynomials $Q_j(\mathbf{z}, \boldsymbol{\theta})$ are odd in \mathbf{z} for odd j . The size of the neighborhood \mathcal{V} depends, *via* the implicit function theorem, on the distribution of \mathbf{W} , and it also depends on δ , as one needs \mathbf{t} to be small enough for $\boldsymbol{\theta}_{\mathbf{t}}$ to be in $B(\delta)$.

By what has just been shown, taking $m = m_\delta = 2$ in (3.3) and (3.4), one obtains the Edgeworth expansion (3.5) for $\boldsymbol{\theta}$ in the neighborhood $B(\delta)$ of $\mathbf{0}$ with $\delta = \frac{1}{2}\delta_0$, that is, in the neighborhood twice smaller than the “original” neighborhood $B(\delta_0)$, in which condition (3.1) was assumed to hold; the corresponding constant $K_m = K_2$ may be taken to be no greater than εCM (in fact, even no greater than $2CM$).

If, however, one wants the Edgeworth expansion (3.5) to hold for all $\boldsymbol{\theta} \in B(\delta)$ with some δ greater than $\frac{1}{2}\delta_0$, then m has to be increased from 2 to $m_\delta := \lceil \frac{\delta_0}{\delta_0 - \delta} \rceil$, which goes to ∞ as $\delta \uparrow \delta_0$, and then of course K_{m_δ} will go to ∞ as well. So, one could expect that the quality of the Edgeworth expansion may deteriorate as $\boldsymbol{\theta}$ gets closer to the boundary of the neighborhood $B(\delta_0)$, in which condition (3.1) was assumed to hold. That appears only natural. However, if $\delta_0 = \infty$, so that condition (3.1) holds for all $\boldsymbol{\theta} \in \mathbb{R}^k$, then $m = 2$ and $K_m = K_2 = 2CM$ will be enough for the Edgeworth expansion to be good for all $\boldsymbol{\theta} \in \mathbb{R}^k$.

In a yet unpublished manuscript we use Corollary 2.3 to obtain other saddle-point approximations under similarly relaxed conditions.

The considerations presented above in this section constituted the original motivation for the present work.

The proof of Proposition 2.2 (given in the next section) shows that probability densities with the deficiencies most resistant to convolution are mixtures of infinitely many mutually (almost) singular densities, spaced regularly enough (see Fig. 1 on p. 90). Such “exponentially deficient” distributions can be contrasted with the well-studied classes of regularly behaving distributions with nearly exponential tails; see *e.g.* [4,6,8].

4. PROOFS

Proof of Theorem 2.1. To begin, note that for $n = 1$ the inequality (2.4) with $K_1 := C_1$ is the same as (2.2). Next, a trivial remark is that (2.1) implies $\mathbf{E} e^{\lambda(\mathbf{X}_1 + \dots + \mathbf{X}_{n-1})} = M_1 \cdots M_{n-1} < \infty$. Note also that (2.5) can be rewritten in an additive form, as

$$\frac{1}{\varepsilon^{(n)}} = \frac{1}{\varepsilon_1} + \dots + \frac{1}{\varepsilon_n}.$$

So, by induction, it suffices to prove Theorem 2.1 for $n = 2$. For such a case, let us simplify the notation by writing p and q instead of p_1 and p_2 , M and N instead of M_1 and M_2 , C and D instead of C_1 and C_2 , μ and ν instead of μ_1 and μ_2 , and ε and δ instead of ε_1 and ε_2 .

Next, without loss of generality, $\mathbf{e} = (1, 0, \dots, 0) \in \mathbb{R}^k$. Then, identifying any vector $\mathbf{x} \in \mathbb{R}^k$ with the corresponding pair $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$, one has $\mathbf{e}\mathbf{x} = x$, so that (2.2) can in this case be rewritten as

$$p(x, \mathbf{y}) \leq C e^{-\mu x} \quad \text{and} \quad q(x, \mathbf{y}) \leq D e^{-\nu x} \quad (4.1)$$

for all $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Also, conditions (2.1) imply

$$\int_x^\infty du \tilde{p}(u) \leq M e^{-\lambda x} \quad \text{and} \quad \int_x^\infty du \tilde{q}(u) \leq N e^{-\lambda x} \quad (4.2)$$

for all $x \in \mathbb{R}$, where

$$\tilde{p}(u) := \int_{\mathbb{R}^k} d\mathbf{v} p(u, \mathbf{v}) \quad \text{and} \quad \tilde{q}(u) := \int_{\mathbb{R}^k} d\mathbf{v} q(u, \mathbf{v})$$

for all $u \in \mathbb{R}$, the densities of the random variables \mathbf{eX}_1 and \mathbf{eX}_2 , respectively.

Fix now any $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Take, for a moment, any $\alpha \in (0, 1)$ and let $\beta := 1 - \alpha$. Then

$$(p * q)(x, \mathbf{y}) = \int_{\mathbb{R}} du \int_{\mathbb{R}^{k-1}} d\mathbf{v} p(x - u, \mathbf{y} - \mathbf{v}) q(u, \mathbf{v}) \leq D I_1 + C I_2 \quad (4.3)$$

by (4.1), where

$$\begin{aligned} I_1 &:= \int_{-\infty}^{\alpha x} du \int_{\mathbb{R}^{k-1}} d\mathbf{v} p(x - u, \mathbf{y} - \mathbf{v}) e^{-\nu u} = \int_{-\infty}^{\alpha x} du \tilde{p}(x - u) e^{-\nu u}, \\ I_2 &:= \int_{\alpha x}^\infty du \int_{\mathbb{R}^{k-1}} d\mathbf{v} p(x - u, \mathbf{y} - \mathbf{v}) q(u, \mathbf{v}) = \int_{-\infty}^{\beta x} dz \tilde{q}(x - z) e^{-\mu z}. \end{aligned}$$

Next, in view of (4.2),

$$\begin{aligned} I_1 &= \int_{-\infty}^{\alpha x} du \tilde{p}(x - u) \int_u^\infty \nu dz e^{-\nu z} \\ &= \int_{-\infty}^{\alpha x} \nu dz e^{-\nu z} \int_{-\infty}^z du \tilde{p}(x - u) + \int_{\alpha x}^\infty \nu dz e^{-\nu z} \int_{-\infty}^{\alpha x} du \tilde{p}(x - u) \\ &= \int_{-\infty}^{\alpha x} \nu dz e^{-\nu z} \int_{x-z}^\infty dw \tilde{p}(w) + \int_{\alpha x}^\infty \nu dz e^{-\nu z} \int_{(1-\alpha)x}^\infty dw \tilde{p}(w) \\ &\leq \int_{-\infty}^{\alpha x} \nu dz e^{-\nu z} M e^{-\lambda(x-z)} + \int_{\alpha x}^\infty \nu dz e^{-\nu z} M e^{-\lambda(1-\alpha)x} \\ &= M \frac{\lambda}{\lambda - \nu} e^{-(\lambda - (\lambda - \nu)\alpha)x}. \end{aligned} \quad (4.4)$$

Note that this derivation of the upper bound (4.5) on I_1 is valid only for $\nu \neq 0$. However, if $\nu = 0$, then

$$I_1 = \int_{-\infty}^{\alpha x} du \tilde{p}(x - u) = \int_{(1-\alpha)x}^\infty dz \tilde{p}(z) \leq M e^{-\lambda(1-\alpha)x},$$

so that the bound (4.5) on I_1 holds for $\nu = 0$ as well. Recall now that $\varepsilon = \lambda - \mu$ and $\delta = \lambda - \nu$, and choose $\alpha := \frac{\varepsilon}{\varepsilon + \delta}$. Then (4.5) can be rewritten as

$$I_1 \leq M \frac{\lambda}{\delta} e^{-(\lambda - \varepsilon^{(2)})x}, \quad (4.6)$$

with $\varepsilon^{(2)} = \frac{\varepsilon\delta}{\varepsilon + \delta} = \frac{1}{1/\varepsilon + 1/\delta}$, in accordance with the definition (2.5) of $\varepsilon^{(n)}$. Quite similarly,

$$I_2 \leq N \frac{\lambda}{\varepsilon} e^{-(\lambda - \varepsilon^{(2)})x}. \quad (4.7)$$

Collecting now (4.3), (4.6), and (4.7), one sees that

$$(p * q)(x, \mathbf{y}) \leq K_2 e^{-(\lambda - \varepsilon^{(2)})x} \quad (4.8)$$

for all $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$, where

$$K_2 \leq D M \frac{\lambda}{\delta} + C N \frac{\lambda}{\varepsilon}. \quad (4.9)$$

Thus, Theorem 2.1 is proved for $n = 2$ and, thereby, for all natural n , except that one has yet to specify the constant K_n .

Since the convolution operation is commutative and associative, one can get many different upper bounds on K_n , depending on the choices of a permutation and of a sequence of dyadic partitions of the set $\{1, \dots, n\}$. For example, for $n = 5$ one can represent the convolution $p_1 * \dots * p_5$ either in the “sequential” form $((p_1 * p_2) * p_3) * p_4 * p_5$ or a balanced-tree form such as $(p_3 * p_4) * ((p_1 * p_2) * p_5)$.

For any $n \geq 2$, choosing the sequential representation $(\dots((p_1 * p_2) * p_3) * \dots) * p_n$ of $p_1 * \dots * p_n$ and referring to (4.9), one obtains by induction on n :

$$K_n \leq \lambda^{n-1} \left(\prod_{j=1}^n M_j \right) \sum_{j=1}^n B_j \prod_{i=j}^{n-1} \frac{1}{\varepsilon^{(i)}}, \quad (4.10)$$

where $B_j := \frac{C_j}{\lambda^{j-2} \varepsilon_j M_j}$ and (as before) $\frac{1}{\varepsilon^{(j)}} = \frac{1}{\varepsilon_1} + \dots + \frac{1}{\varepsilon_j}$ for $j = 2, \dots, n$, and $B_1 := \frac{C_1}{M_1}$; we let $\prod_{i=n}^{n-1} \frac{1}{\varepsilon^{(i)}}$ to be 1, as usually done with the product of an empty set of factors.

In particular, if

$$\varepsilon_j \geq \varepsilon > 0, \quad C_j \leq C, \quad \text{and} \quad M_j \leq M \quad \text{for all } j = 1, \dots, n, \quad (4.11)$$

then

$$K_n \leq (n-1)! C \left(\frac{\lambda M}{\varepsilon} \right)^{n-1} \left(1 + \sum_{i=0}^{n-2} \frac{\varepsilon^i}{\lambda^i (i+1)!} \right); \quad (4.12)$$

in particular, if $\mu = 0$ and hence $\varepsilon = \lambda$, one has

$$K_n \leq (n-1)! C M^{n-1} e. \quad (4.13)$$

One may conjecture that, under the same assumptions (4.11), an upper bound on K_n better than that in (4.12) can be obtained using maximally-balanced-tree representations rather than the sequential one; for example, for $n = 5$ the representation $((p_1 * p_2) * p_3) * (p_4 * p_5)$ results in an upper bound on K_n which is less than the one given by (4.12) for any positive λ , ε , M , and C . \square

Let us now turn to the proof of Proposition 2.2, which rests on Lemma 4.1 below. To state the lemma, for any $\lambda \in (0, \infty)$ and $\varepsilon \in (0, \lambda]$ introduce the class $\mathcal{P}_{\lambda, \varepsilon}$ of all probability densities p on \mathbb{R} such that

- (i) $\int_{\mathbb{R}} e^{\lambda x} p(x) dx < \infty$ and
- (ii) $p(x) \geq c p_{\lambda, \varepsilon, \kappa, \alpha}(x)$ for some $c \in (0, \infty)$, $\kappa \in (0, \infty)$, $\alpha \in (\frac{1}{2}, \infty)$, and all $x \in \mathbb{R}$, where

$$p_{\lambda, \varepsilon, \kappa, \alpha}(x) := \sum_{j=-\infty}^{\infty} W_j(x), \quad (4.14)$$

$$W_j(x) := W_{j; \lambda, \varepsilon, \kappa, \alpha}(x) := w_j f_{j, \kappa e^{-\varepsilon|j|}}(x), \quad (4.15)$$

$$w_j := w_{j; \lambda, \alpha} := \frac{e^{-\lambda|j|}}{(j^2 + 1)^\alpha}, \quad (4.16)$$

$$f_{a,b}(x) := \frac{1}{b} \varphi\left(\frac{x-a}{b}\right), \quad \varphi(u) := \frac{1}{\sqrt{2\pi}} e^{-u^2/2};$$

of course, $f_{a,b}$ is the density of the normal distribution with mean a and variance b^2 .

(One could similarly, and even a little more easily, deal with the “asymmetric” version of the class $\mathcal{P}_{\lambda,\varepsilon}$, having $\sum_{j=-\infty}^{\infty}$ in (4.14) replaced by $\sum_{j=0}^{\infty}$.)

Lemma 4.1. *Take any $\lambda \in (0, \infty)$, $\varepsilon \in (0, \lambda]$, $\kappa \in (0, \infty)$, and $\alpha \in (\frac{1}{2}, \infty)$.*

(I) *There exists some $c_{\lambda,\varepsilon,\kappa,\alpha} \in (0, \infty)$ such that $\tilde{p}_{\lambda,\varepsilon,\kappa,\alpha} := \frac{p_{\lambda,\varepsilon,\kappa,\alpha}}{c_{\lambda,\varepsilon,\kappa,\alpha}} \in \mathcal{P}_{\lambda,\varepsilon}$. In particular, it follows that*

$$\mathcal{P}_{\lambda,\varepsilon} \neq \emptyset.$$

(II) *There exists some $C = C_{\lambda,\varepsilon,\kappa,\alpha} \in (0, \infty)$ such that for $p = p_{\lambda,\varepsilon,\kappa,\alpha}$ and $\mu := \lambda - \varepsilon$*

$$p(x) \leq C e^{-\mu x} \quad \text{for all } x \in \mathbb{R}. \quad (4.17)$$

(III) *For any $p \in \mathcal{P}_{\lambda,\varepsilon}$ and any $C \in (0, \infty)$, relation (4.17) does not hold with any $\mu^\diamond \in (\lambda - \varepsilon, \infty)$ in place of μ .*

(IV) *In addition to ε , take any $\delta \in (0, \lambda]$. Then, for any $p \in \mathcal{P}_{\lambda,\varepsilon}$ and $q \in \mathcal{P}_{\lambda,\delta}$, one has $p * q \in \mathcal{P}_{\lambda,\tilde{\varepsilon}}$, where*

$$\tilde{\varepsilon} := \frac{1}{\frac{1}{\varepsilon} + \frac{1}{\delta}} = \frac{\varepsilon\delta}{\varepsilon + \delta}. \quad (4.18)$$

The (symmetric about 0) probability density $\tilde{p}_{\lambda,\varepsilon,\kappa,\alpha}$ as in part (I) of this lemma is illustrated in Figure 1 on page 90.

Let us postpone the proof of Lemma 4.1, which is somewhat long, and proceed now to the Proof of Proposition 2.2.

Proof of Proposition 2.2. Take indeed any natural k and n , any $\lambda \in (0, \infty)$, and any μ_1, \dots, μ_n in $[0, \lambda]$. In accordance with (2.5), let $\varepsilon_i := \lambda - \mu_i$, so that $\varepsilon_i \in (0, \lambda]$ for all $i = 1, \dots, n$. For each $i = 1, \dots, n$, take any density $q_i \in \mathcal{P}_{\lambda,\varepsilon_i}$ such that

$$q_i(x) \leq C_i e^{-\mu_i x} \quad (4.19)$$

for some finite positive real constant C_i and all $x \in \mathbb{R}$; by parts (I) and (II) of Lemma 4.1, such q_i 's do exist.

As in the proof of Theorem 2.1, let $\mathbf{e} = (1, 0, \dots, 0) \in \mathbb{R}^k$ and identify any vector $\mathbf{x} \in \mathbb{R}^k$ with $(x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Then, for each $i = 1, \dots, n$, introduce the densities

$$p_i(\mathbf{x}) = p_i(x, \mathbf{y}) := q_i(x) \varphi_{k-1}(\mathbf{y}) \quad (4.20)$$

for all $\mathbf{x} = (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$, where $\varphi_{k-1}(\mathbf{y}) := (2\pi)^{-(k-1)/2} e^{-\mathbf{y}\mathbf{y}/2}$ for all $\mathbf{y} \in \mathbb{R}^{k-1}$; then

$$\int_{\mathbb{R}^k} e^{\lambda \mathbf{e}\mathbf{x}} p_i(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}} e^{\lambda x} q_i(x) dx < \infty,$$

since $q_i \in \mathcal{P}_{\lambda,\varepsilon_i}$; also, by (4.19),

$$p_i(\mathbf{x}) = q_i(x) \varphi_{k-1}(\mathbf{y}) \leq (2\pi)^{-(k-1)/2} q_i(x) \leq C_i e^{-\mu_i x}$$

for all $\mathbf{x} = (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. So, conditions (2.1) and (2.2) hold.

Next, introduce

$$q^{(n)} := q_1 * \dots * q_n,$$

so that, by (2.3) and (4.20),

$$p^{(n)}(\mathbf{x}) = p^{(n)}(x, \mathbf{y}) = q^{(n)}(x) \varphi_{k-1}^{*n}(\mathbf{y}) \quad (4.21)$$

for all $\mathbf{x} = (x, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Moreover, recalling the conditions $q_i \in \mathcal{P}_{\lambda,\varepsilon_i}$ for $i = 1, \dots, n$ and using part (IV) of Lemma 4.1, by induction one concludes that $q^{(n)} \in \mathcal{P}_{\lambda,\varepsilon^{(n)}}$.

Now, to obtain a contradiction, assume that (2.4) holds with some “deficiency” ε^\diamond in place of $\varepsilon^{(n)}$ such that $\varepsilon^\diamond < \varepsilon^{(n)}$. Then, by (4.21), for $\mu^\diamond := \lambda - \varepsilon^\diamond$

$$q^{(n)}(x)\varphi_{k-1}^{*n}(\mathbf{0}) \leq K_n e^{-\mu^\diamond x}$$

for some constant K_n and all $x \in \mathbb{R}$. But this contradicts part (III) of Lemma 4.1, since $\mu^\diamond \in (\lambda - \varepsilon^{(n)}, \infty)$, $q^{(n)} \in \mathcal{P}_{\lambda, \varepsilon^{(n)}}$, and $\varphi_{k-1}^{*n}(\mathbf{0}) = (2\pi n)^{-(k-1)/2} > 0$. This concludes the Proof of Proposition 2.2, except that one still needs to prove Lemma 4.1. \square

Proof of Lemma 4.1.

(I) Obviously, $p_{\lambda, \varepsilon, \kappa, \alpha} > 0$ and $c_{\lambda, \varepsilon, \kappa, \alpha} := \int_{\mathbb{R}} p_{\lambda, \varepsilon, \kappa, \alpha}(x) dx = \sum_{j=-\infty}^{\infty} w_j < \infty$ for any $\lambda \in (0, \infty)$, $\varepsilon \in (0, \lambda]$, $\kappa \in (0, \infty)$, and $\alpha \in (\frac{1}{2}, \infty)$. So, $\tilde{p}_{\lambda, \varepsilon, \kappa, \alpha}$ is a probability density. Moreover,

$$\int_{\mathbb{R}} e^{\lambda x} p_{\lambda, \varepsilon, \kappa, \alpha}(x) dx = \sum_{j=-\infty}^{\infty} w_j \exp\left(\lambda j + \frac{1}{2} \lambda^2 \kappa^2 e^{-2\varepsilon|j|}\right) \leq \sum_{j=-\infty}^{\infty} \frac{e^{\lambda^2 \kappa^2 / 2}}{(j^2 + 1)^\alpha} < \infty. \quad (4.22)$$

Thus, part (I) of Lemma 4.1 is verified.

(II) Note that

$$W_j(x) = \frac{1}{\kappa\sqrt{2\pi}} \frac{e^{-(\lambda-\varepsilon)|j|}}{(j^2 + 1)^\alpha} \exp\left(-\frac{(x-j)^2 e^{2\varepsilon|j|}}{2\kappa^2}\right). \quad (4.23)$$

Hence and because $\varepsilon \in (0, \lambda]$, one has $p_{\lambda, \varepsilon, \kappa, \alpha}(x) \leq C := \sum_{j=-\infty}^{\infty} W_j(j) < \infty$ for all $x \in \mathbb{R}$. So, $p_{\lambda, \varepsilon, \kappa, \alpha}(x) \leq C e^{-\mu x}$ for all $x \in (-\infty, 0]$; that is, (4.17) holds for $p = p_{\lambda, \varepsilon, \kappa, \alpha}$, $\mu = \lambda - \varepsilon$, and all $x \in (-\infty, 0]$.

Take now any $x \in (0, \infty)$. Introduce $j_x := \lfloor x \rfloor$, so that $0 \leq j_x \leq x < j_x + 1$ and for $j \geq j_x$ one has $|j| = j > x - 1$. Then, in view of (4.23),

$$\sum_{j=j_x}^{\infty} W_j(x) \leq \sum_{j=j_x}^{\infty} W_j(j) \leq \frac{e^{-(\lambda-\varepsilon)(x-1)}}{\kappa\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{1}{(j^2 + 1)^\alpha} = c_1 e^{-(\lambda-\varepsilon)x}; \quad (4.24)$$

in this proof of part (II) of the lemma, let c_1, c_2, \dots denote finite positive constants depending only on $\lambda, \varepsilon, \kappa, \alpha$. Next, for $r_x := \lceil \kappa\sqrt{2(\lambda-\varepsilon)x} \rceil$ and $j \in (-\infty, j_x - r_x]$, one has $x - j \geq r_x$, whence

$$\sum_{j=-\infty}^{j_x - r_x} W_j(x) \leq \sum_{j=-\infty}^{j_x - r_x} W_j(j) \exp\left(-\frac{r_x^2}{2\kappa^2}\right) \leq c_2 \exp\left(-\frac{r_x^2}{2\kappa^2}\right) \leq c_2 e^{-(\lambda-\varepsilon)x}. \quad (4.25)$$

Further, for $j \in [j_x - r_x + 1, j_x - 1]$ one has $x - j \geq 1$ and $|j| \geq j \geq j_x - r_x + 1 > x - r_x \geq \frac{x}{2} - c_3$, whence

$$\sum_{j=j_x - r_x + 1}^{j_x - 1} W_j(x) \leq \sum_{j=j_x - r_x + 1}^{j_x - 1} W_j(j) \exp\left(-\frac{e^{2\varepsilon(x-r_x)}}{2\kappa^2}\right) \leq c_4 \exp\left(-\frac{e^{\varepsilon(x-2c_3)}}{2\kappa^2}\right) \leq c_5 e^{-(\lambda-\varepsilon)x}. \quad (4.26)$$

So, by (4.14)–(4.26), the relation (4.17) (with $\mu = \lambda - \varepsilon$) holds for $p = p_{\lambda, \varepsilon, \kappa, \alpha}$ and all $x \in (0, \infty)$ as well. This completes the verification of part (II) of the lemma.

(III) Take any $p \in \mathcal{P}_{\lambda, \varepsilon}$, so that $p \geq c p_{\lambda, \varepsilon, \kappa, \alpha}$ for some $c \in (0, \infty)$, $\kappa \in (0, \infty)$, and $\alpha \in (\frac{1}{2}, \infty)$. Then

$$p(j) \geq c p_{\lambda, \varepsilon, \kappa, \alpha}(j) \geq c W_j(j) = \frac{c}{\kappa\sqrt{2\pi}} \frac{e^{-(\lambda-\varepsilon)j}}{(j^2 + 1)^\alpha} > C e^{-\mu^\diamond j}$$

for any $\mu^\diamond \in (\lambda - \varepsilon, \infty)$, any $C \in (0, \infty)$, and all large enough natural j . This proves part (III) of the lemma.

(IV) Take any $p \in \mathcal{P}_{\lambda, \varepsilon}$ and $q \in \mathcal{P}_{\lambda, \delta}$, so that $p \geq c p_{\lambda, \varepsilon, \kappa, \alpha}$ and $q \geq \tilde{c} p_{\lambda, \delta, \xi, \beta}$ for some $c \in (0, \infty)$, $\kappa \in (0, \infty)$, $\alpha \in (\frac{1}{2}, \infty)$, $\tilde{c} \in (0, \infty)$, $\xi \in (0, \infty)$, and $\beta \in (\frac{1}{2}, \infty)$.

Choose for a moment any $m \in \{0, 1, \dots\}$ and let

$$i_m := \left\lfloor m \frac{\delta}{\varepsilon + \delta} \right\rfloor \quad \text{and} \quad j_m := \left\lfloor m \frac{\varepsilon}{\varepsilon + \delta} \right\rfloor = m - i_m, \quad (4.27)$$

so that $m \frac{\delta}{\varepsilon + \delta} - 1 \leq i_m \leq m \frac{\delta}{\varepsilon + \delta}$ and $m \frac{\varepsilon}{\varepsilon + \delta} \leq j_m \leq m \frac{\varepsilon}{\varepsilon + \delta} + 1$. Next, introduce

$$\begin{aligned} \sigma_m &:= \sqrt{\kappa^2 e^{-2\varepsilon|i_m|} + \xi^2 e^{-2\delta|j_m|}} = \sqrt{\kappa^2 e^{-2\varepsilon i_m} + \xi^2 e^{-2\delta j_m}}, \\ \zeta &:= \sqrt{\kappa^2 + \xi^2 e^{-2\delta}}, \quad \tilde{\zeta} := \sqrt{\kappa^2 e^{2\varepsilon} + \xi^2}, \end{aligned}$$

and observe that

$$\begin{aligned} \frac{\tilde{\zeta}}{\zeta} &\leq e^{\varepsilon \vee \delta}, \\ \sigma_m^2 &\geq \kappa^2 \exp\left\{-2\varepsilon m \frac{\delta}{\varepsilon + \delta}\right\} + \xi^2 \exp\left\{-2\delta \left(m \frac{\varepsilon}{\varepsilon + \delta} + 1\right)\right\} = \zeta^2 e^{-2\tilde{\varepsilon} m}, \\ \sigma_m^2 &\leq \kappa^2 \exp\left\{-2\varepsilon \left(m \frac{\delta}{\varepsilon + \delta} - 1\right)\right\} + \xi^2 \exp\left\{-2\delta m \frac{\varepsilon}{\varepsilon + \delta}\right\} = \tilde{\zeta}^2 e^{-2\tilde{\varepsilon} m}, \end{aligned}$$

where $\tilde{\varepsilon}$ is as in (4.18). Also, recall that here $m \geq 0$, $i_m \geq 0$, and $j_m \geq 0$. It follows that for all $x \in \mathbb{R}$

$$\begin{aligned} (f_{i_m, \kappa e^{-\varepsilon|i_m|}} * f_{j_m, \xi e^{-\delta|j_m|}})(x) &= f_{m, \sigma_m^2}(x) = \frac{1}{\sigma_m \sqrt{2\pi}} \exp\left\{-\frac{(x-m)^2}{2\sigma_m^2}\right\} \geq \frac{\zeta e^{-\tilde{\varepsilon} m}}{\sigma_m} f_{m, \zeta} e^{-\tilde{\varepsilon} m}(x) \\ &\geq \frac{\zeta}{\tilde{\zeta}} f_{m, \zeta} e^{-\tilde{\varepsilon} m}(x) \geq e^{-(\varepsilon \vee \delta)} f_{m, \zeta} e^{-\tilde{\varepsilon}|m|}(x). \end{aligned}$$

Quite similarly (or by symmetry), one has

$$f_{i_m, \kappa e^{-\varepsilon|i_m|}} * f_{j_m, \xi e^{-\delta|j_m|}} \geq e^{-(\varepsilon \vee \delta)} f_{m, \zeta} e^{-\tilde{\varepsilon}|m|}$$

for any $m \in \{-1, -2, \dots\}$, letting now $i_m := -i_{-m} = \left\lfloor m \frac{\delta}{\varepsilon + \delta} \right\rfloor$ and $j_m := -j_{-m} = \left\lfloor m \frac{\varepsilon}{\varepsilon + \delta} \right\rfloor$, so that still $i_m + j_m = m$.

On recalling the conditions $p \geq c p_{\lambda, \varepsilon, \kappa, \alpha}$, $q \geq \tilde{c} p_{\lambda, \delta, \xi, \beta}$, (4.14)–(4.16) and (4.27), it follows that

$$\begin{aligned} p * q &\geq c \tilde{c} \sum_{m=-\infty}^{\infty} w_{i_m; \lambda, \alpha} w_{j_m; \lambda, \beta} f_{i_m, \kappa e^{-\varepsilon|i_m|}} * f_{j_m, \xi e^{-\delta|j_m|}} \\ &\geq c_1 \sum_{m=-\infty}^{\infty} w_{i_m; \lambda, \alpha} w_{j_m; \lambda, \beta} f_{m, \zeta} e^{-\tilde{\varepsilon}|m|} \\ &\geq c_2 \sum_{m=-\infty}^{\infty} w_{m; \lambda, \alpha + \beta} f_{m, \zeta} e^{-\tilde{\varepsilon}|m|} = c_2 p_{\lambda, \tilde{\varepsilon}, \zeta, \alpha + \beta}, \end{aligned}$$

where c_1 and c_2 are strictly positive constants depending only on $\lambda, \varepsilon, \delta, \kappa, \xi, \alpha, \beta$.

Also, $\int_{\mathbb{R}} e^{\lambda x} (p * q)(x) dx = \int_{\mathbb{R}} e^{\lambda x} p(x) dx \int_{\mathbb{R}} e^{\lambda x} q(x) dx < \infty$. Thus, it has been shown that $p * q \in \mathcal{P}_{\lambda, \tilde{\varepsilon}}$. This completes the verification of part (IV). The lemma is now completely proved. \square

Proof of Corollary 2.4. This follows because $\tilde{p}_t^{*n}(\mathbf{x}) = \frac{e^{t \mathbf{e} \mathbf{x}} p^{*n}(\mathbf{x})}{(\mathbb{E} e^{t \mathbf{e} \mathbf{X}})^n}$ for all $\mathbf{x} \in \mathbb{R}^k$. \square

Proof of Corollary 2.5. Take any $t \in (0, \lambda)$. Then, by Corollary 2.4, $\tilde{p}_t^{*n_t}$ is bounded by some constant $K < \infty$. Then, by the Plancherel isometry (see *e.g.* ([2], Thm. 4.2)), for all $\gamma \geq 2n_t$

$$\int_{\mathbb{R}^k} |\tilde{f}_t(\mathbf{s})|^\gamma \, d\mathbf{s} \leq \int_{\mathbb{R}^k} |\tilde{f}_t(\mathbf{s})|^{2n_t} \, d\mathbf{s} = (2\pi)^k \int_{\mathbb{R}^k} \tilde{p}_t^{*n_t}(\mathbf{x})^2 \, d\mathbf{x} \leq (2\pi)^k K < \infty.$$

Vice versa, assume that (2.8) holds for all $\gamma \geq \gamma_t$; then \tilde{p}_t^{*n} is bounded for all natural $n \geq \gamma_t$ by the Fourier inversion formula (see *e.g.* [2], Thm. 4.1(iv)), since the characteristic function of \tilde{p}_t^{*n} is $\tilde{f}_t(\mathbf{s})^n$. \square

Remark 4.2. Weaker results than the one given by Theorem 2.1 or even Corollary 2.3 (but which still be enough to deduce Corollaries 2.4 and 2.5) can be obtained more simply modulo the Plancherel isometry. Indeed, if conditions (1.1) and (1.3) hold, then

$$\int_{\mathbb{R}^k} |\tilde{f}_t(\mathbf{s})|^2 \, d\mathbf{s} = (2\pi)^k \int_{\mathbb{R}^k} \tilde{p}_t(\mathbf{x})^2 \, d\mathbf{x} = \frac{(2\pi)^k}{(\mathbb{E} e^{t \mathbf{eX}})^2} \int_{\mathbb{R}^k} e^{2t \mathbf{eX}} p(\mathbf{x})^2 \, d\mathbf{x} < \infty$$

for $t = \lambda - \varepsilon/2$ and $\varepsilon := \lambda - \mu$, since $e^{2(\lambda - \varepsilon/2) \mathbf{eX}} p(\mathbf{x})^2 \leq C e^{\lambda \mathbf{eX}} p(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}$. Also, by the Fourier inversion formula, again with $t = \lambda - \varepsilon/2$,

$$\tilde{p}_t^{*2}(\mathbf{x}) \leq (2\pi)^{-k} \int_{\mathbb{R}^k} |\tilde{f}_t(\mathbf{s})|^2 \, d\mathbf{s} < \infty \quad \text{for all } \mathbf{x} \in \mathbb{R}^k,$$

which yields (2.6) for $n = 2$. Thus, by induction, one can obtain (2.6) for $n = 2^j$, where j is any natural number.

However, it is unclear whether such an approach, *via* the Plancherel isometry, could be extended to yield Theorem 2.1 or, at least, Corollary 2.3 for all natural n . Anyway, it might be not worthwhile to exert efforts in such a direction, as the direct probabilistic proof of Theorem 2.1 given above is rather simple already and yet produces the best possible bound on the exponential deficiency.

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