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FURTHER REFINEMENT OF SELF-NORMALIZED CRAMÉR-TYPE MODERATE DEVIATIONS*

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Abstract. In this paper, we study the self-normalized Cramér-type moderate deviations for centered independent random variables $X_1, X_2, \ldots$ with $0 < E|X_i|^3 < \infty$. The main results refine Theorems 1.1 and 1.2 of Wang [Q. Wang, *J. Theoret. Probab.* **24** (2011) 307–329], the Berry−Esseen bound (2.11) and Corollaries 2.2 and 2.3 of Jing, *et al.* [B.Y. Jing, Q.M. Shao and Q. Wang, *Ann. Probab.* **31** (2003) 2167–2215] under stronger moment conditions.

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1. Introduction

Let $X_1, X_2, \ldots$ be independent random variables with $EX_i = 0$ and $0 < EX_i^2 < \infty$. Set

$$S_n = \sum_{i=1}^n X_i, \quad V_n^2 = \sum_{i=1}^n X_i^2, \quad \text{and} \quad B_n^2 = \sum_{i=1}^n EX_i^2.$$

The last two decades have witnessed a significant development on the limit theorems in the self-normalized form S_n/V_n, including central limit theorem, weak invariance principle, law of the iterated logarithm, Berry−Esseen inequality, large and moderate deviation probabilities. The last two in this list are the main approaches for estimating the error of the normal approximation of the self-normalized probabilities. One advantage of these self-normalized limit theorems is that they usually require less moment conditions than those for the corresponding regular limit theorems. An incomplete list of reference includes Griffin and Kuelbs [4, 5], Giné, Götze and Mason [3], Shao [11,13], Wang and Jing [18], Csörgő, Szyszkowicz and Wang [1], Jing, Shao and Wang [6], Jing, Shao and Zhou [7], Robinson and Wang [10], Wang [16, 17], the survey papers of Shao [12, 14] and Shao and Wang [15]. A systematic treatment of self-normalized limit theory is also collected in the book by de la Peña, Lai and Shao [2].

Keywords and phrases. Cramér-type moderate deviations, self-normalized sums, normal approximation.

* *This paper is dedicated to the memory of Evarist Giné.*

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The focus of this paper is on the self-normalized Cramér-type moderate deviations. Let $b = x/B_n$ and $\tau = B_n/\max\{1, x\}$ with $x \geq 0$. Set

$$L_{kn} = B_n^{-k} \sum_{i=1}^n EX_i^k \quad \text{and} \quad \bar{L}_{kn} = B_n^{-k} \sum_{i=1}^n EX_i^k I(|bX_i| \leq 1), \quad k \geq 2,$$

$$\mathcal{L}_{kn} = B_n^{-k} \sum_{i=1}^n E|X_i|^k \quad \text{and} \quad \bar{\mathcal{L}}_{kn} = B_n^{-k} \sum_{i=1}^n E|X_i|^k I(b|X_i| \leq 1), \quad k \geq 2,$$

$$\Delta_{n,x}^{j,k} = \tau^{-j} \sum_{i=1}^n E|X_i|^j I(|X_i| > \tau) + \tau^{-k} \sum_{i=1}^n E|X_i|^k I(|X_i| \leq \tau), \quad k > j \geq 2.$$

For the i.i.d. case, Shao [13] proved that as $n \to \infty$,

$$\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} \to 1, \qquad \frac{P(S_n/V_n \leq -x)}{\Phi(-x)} \to 1$$

holds uniformly for $x \in [0, o(n^{\delta/(4+2\delta)})]$ under the conditions $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^{2+\delta} < \infty$ for $0 < \delta \leq 1$. Here $\Phi(x)$ is the distribution function of the standard normal random variables.

For independent random variables $X_1, X_2, \dots$ with $EX_i = 0$, $0 < EX_i^2 < \infty$, $x^2 \max_i EX_i^2 \leq B_n^2$, and $\Delta_{n,x}^{2,3} \leq (1+x)^2/A$ where A is a constant sufficiently large, Jing, Shao and Wang [6] established a Cramér-type deviation result for self-normalized sums

$$\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} = \mathrm{e}^{O(1)\Delta_{n,x}^{2,3}}.$$

Wang [17] developed new techniques to refine the self-normalized Cramér-type deviation results and simplified the proof. Theorem 1.2 there states that if $EX_i = 0$, $0 < E|X_i|^3 < \infty$, $x\mathcal{L}_{3n} \leq 1/A$, $x^3 \max_i E|X_i|^3 \leq B_n^3/27$, and $|c| \leq x/5$, then

$$P(S_n \geq xV_n + cB_n) = \tilde{\Psi}_x(\lambda_1, c)(1 - \Phi(x+c))\mathrm{e}^{O(1)\Delta_{n,x}^{3,4}}\{1 + O(1)(1+x)\mathcal{L}_{3n}\},$$

where $\tilde{\Psi}_x(t, c) = \mathrm{e}^{m(t)+(x+c)^2/2 - t(x^2+2xc)}$ for $m(t) = \sum_{i=1}^n \log E\mathrm{e}^{t(2bX_i - b^2X_i^2)}$, and λ_1 is the solution of λ for $m'(\lambda) = x^2 + 2cx$. His Theorem 1.1 states that if $EX_i = 0$, $0 < EX_i^4 < \infty$ and $x \leq \mathcal{L}_{4n}^{-1/4}$, then

$$P(S_n/V_n \geq x) = \tilde{\Psi}_x(\lambda_0, 0)(1 - \Phi(x))\{1 + O(1)(1+x)\mathcal{L}_{3n} + O(1)(1+x^4)\mathcal{L}_{4n}\},$$

where λ_0 is the solution of λ for $m'(\lambda) = x^2$.

In this paper, we further refine the proofs and results of Wang [17]. Section 2 gives the main results including a theorem and three corollaries. The theorem refines Theorems 1.1 and 1.2 of Wang [17] under stronger moment conditions. The corollaries refine the Berry−Esseen bound (2.11) and Corollaries 2.2 and 2.3 of Jing, Shao and Wang [6] under stronger moment conditions. The proof of the theorem is also given in Section 2 using the propositions in Section 3. In Proposition 3.4, we obtain a formula for the probability $P\{2bS_n - (\alpha_0 - \alpha_3 x\bar{L}_{3n})b^2V_n^2 \geq (2-\alpha_0)x^2 + \delta(x)\}$ where α_0 and α_3 are constants with different values in different situations, and $\delta(x)$ is a function of x. This probability generalizes the probability $P(2bS_n - b^2V_n^2 \geq x^2)$ given by Jing, Shao and Wang [6] and the probability $P\{2bS_n - b^2V_n^2 \geq x^2 + \delta(x)\}$ given by Wang [17]. This generalized probability is necessary to produce the desired accuracy in Theorem 2.1, in particular, the exponent $-x^3L_{3n}/3 - x^4L_{4n}/12$ in equations (2.5) and (2.6).

2. Main results

Theorem 2.1. *Let $X_1, X_2, \ldots$ be independent random variables with $EX_i = 0$ and $0 < E|X_i|^3 < \infty$. Assume that there exists a constant $1 < A < \infty$ sufficiently large such that for $x \geq 0$,*

$$x\mathcal{L}_{3n} \leq 1/A \tag{2.1}$$

and

$$x^2 B_n^{-2} \max_{1\leq i\leq n} EX_i^2 \leq 1/12. \tag{2.2}$$

Then

$$\frac{P(S_n \geq xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3\bar{L}_{3n}}{3} - \frac{x^4\bar{L}_{4n}}{12} + O(1)\Delta_{n,x}^{3,5}\right)\{1+O(1)(1+x)\mathcal{L}_{3n}\}. \tag{2.3}$$

Consequently,

$$\frac{P(S_n \geq xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3 L_{3n}}{3} + O(1)\Delta_{n,x}^{3,4}\right)\Big\{1+O(1)(1+x)\mathcal{L}_{3n}\Big\}. \tag{2.4}$$

If $0 < EX_i^4 < \infty$, then

$$\frac{P(S_n \geq xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3 L_{3n}}{3} - \frac{x^4 L_{4n}}{12} + O(1)\Delta_{n,x}^{4,5}\right)\{1+O(1)(1+x)\mathcal{L}_{3n}\}. \tag{2.5}$$

If $0 < E|X_i|^{4+\delta} < \infty$ with $0 < \delta \leq 1$, then for $0 \leq x \leq \mathcal{L}_{(4+\delta)n}^{-1/(4+\delta)}$,

$$\frac{P(S_n \geq xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3 L_{3n}}{3} - \frac{x^4 L_{4n}}{12}\right)\Big\{1+O(1)(1+x)\mathcal{L}_{3n} + O(1)(1+x)^{4+\delta}\mathcal{L}_{(4+\delta)n}\Big\}. \tag{2.6}$$

Proof. First we prove the lower bound. By Lemma 3.1(i), we have $x^3\bar{\mathcal{L}}_{3n}^3 \leq x^3\bar{\mathcal{L}}_{5n} \leq \Delta_{n,x}^{3,5}/x^2$. Since $|(1-s)^{-1} - 1 - s - s^2| \leq 2|s|^3$ for $|s| \leq 1/2$, then

$$\left|\frac{1}{1-x\bar{L}_{3n}/2} - 1 - \frac{x\bar{L}_{3n}}{2} - \frac{x^2\bar{L}_{3n}^2}{4}\right| \leq \frac{x^3\bar{\mathcal{L}}_{3n}^3}{4} \leq \frac{\Delta_{n,x}^{3,5}}{4x^2}. \tag{2.7}$$

Hence

$$\begin{aligned} P(S_n \geq xV_n) &= P\Bigg\{2bS_n - (1-x\bar{L}_{3n}/2)b^2V_n^2 \geq \frac{x^2}{1-x\bar{L}_{3n}/2} \\ &\qquad - \left(\frac{x}{(1-x\bar{L}_{3n}/2)^{1/2}} - bV_n(1-x\bar{L}_{3n}/2)^{1/2}\right)^2\Bigg\} \\ &\geq P\left\{2bS_n - (1-x\bar{L}_{3n}/2)b^2V_n^2 \geq \frac{x^2}{1-x\bar{L}_{3n}/2}\right\} \\ &\geq P\left\{2bS_n - (1-x\bar{L}_{3n}/2)b^2V_n^2 \geq x^2 + \frac{x^3\bar{L}_{3n}}{2} + \frac{x^4\bar{L}_{3n}^2}{4} + \Delta_{n,x}^{3,5}\right\}. \end{aligned}$$

By Proposition 3.4 with $\alpha_0 = 1,\ \alpha_3 = 1/2,\ \beta_3 = 1/2,\ \beta_4 = 0$ and $\beta_5 = 1/4$,

$$P(S_n \geq xV_n) \geq \exp\Big\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} - A\Delta_{n,x}^{3,5}\Big\}(1-\Phi(x))(1-Ax\mathcal{L}_{3n}).$$

Next we prove the upper bound. First we consider $0 \le x \le 2$. Observe that $x^4 \bar{L}_{4n} \le x^3 \bar{\mathcal{L}}_{3n} \le x^2/A \le 1$ by condition (2.1). Since $|e^s - 1| \le e^{s\vee 0}|s|$, then

$$\left|\exp\left(-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n}\right) - 1\right| \le \left(\frac{1}{3} + \frac{1}{12}\right) e^{1/3} x^3 \bar{\mathcal{L}}_{3n} \le 3x\bar{\mathcal{L}}_{3n}.$$

Hence by condition (2.1),

$$\exp\left(-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n}\right)\{1 + 2A(1+x)\bar{\mathcal{L}}_{3n}\} \ge (1 - 3x\bar{\mathcal{L}}_{3n})\{1 + 2A(1+x)\bar{\mathcal{L}}_{3n}\} \ge 1 + A(1+x)\bar{\mathcal{L}}_{3n}. \quad (2.8)$$

By (2.16) of Wang [17], $|P(S_n \ge xV_n) - (1 - \Phi(x))| \le A\mathcal{L}_{3n}$. Then for $0 \le x \le 2$,

$$\frac{P(S_n \ge xV_n)}{1 - \Phi(x)} \le 1 + A(1+x)\mathcal{L}_{3n}. \quad (2.9)$$

Combining (2.8) and (2.9), we obtain the upper bound. For $x \ge 2$,

$$P(S_n \ge xV_n) = P\left(S_n \ge xV_n,\ \max_{1\le i\le n}|X_i| > \tau\right) + P(\bar{S}_n \ge x\bar{V}_n)$$

where $\tau = B_n/x$, $\bar{S}_n = \sum_{i=1}^n X_i I(|X_i| \le \tau)$ and $\bar{V}_n^2 = \sum_{i=1}^n X_i^2 I(|X_i| \le \tau)$. Page 2181 of Jing, Shao and Wang [6] shows that

$$P\left(S_n \ge xV_n,\ \max_{1\le i\le n}|X_i| > \tau\right) \le \sum_{i=1}^n P\left(S_n^{(i)} \ge (x^2-1)^{1/2}V_n^{(i)}\right) P(|X_i| > \tau).$$

Since $\sum_{i=1}^n P(|X_i| > \tau) \le \Delta_{n,x}^{3,5}$, then the upper bound follows from Propositions 3.5 and 3.3. □

Note that equations (2.5) and (2.6) refine Theorems 1.2 and 1.1 (when $c = 0$) of Wang [17], respectively, under higher moment conditions.

Corollary 2.2. *Suppose that* $\max_{1\le i\le n} EX_i^4 \le C$ *for some* $C < \infty$ *and* $B_n^2 \ge cn$ *for some* $c > 0$. *If* $\sum_{i=1}^n \mathbb{E}X_i^3 = O(n^\gamma)$ *for* $0 \le \gamma \le 1$, *then*

$$P(S_n \ge xV_n) - (1 - \Phi(x)) = O\left(\frac{x^2}{n^{3/2-\gamma}} + \frac{x^3}{n} + \frac{1}{\sqrt{n}}\right) e^{-x^2/2}$$

for $0 \le x \le \min\{O(n^{1/2-\gamma/3}),\ O(n^{1/4})\}$.

Proof. Let $y = -x^3L_{3n}/3 - x^4L_{4n}/12 + O(1)\Delta_{n,x}^{4,5}$ in (2.5). Note that $\Delta_{n,x}^{4,5} \le (1+x)^4L_{4n}$. Then y is bounded for $x = \min\{O(n^{1/2-\gamma/3}),\ O(n^{1/4})\}$. Since $|e^y - 1| \le e^{y\vee 0}|y|$, then $e^y = 1 + O(1)y$. Hence $P(S_n \ge xV_n) = (1 - \Phi(x))\{1 + O(1)y + O(1)(1+x)\mathcal{L}_{3n}\}$ by (2.5). Note that $1 - \Phi(x) \le 2e^{-x^2/2}/(1+x)$ for $x \ge 0$. Then $x^3L_{3n}(1-\Phi(x)) = O(x^2/n^{3/2-\gamma})e^{-x^2/2}$ and $x^4L_{4n}(1-\Phi(x)) = O(x^3/n)e^{-x^2/2}$. Moreover, $(1+x)\mathcal{L}_{3n}(1-\Phi(x)) = O(1/\sqrt{n})e^{-x^2/2}$ because $E|X_i|^3 \le (EX_i^4)^{3/4} \le C^{3/4}$. Therefore, the corollary follows. □

Compared with the Berry−Esseen bound (2.11) by Jing, Shao and Wang [6], this corollary shows that the bound can be lowered and the range of x can be extended under stronger moment conditions. For example, if $\max_{1\le i\le n} E|X_i|^3 \le C$ and $B_n^2 \ge cn$, their result shows that $P(S_n > xV_n) - (1-\Phi(x)) = O(1)(1+x)^2e^{-x^2/2}/\sqrt{n}$ for $x = O(n^{1/6})$. However, if $\max_{1\le i\le n} EX_i^4 \le C$ and $B_n^2 \ge cn$, the above corollay shows that $P(S_n > xV_n) - (1 - \Phi(x)) = O(1)(1+x)e^{-x^2/2}/\sqrt{n}$ for $\sum_{i=1}^n \mathbb{E}X_i^3 = O(n^{3/4})$ and $x = O(n^{1/4})$.

Corollary 2.3. *Suppose that* $\max_{1\le i\le n} EX_i^4 \le C$ *for some* $C<\infty$ *and* $B_n^2 \ge cn$ *for some* $c>0$. *If* $\sum_{i=1}^n \mathbb{E}X_i^3 = O(n^\gamma)$ *for* $0\le\gamma\le 1$, *then by* (2.5),

$$\frac{P(S_n \ge xV_n)}{1-\Phi(x)} \to 1$$

uniformly for $0\le x\le \min\{o(n^{1/2-\gamma/3}),\ o(n^{1/4})\}$.

This corollary extends the range of x for Corollary 2.2 of Jing, Shao and Wang [6] where $0\le x\le O(n^{\delta/(4+2\delta)})$ for $0<\delta<1$.

Corollary 2.4. *Let* $X_1, X_2, \ldots$ *be i.i.d. random variables with* $EX_1 = 0$, $\sigma^2 = EX_1^2$ *and* $0 < E|X_1|^3 < \infty$. *Assume that* $0\le x\le \sqrt{n}/A$ *for some sufficiently large constant* $2\sqrt{3}\le A<\infty$. *Then*

$$\frac{P(S_n \ge xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3EX_1^3}{3\sqrt{n}\sigma^3} + O(1)\Delta_{n,x}^{3,4}\right)\left\{1+O(1)\frac{1+x}{\sqrt{n}}\right\},$$

where $\Delta_{n,x}^{3,4} = \frac{(1\vee x)^3 E|X_1|^3 I\{|X_1|>\sqrt{n}\sigma/(1\vee x)\}}{\sqrt{n}\sigma^3} + \frac{(1\vee x)^4 EX_1^4 I\{|X_1|\le\sqrt{n}\sigma/(1\vee x)\}}{n\sigma^4}$.

If $0<EX_1^4<\infty$, *then*

$$\frac{P(S_n \ge xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3EX_1^3}{3\sqrt{n}\sigma^3} - \frac{x^4EX_1^4}{12n\sigma^4} + O(1)\Delta_{n,x}^{4,5}\right)\left\{1+O(1)\frac{1+x}{\sqrt{n}}\right\},$$

where $\Delta_{n,x}^{4,5} = \frac{(1\vee x)^4 EX_1^4 I\{|X_1|>\sqrt{n}\sigma/(1\vee x)\}}{n\sigma^4} + \frac{(1\vee x)^5 E|X_1|^5 I\{|X_1|\le\sqrt{n}\sigma/(1\vee x)\}}{n^{3/2}\sigma^5}$.

If $0<E|X_1|^{4+\delta}<\infty$ *with* $0<\delta\le 1$, *then for* $0\le x\le n^{(2+\delta)/(8+2\delta)}\sigma/(E|X_1|^{4+\delta})^{1/(4+\delta)}$,

$$\frac{P(S_n \ge xV_n)}{1-\Phi(x)} = \exp\left(-\frac{x^3EX_1^3}{3\sqrt{n}\sigma^3} - \frac{x^4EX_1^4}{12n\sigma^4}\right)\left\{1+\frac{O(1)(1+x)}{\sqrt{n}} + \frac{O(1)(1+x)^{4+\delta}}{n^{1+\delta/2}}\right\}.$$

Note that since $X_1, X_2, \ldots$ are i.i.d. random variables, condition (2.1) becomes $x\le\sqrt{n}\sigma^3/(AE|X_1|^3)\le\sqrt{n}/A$, and (2.2) becomes $x\le\sqrt{n/12}$.

3. Lemmas and propositions

From now on, all $|O(1)|\le A$. First we establish some preliminary facts in the following lemma.

Lemma 3.1.

(i)

$$\bar{\mathcal{L}}_{4n}^2 \le \bar{\mathcal{L}}_{3n}\bar{\mathcal{L}}_{5n},\quad \bar{\mathcal{L}}_{3n}\bar{\mathcal{L}}_{4n}\le\bar{\mathcal{L}}_{5n}\quad\textit{and}\quad \bar{\mathcal{L}}_{3n}^3\le\bar{\mathcal{L}}_{5n}.$$

(ii)

$$x^2\bar{\mathcal{L}}_{4n} \le \max\{\bar{\mathcal{L}}_{3n},\ x^4\bar{\mathcal{L}}_{5n}\}.$$

Proof. (i) Let $\bar{X}_i = X_i I(|bX_i|\le 1)$. Then $E\bar{X}_i^4 \le (E|\bar{X}_i|^3)^{1/2}(E|\bar{X}_i|^5)^{1/2}$. Hence

$$\left(\sum_{i=1}^n E\bar{X}_i^4\right)^2 \le \sum_{i=1}^n E|\bar{X}_i|^3 \sum_{i=1}^n E|\bar{X}_i|^5$$

by the Cauchy−Schwarz inequality. Therefore,

$$\bar{\mathcal{L}}_{4n}^2 \le \bar{\mathcal{L}}_{3n}\bar{\mathcal{L}}_{5n}. \tag{3.1}$$

Similarly,

$$\bar{\mathcal{L}}_{3n}^2 \le \bar{\mathcal{L}}_{2n}\bar{\mathcal{L}}_{4n} \le \bar{\mathcal{L}}_{4n}. \tag{3.2}$$

By (3.1) and (3.2), $\bar{\mathcal{L}}_{3n}\bar{\mathcal{L}}_{4n} \le \bar{\mathcal{L}}_{5n}$ which, together with (3.2), implies $\bar{\mathcal{L}}_{3n}^3 \le \bar{\mathcal{L}}_{5n}$.

(ii) If $x^2\bar{\mathcal{L}}_{4n} \le x^4\bar{\mathcal{L}}_{5n}$, then $x^2\bar{\mathcal{L}}_{4n} \le \max\{\bar{\mathcal{L}}_{3n},\ x^4\bar{\mathcal{L}}_{5n}\}$. If $x^2\bar{\mathcal{L}}_{4n} \ge x^4\bar{\mathcal{L}}_{5n}$, then $\bar{\mathcal{L}}_{4n}^2 \le \bar{\mathcal{L}}_{3n}x^{-2}\bar{\mathcal{L}}_{4n}$ by (3.1) and hence $x^2\bar{\mathcal{L}}_{4n} \le \bar{\mathcal{L}}_{3n} \le \max\{\bar{\mathcal{L}}_{3n},\ x^4\bar{\mathcal{L}}_{5n}\}$.

□

Let $b = x/B_n$ with $x > 0$. Define the function

$$\xi(y) = 2by - \big(\alpha_0 - \alpha_3 x\bar{L}_{3n}\big)b^2y^2$$

where α_0 and α_3 are constants such that $1/3 \le \alpha_0 - \alpha_3 x\bar{L}_{3n} \le 7/3$. Here 1/3 and 7/3 are selected from the estimates of I_2 and I_3 in Proposition 3.5. In other situations, we only need $\alpha_0 = 1$ and $\alpha_3 = 0$ or 1/2, and thus $\alpha_0 - \alpha_3 x\bar{L}_{3n}$ is close to 1 by condition (2.1).

Denote that $X_{i,(1)} = X_i$ and $X_{i,(2)} = X_iI(|bX_i| \le 1)$ for $i \ge 1$. For $\lambda > 0$, let $Z_{\lambda,1,(k)}, \ldots, Z_{\lambda,n,(k)}$ be independent random variables with $Z_{\lambda,i,(k)}$ having the distribution function

$$P(Z_{\lambda,i,(k)} \le u) = \frac{\int_{-\infty}^{u} \mathrm{e}^{\lambda\xi(y)}\mathrm{d}P(X_{i,(k)} \le y)}{E\mathrm{e}^{\lambda\xi(X_{i,(k)})}} \tag{3.3}$$

where $k = 1$ or 2. Then for any function f,

$$Ef(Z_{\lambda,i,(k)}) = \frac{\int_{-\infty}^{\infty} f(u)\mathrm{e}^{\lambda\xi(u)}\mathrm{d}P(X_{i,(k)} \le u)}{E\mathrm{e}^{\lambda\xi(X_{i,(k)})}}. \tag{3.4}$$

In particular,

$$E\xi(Z_{\lambda,i,(k)}) = \frac{\int_{-\infty}^{\infty} \xi(u)\mathrm{e}^{\lambda\xi(u)}\mathrm{d}P(X_{i,(k)} \le u)}{E\mathrm{e}^{\lambda\xi(X_{i,(k)})}} = \frac{d\big(\log E\mathrm{e}^{\lambda\xi(X_{i,(k)})}\big)}{\mathrm{d}\lambda} \tag{3.5}$$

and

$$\mathrm{Var}(\xi(Z_{\lambda,i,(k)})) = \frac{d^2\big(\log E\mathrm{e}^{\lambda\xi(X_{i,(k)})}\big)}{\mathrm{d}\lambda^2}. \tag{3.6}$$

Lemma 3.2. *Let $b = x/B_n$ with $x > 0$, $X_{i,(1)} = X_i$ and $X_{i,(2)} = X_iI(|bX_i| \le 1)$. Under conditions (2.1) and (2.2), for $7/16 \le \lambda \le 9/16$ and $k = 1$ or 2,*

$$\begin{aligned}\sum_{i=1}^{n} \log E\mathrm{e}^{\lambda\xi(X_{i,(k)})} &= (2\lambda^2 - \lambda\alpha_0)x^2 + \left(\frac{4\lambda^3}{3} - 2\lambda^2\alpha_0 + \lambda\alpha_3\right)x^3\bar{L}_{3n} \\ &\quad + \left(\frac{2\lambda^4}{3} - 2\lambda^3\alpha_0 + \frac{\lambda^2\alpha_0^2}{2}\right)x^4\bar{L}_{4n} + 2\lambda^2\alpha_3x^4\bar{L}_{3n}^2 \\ &\quad - \frac{\big(2\lambda^2 - \lambda\alpha_0\big)^2}{2}\frac{x^4\sum_{i=1}^{n}(E\bar{X}_i^2)^2}{B_n^4} + O(1)\Delta_{n,x}^{3,5}.\end{aligned} \tag{3.7}$$

Consequently,

$$\begin{aligned}\sum_{i=1}^{n} E\xi(Z_{\lambda,i,(k)}) &= (4\lambda-\alpha_0)x^2 + (4\lambda^2 - 4\lambda\alpha_0 + \alpha_3)x^3\bar{L}_{3n} \\ &+ \left(\frac{8\lambda^3}{3} - 6\lambda^2\alpha_0 + \lambda\alpha_0^2\right) x^4 \bar{L}_{4n} + 4\lambda\alpha_3 x^4 \bar{L}_{3n}^2 \\ &+ \left(-8\lambda^3 + 6\lambda^2\alpha_0 - \lambda\alpha_0^2\right) \frac{x^4 \sum_{i=1}^{n}(E\bar{X}_i^2)^2}{B_n^4} \\ &+ O(1)\Delta_{n,x}^{3,5} \end{aligned} \tag{3.8}$$

and

$$\sum_{i=1}^{n} \mathrm{Var}(\xi(Z_{\lambda,i,(k)})) = 4x^2 + O(1)x^3\mathcal{L}_{3n}. \tag{3.9}$$

Moreover, for $m \geq 2$ and $|O(1)| \leq A^{1/5}$,

$$EZ_{\lambda,i,(2)}^m = E\bar{X}_i^m + 2\lambda b E\bar{X}_i^{m+1} + O(1)b^2 E|\bar{X}_i|^{m+2}. \tag{3.10}$$

Proof. We follow some proofs in Lemmas 6.1 and 6.2 of Jing, Shao and Wang [6]. Let $\gamma = 2\lambda$ and $\theta = \lambda(\alpha_0 - \alpha_3 x\bar{L}_{3n})$. Then $\lambda\xi(X_i) = \gamma b X_i - \theta b^2 X_i^2$. Hence

$$E(\mathrm{e}^{\lambda\xi(X_{i,(k)})} - 1)I(|bX_i| > 1) \leq \mathrm{e}^{\gamma^2/(4\theta)} bE|X_i|I(|bX_i| > 1). \tag{3.11}$$

Since $|\mathrm{e}^s - 1 - s - s^2/2 - s^3/6 - s^4/24| \leq |s|^5\mathrm{e}^{s\vee 0}/120$ for $s \in R$, then

$$\begin{aligned} E(\mathrm{e}^{\lambda\xi(X_{i,(k)})} - 1)I(|bX_i| \leq 1) &= E\left(\gamma b\bar{X}_i - \theta b^2\bar{X}_i^2\right) + \frac{1}{2}E\left(\gamma b\bar{X}_i - \theta b^2\bar{X}_i^2\right)^2 + \frac{1}{6}E\left(\gamma b\bar{X}_i - \theta b^2\bar{X}_i^2\right)^3 \\ &+ \frac{1}{24}E\left(\gamma b\bar{X}_i - \theta b^2\bar{X}_i^2\right)^4 + O(1)\frac{\mathrm{e}^{\gamma^2/(4\theta)}}{120}E\left|\gamma b\bar{X}_i - \theta b^2\bar{X}_i^2\right|^5 \\ &= \gamma bE\bar{X}_i + \left(\frac{\gamma^2}{2} - \theta\right)b^2E\bar{X}_i^2 + \left(\frac{\gamma^3}{6} - \gamma\theta\right)b^3E\bar{X}_i^3 \\ &+ \left(\frac{\gamma^4}{24} - \frac{\gamma^2\theta}{2} + \frac{\theta^2}{2}\right)b^4E\bar{X}_i^4 + O'_{\gamma,\theta}b^5E|\bar{X}_i|^5, \end{aligned} \tag{3.12}$$

where

$$|O'_{\gamma,\theta}| \leq \left\{\frac{1}{6}(\gamma+\theta)^3 + \frac{1}{24}(\gamma+\theta)^4 + \frac{\mathrm{e}^{\gamma^2/(4\theta)}}{120}(\gamma+\theta)^5\right\}. \tag{3.13}$$

By (3.11) and (3.12),

$$\begin{aligned} E\mathrm{e}^{\lambda\xi(X_{i,(k)})} &= 1 + \gamma bE\bar{X}_i + \left(\frac{\gamma^2}{2} - \theta\right)b^2E\bar{X}_i^2 + \left(\frac{\gamma^3}{6} - \gamma\theta\right)b^3E\bar{X}_i^3 \\ &+ \left(\frac{\gamma^4}{24} - \frac{\gamma^2\theta}{2} + \frac{\theta^2}{2}\right)b^4E\bar{X}_i^4 \\ &+ O(1)\mathrm{e}^{\gamma^2/(4\theta)}bE|X_i|I(|bX_i| > 1) + O'_{\gamma,\theta}b^5E|\bar{X}_i|^5, \end{aligned} \tag{3.14}$$

where $|O(1)| \le 1$. We want to show that $|E\mathrm{e}^{\lambda\xi(X_{i,(k)})} - 1| \le 11/12$. Since $7/16 \le \lambda \le 9/16$ and $1/3 \le \alpha_0 - \alpha_3 x\bar{L}_{3n} \le 7/3$, then $7/8 \le \gamma \le 9/8$ and $7/48 \le \theta \le 63/48$. Hence by (3.13),

$$\begin{aligned}
&\left|\gamma^2/2-\theta\right| + \left|\frac{\gamma^3}{6} - \gamma\theta\right| + \left|\frac{\gamma^4}{24} - \frac{\gamma^2\theta}{2} + \frac{\theta^2}{2}\right| + |O'_{\gamma,\theta}| \\
\le{}& \left|(7/8)^2/2 - 63/48\right| + \left|(7/8)^3/6 - (9/8)(63/48)\right| \\
&+ \left|(9/8)^4/24 - (7/8)^2(7/48)/2 + (63/48)^2/2\right| \\
&+ \left|(9/8+63/48)^3/6 + (9/8+63/48)^4/24 + \mathrm{e}^{27/16}(9/8+63/48)^5/120\right| \\
\le{}& 0.93 + 1.37 + 0.88 + 7.77 = 10.95.
\end{aligned}$$

Note that $\gamma b E\bar{X}_i = -\gamma b E X_i I(|bX_i| > 1)$. Since $\gamma^2/(4\theta) = \lambda/(\alpha_0 - \alpha_3 x\bar{L}_{3n}) \le 27/16$, then

$$\begin{aligned}
\left|\gamma + \mathrm{e}^{\gamma^2/(4\theta)}\right| bE|X_i| I(|bX_i|>1) &\le (9/8 + \mathrm{e}^{27/16}) b^2 E X_i^2 I(|bX_i|>1) \\
&\le 6.54 b^2 E X_i^2 I(|bX_i|>1).
\end{aligned}$$

From the above, $|E\mathrm{e}^{\lambda\xi(X_{i,(k)})} - 1| \le 10.95 b^2 E X_i^2 \le 11/12$ by condition (2.2). Note that $|\log(1+y) - y + y^2/2| \le 12^3|y|^3/3$ for $|y| \le 11/12$. Since $\gamma = 2\lambda$ and $\theta = \lambda(\alpha_0 - \alpha_3 x\bar{L}_{3n})$, then by (3.14) and Lemma 3.1(i),

$$\begin{aligned}
&\log E\mathrm{e}^{\lambda\xi(X_{i,(k)})} = \log\left(1 + \left\{E\mathrm{e}^{\lambda\xi(X_{i,(k)})} - 1\right\}\right) \\
={}& \left\{2\lambda^2 - \lambda(\alpha_0 - \alpha_3 x\bar{L}_{3n})\right\} b^2 E\bar{X}_i^2 + \left\{\frac{4\lambda^3}{3} - 2\lambda^2(\alpha_0 - \alpha_3 x\bar{L}_{3n})\right\} b^3 E\bar{X}_i^3 \\
&+ \left\{\frac{2\lambda^4}{3} - 2\lambda^3(\alpha_0 - \alpha_3 x\bar{L}_{3n}) + \frac{\lambda^2(\alpha_0 - \alpha_3 x\bar{L}_{3n})^2}{2}\right\} b^4 E\bar{X}_i^4 \\
&- \frac{1}{2}\left\{2\lambda^2 - \lambda(\alpha_0 - \alpha_3 x\bar{L}_{3n})\right\}^2 b^4 (E\bar{X}_i^2)^2 + O(1) b^3 E|X_i|^3 I(|bX_i|>1) \\
&+ O(1) b^5 E|\bar{X}_i|^5.
\end{aligned}$$

Hence (3.7) follows from Lemma 3.1(i). Taking derivative with respect to λ on both sides of (3.7) and by (3.5), we obtain (3.8). Taking derivative with respect to λ on both sides of (3.8) and by (3.6), we obtain (3.9). Similar to (3.14),

$$E\bar{X}_i^m \mathrm{e}^{\lambda\xi(X_{i,(2)})} = E\bar{X}_i^m + \gamma b E\bar{X}_i^{m+1} + O(1) b^2 E|\bar{X}_i|^{m+2} \tag{3.15}$$

for $m \ge 2$. Since $|(1+y)^{-1} - 1| \le 12^2|y|$ for $|y| \le 11/12$ and $\gamma = 2\lambda$, then by (3.4), (3.14) and (3.15),

$$E Z_{\lambda,i,(2)}^m = \frac{E\bar{X}_i^m \mathrm{e}^{\lambda\xi(\bar{X}_{i,(2)})}}{E\mathrm{e}^{\lambda\xi(X_{i,(2)})}} = E\bar{X}_i^m + 2\lambda b E\bar{X}_i^{m+1} + O(1) b^2 E|\bar{X}_i|^{m+2}. \qquad \square$$

Lemma 3.3. *Let $x > 0$, $|\delta(x)| \le x^2/5$ and $k = 1$ or 2. Under conditions (2.1) and (2.2), for $\sum_{i=1}^n E\xi(Z_{\lambda,i,(k)})$ in (3.8), the equation*

$$\sum_{i=1}^n E\xi(Z_{\lambda,i,(k)}) = (2 - \alpha_0)x^2 + \delta(x) \tag{3.16}$$

has a unique solution of λ, denoted as λ_δ, with $7/16 \le \lambda_\delta \le 9/16$. If $\delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}$ with $|\beta_j| \le \sqrt{A}$, then

$$\lambda_\delta = \frac{1}{2} + \left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right) x\bar{L}_{3n} + \left(\frac{\beta_4}{4} + \frac{3\alpha_0}{8} - \frac{\alpha_0^2}{8} - \frac{1}{12}\right) x^2 \bar{L}_{4n}$$
$$+ \left\{\frac{\beta_5}{4} - \frac{\alpha_3}{2} + (\alpha_0 - 1)\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)\right\} x^2 \bar{L}_{3n}^2$$
$$+ \left(\frac{1}{4} - \frac{3\alpha_0}{8} + \frac{\alpha_0^2}{8}\right) \frac{x^2 \sum_{i=1}^n (E\bar{X}_i^2)^2}{B_n^4} + \frac{O(1)\Delta_{n,x}^{3,5}}{x^2}. \tag{3.17}$$

Proof. Since the left-hand side of (3.9) is equal to $d(\sum_{i=1}^n E\xi(Z_{\lambda,i,(k)}))/\mathrm{d}\lambda$ and the right-hand side is positive, then $\sum_{i=1}^n E\xi(Z_{\lambda,i})$ increases strictly as λ increases. By (3.8), condition (2.1) and $|\delta(x)| \le x^2/5$,

$$\sum_{i=1}^n E\xi(Z_{7/16,i,(k)}) \le (2-\alpha_0)x^2 + \delta(x) \le \sum_{i=1}^n E\xi(Z_{9/16,i,(k)}).$$

Hence (3.16) has a unique soluion λ_δ and $7/16 \le \lambda_\delta \le 9/16$. By (3.8) and (3.16),

$$\lambda_\delta = \frac{1}{2} + \left(-\lambda_\delta^2 + \lambda_\delta\alpha_0 - \frac{\alpha_3}{4}\right) x\bar{L}_{3n} + \left(-\frac{2\lambda_\delta^3}{3} + \frac{3\lambda_\delta^2\alpha_0}{2} - \frac{\lambda_\delta\alpha_0^2}{4}\right) x^2\bar{L}_{4n}$$
$$-\lambda_\delta\alpha_3 x^2 \bar{L}_{3n}^2 + \left(2\lambda_\delta^3 - \frac{3\lambda_\delta^2\alpha_0}{2} + \frac{\lambda_\delta\alpha_0^2}{4}\right) \frac{x^2 \sum_{i=1}^n (E\bar{X}_i^2)^2}{B_n^4} + \frac{O(1)\Delta_{n,x}^{3,5}}{x^2} + \frac{\delta(x)}{4x^2}.$$

If $\delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}$, then

$$\lambda_\delta = \frac{1}{2} + \left(\frac{\beta_3}{4} - \lambda_\delta^2 + \lambda_\delta\alpha_0 - \frac{\alpha_3}{4}\right) x\bar{L}_{3n}$$
$$+ \left(\frac{\beta_4}{4} - \frac{2\lambda_\delta^3}{3} + \frac{3\lambda_\delta^2\alpha_0}{2} - \frac{\lambda_\delta\alpha_0^2}{4}\right) x^2\bar{L}_{4n} + \left(\frac{\beta_5}{4} - \lambda_\delta\alpha_3\right) x^2\bar{L}_{3n}^2$$
$$+ \left(2\lambda_\delta^3 - \frac{3\lambda_\delta^2\alpha_0}{2} + \frac{\lambda_\delta\alpha_0^2}{4}\right) \frac{x^2 \sum_{i=1}^n (E\bar{X}_i^2)^2}{B_n^4} + \frac{O(1)\Delta_{n,x}^{3,5}}{x^2}. \tag{3.18}$$

By Lemma 3.1(i) and (3.18),

$$\lambda_\delta^2 x\bar{L}_{3n} = \frac{1}{4}x\bar{L}_{3n} + \left(\frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4}\right) x^2\bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3,5}}{x^2}$$

and

$$\lambda_\delta\alpha_0 x\bar{L}_{3n} = \frac{\alpha_0}{2}x\bar{L}_{3n} + \alpha_0\left(\frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4}\right) x^2\bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3,5}}{x^2}.$$

Hence the second term on the right-hand side of (3.18)

$$\left(\frac{\beta_3}{4} - \lambda_\delta^2 + \lambda_\delta\alpha_0 - \frac{\alpha_3}{4}\right) x\bar{L}_{3n} = \left(\frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4}\right) x\bar{L}_{3n} + (\alpha_0 - 1)\left(\frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4}\right) x^2\bar{L}_{3n}^2$$
$$+ \frac{O(1)\Delta_{n,x}^{3,5}}{x^2}.$$

For the other terms in (3.18) involving $x^2\bar{L}_{4n}$, $x^2\bar{L}_{3n}^2$ and $x^2 B_n^{-4}\sum_{i=1}^n (E\bar{X}_i^2)^2$, by Lemma 3.1(i), λ_δ can be replaced by $1/2$ with a difference of $O(1)\Delta_{n,x}^{3,5}/x^2$. Therefore (3.17) follows. □

Proposition 3.4. *Let* $S_{n,(k)} = \sum_{i=1}^n X_{i,(k)}$ *and* $V_{n,(k)}^2 = \sum_{i=1}^n X_{i,(k)}^2$*, where* $X_{i,(1)} = X_i$ *and* $X_{i,(2)} = X_i I(|bX_i| \le 1)$*. Under conditions* (2.1) *and* (2.2)*, for* $x > 0$*,* $|\delta(x)| \le x^2/5$ *and* $k = 1$ *or* 2*,*

$$P\Big\{2bS_{n,(k)} - (\alpha_0 - \alpha_3 x\bar{L}_{3n})b^2 V_{n,(k)}^2 \ge (2-\alpha_0)x^2 + \delta(x)\Big\}$$
$$= \exp\left\{\frac{x^2}{2} + \sum_{i=1}^n \log E\mathrm{e}^{\lambda_\delta \xi(X_{i,(k)})} - \lambda_\delta(2-\alpha_0)x^2 - \lambda_\delta\delta(x)\right\}(1-\Phi(x)) \times (1 + O(1)x\mathcal{L}_{3n}). \qquad (3.19)$$

If $\delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}$ *with* $|\beta_j| \le \sqrt{A}$*, then*

$$\frac{x^2}{2} + \sum_{i=1}^n \log E\mathrm{e}^{\lambda_\delta \xi(X_{i,(k)})} - \lambda_\delta(2-\alpha_0)x^2 - \lambda_\delta\delta(x)$$
$$= \left(\frac{\alpha_3 - \beta_3}{2} + \frac{1-3\alpha_0}{6}\right)x^3\bar{L}_{3n} + \left(-\frac{\beta_4}{2} + \frac{(\alpha_0-1)^2}{8} - \frac{1}{12}\right)x^4\bar{L}_{4n}$$
$$+ \left(\frac{\alpha_3 - \beta_5}{2} - \frac{\left(\beta_3 - \alpha_3 + 2\alpha_0 - 1\right)^2}{8}\right)x^4\bar{L}_{3n}^2$$
$$- \frac{(1-\alpha_0)^2}{8}\frac{x^4\sum_{i=1}^n (EX_i^2)^2}{B_n^4} + O(1)\Delta_{n,x}^{3,5}. \qquad (3.20)$$

Proof. By the conjugate method (*e.g.*, (4.9) of Petrov [8] or (2.11) of Petrov [9]) together with (3.3) and (3.16),

$$P\Big\{2bS_{n,(k)} - (\alpha_0 + \alpha_3 x\bar{L}_{3n})b^2 V_{n,(k)}^2 \ge (2-\alpha_0)x^2 + \delta(x)\Big\}$$
$$= \int\cdots\int I\Big\{\sum_{i=1}^n \xi(x_i) \ge (2-\alpha_0)x^2 + \delta(x)\Big\}\prod_{i=1}^n \mathrm{d}P(X_{i,(k)} \le x_i)$$
$$= \int\cdots\int \mathrm{e}^{\sum_i \log E\mathrm{e}^{\lambda_\delta\xi(X_{i,(k)})} - \lambda_\delta\sum_i \xi(x_i)}$$
$$\times I\Big\{\sum_{i=1}^n \xi(x_i) \ge (2-\alpha_0)x^2 + \delta(x)\Big\}\prod_{i=1}^n \mathrm{d}P(Z_{\lambda_\delta,i,(k)} \le x_i)$$
$$= \mathrm{e}^{\sum_i \log E\mathrm{e}^{\lambda_\delta\xi(X_{i,(k)})} - \lambda_\delta(2-\alpha_0)x^2 - \lambda_\delta\delta(x)} E\Bigg\{\mathrm{e}^{-\lambda_\delta\sum_i\{\xi(Z_{\lambda_\delta,i,(k)}) - E\xi(Z_{\lambda_\delta,i,(k)})\}}$$
$$\times I\left(\sum_{i=1}^n (\xi(Z_{\lambda_\delta,i,(k)}) - E\xi(Z_{\lambda_\delta,i,(k)})) \ge 0\right)\Bigg\}. \qquad (3.21)$$

Let $W_i = \xi(Z_{\lambda_\delta,i,(k)})$ and $S = \sum_{i=1}^n (W_i - EW_i)$. By (3.9), we have

$$B_n'^2 := \sum_{i=1}^n E(W_i - EW_i)^2 = \sum_{i=1}^n \mathrm{Var}(\xi(Z_{\lambda_\delta,i,(k)})) = 4x^2 + O(1)x^3\mathcal{L}_{3n}.$$

Then by the proof of Proposition 2.2 of Wang [17] with $\theta = 0$,

$$E\left\{\mathrm{e}^{-\lambda_\delta\sum\{\xi(Z_{\lambda_\delta,i,(k)}) - E\xi(Z_{\lambda_\delta,i,(k)})\}} \times I\left(\sum_{i=1}^n (\xi(Z_{\lambda_\delta,i,(k)}) - E\xi(Z_{\lambda_\delta,i,(k)})) \ge 0\right)\right\}$$
$$= E\mathrm{e}^{-\lambda_\delta S} I(S \ge 0)$$
$$= \mathrm{e}^{x^2/2}(1-\Phi(x))(1 + O(1)x\mathcal{L}_{3n}). \qquad (3.22)$$

Hence (3.19) follows from (3.21) and (3.22). By (3.7),

$$\sum_{i=1}^{n} \log E\mathrm{e}^{\lambda_\delta \xi(X_{i,(k)})} - \lambda_\delta(2-\alpha_0)x^2 = (2\lambda_\delta^2 - 2\lambda_\delta)x^2 + \left(\frac{4\lambda_\delta^3}{3} - 2\lambda_\delta^2\alpha_0 + \lambda_\delta\alpha_3\right)x^3\bar{L}_{3n} + \left(\frac{2\lambda_\delta^4}{3} - 2\lambda_\delta^3\alpha_0 + \frac{\lambda_\delta^2\alpha_0^2}{2}\right)x^4\bar{L}_{4n} + 2\lambda_\delta^2\alpha_3 x^4\bar{L}_{3n}^2 - \frac{\left(2\lambda_\delta^2 - \lambda_\delta\alpha_0\right)^2}{2}\frac{x^4\sum_{i=1}^n (E\bar{X}_i^2)^2}{B_n^4} + O(1)\Delta_{n,x}^{3,5}. \tag{3.23}$$

From the expression of λ_δ in (3.17) and by Lemma 3.1(i), we have

$$(2\lambda_\delta^2 - 2\lambda_\delta)x^2 = -\frac{x^2}{2} + 2\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)^2 x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5} \tag{3.24}$$

and

$$\left(\frac{4\lambda_\delta^3}{3} - 2\lambda_\delta^2\alpha_0 + \lambda_\delta\alpha_3\right)x^3\bar{L}_{3n} = \left(\frac{1}{6} - \frac{\alpha_0}{2} + \frac{\alpha_3}{2}\right)x^3\bar{L}_{3n} + (1-2\alpha_0+\alpha_3)\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}. \tag{3.25}$$

Applying (3.24) and (3.25) to (3.23) and by Lemma 3.1(i),

$$\sum_{i=1}^{n} \log E\mathrm{e}^{\lambda_\delta \xi(X_{i,(k)})} - \lambda_\delta(2-\alpha_0)x^2 = -\frac{x^2}{2} + \left(\frac{1}{6} - \frac{\alpha_0}{2} + \frac{\alpha_3}{2}\right)x^3\bar{L}_{3n} + \left(\frac{1}{24} - \frac{\alpha_0}{4} + \frac{\alpha_0^2}{8}\right)x^4\bar{L}_{4n} + 2\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)^2 x^4\bar{L}_{3n}^2 + (1-2\alpha_0+\alpha_3)\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)x^4\bar{L}_{3n}^2 + \frac{\alpha_3}{2}x^4\bar{L}_{3n}^2 - \frac{(1-\alpha_0)^2}{8}\frac{x^4\sum_{i=1}^n (E\bar{X}_i^2)^2}{B_n^4} + O(1)\Delta_{n,x}^{3,5}. \tag{3.26}$$

Since $\delta(x) = \beta_3 x^3\bar{L}_{3n} + \beta_4 x^4\bar{L}_{4n} + \beta_5 x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}$, then

$$-\lambda_\delta\delta(x) = -\frac{\beta_3}{2}x^3\bar{L}_{3n} - \frac{\beta_4}{2}x^4\bar{L}_{4n} - \left\{\frac{\beta_5}{2} + \beta_3\left(\frac{\beta_3}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} - \frac{1}{4}\right)\right\}x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}. \tag{3.27}$$

Combining (3.26) and (3.27), we obtain (3.20). □

Proposition 3.5. *Under conditions (2.1) and (2.2), for $x \geq 2$,*

$$P(S_n \geq xV_n) \leq A\exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\} \times(1-\Phi(x))(1+Ax\mathcal{L}_{3n}). \tag{3.28}$$

Consequently, for $S_n^{(i)} = S_n - X_i$ and $V_n^{(i)} = (V_n^2 - X_i^2)^{1/2}$,

$$P\left(S_n^{(i)} \geq (x^2-1)^{1/2}V_n^{(i)}\right) \leq A\exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\} \times(1-\Phi(x))(1+Ax\mathcal{L}_{3n}). \tag{3.29}$$

Proof. Let $P(S_n \geq xV_n) = I_1 + I_2 + I_3 + I_4$ where

$$I_1 = P\left\{S_n \geq xV_n,\ \ |V_n/B_n - 1| \leq |x\bar{L}_{3n}|/2 + x^2\bar{L}_{4n}\right\},$$
$$I_2 = P\left\{S_n \geq xV_n,\ \ 0 \leq V_n/B_n \leq 1 - |x\bar{L}_{3n}|/2 - x^2\bar{L}_{4n}\right\},$$
$$I_3 = P\left\{S_n \geq xV_n,\ \ 1 + |x\bar{L}_{3n}|/2 + x^2\bar{L}_{4n} < V_n/B_n \leq 3\right\},$$
$$I_4 = P\left\{S_n \geq xV_n,\ \ V_n/B_n > 3\right\}.$$

Observe that for any real valued set B and any constants $\gamma > 0$ and $\theta > 0$,

$$P\left\{S_n \geq xV_n,\ \frac{V_n}{B_n} \in B\right\} \leq P\left\{\gamma b S_n - \theta b^2 V_n^2 \geq \inf_{V_n/B_n \in B}\left(\gamma x \frac{xV_n}{B_n} - \theta \frac{x^2V_n^2}{B_n^2}\right)\right\}. \tag{3.30}$$

Let $t = xV_n/B_n$. Then $f_x(t) := 2xt - t^2$ has minimum at a for $t \in [a, x]$ and has minimum at b for $t \in [x, b]$. Hence by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 1,\ \alpha_3 = 0,\ \beta_3 = 0,\ \beta_4 = 0$ and $\beta_5 = -1/4$,

$$\begin{aligned}
I_1 &\leq P\left\{2bS_n - b^2V_n^2 \geq 2x^2\left(1 - \frac{|x\bar{L}_{3n}|}{2} - x^2\bar{L}_{4n}\right) - x^2\left(1 - \frac{|x\bar{L}_{3n}|}{2} - x^2\bar{L}_{4n}\right)^2\right\}\\
&\quad + P\left\{2bS_n - b^2V_n^2 \geq 2x^2\left(1 + \frac{|x\bar{L}_{3n}|}{2} + x^2\bar{L}_{4n}\right) - x^2\left(1 + \frac{|x\bar{L}_{3n}|}{2} + x^2\bar{L}_{4n}\right)^2\right\}\\
&= 2P\left\{2bS_n - b^2V_n^2 \geq x^2 - \frac{1}{4}x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}\right\}\\
&\leq 2\exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n}).
\end{aligned}$$

To estimate I_2, note that $f_x(t) := 2xt - 7t^2/3$ has minimum at a for $t \in [0, a]$ with $a \geq 6x/7$. Then by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 7/3,\ \alpha_3 = 0,\ \beta_4 = 8/3,\ \beta_5 = -7/12$, and $\beta_3 = 4/3$ if $x^3\bar{L}_{3n} \geq 0$ and $\beta_3 = -4/3$ if $x^3\bar{L}_{3n} < 0$,

$$\begin{aligned}
I_2 &\leq P\left\{2bS_n - \frac{7}{3}b^2V_n^2 \geq 2x^2\left(1 - \frac{|x\bar{L}_{3n}|}{2} - x^2\bar{L}_{4n}\right)\right.\\
&\qquad \left. - \frac{7}{3}x^2\left(1 - \frac{|x\bar{L}_{3n}|}{2} - x^2\bar{L}_{4n}\right)^2\right\}\\
&= P\left\{2bS_n - \frac{7}{3}b^2V_n^2 \geq -\frac{x^2}{3} + \frac{4}{3}|x^3\bar{L}_{3n}| + \frac{8}{3}x^4\bar{L}_{4n} - \frac{7}{12}x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}\right\}\\
&\leq \exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n})
\end{aligned}$$

To estimate I_3, note that $f_x(t) := 2xt - t^2/3$ has minimum at a for $t \in [a, 3x]$. Then by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 1/3,\ \alpha_3 = 0,\ \beta_4 = 4/3,\ \beta_5 = -1/12$, and $\beta_3 = 2/3$ if $x^3\bar{L}_{3n} \geq 0$ and $\beta_3 = -2/3$ if $x^3\bar{L}_{3n} < 0$,

$$\begin{aligned}
I_3 &\leq P\left\{2bS_n - \frac{1}{3}b^2V_n^2 \geq 2x^2\left(1 + \frac{|x\bar{L}_{3n}|}{2} + x^2\bar{L}_{4n}\right)\right.\\
&\qquad \left. - \frac{1}{3}x^2\left(1 + \frac{|x\bar{L}_{3n}|}{2} + x^2\bar{L}_{4n}\right)^2\right\}\\
&= P\left\{2bS_n - \frac{1}{3}b^2V_n^2 \geq \frac{5x^2}{3} + \frac{2}{3}|x^3\bar{L}_{3n}| + \frac{4}{3}x^4\bar{L}_{4n} - \frac{1}{12}x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5}\right\}\\
&\leq \exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n}),
\end{aligned}$$

where we also use the fact that $\bar{L}_{3n}^2 \le \bar{L}_{4n}$.

Finally by the proof of Lemma 8.1 of Jing, Shao and Wang [6], for $x \ge 2$,

$$I_4 \le 2\mathrm{e}^{-x^2} \le A \exp\left\{ -\frac{1}{3}x^3 \bar{L}_{3n} - \frac{1}{12}x^4 \bar{L}_{4n} + A\Delta_{n,x}^{3,5} \right\}(1-\Phi(x)).$$

Therefore, (3.28) follows from all of the above. □

For $x \ge 2$, let $\tau = B_n/x$ and set

$$\begin{aligned} \bar{X}_i = X_i I(|X_i| \le \tau), \quad \bar{Z}_{\lambda_\delta,i} = Z_{\lambda_\delta,i,(2)}, \quad \bar{S}_n = \sum_{i=1}^n \bar{X}_i, \quad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_i^2, \\ \bar{V}_{n,\delta}^2 = \sum_{i=1}^n \bar{Z}_{\lambda_\delta,i}^2, \quad u_i = \bar{Z}_{\lambda_\delta,i}^2 - E\bar{Z}_{\lambda_\delta,i}^2, \quad \triangledown_n = \frac{x^2}{B_n^4}\left(\sum_{i=1}^n u_i\right)^2. \end{aligned} \tag{3.31}$$

Proposition 3.6. *Under conditions (2.1) and (2.2), for $x \ge 2$,*

$$P(\bar{S}_n \ge x\bar{V}_n) \le \exp\left\{-\frac{1}{3}x^3\bar{L}_{3n} - \frac{1}{12}x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5}\right\}(1-\Phi(x))(1+Ax\mathcal{L}_{3n}).$$

Proof. Let $\xi(y) = 2by - (1 - x\bar{L}_{3n}/2)b^2y^2$ and

$$\delta(x) = \frac{x^3\bar{L}_{3n}}{2} + \frac{x^4\bar{L}_{3n}^2}{4} - 2A\Delta_{n,x}^{3,5}. \tag{3.32}$$

By the conjugate method similar to (3.21),

$$\begin{aligned} P(\bar{S}_n \ge x\bar{V}_n) &= \int\cdots\int I\left\{\sum_{i=1}^n x_i \ge x\left(\sum_{i=1}^n x_i^2\right)^{1/2}\right\}\prod_{i=1}^n \mathrm{d}P(\bar{X}_i \le x_i) \\ &= \int\cdots\int \mathrm{e}^{\sum_i \log E\mathrm{e}^{\lambda_\delta \xi(\bar{X}_i)} - \lambda_\delta\sum_i \xi(x_i)} I(s_n \ge xv_n)\prod_{i=1}^n \mathrm{d}P(\bar{Z}_{\lambda_\delta,i} \le x_i), \end{aligned} \tag{3.33}$$

where $s_n = \sum_{i=1}^n x_i$ and $v_n^2 = \sum_{i=1}^n x_i^2$. We can write

$$v_n = \frac{B_n}{1 - x\bar{L}_{3n}/2}(1+y_n)^{1/2}, \tag{3.34}$$

where

$$y_n = \frac{(1-x\bar{L}_{3n}/2)^2 v_n^2 - B_n^2}{B_n^2}. \tag{3.35}$$

Since $\bar{L}_{3n}^2 \le \bar{L}_{4n}$, then $\lambda_\delta = 1/2 + O(1)x\bar{L}_{3n} + O(1)x^{-2}\Delta_{n,x}^{3,5}$ by (3.17). Hence by (3.10) and (3.31),

$$\begin{aligned} E\bar{V}_{n,\delta}^2 &= \sum_{i=1}^n E\bar{X}_i^2 + \frac{x}{B_n}\sum_{i=1}^n E\bar{X}_i^3 + O(1)\frac{x^2}{B_n^2}\sum_{i=1}^n E\bar{X}_i^4 + O(1)x^{-2}\Delta_{n,x}^{3,5}B_n^2 \\ &= B_n^2 + x\bar{L}_{3n}B_n^2 + O(1)x^2\bar{L}_{4n}B_n^2 + O(1)x^{-2}\Delta_{n,x}^{3,5}B_n^2. \end{aligned}$$

Since $(1-x\bar{L}_{3n}/2)^2 = 1 - x\bar{L}_{3n} + x^2\bar{L}_{3n}^2/4$. Then

$$(1-x\bar{L}_{3n}/2)^2 E\bar{V}_{n,\delta}^2 = B_n^2 + O(1)x^2\bar{L}_{4n}B_n^2 + O(1)x^{-2}\Delta_{n,x}^{3,5}B_n^2. \tag{3.36}$$

Applying (3.36) to (3.35), we have

$$\begin{aligned} y_n &= \frac{(1-x\bar{L}_{3n}/2)^2(v_n^2 - E\bar{V}_{n,\delta}^2)}{B_n^2} + \frac{(1-x\bar{L}_{3n}/2)^2 E\bar{V}_{n,\delta}^2 - B_n^2}{B_n^2} \\ &= \frac{(1-x\bar{L}_{3n}/2)^2(v_n^2 - E\bar{V}_{n,\delta}^2)}{B_n^2} + O(1)x^2\bar{L}_{4n} + O(1)x^{-2}\Delta_{n,x}^{3,5}. \end{aligned} \tag{3.37}$$

Note that $x\bar{L}_{4n}^2 \le x\bar{\mathcal{L}}_{3n}\bar{\mathcal{L}}_{5n} \le \bar{\mathcal{L}}_{5n}/A$ by Lemma 3.1(i). Then

$$y_n^2 \le \frac{2(1-x\bar{L}_{3n}/2)^4(v_n^2 - E\bar{V}_{n,\delta}^2)^2}{B_n^4} + Ax^{-2}\Delta_{n,x}^{3,5}. \tag{3.38}$$

Observe that $(1+y)^{1/2} \ge 1 + y/2 - y^2/2$ for any $y \ge -1$ and $(1+y)^{1/2} \ge 1 + y/2 - y^2/m$ for $y \ge m/2$ with $m > 0$. By (3.37), $B_n^{-2}(v_n^2 - E\bar{V}_{n,\delta}^2) \ge -3/2$ because $y_n \ge -1$. Moreover, $B_n^{-2}(v_n^2 - E\bar{V}_{n,\delta}^2) \ge m$ implies that $y_n \ge m/2$ for $m \ge 2$. Hence

$$(1+y_n)^{1/2} \ge 1 + \frac{y_n}{2} - \theta y_n^2, \tag{3.39}$$

where

$$\theta = \begin{cases} 1/2 & \text{if} \quad B_n^{-2}|v_n^2 - E\bar{V}_{n,\delta}^2| \le 2 \\ 1/m & \text{if} \quad m < B_n^{-2}\left(v_n^2 - E\bar{V}_{n,\delta}^2\right) \le m+1 \text{ for } m \ge 2. \end{cases}$$

Since $4(1-x\bar{L}_{3n}/2)^3 \le 5$, then combining (3.34), (3.35), (3.38) and (3.39), we have

$$\begin{aligned} I(s_n \ge xv_n) &= I\left\{2bs_n \ge \frac{2x^2}{1-x\bar{L}_{3n}/2}(1+y_n)^{1/2}\right\} \\ &\le I\bigg\{2bs_n - (1-x\bar{L}_{3n}/2)b^2v_n^2 + \frac{5\theta x^2(v_n^2 - E\bar{V}_{n,\delta}^2)^2}{B_n^4} \\ &\ge \frac{x^2}{1-x\bar{L}_{3n}/2} - A\Delta_{n,x}^{3,5}\bigg\}. \end{aligned} \tag{3.40}$$

From (2.7) and (3.32),

$$\frac{x^2}{1-x\bar{L}_{3n}/2} - A\Delta_{n,x}^{3,5} \ge x^2 + \delta(x). \tag{3.41}$$

By (3.33), (3.40), (3.41) and the definition of $\bigtriangledown_n$ in (3.31),

$$P(\bar{S}_n \ge x\bar{V}_n) \le E\mathrm{e}^{\sum_i \log E\mathrm{e}^{\lambda_\delta \xi(\bar{X}_i)} - \lambda_\delta \sum_i \xi(\bar{Z}_{\lambda_\delta,i})} I\left\{\sum_{i=1}^n \xi(\bar{Z}_{\lambda_\delta,i}) + 5\theta\bigtriangledown_n \ge x^2 + \delta(x)\right\}. \tag{3.42}$$

Let

$$T_n = \sum_{i=1}^n (\xi(\bar{Z}_{\lambda_\delta,i}) - E\xi(\bar{Z}_{\lambda_\delta,i})) + 5\theta \bigtriangledown_n .$$

Note that $\sum_{i=1}^n E\xi(\bar{Z}_{\lambda_\delta,i}) = x^2 + \delta(x)$ in (3.16). Hence

$$P(\bar{S}_n \geq x\bar{V}_n) \leq \exp\left\{\sum \log E\mathrm{e}^{\lambda_\delta \xi(\bar{X}_i)} - \lambda_\delta x^2 - \lambda_\delta \delta(x)\right\} \times E\mathrm{e}^{-\lambda_\delta \sum_i (\xi(\bar{Z}_{\lambda_\delta,i}) - E\xi(\bar{Z}_{\lambda_\delta,i}))} I\left(T_n \geq 0\right). \tag{3.43}$$

Since $|\mathrm{e}^s - 1| \leq \mathrm{e}^{0\vee s}|s|$, then

$$\begin{aligned} E\mathrm{e}^{-\lambda_\delta \sum_i (\xi(\bar{Z}_{\lambda_\delta,i}) - E\xi(\bar{Z}_{\lambda_\delta,i}))} I\left(T_n \geq 0\right) &= E\mathrm{e}^{5\theta\lambda_\delta \bigtriangledown_n - \lambda_\delta T_n} I\left(T_n \geq 0\right) \\ &\leq E\mathrm{e}^{-\lambda_\delta T_n} I\left(T_n \geq 0\right) + 5\theta\lambda_\delta E \bigtriangledown_n \mathrm{e}^{5\theta\lambda_\delta \bigtriangledown_n - \lambda_\delta T_n} I\left(T_n \geq 0\right) \\ &\leq E\mathrm{e}^{-\lambda_\delta T_n} I\left(T_n \geq 0\right) + 5\theta\lambda_\delta E \bigtriangledown_n \mathrm{e}^{5\theta\lambda_\delta \bigtriangledown_n} \\ &:= J_1 + J_2. \end{aligned} \tag{3.44}$$

To estimate J_1, let $B_n'^2 = \sum_{i=1}^n (K_i - EK_i)^2$ where $K_i = \xi(\bar{Z}_{\lambda_\delta,i}) + 5\theta x^2 B_n^{-4} u_i^2$. Then $B_n'^2 = 4x^2 + O(x^3 \mathcal{L}_{3n})$ by (3.9) and (3.10). Following the proof of Proposition 2.2 of Wang [17], we have

$$J_1 = \mathrm{e}^{x^2/2}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n}). \tag{3.45}$$

By Lemma 3.7 below and Lemma 3.1(ii),

$$J_2 \leq Ax^2 \bar{\mathcal{L}}_{4n} \leq A(\bar{\mathcal{L}}_{3n} + x^{-1}\Delta_{n,x}^{3,5}). \tag{3.46}$$

Note that $(2\pi)^{-1/2}(x^{-1} - x^{-3})\mathrm{e}^{-x^2/2} \leq (1 - \Phi(x))$ for $x \geq 2$. Then by (3.44)–(3.46),

$$\begin{aligned} E\mathrm{e}^{-\lambda_\delta \sum_i \xi(Z_{\lambda_\delta,i,(2)}) - E\xi(Z_{\lambda_\delta,i,(2)}))} I\left(T_n \geq 0\right) &\leq \mathrm{e}^{x^2/2}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n} + A\Delta_{n,x}^{3,5}) \\ &\leq \mathrm{e}^{x^2/2 + A\Delta_{n,x}^{3,5}}(1 - \Phi(x))(1 + Ax\mathcal{L}_{3n}). \end{aligned} \tag{3.47}$$

By (3.20) in Proposition 3.4 and the $\delta(x)$ in (3.32),

$$\frac{x^2}{2} + \sum_{i=1}^n \log E\mathrm{e}^{\lambda_\delta \xi(\bar{X}_i)} - \lambda_\delta x^2 - \lambda_\delta \delta(x) \leq -\frac{1}{3}x^3 \bar{L}_{3n} - \frac{1}{12}x^4 \bar{L}_{4n} + A\Delta_{n,x}^{3,5}. \tag{3.48}$$

Therefore, the proposition follows from (3.43), (3.47) and (3.48). □

Lemma 3.7. *Suppose that $\theta = 1/2$ if $B_n^{-2}|\sum_{i=1}^n u_i| \leq 2$ and $\theta = 1/m$ if $m < B_n^{-2}(\sum_{i=1}^n u_i) \leq m+1$ for $m \geq 2$. Then for $x \geq 2$ and $\bigtriangledown_n$ defined in* (3.31),

$$E \bigtriangledown_n \mathrm{e}^{5\theta \bigtriangledown_n} \leq Ax^2 \bar{\mathcal{L}}_{4n}.$$

Proof. By the definition of $\bigtriangledown_n$ in (3.31),

$$E \bigtriangledown_n \mathrm{e}^{5\theta \bigtriangledown_n} = \frac{x^2}{B_n^4} \sum_{i=1}^n E u_i^2 \mathrm{e}^{5\theta \bigtriangledown_n} + \frac{x^2}{B_n^4} \sum_{1 \leq i,j \leq n, i \neq j} E u_i u_j \mathrm{e}^{5\theta \bigtriangledown_n}. \tag{3.49}$$

First, we want to show that for $d = \{1, \ldots, n\}\backslash\{i\}$ or $d = \{1, \ldots, n\}\backslash\{i, j\}$,

$$E\mathrm{e}^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} \leq A. \tag{3.50}$$

Since $B_n^{-2}|u_k| \le 1/x^2$ and since $B_n^{-2}(\sum_{k=1}^n u_k) \ge -3/2$ as mentioned after (3.38), then $B_n^{-2}(\sum_{k\in d}^n u_k) \ge -3/2 - 2/x^2 \ge -2$. Hence

$$\begin{aligned} Ee^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} &= Ee^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} I\left(xB_n^{-2}\left|\sum_{k\in d} u_k\right| \le 2\right) \\ &\quad + Ee^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} I\left(2 < xB_n^{-2}\left|\sum_{k\in d} u_k\right| \le 2x\right) \\ &\quad + \sum_{m=2}^{\infty} Ee^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} I\left(mx < xB_n^{-2}\sum_{k\in d} u_k \le (m+1)x\right) \\ &:= K_1 + K_2 + K_3. \end{aligned} \tag{3.51}$$

Since $\theta \le 1/2$, then

$$K_1 \le e^{20}. \tag{3.52}$$

Let $[2x]$ be the largest integer smaller than or equal to x. Then

$$\begin{aligned} K_2 &\le \sum_{l=2}^{[2x]} Ee^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} I\left(l < xB_n^{-2}\left|\sum_{k\in d} u_k\right| \le l+1\right) \\ &\le \sum_{l=2}^{[2x]} e^{5(l+1)^2} P\left(xB_n^{-2}\left|\sum_{k\in d} u_k\right| > l\right). \end{aligned} \tag{3.53}$$

Since $|e^s - 1 - s| \le e^{0\vee s} s^2/2$ and $Eu_k = 0$, then for $t > 0$,

$$\begin{aligned} P\left(xB_n^{-2}\left|\sum_{k\in d} u_k\right| > l\right) &\le e^{-tl}\prod_{k\in d} Ee^{txB_n^{-2}u_k} + e^{-tl}\prod_{k\in d} Ee^{-txB_n^{-2}u_k} \\ &\le 2e^{-tl}\prod_{k\in d}\left(1 + \frac{1}{2}e^{txB_n^{-2}|u_k|}t^2x^2B_n^{-4}Eu_k^2\right). \end{aligned}$$

Let $t = (\log A^{1/2})l/2$. Since $xB_n^{-2}|u_k| \le 1/x$, then $e^{txB_n^{-2}|u_k|} \le A^{1/2}$ for $l \le 2x$. Note that $Eu_k^2 \le E\bar{Z}_{\lambda_\delta,i}^4 \le A^{1/4}E\bar{X}_k^4$ by (3.10) and $x^2\bar{\mathcal{L}}_{4n} \le 1/A$. Then

$$P\left(xB_n^{-2}\left|\sum_{k\in d} u_k\right| > l\right) \le 2e^{-tl}e^{A^{1/2}t^2A^{1/4}x^2\bar{\mathcal{L}}_{4n}} \le 2e^{-(\log A)l^2/4 + l^2}. \tag{3.54}$$

Applying (3.54) to (3.53), we have

$$K_2 \le \sum_{l=2}^{[2x]} e^{-l^2} \le A. \tag{3.55}$$

For K_3, if $mx < xB_n^{-2}\sum_{k\in d} u_k \le (m+1)x$, then $m - 2/x^2 < B_n^{-2}\sum_{k=1}^n u_k \le m + 1 + 2/x^2$. By the assumptions of the lemma, $\theta \le 1/(m-1)$ for $mx < xB_n^{-2}\sum_{k\in d} u_k \le (m+1)x$ with $m \ge 2$. Hence

$$K_3 \le \sum_{m=2}^{\infty} e^{10x^2(m+1)^2/(m-1)} P\left(B_n^{-2}\sum_{k\in d} u_k > m\right).$$

Let $t = (\log A^{1/2})x^2$. Then

$$P\left(B_n^{-2}\sum_{k\in d} u_k > m\right) \le \mathrm{e}^{-tm}\prod_{k\in d}\left(1+\frac{1}{2}\mathrm{e}^{tB_n^{-2}|u_k|}t^2B_n^{-4}Eu_k^2\right)$$
$$\le \mathrm{e}^{-(\log A)mx^2/2+x^2}.$$

Hence

$$K_3 \le A\sum_{m=2}^{\infty}\mathrm{e}^{-mx^2} \le A. \tag{3.56}$$

Therefore, (3.50) follows from (3.51), (3.52), (3.55) and (3.56).

Now we estimate the first term on the right-hand side of (3.49). Since $\theta \le 1/2$, then for each i and $d = \{1,\dots,n\}\backslash\{i\}$,

$$\begin{aligned} Eu_i^2\mathrm{e}^{5\theta\nabla_n} &\le Eu_i^2\mathrm{e}^{10x^2B_n^{-4}u_i^2+10x^2B_n^{-4}(\sum_{k\in d}u_k)^2} \\ &= E\mathrm{e}^{10x^2B_n^{-4}(\sum_{k\in d}u_k)^2}Eu_i^2\mathrm{e}^{10x^2B_n^{-4}u_i^2}. \end{aligned} \tag{3.57}$$

Since $x^2B_n^{-2}|u_i| \le 1$, then applying (3.50) to (3.57), we have

$$\frac{x^2}{B_n^4}\sum_{i=1}^n Eu_i^2\mathrm{e}^{5\theta\nabla_n} \le \frac{Ax^2}{B_n^4}\sum_{i=1}^n Eu_i^2 \le Ax^2\bar{\mathcal{L}}_{4n}. \tag{3.58}$$

Next we estimate the second term on the right-hand side of (3.49). For each i and j with $d = \{1,\dots,n\}\backslash\{i,j\}$,

$$Eu_iu_j\mathrm{e}^{5\theta\nabla_n} = Eu_iu_j\mathrm{e}^{5\theta x^2B_n^{-4}(u_i+u_j)^2+10\theta x^2B_n^{-4}(u_i+u_j)\sum_{k\in d}u_k+5\theta x^2B_n^{-4}(\sum_{k\in d}u_k)^2}.$$

Since $|\mathrm{e}^s-1-s| \le \mathrm{e}^{0\vee s}s^2/2$ for $s = 10\theta x^2B_n^{-4}(u_i+u_j)\sum_{k\in d}u_k$ and since $10\theta x^2B_n^{-4}|(u_i+u_j)\sum_{k\in d}u_k| \le 10B_n^{-2}|\sum_{k\in d}u_k|$, then

$$\begin{aligned} \frac{x^2}{B_n^4}\sum_{i\ne j}Eu_iu_j\mathrm{e}^{5\theta\nabla_n} \le\ & \frac{x^2}{B_n^4}\sum_{i\ne j}Eu_iu_j\mathrm{e}^{5\theta x^2B_n^{-4}(u_i+u_j)^2}\mathrm{e}^{5\theta x^2B_n^{-4}(\sum_{k\in d}u_k)^2} \\ &+\frac{10\theta x^4}{B_n^8}\sum_{i\ne j}E\left\{u_iu_j(u_i+u_j)\mathrm{e}^{5\theta x^2B_n^{-4}(u_i+u_j)^2}\times\left(\sum_{k\in d}u_k\right)\mathrm{e}^{5\theta x^2B_n^{-4}(\sum_{k\in d}u_k)^2}\right\} \\ &+\frac{50\theta^2x^6}{B_n^{12}}\sum_{i\ne j}E|u_iu_j|(u_i+u_j)^2\mathrm{e}^{5x^2B_n^{-4}(u_i+u_j)^2} \\ &\times E\left(\sum_{k\in d}u_k\right)^2\mathrm{e}^{10B_n^{-2}|\sum_{k\in d}u_k|+5x^2B_n^{-4}(\sum_{k\in d}u_k)^2} \\ :=\ & L_1+L_2+L_3. \end{aligned} \tag{3.59}$$

For L_1, since $|\mathrm{e}^s-1| \le \mathrm{e}^{s\vee 0}|s|$, $Eu_iu_j = 0$ and $\theta \le 1/2$, then

$$\left|E\left(u_iu_j\mathrm{e}^{5\theta x^2B_n^{-4}(u_i+u_j)^2}\,\middle|\,u_k, k\in d\right)\right| \le Ax^2B_n^{-4}E|u_iu_j|(u_i+u_j)^2.$$

Then by (3.50),

$$L_1 = \frac{x^2}{B_n^4}\sum_{i\neq j} E\left\{ e^{5\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} E\left(u_i u_j e^{5\theta x^2 B_n^{-4}(u_i+u_j)^2} \Big| u_k, k\in d\right)\right\}$$
$$\leq \frac{Ax^4}{B_n^8}\sum_{i\neq j} E|u_i u_j|(u_i+u_j)^2 \leq Ax^2\bar{\mathcal{L}}_{4n}. \tag{3.60}$$

For L_2, since $|e^s - 1| \leq e^{0\vee s}|s|$ and $Eu_iu_j(u_i+u_j) = 0$, then

$$\left| E\left(u_i u_j (u_i+u_j) e^{5\theta x^2 B_n^{-4}(u_i+u_j)^2} \Big| u_k, k\in d\right)\right| \leq Ax^2 B_n^{-4} E|u_iu_j||u_i+u_j|^3. \tag{3.61}$$

By Hölder's inequality and (3.50),

$$\frac{x}{B_n^2} E\left|\sum_{k\in d} u_k\right| e^{5\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} \leq \left\{ \frac{x^2}{B_n^4} E\Big(\sum_{k\in d} u_k\Big)^2\right\}^{1/2} \left\{ E^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2}\right\}^{1/2} \leq A. \tag{3.62}$$

By (3.61) and (3.62),

$$L_2 = \frac{10\theta x^4}{B_n^8}\sum_{i\neq j} E\Bigg\{ \left(\sum_{k\in d} u_k\right) e^{5\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2}$$
$$\times E\left(u_i u_j (u_i+u_j) e^{5\theta x^2 B_n^{-4}(u_i+u_j)^2} \Big| u_k, k\in d\right)\Bigg\}$$
$$\leq \frac{Ax^5}{B_n^{10}}\sum_{i\neq j} E|u_iu_j||u_i+u_j|^3 \leq Ax^2\bar{\mathcal{L}}_{4n}. \tag{3.63}$$

For L_3,

$$E\Big(|u_iu_j|(u_i+u_j)^2 e^{5\theta x^2 B_n^{-4}(u_i+u_j)^2} \Big| u_k, k\in d\Big) \leq AE(|u_iu_j|(u_i^2+u_i^2)). \tag{3.64}$$

Applying Hölder's inequality twice and by (3.50),

$$\frac{x^2}{B_n^4} E\left(\sum_{k\in d} u_k\right)^2 e^{20B_n^{-2}|\sum_{k\in d} u_k| + 5\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2}$$
$$\leq \frac{x^2}{B_n^4}\left\{ E\left(\sum_{k\in d} u_k\right)^4 e^{40B_n^{-2}|\sum_{k\in d} u_k|}\right\}^{1/2} \left\{ E^{10\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2}\right\}^{1/2}$$
$$\leq A\left\{ \frac{x^8}{B_n^{16}} E\left(\sum_{k\in d} u_k\right)^8\right\}^{1/4} \left\{ Ee^{80B_n^{-2}|\sum_{k\in d} u_k|}\right\}^{1/4}.$$

Since $|e^s - 1 - s| \leq e^{0\vee s}s^2/2$ and $Eu_k = 0$, then

$$Ee^{80B_n^{-2}|\sum_{k\in d} u_k|} \leq Ee^{80B_n^{-2}\sum_{k\in d} u_k} + Ee^{-80B_n^{-2}\sum_{k\in d} u_k}$$
$$\leq 2\prod_{k\in d}(1 + AB_n^{-4}Eu_k^2) \leq 2e^{AB_n^{-4}\bar{\mathcal{L}}_{4n}} \leq 2e.$$

It is easy to check that $x^8 B_n^{-16} E(\sum_{k\in d} u_k)^8 \le Ax^2\bar{\mathcal{L}}_{4n} \le 1$. Then from the above, we have

$$\frac{x^2}{B_n^4} E\left(\sum_{k\in d} u_k\right)^2 \mathrm{e}^{20B_n^{-2}|\sum_{k\in d} u_k| + 5\theta x^2 B_n^{-4}(\sum_{k\in d} u_k)^2} \le A. \tag{3.65}$$

By (3.64) and (3.65),

$$L_3 \le \frac{Ax^4}{B_n^8} \sum_{i\ne j} E(|u_i u_j|(u_j^2 + u_i^2)) \le Ax^2\bar{\mathcal{L}}_{4n}. \tag{3.66}$$

Combining (3.59), (3.60), (3.63) and (3.66), we obtain

$$\frac{x^2}{B_n^4} \sum_{i\ne j} E u_i u_j \mathrm{e}^{5\theta\nabla_n} \le Ax^2\bar{\mathcal{L}}_{4n}. \tag{3.67}$$

Therefore, the lemma follows from (3.49), (3.58) and (3.67). □

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