ESAIM: PS 22 (2018) 129–162
https://doi.org/10.1051/ps/2017022

ESAIM: Probability and Statistics
www.esaim-ps.org

UNIFORM CONVERGENCE OF PENALIZED TIME-INHOMOGENEOUS MARKOV PROCESSES

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Abstract. We provide a general criterion ensuring the exponential contraction of Feynman–Kac semi-groups of penalized processes. This criterion applies to time-inhomogeneous Markov processes with absorption and killing through penalization. We also give the asymptotic behavior of the expected penalization and provide results of convergence in total variation of the process penalized up to infinite time. For exponential convergence of penalized semi-groups with bounded penalization, a converse result is obtained, showing that our criterion is sharp in this case. Several cases are studied: we first show how our criterion can be simply checked for processes with bounded penalization, and we then study in detail more delicate examples, including one-dimensional diffusion processes conditioned not to hit 0 and penalized birth and death processes evolving in a quenched random environment.

Mathematics Subject Classification. 60B10, 60F99, 60J57, 37A25

Received 14 November 2017. Accepted 12 December 2017.

1. Introduction

In [5], we developed a probabilistic framework to study Markov processes with absorption conditionned on non-absorption. The main result is a necessary and sufficient condition for the exponential convergence of conditional distributions to a unique quasi-stationary distribution. Our approach is based on coupling estimates (Doeblin condition and Dobrushin coefficient) which allow to use probabilistic methods to check the criteria in various classes of models, such as one-dimensional diffusions [4, 8], multi-dimensional diffusions [7, 9] or multi-dimensional birth and death processes [6].

The present paper studies the extension of the previous results to the time-inhomogeneous setting in discrete and continuous time in the general framework of Feynman–Kac semi-groups of penalized processes developed by Del Moral and Miclo [13] and Del Moral and Guionnet [12]. The literature on the topic is vast and closely related to the study of genealogical and interacting particle systems. For more details, we refer the reader to the two textbooks [10, 11] and the numerous references therein. This approach allows us to prove non-exponential convergence also in time-homogeneous models which are not covered by the criteria of [5].

The present paper can be seen as a complement and an extension of the results on the contraction of Feynman–Kac semi-groups gathered in ([11], Chap. 12). The main novelties of our work lies in the facts that our criteria

Keywords and phrases: Feynman–Kac formula, time-inhomogeneous Markov processes, penalized processes, one-dimensional diffusions with absorption, birth and death processes in random environment with killing, asymptotic stability, uniform exponential mixing, Dobrushin's ergodic coefficient.

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can be proved to be necessary and sufficient for exponential convergence of the Feynman–Kac semi-groups in the soft obstacle setting (in Sect. 4.1), that our work allows to study models with hard obstacles (in Sect. 5) and that we study examples of time-inhomogeneous Markov processes in quenched environments (in Sect. 6.2).

In our applications, we first explain how our criteria can be checked in the special case of Feynman–Kac semi-groups with bounded penalization rate. To show the novelty of our criteria and how to apply the methods developed in [4, 5, 8], we provide a detailed study of two natural classes of models: time-inhomogeneous diffusion processes in dimension 1 with hard obstacles and time-inhomogeneous penalized one-dimensional birth and death processes. In particular, we study non-periodic diffusions with non-regular coefficients, hence improving the results of [14]. We also consider the case of birth and death processes evolving in a quenched random environment, alternating phases of growth and decay, under general assumptions on the environment. Similar questions are studied for other types of applications in [1].

In Section 2, we present the general class of models considered in this paper and state our main result on the contraction of Feynman–Kac semi-groups (Thm. 2.1). Section 3 is concerned with original results on the limiting behavior of the expectation of the penalization (Prop. 3.1) with consequences on uniqueness on time-inhomogeneous stationary evolution problems with growth conditions at infinity, and on the existence and asymptotic mixing of the Markov process penalized up to infinite time (Thm. 3.4). Section 4 contains a study of Feynman–Kac semi-groups with bounded penalization rate. We first give in subsection 4.1 a converse result for exponential convergence of such semigroups, showing that our criterion is sharp. We then show in subsection 4.2 how our criteria can be easily checked when the process satisfies uniform irreducibility properties and uniform exponential moments for the entrance time in compact sets. Sections 5 and 6 are devoted to the application of our criteria to more complex situations. We first study the case of time-inhomogeneous diffusions on $[0,+\infty)$ absorbed at 0 and conditioned to non-absorption (that is, with infinite penalization at 0) in Section 5. Section 6 is devoted to the study of penalized continuous time inhomogeneous birth and death processes in $\mathbb{N}$, for which the penalization rate is bounded but where exponential moments of entrance times in compact sets do not hold uniformly in time. We first give a general criterion in subsection 6.1 and then study the case of birth and death processes in quenched environment alternating phases of growth and decay (close to infinity) in subsection 6.2. The proof of Theorem 2.1 is given in Section 7. Proposition 3.1 and Theorem 3.4 are proved respectively in Sections 8 and 9. Finally, Theorems 4.1 and 4.4 are proved respectively in Sections 10 and 11.

2. Main result

Let $(\Omega,(\mathcal{F}_{s,t})_{0\leq s\leq t\in I},\mathbb{P},(X_t)_{t\in I})$ be a Markov process evolving in a measurable space $(E,\mathcal{E})$, where the time space is $I=[0,+\infty)$ or $I=\mathbb{N}$ and X can be time-inhomogeneous, such that X_t is $\mathcal{F}_{s,r}$-measurable for all $s\leq t\leq r$. Let $Z=\{Z_{s,t};0\leq s\leq t,\ s,t\in I\}$ be a collection of multiplicative nonnegative random variables such that, for any $s\leq t$, $Z_{s,t}$ is a $\mathcal{F}_{s,t}$-measurable random variable and

$$\mathbb{E}_{s,x}(Z_{s,t})>0\quad\text{and}\quad\sup_{y\in E}\mathbb{E}_{s,y}(Z_{s,t})<\infty\quad\forall s\leq t\in I\ \forall x\in E. \tag{2.1}$$

By multiplicative, we mean that, for all $s\leq r\leq t\in I$,

$$Z_{s,r}Z_{r,t}=Z_{s,t}.$$

We define the non linear semi-group $\Phi=\{\Phi_{s,t};\ 0\leq s\leq t\}$ on the set $M_1(E)$ of all probability measures on E by setting, for any distribution $\mu\in M_1(E)$, $\Phi_{s,t}(\mu)$ as the probability measure on E such that, for any bounded and $\mathcal{E}$-measurable function $f:E\to\mathbb{R}$,

$$\Phi_{s,t}(\mu)(f):=\frac{\mathbb{E}_{s,\mu}(f(X_t)Z_{s,t})}{\mathbb{E}_{s,\mu}(Z_{s,t})}, \tag{2.2}$$

where $((X_t)_{t\geq s}, \mathbb{P}_{s,\mu})$ denotes the Markov process X on $[s,+\infty)$ starting with initial distribution μ at time s. It is straightforward to check that the nonlinear map $\Phi_{s,t}$ on the set of probability measures on E satisfies the semigroup property

$$\Phi_{r,t} = \Phi_{s,t} \circ \Phi_{r,s}, \quad \forall r \leq s \leq t. \tag{2.3}$$

Typical examples of penalizations with hard or soft obstacles are given by, respectively,

$$Z_{s,t} = \mathbb{1}_{X_t \notin D} \quad \text{or} \quad Z_{s,t} = e^{\int_s^t \kappa(u,X_u)du}, \tag{2.4}$$

where $D \subset E$ is some absorbing set for the process X or κ is a measurable function from $\mathbb{R}_+ \times E$ to $\mathbb{R}$. In the first case, $\Phi_{s,t}(\mu)$ is simply the conditional distribution of X_t with distribution μ at time s, given it is not absorbed in D at time t. In the second case, if $\kappa(t,x) \leq 0$ for all $t \geq 0$ and $x \in E$, then $-\kappa(t,x)$ can be interpreted as a killing rate at time t in position x and $\Phi_{s,t}(\mu)$ is the conditional distribution of X_t with distribution μ at time s, given it is not killed before time t. Note that if κ is bounded from above by a finite constant $\bar{\kappa}$, then we can replace κ by $\kappa - \bar{\kappa}$ without modifying $\Phi_{s,t}(\mu)$ and hence recover the previous interpretation of $\bar{\kappa} - \kappa$ as a killing rate. If $\kappa \geq 0$, it can be interpreted as a branching rate in branching particle systems.

For all $s \geq 1$ and all $x_1, x_2 \in E$, we define the non-negative measure on E

$$\nu_{s,x_1,x_2} = \min_{i=1,2} \Phi_{s-1,s}(\delta_{x_i}),$$

where the minimum between two measures is understood as usual as the largest measure smaller than both measures, and the real constant

$$d_s = \inf_{t\geq 0, x_1,x_2\in E} \frac{\mathbb{E}_{s,\nu_{s,x_1,x_2}}(Z_{s,s+t})}{\sup_{x\in E} \mathbb{E}_{s,x}(Z_{s,s+t})}.$$

Similarly, we define

$$\nu_s = \min_{x\in E} \Phi_{s-1,s}(\delta_x) \tag{2.5}$$

and the real constant

$$d'_s = \inf_{t\geq 0} \frac{\mathbb{E}_{s,\nu_s}(Z_{s,s+t})}{\sup_{x\in E} \mathbb{E}_{s,x}(Z_{s,s+t})}. \tag{2.6}$$

Note that $\nu_s \leq \nu_{s,x_1,x_2}$ and $d'_s \leq d_s$.

Let us define, for all $0 \leq s \leq t \leq T$ the linear operator $K^T_{s,t}$ on the set of bounded measurable function on E by

$$K^T_{s,t} f(x) = \frac{\mathbb{E}_{s,x}(f(X_t) Z_{s,T})}{\mathbb{E}_{s,x}(Z_{s,T})}. \tag{2.7}$$

We extend as usual this definition to any initial distribution μ on E as

$$\mu K^T_{s,t} f = \int_E K^T_{s,t} f(x)\, \mu(\mathrm{d}x).$$

Note that $K^t_{s,t} f(x) = \Phi_{s,t}(\delta_x)(f)$ but $\mu K^t_{s,t} f \neq \Phi_{s,t}(\mu)(f)$ in general.

Theorem 2.1. *For all probability measures μ_1, μ_2 on E and for all $0 \leq s \leq s+1 \leq t \leq T \in I$, we have*

$$\left\|\mu_1 K^T_{s,t} - \mu_2 K^T_{s,t}\right\|_{TV} \leq \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}) \left\|\mu_1 - \mu_2\right\|_{TV} \tag{2.8}$$

and

$$\left\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\right\|_{TV} \leq 2 \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}), \tag{2.9}$$

where $\|\cdot\|_{TV}$ denotes the usual total variation distance: for all signed finite measure μ on E,

$$\|\mu\|_{TV} = \sup_{A\in\mathcal{E}} \mu(A) - \inf_{A\in\mathcal{E}} \mu(A).$$

In particular, if $\limsup_{t\to\infty} d_t > 0$, there is convergence in (2.8) and (2.9) when $t \to +\infty$, and if $\inf_{s\in I} d_s > 0$, or more generally if $\limsup_{t\to\infty} \frac{1}{t}\sum_{s\leq t}\log(1-d_s) < 0$, we have geometric convergence in (2.8) and (2.9). There is also convergence for example if $d_t \geq ct^{-1}$ for t large enough for some $c > 0$.

Note that Theorem 2.1 gives uniform convergence with respect to the initial distribution. Hence, the process necessarily comes down from infinity in some sense and one expects some form of uniform domination property for the time-inhomogeneous process. In Section 4.2, we propose a domination uniform in time, in Section 5 a domination up to a time-change and in Section 6 an intermittent domination.

Remark 2.2. For a fixed $t > 0$, considering the processes $\bar{X}_r := X_{r\wedge t}$ and $\bar{Z}_{r_1,r_2} = Z_{r_1\wedge t, r_2\wedge t}$, it is clear that the bounds (2.8) and (2.9) are actually valid replacing d_s by

$$\bar{d}^{(t)}_s := \inf_{s\leq u\leq t, x_1,x_2\in E} \frac{\mathbb{E}_{s,\nu_{s,x_1,x_2}}(Z_{s,u})}{\sup_{x\in E}\mathbb{E}_{s,x}(Z_{s,u})}.$$

For example, the case considered in Section 3.1, [2] (in discrete time) fits to our settings, with $d_s = 0$ for all $s \geq 0$ but $\bar{d}^{(t)}_s \geq \frac{C}{1+t-s}$ for $0 \leq s \leq t$. In this case, our result entails a polynomial speed of convergence to 0 of $\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV}$. Note that this time-homogeneous case is not covered by the results of [5].

Remark 2.3. Note that, in the definition of ν_{s,x_1,x_2} and ν_s, the time increments of -1 are not restrictive, since we could change the time-scale in the definition of the time-inhomogeneous Markov process X and the penalization Z using any deterministic increasing function. In particular, given $s = s_0 < t_0 \leq s_1 < t_1 \leq \cdots \leq s_n < t_n \leq t$ in I, we may define for all $i = 0, \ldots, n$ and all $x_1, x_2 \in E$,

$$\nu_{s_i,t_i,x_1,x_2} = \min_{j=1,2} \Phi_{s_i,t_i}(\delta_{x_j}),$$

and the real constant

$$d_{s_i,t_i} = \inf_{t\geq 0, x_1,x_2\in E} \frac{\mathbb{E}_{t_i,\nu_{s_i,t_i,x_1,x_2}}(Z_{t_i,t_i+t})}{\sup_{x\in E}\mathbb{E}_{t_i,x}(Z_{t_i,t_i+t})}.$$

Then it is straightforward that, for all probability measures μ_1, μ_2 on E and all $T \geq t$, we have

$$\left\|\mu_1 K^T_{s,t} - \mu_2 K^T_{s,t}\right\|_{TV} \leq \prod_{k=0}^{n} (1 - d_{s_k,t_k}) \left\|\mu_1 - \mu_2\right\|_{TV}$$

and

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV} \le 2 \prod_{k=0}^{n} (1 - d_{s_k,t_k}) .$$

This remark also applies to the next results (Prop. 3.1 and Thm. 3.4), where ν_s and d'_s can also be modified accordingly.

Note also that our result is optimal in the time-homogeneous setting, in the sense that the exponential contraction in (2.9) is equivalent to the property $d_0 > 0$ (see [5], Thm. 2.1). We discuss the extension of this optimality result to the time-inhomogeneous case in subsection 4.1.

3. Convergence of the expected penalization and penalized process up to infinite time

In the absorbed time-homogeneous setting of [5] (with $Z_{s,t} = \mathbb{1}_{X_t \notin D}$ as in (2.4)), we also obtained complementary results on the limiting behavior of the probability of survival $\mathbb{P}_x(X_t \notin D) = \mathbb{E}_x(Z_{s,t})$ when $t \to \infty$ and on the process conditioned to never be extinct. Both statements can be extended to the present time-inhomogeneous penalized framework, as stated in the following two results.

Proposition 3.1. *For all $y \in E$ and $s \in I$ such that $d'_s > 0$, there exists a finite constant $C_{s,y}$ only depending on s and y such that, for all $x \in E$ and $t, u \ge s+1$ with $t \le u$,*

$$\left| \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})} - \frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})} \right| \le C_{s,y} \inf_{v \in [s+1,t]} \frac{1}{d'_v} \prod_{k=0}^{\lfloor v-s \rfloor -1} (1 - d_{v-k}) . \tag{3.1}$$

In particular, if

$$\liminf_{t \in I,\ t \to +\infty} \frac{1}{d'_t} \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}) = 0, \tag{3.2}$$

for all $s \ge 0$, there exists a positive bounded function $\eta_s : E \to (0, +\infty)$ such that

$$\lim_{t \to \infty} \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})} = \frac{\eta_s(x)}{\eta_s(y)}, \quad \forall x, y \in E, \tag{3.3}$$

where, for any fixed y, the convergence holds uniformly in x, and such that, for all $x \in E$ and $s \le t \in I$,

$$\mathbb{E}_{s,x}(Z_{s,t} \eta_t(X_t)) = \eta_s(x). \tag{3.4}$$

In addition, the function $s \mapsto \|\eta_s\|_\infty$ is locally bounded on $[0, +\infty)$.

Since $d'_t \le d_t$, there is convergence to 0 in (3.1) if $\limsup d'_t > 0$, and the convergence is geometric if $\inf_{t \ge 0} d'_t > 0$. There is also convergence to 0 for example if $d'_t \ge c t^{-1}$ for t large enough and for some $c > 1$.

The last theorem also implies uniqueness results on equation (3.4) and on associated PDE problems.

Corollary 3.2. *Assume that*

$$\liminf_{t \in I,\ t \to +\infty} \frac{1}{d'_t} \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}) = 0,$$

Then the function $(s,x)\mapsto \eta_s(x)$ of the last proposition is the unique solution $(s,x)\mapsto f_s(x)$, up to a multiplicative constant, of

$$\mathbb{E}_{s,x}(Z_{s,t}f_t(X_t)) = f_s(x) \tag{3.5}$$

such that f_s is bounded for all $s\geq 0$ and for some $x_0\in E$,

$$\|f_t\|_\infty = o\left(\frac{\prod_{k=0}^{\lfloor t\rfloor -1}(1-d_{t-k})^{-1}}{\mathbb{E}_{0,x_0}(Z_{0,t})}\right) \tag{3.6}$$

when $t\to+\infty$. Moreover, this unique solution f_s of (3.5) can be chosen positive.

Remark 3.3. The last result also gives uniqueness properties for stationary time-inhomogeneous evolution equations with growth conditions at infinity. Namely, let us assume that the semigroup $P_{s,t}f(x)=\mathbb{E}_{s,x}[Z_{s,t}f(X_t)]$ admits as time-inhomogeneous infinitesimal generator $(L_t, t\geq 0)$ (as defined *e.g.* in [17], Chap. 5). We can also define the (time-homogeneous) semigroup on $[0,+\infty)\times E$ by $T_tf(s,x)=P_{s,s+t}f(s+t,x)$. Then (3.5) writes $T_t\eta=\eta$ and hence can be interpreted as some form of weak solution of the evolution equation

$$\partial_t f_t(x) + L_t f_t(x) = 0,\quad \forall (s,x)\in[0,+\infty)\times E, \tag{3.7}$$

for which Proposition 3.1 and Corollary 3.2 give existence and uniqueness under condition (3.6).

Theorem 3.4. *Assume that*

$$\liminf_{t\in I,\ t\to+\infty}\frac{1}{d'_t}\prod_{k=0}^{\lfloor t-s\rfloor -1}(1-d_{t-k}) = 0.$$

Then, for all $s\in I$, the family $(\mathbb{Q}_{s,x})_{s\in I, x\in E}$ of probability measures on Ω defined by

$$\mathbb{Q}_{s,x}(A)=\lim_{T\to+\infty}\mathbb{P}_{s,x}(A\mid T<\tau_\partial),\ \forall A\in\mathcal{F}_{s,u},\ \forall u\geq s,$$

is well defined and given by

$$\left.\frac{d\mathbb{Q}_{s,x}}{d\mathbb{P}_{s,x}}\right|_{\mathcal{F}_{s,u}} = \frac{Z_{s,u}\eta_u(X_u)}{\mathbb{E}_{s,x}[Z_{s,u}\eta_u(X_u)]},$$

and the process $(\Omega,(\mathcal{F}_{s,t})_{t\geq s},(X_t)_{t\geq 0},(\mathbb{Q}_{s,x})_{s,\in I,x\in E})$ is an E-valued time-inhomogeneous Markov process. In addition, this process is asymptotically mixing in the sense that, for any $s\leq t\in I$ and $x\in E$,

$$\|\mathbb{Q}_{s,x}(X_t\in\cdot)-\mathbb{Q}_{s,y}(X_t\in\cdot)\|_{TV}\leq 2\prod_{k=0}^{\lfloor t-s\rfloor -1}(1-d_{t-k}). \tag{3.8}$$

Remark 3.5. In the case where $Z_{s,u}$ admits a regular conditional probability given X_u for all $s\leq u$ (for example if E is a Polish space), we can express as in [5] the transition kernel of X under $(\mathbb{Q}_{s,x})_{s,x}$ in terms of the transition kernel p of the process X under $(\mathbb{P}_{s,x})_{s,x}$ as

$$\tilde p(s,x;u,\mathrm{d}y) = \frac{\mathbb{E}_{s,x}(Z_{s,u}\mid X_u=y)\,\eta_u(y)}{\mathbb{E}_{s,x}(Z_{s,u}\eta_u(X_u))}p(s,x;u,\mathrm{d}y).$$

4. First applications for bounded penalization rates

In all this section, we consider the case where there exists a bounded measurable function $\kappa : \mathbb{R}_+ \times E \to \mathbb{R}$ such that

$$Z_{s,t} = \exp\left(-\int_s^t \kappa(u, X_u)\mathrm{d}u\right), \quad \forall\, 0 \le s \le t. \tag{4.1}$$

4.1. On the necessity of our assumptions

Due to the large variety of situations that can be covered by our framework, we cannot expect in the general case to have a converse to Theorem 2.1. Actually, Remark 2.2 in Section 2 shows that one can have both $\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV} \to 0$ as $t - s \to +\infty$ and $d_s = 0$ for all $s \ge 0$. Therefore, we restrict in the next result to the case of bounded penalization rates, and show that this situation cannot occur if the convergence in (2.9) is fast enough, and that our criterion is sharp in the case of uniform exponential convergence.

Theorem 4.1. *Assume (4.1) holds for a bounded measurable κ and, defining for all $0 \le s \le t$*

$$\varepsilon_{s,t} := \sup_{\mu_1,\mu_2 \in \mathcal{P}(E)} \|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV}, \tag{4.2}$$

that

$$\sup_{s\ge 0} \varepsilon_{s,s+t} \to 0 \quad \text{when } t \to +\infty \tag{4.3}$$

and

$$\sup_{s\ge 0} \sum_{k\ge 0} \varepsilon_{s,s+k} < \infty. \tag{4.4}$$

Then, there exist $n_0 \in \mathbb{N}$ and $\underline{d} > 0$ such that $\bar{d}'_s \ge \underline{d}$ for all $s \ge n_0$, where

$$\bar{d}'_s := \inf_{t\ge 0} \frac{\mathbb{E}_{s,\bar{\nu}_s}(Z_{s,s+t})}{\sup_{x\in E} \mathbb{E}_{s,x}(Z_{s,s+t})},$$

and, for all $s \ge n_0$,

$$\bar{\nu}_s := \min_{x\in E} \Phi_{s-n_0,s}(\delta_x).$$

In particular, it follows from (3.8) (with a linear change of time) that $\varepsilon_{s,t} \le Ce^{-\gamma(t-s)}$ for some constants C and $\gamma > 0$.

Remark 4.2. It will appear in the proof that the assumption that κ is bounded may be relaxed, provided that $\varepsilon_{s,t}$ can be controlled appropriately; for example, one can replace (4.4) with

$$\sup_{s\ge 0} \sum_{k\ge 0} \varepsilon_{s,s+k} \exp\left(\mathrm{osc}(\kappa, [s+k, s+k+1])\right) < \infty,$$

where, for any interval $I \subset \mathbb{R}_+$, $\mathrm{osc}(\kappa, I) = \sup_{t\in I, x\in E} \kappa(t,x) - \inf_{t\in I, x\in E} \kappa(t,x)$.

Remark 4.3. The last result covers the case of exponential convergence, and it can also be adapted to cases of slower convergence, similarly as in Remark 2.2. With the notations of this remark, adapting the proof of Theorem 4.1, we can prove that, if (4.1) holds for a bounded measurable κ, then the property

$$\varepsilon_{s,t} \leq \frac{C}{(t-s+1)\log^{\beta}(t-s+2)}, \quad \forall 0 \leq s < t \tag{4.5}$$

for some $0 < \beta < 1$ implies that

$$\bar{d}_s^{(t)} \geq C' e^{-C'' \log^{1-\beta}(t-s+1)} \tag{4.6}$$

for some positive constants C' and C''. Note that (4.5) is not compatible with (4.4), so that Theorem 4.1 does not apply here, but since (4.6) implies that $\bar{d}_s^{(t)} \geq C'/(t-s+1)$ for $t-s$ large enough, we recover from (2.9) a polynomial bound on $\varepsilon_{s,t}$.

4.2. Irreducible Markov processes with exponential moments

Our goal in this section is to show how the condition $\inf_{s\geq 1} \bar{d}'_s > 0$ (possibly after a linear scaling of time, see Sect. 11), implying exponential convergence in (2.8), can be checked under simple conditions when the penalization rate is bounded. We consider a time-inhomogeneous Markov process in a finite or countable state space, uniformly irreducible, meaning that, for all $x, y \in E$,

$$\inf_{s\geq 0,\, u\in[1,2]} \mathbb{P}_{s,x}(\mathbb{1}_{X_{s+u}=y}) > 0. \tag{4.7}$$

We also assume uniform exponential moments for return times in finite sets, in the sense that, for all $\lambda > 0$, there exists a finite set $K \subset E$ such that

$$\sup_{x\in E,\, s\geq 0} \mathbb{E}_{s,x}\left(e^{\lambda(T_K - s)}\right) < \infty, \tag{4.8}$$

where $T_K = \inf\{t \geq s,\ X_t \in K\}$.

The irreducibility condition is classical for Markov processes in discrete state spaces, such as $\mathbb{N}^d$ ($d \geq 1$), and the existence of moments of all orders can easily be obtained by exhibiting judicious Lyapunov functions (see Example 4.5 below).

Theorem 4.4. *Assume that κ is uniformly bounded and that conditions (4.7) and (4.8) are satisfied, then there exist two positive constants C and γ such that, for all initial distributions μ_1 and μ_2,*

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV} \leq C\, e^{-\gamma(t-s)}.$$

Example 4.5. We consider a time-inhomogeneous multidimensional birth and death process evolving in $\mathbb{N}^d$ ($d \geq 1$) with transition rates $(q^s_{xy})_{x,y\in\mathbb{N}^d, s\geq 0}$ given, for all $x \neq y \in \mathbb{N}^d$, by

$$q^s_{xy} = \begin{cases} \lambda^s_i(x) & \text{if } y = x + e_i \\ \mu^s_i(x) & \text{if } y = x - e_i \\ 0 & \text{otherwise,} \end{cases}$$

where $(e_i)_{i=1,\dots,d}$ is the canonical basis of $\mathbb{N}^d$ and λ^s_i, μ^s_i are measurable functions satisfying

$$0 < \lambda^s_i(x) \leq \bar{\lambda}|x| \text{ and } \underline{\mu}|x|^2 \leq \mu^s_i(x)$$

for some positive constants $\bar{\lambda}$ and $\underline{\mu}$ (here $|(x_1,\dots,x_d)| = x_1 + x_2 + \cdots + x_d$). The penalization $Z_{s,t}$ is assumed to be given by (4.1) with κ uniformly bounded. Then the assumptions of Theorem 4.4 hold true.

Indeed, assumption (4.7) is clearly satisfied and assumption (4.8) is a consequence of the following classical argument, based on Lyapunov functions: setting $\varphi(x) = \sum_{k=1}^{|x|} 1/k^{3/2}$ one easily checks that there exists a constant $c > 0$ such that, for all $s \geq 0$ and all $x \in \mathbb{N}^d$,

$$\sum_{y\in\mathbb{N}^d,\, y\neq x} q^s_{xy}(\varphi(y) - \varphi(x)) \leq -c\left(\underline{\mu}\sqrt{|x|} - \bar{\lambda}\right)\varphi(|x|).$$

It is standard to deduce from this and Dynkin's formula that, for any $\lambda > 0$, the finite set $K = \{x \in \mathbb{N}^d,\ c\left(\underline{\mu}\sqrt{|x|} - \bar{\lambda}\right) > \lambda\}$ satisfies (4.8).

5. One-dimensional diffusions with time-dependent coefficients

To illustrate the possible applications of our criterion, this section and the next one are devoted to the study of examples where the conditions of subsection 4.2 (*i.e.* uniform penalization rate (4.1), uniform irreducibility (4.7) and uniform exponential moments of entrance times in compact sets (4.8)) are not satisfied.

Our first example deals with one-dimensional diffusion processes conditioned not to hit some absorbing point ∂, *i.e.*

$$Z_{s,t} = \mathbb{1}_{t<\tau_\partial},$$

where τ_∂ is the hitting time of ∂. This corresponds to a case of hard obstacles in the terminology of [11], where (4.1) is not satisfied. This is the setting of [5], but we study here the time-inhomogeneous case.

More precisely, we consider a time inhomogeneous diffusion process X on $[0,+\infty)$ stopped when it hits 0 at time $T_0^X = \inf\{t \geq 0,\ X_{t-} = 0\}$ assumed almost surely finite and solution, on $[s, T_0^X)$ to

$$\mathrm{d}X_t = \sigma(t, X_t)dB_t, \quad X_0 \in (0,+\infty), \tag{5.1}$$

where B is a standard one-dimensional Brownian motion and σ is a measurable function on $[0,+\infty)\times(0,+\infty)$ to $(0,+\infty)$. Note that our result could of course also apply to time-inhomogeneous diffusions with drift using the usual trick of change of spatial scale (here, a time-dependent change of scale solution to a parabolic problem). We assume that

$$\sigma_*(x) \leq \sigma(t,x) \leq \sigma^*(x), \tag{5.2}$$

for some measurable functions σ^* and σ_* from $(0,+\infty)$ to $[0,+\infty]$ satisfying

$$\int_{(0,+\infty)} \frac{x\,\mathrm{d}x}{\sigma_*(x)^2} < \infty \quad \text{and} \quad \int_{(a,b)} \frac{\mathrm{d}x}{\sigma^*(x)^2} > 0,\ \forall 0 < a < b < \infty.$$

Note that the former condition means that the time-homogeneous diffusion $dY_t = \sigma_*(Y_t)dB_t$ on $(0,\infty)$ stopped when it hits 0 at time T_0^Y admits $+\infty$ as entrance boundary (*i.e.* Y comes down from infinity, as defined in [3]) and that $T_0^Y < \infty$ almost surely (see *e.g.* [15]). As will appear in the proof of our result below, condition (5.2) means that the process X is a random time change of the process Y such that X is absorbed faster than Y, and that the diffusion $dZ_t = \sigma_*(Z_t)dB_t$ is absorbed faster than X. The next result shows that this (apparently non-related) property implies exponential contraction of the conditional distributions of X, provided that the time-homogeneous diffusion process Y satisfies, for some constants $t_1 > 0$ and $A > 0$,

$$\mathbb{P}_y(t_1 < T_0^Y) \leq Ay, \ \forall y > 0. \tag{5.3}$$

Up to a linear transformation of time (*i.e.* multiplying $\sigma(t,x)$ by some positive constant), we can—and will—assume without loss of generality that $t_1 < 1$. Explicit conditions on σ_* ensuring the last assumption are given in [4] (Thms 4.3 and 4.6). For instance, these conditions are fulfilled if $\sigma_*(x) \geq Cx \log^{\frac{1+\varepsilon}{2}} \frac{1}{x}$ for some constants $C > 0$ and $\varepsilon > 0$ in a neighborhood of 0. Note that this condition is not very restrictive since, if $\varepsilon = 0$, the condition $\int_{0+} \frac{x\,\mathrm{d}x}{\sigma_*(x)^2} < \infty$ might not be satisfied in which case the diffusion Y would not hit 0 in finite time.

Theorem 5.1. *Under the above assumptions,*

$$\inf_{s\geq 1} d'_s > 0.$$

In particular, we obtain exponential convergence in (2.8), (2.9)*. Moreover, the assumptions of Proposition 3.1 and Theorem 3.4 are satisfied.*

Proof of Theorem 5.1. The proof follows the same steps (as in [4], Sect. 5.1), making use of the next lemma.

Lemma 5.2. *There exist constants $t_1 \in]0,1[$ and $A > 0$ such that, for all $s \geq 0$ and $x > 0$,*

$$\mathbb{P}_{s,x}(s + t_1 < T_0^X) \leq Ax \quad \textit{and} \quad \inf_{s\geq 0} \mathbb{P}_{s,x}(s + t < T_0^X) > 0, \ \forall t \geq 0. \tag{5.4}$$

Moreover, for all $t_2 > 0$,

$$\inf_{s\geq 0, x>0} \mathbb{P}_{s,x}(T_0^X < s + t_2) > 0, \tag{5.5}$$

for all $\rho > 0$, there exists $b_\rho > 0$ such that

$$\sup_{s\geq 0, x\geq b_\rho} \mathbb{E}_{s,x}(e^{\rho(T_{b_\rho} - s)}) < +\infty, \tag{5.6}$$

for all $a > 0$ and $t \geq 0$,

$$\inf_{s\geq 0} \mathbb{P}_{s,a}(t + s < T_{a/2}^X) > 0 \tag{5.7}$$

and for all $a, b > 0$, there exists $t_{a,b} > 0$ such that for all $t \geq t_{a,b}$,

$$\inf_{s\geq 0} \mathbb{P}_{s,a}(X_{s+t} \geq b) > 0. \tag{5.8}$$

We admit for the moment this result and extend the main steps of ([4], Sect. 5.1) to our new setting.

Step 1: *The conditioned process escapes a neighborhood of 0 in finite time.*
The goal of this step is to prove that there exists $\varepsilon, c > 0$ such that

$$\mathbb{P}_{s,x}(X_{s+t_1} \geq \varepsilon \mid s + t_1 < T_0^X) \geq c, \quad \forall s \geq 0, \ x > 0. \tag{5.9}$$

To prove this, we first observe that, since X is a local martingale and since $X_{(s+t_1)\wedge T_1^X}=0$ on the event $T_0^X\leq (s+t_1)\wedge T_1^X$, for all $x\in(0,1)$,

$$\begin{aligned} x &= \mathbb{E}_{s,x}(X_{(s+t_1)\wedge T_1^X}) = \mathbb{E}_{s,x}\left(X_{(s+t_1)\wedge T_1^X}\mathbb{1}_{(s+t_1)\wedge T_1^X<T_0^X}\right)\\ &= \mathbb{P}_{s,x}(s+t_1<T_0^X)\mathbb{E}_{s,x}(X_{(s+t_1)\wedge T_1^X}\mid s+t_1<T_0^X)+\mathbb{P}_{s,x}(T_1^X<T_0^X\leq s+t_1). \end{aligned}$$

By the Markov property,

$$\begin{aligned} \mathbb{P}_{s,x}(T_1^X<T_0^X\leq s+t_1) &\leq \mathbb{E}_{s,x}\left[\mathbb{1}_{T_1^X<T_0^X\wedge(s+t_1)}\mathbb{P}_{T_1^X,1}(T_0^X\leq s+t_1)\right]\\ &\leq \mathbb{P}_{s,x}(T_1^X<T_0^X)\sup_{u\in[s,s+t_1]}\mathbb{P}_{u,1}(T_0^X\leq s+t_1)\\ &\leq \mathbb{P}_{s,x}(T_1^X<T_0^X)\sup_{u\in[s,s+t_1]}\mathbb{P}_{u,1}(T_0^X\leq u+t_1)\\ &= x\sup_{u\in[s,s+t_1]}\mathbb{P}_{u,1}(T_0^X\leq u+t_1). \end{aligned}$$

The second part of equation (5.4) of Lemma 5.2 entails that $\sup_{u\geq 0}\mathbb{P}_{u,1}(T_0^X\leq u+t_1)<1$ and therefore, using the first part of equation (5.4) of Lemma 5.2,

$$\mathbb{E}_{s,x}\left(1-X_{(s+t_1)\wedge T_1^X}\mid s+t_1<T_0^X\right)\leq 1-\frac{1}{A'},$$

with $A'=A/(1-\sup_{u\geq 0}\mathbb{P}_{u,1}(T_0^X\leq u+t_1))$. Markov's inequality then implies that, for all $x\in(0,1)$,

$$\mathbb{P}_{s,x}\left(X_{(s+t_1)\wedge T_1^X}\leq\frac{1}{2A'-1}\,\middle|\, s+t_1<T_0^X\right)\leq\frac{1-1/A'}{1-1/(2A'-1)}=1-\frac{1}{2A'}. \tag{5.10}$$

Set $\varepsilon:=1/(2(2A'-1))$ and assume, without loss of generality, that A' is big enough so that $2\varepsilon\in(0,1)$. Applying the second part of (5.4) to the diffusion $dZ_t=\sigma_*(t,Z_t+\varepsilon)$ (which satisfies the above assumptions since $\int_0^\infty\frac{x\,\mathrm{d}x}{\sigma_*(\varepsilon+x)^2}\leq\int_\varepsilon^\infty\frac{x\,\mathrm{d}x}{\sigma_*(x)^2}<\infty$), we have

$$\inf_{t\geq 0}\mathbb{P}_{t,2\varepsilon}\left(t+t_1<T_\varepsilon^X\right)>0.$$

Hence, for all $x\in(0,2\varepsilon)$,

$$\begin{aligned} \mathbb{P}_{s,x}(X_{s+t_1}\geq\varepsilon) &\geq \mathbb{P}_{s,x}\left(T_{2\varepsilon}^X<s+t_1\right)\inf_{t\geq 0}\mathbb{P}_{t,2\varepsilon}\left(t+t_1<T_\varepsilon^X\right)\\ &\geq \mathbb{P}_{s,x}\left(X_{(s+t_1)\wedge T_1^X}\geq 2\varepsilon\right)\inf_{t\geq 0}\mathbb{P}_{t,2\varepsilon}\left(t+t_1<T_\varepsilon\right)\\ &\geq \frac{\mathbb{P}_{s,x}\left(s+t_1<T_0^X\right)}{2A'}\inf_{t\geq 0}\mathbb{P}_{t,2\varepsilon}\left(t+t_1<T_\varepsilon\right) \end{aligned}$$

by (5.10). This ends the proof of (5.9) for $x<2\varepsilon$. For $x\geq 2\varepsilon$, standard coupling arguments entail

$$\mathbb{P}_{s,x}(X_{t_1}>\varepsilon\mid t_1<\tau_\partial)\geq\mathbb{P}_{s,x}(X_{t_1}>\varepsilon)\geq\mathbb{P}_{s,x}(t_1<T_\varepsilon^X)\geq\mathbb{P}_{s,2\varepsilon}(T_\varepsilon^X>t_1)>0.$$

Hence (5.9) is proved.

Step 2: *Construction of coupling measures for the unconditioned process.*
Set $t_2 = 1 - t_1 > 0$. Our goal is to prove that there exists a constant $c_1 > 0$ such that, for all $s \geq 0$ and $x \geq \varepsilon$,

$$\mathbb{P}_{s,x}(X_{s+t_2} \in \cdot) \geq c_1 \pi_s(\cdot), \tag{5.11}$$

where

$$\pi_s(\cdot) = \mathbb{P}_{s,\varepsilon}(X_{s+t_2} \in \cdot \mid s + t_2 < T_0^X).$$

Fix $s \geq 0$ and $x \geq \varepsilon$ and construct two independent diffusions $X^{s,\varepsilon}$ and $X^{s,x}$ solution to (5.1) with initial values at time s given by ε and x respectively. Let $\theta = \inf\{t \geq s : X_t^{s,\varepsilon} = X_t^{s,x}\}$. By the strong Markov property, the process

$$Y_t^{s,x} = \begin{cases} X_t^{s,x} & \text{if } t \in [s, \theta], \\ X_t^{s,\varepsilon} & \text{if } t > \theta \end{cases}$$

has the same law as $X^{s,x}$. Since $\theta \leq T_0^{s,x} := \inf\{t \geq s : X_t^{s,x} = 0\}$, for all $t > s$, $\mathbb{P}(\theta < t) \geq \mathbb{P}(T_0^{s,x} < t)$. Using equation (5.5) of Lemma 5.2, we have

$$c_1' := \inf_{s \geq 0, y > 0} \mathbb{P}_{s,y}(T_0^{s,x} < s + t_2) > 0.$$

Hence

$$\mathbb{P}_{s,x}(X_{s+t_2} \in \cdot) = \mathbb{P}(Y_{s+t_2}^{s,x} \in \cdot) \geq \mathbb{P}(X_{s+t_2}^{s,\varepsilon} \in \cdot,\ T_0^{s,x} < s + t_2) \geq c_1' \mathbb{P}_{s,\varepsilon}(X_{s+t_2} \in \cdot).$$

Therefore, (5.11) is proved with $c_1 = c_1' \inf_{s \geq 0} \mathbb{P}_{s,\varepsilon}(s + t_2 < T_0^X)$, which is positive by (5.4) of Lemma 5.2.

Step 3: *Proof that* $\nu_s \geq c_1 c \pi_{s-1+t_1}$.
Recall that $t_1 + t_2 = 1$. Using successively the Markov property, Step 2 and Step 1, we have for all $s \geq 1$ and $x > 0$

$$\begin{aligned}
\mathbb{P}_{s-1,x}(X_{s-1+t_1+t_2} \in \cdot \mid s - 1 + t_1 + t_2 < T_0^X) &\geq \mathbb{P}_{s-1,x}(X_s \in \cdot \mid s - 1 + t_1 < T_0^X) \\
&\geq \int_\varepsilon^\infty \mathbb{P}_{s-1+t_1,y}(X_s \in \cdot) \mathbb{P}_{s-1,x}(X_{s-1+t_1} \in \mathrm{d}y \mid s - 1 + t_1 < T_0^X) \\
&\geq c_1 \int_\varepsilon^\infty \pi_{s-1+t_1}(\cdot) \mathbb{P}_{s-1,x}(X_{s-1+t_1} \in \mathrm{d}y \mid s - 1 + t_1 < T_0^X) \\
&= c_1 \pi_{s-1+t_1}(\cdot) \mathbb{P}_{s-1,x}(X_{s-1+t_1} \geq \varepsilon \mid s - 1 + t_1 < T_0^X) \geq c_1 c \pi_{s-1+t_1}(\cdot).
\end{aligned}$$

This entails $\nu_s \geq c_1 c \pi_{s-1+t_1}$, where ν_s is defined in (2.5).

Step 4: *Proof that* $\inf_{s \geq 1} d_s' > 0$.
We set $a = \varepsilon/2$. Using the definition of π_s , we have

$$\begin{aligned}
\pi_s([a, +\infty[) &\geq \mathbb{P}_{s,2a}(T_a^X \geq s + t_2 \mid s + t_2 < T_0^X) \\
&\geq \mathbb{P}_{s,2a}(T_a^X \geq s + t_2).
\end{aligned}$$

Inequality (5.7) allows us to conclude that $\inf_{s \geq 1} \nu_s([a, +\infty)) > 0$.

We also deduce from (5.8) that, setting $t_3 = t_{a,a}$, there exists $\rho > 0$ such that

$$\inf_{s\geq 0} \mathbb{P}_{s,a}(X_{s+t_3} \geq a) \geq e^{-\rho t_3}.$$

From (5.6), one can choose $b > a$ large enough so that

$$A := \sup_{s\geq 0, x\geq b} \mathbb{E}_{s,x}\left(e^{\rho(T_b^X - s)}\right) < \infty.$$

Then, defining $T_{[0,b]}^X$ as the first hitting time of $[0, b]$ by the process X and by θ_t the shift operator of time t, Markov's property entails

$$\sup_{s\geq 0, x\geq b} \mathbb{E}_{s,x}\left(e^{\rho(T_{[0,b]}^X\circ\theta_t - s - t)}\right) \leq A, \tag{5.12}$$

where, under $\mathbb{P}_{s,x}$, $T_{[0,b]}^X \circ \theta_t$ is the first hitting time of $[0, b]$ after time $s + t$ by the process X. Note that, in particular, $T_{[0,b]}^X \circ \theta_t = s + t$ if $T_0^X \leq s + t$.

Then, setting $t_4 = t_{a,b}$, for all $u \geq s + t_4$, defining k as the unique integer such that $s + kt_3 + t_4 \leq u < s + (k + 1)t_3 + t_4$, we have by Markov's property

$$\begin{aligned}
\mathbb{P}_{s,a}(X_u \geq b) &\geq \mathbb{P}_{s,a}(X_{s+t_3} \geq a,\ X_{s+2t_3} \geq a, \dots, X_{s+kt_3} \geq a,\ X_u \geq b)\\
&\geq e^{-\rho k t_3} \inf_{v\geq 0} \mathbb{P}_{v,a}(X_{v+u-s-kt_3} \geq b)\\
&\geq c e^{-\rho(u-s)}
\end{aligned}$$

where $c > 0$ by (5.8). Therefore, for all $t \geq u \geq s + t_4$, making use of the monotonicity of $x \mapsto \mathbb{P}_{s,x}(t < T_0^X)$,

$$c e^{-\rho(u-s)} \mathbb{P}_{u,b}(t < T_0^X) \leq \mathbb{P}_{s,a}(X_u \geq b)\mathbb{P}_{u,b}(t < T_0^X) \leq \mathbb{P}_{s,a}(t < T_0^X). \tag{5.13}$$

Then, for all $x \geq b$ and all $t \geq s + t_4$, using successively the strong Markov property, equation (5.12) with $t = t_4$, (5.13) with $u = t$, (5.13) with $u \geq s + t_4$, and (5.12) again,

$$\begin{aligned}
\mathbb{P}_{s,x}(t < T_0^X) &\leq \mathbb{P}_{s,x}(t < T_{[0,b]}^X \circ \theta_{t_4}) + \int_{s+t_4}^t \sup_{y\in[0,b]} \mathbb{P}_{u,y}(t < T_0^X)\, \mathbb{P}_{s,x}(T_{[0,b]}^X \circ \theta_{t_4} \in du)\\
&\leq A e^{-\rho(t-s-t_4)} + \int_{s+t_4}^t \mathbb{P}_{u,b}(t < T_0^X)\, \mathbb{P}_{s,x}(T_{[0,b]}^X \circ \theta_{t_4} \in du)\\
&\leq c^{-1} A e^{\rho t_4} \mathbb{P}_{s,a}(t < T_0^X) + c^{-1}\mathbb{P}_{s,a}(t < T_0^X) \int_{s+t_4}^t e^{\rho(u-s)}\, \mathbb{P}_x(T_{[0,b]}^X \circ \theta_{t_4} \in du)\\
&\leq 2c^{-1} A e^{\rho t_4} \mathbb{P}_{s,a}(t < T_0^X).
\end{aligned}$$

In the case where $t \in [s, s + t_4]$,

$$\mathbb{P}_{s,x}(t < T_0^X) \leq 1 \leq \frac{\mathbb{P}_{s,a}(s + t_4 < T_0^X)}{\inf_{s\geq 0} \mathbb{P}_{s,a}(s + t_4 < T_0^X)} \leq \frac{\mathbb{P}_{s,a}(t < T_0^X)}{\inf_{s\geq 0} \mathbb{P}_{s,a}(s + t_4 < T_0^X)}.$$

We deduce from inequality (5.4) of Lemma 5.2 that there exists a constant $C > 0$ such that, for all $s \geq 0$ and $t \geq s$,

$$\sup_{x>0} \mathbb{P}_{s,x}(t < T_0^X) = \sup_{x \geq b} \mathbb{P}_{s,x}(t < T_0^X) \leq C\mathbb{P}_{s,a}(t < T_0^X).$$

Since $\inf_{x \geq a} \mathbb{P}_{s,x}(t < T_0^X) = \mathbb{P}_{s,a}(t < T_0^X)$ and $\inf_{s \geq 1} \nu_s([a, +\infty)) > 0$, we obtain

$$\inf_{s \geq 1} d'_s > 0.$$

This concludes the proof of Theorem 5.1. □

Proof of Lemma 5.2. We assume in the whole proof that $s = 0$ and $X_0 = x$. Since the statements of Lemma 5.2 are obtained from comparisons with time-homogeneous diffusions, the result will follow from the study of the case $s = 0$ only. For all $t \geq 0$, let

$$b(s) = \int_0^s \sigma^2(u, X_u)\,\mathrm{d}u.$$

Note that b is continuous and increasing. The equality $X_t = W_{b(t)}$ for all $t < T_0^X$ defines a Brownian motion W started at $W_0 = x$ and stopped at its first hitting time of 0 denoted by $T_0^W = b(T_0^X)$. This is a classical consequence of Levy's characterization of the Brownian motion, see for instance [18]. Note that, since a one dimensional Brownian motion hits 0 in finite time almost surely, there exists $t \geq 0$ such that $W_{b(t)} = 0$ and hence $b(T_0^X) < \infty$ almost surely.

Let Y be the time-homogeneous diffusion process stopped at 0 defined as $Y_t = W_{b_*(t)}$, where

$$b_*(t) = \inf\{s \geq 0,\ a_*(s) \geq t\}, \qquad \text{with } a_*(s) = \int_0^s \frac{du}{\sigma_*(W_u)^2}.$$

And similarly for $Z_t = W_{b^*(t)}$, replacing σ_* by σ^*. In particular, $Y_{a_*(t)} = W_t$, $Z_{a^*(t)} = W_t$, and hence $T_0^Y = a_*(T_0^W)$ and $T_0^Z = a^*(T_0^W)$. Note that Y and Z are solutions of the time-homogeneous SDEs

$$dY_t = \sigma_*(Y_t)dB_t^Y \quad \text{and} \quad dZ_t = \sigma^*(Z_t)dB_t^Z, \quad \text{with } Y_0 = Z_0 = x,$$

for some Brownian motions B^Y and B^Z with $Y_0 = Z_0 = x$. The interest of this construction is that the processes Y and Z are both obtained from a random time change of X: Y is obtained by a slowing down of X, and Z by a speeding up of X. In particular, it is easy to check that

$$a_*(b(t)) \geq t \quad \text{and} \quad a^*(b(t)) \leq t,\ \forall t \geq 0.$$

Hence $T_0^Z \leq T_0^X \leq T_0^Y$ almost surely. Therefore, the first inequality of (5.4) follows from the same property for Y, as assumed in (5.3). Similarly, the second inequality in (5.4) and (5.5) follow from the same property for Z and Y, respectively, which are standard properties of time-homogeneous diffusion processes (see for instance [15]).

Using the previous argument, we also deduce that, $T_a^Z \leq T_a^X \leq T_a^Y$ almost surely for all $a \leq x$. Hence (5.6) follows from the same property for Y, which is classical because infinity is an entrance boundary for Y (see for instance [3, 4]). Inequality (5.7) also follows from the same comparison of hitting times and standard regularity properties of the time-homogeneous diffusion Z.

Finally, if $b \leq a/2$ (5.8) follows directly from (5.7), and if $b > a/2$, we use the comparison with Y and the fact that $\mathbb{P}_a(T_{2b}^Y < t_0) > 0$ for some $t_0 > 0$ to see that X hits $2b$ before time t_0 with probability under $\mathbb{P}_{s,a}$ uniformly bounded from below with respect to $s \geq 0$. Next we use the comparison with Z (as we did to prove the second inequality in (5.4)) to see that, under $\mathbb{P}_{s,2b}$, for any $t \geq 0$, there is a uniformly (with respect to s) positive probability that X does not hit $a < 2b$ before time t. Combining these two facts entails (5.8) with $t_{a,b} = t_0$. □

6. Penalized time-inhomogeneous birth and death processes

In the last section, we gave an example where the penalization rate is unbounded, using an assumption of uniform domination by a process coming down from infinity. Our goal in this section is to provide examples of applications to inhomogeneous Markov processes alternating periods of uniform domination and periods without any domination. In particular, the penalization rate is bounded as in (4.1), and we shall assume uniform irreducibility as in (4.7), but no uniform exponential moments for the return times in compact sets as in (4.8) are required.

This situation is for example natural for a birth and death process in random environment, where the environment alternates periods favorable to growth and periods where the population has a tendency to decrease. The quasi-stationary behavior of such a population can be studied in two different ways: the convergence of the distribution of the population conditional on non-extinction 1) when expectations are taken with respect to the law of the environment and of the birth and death process (so-called *annealed* quasi-stationary behavior), and 2) when expectations are taken only with respect to the law of the birth and death process, for any fixed realization of the environment (so-called *quenched* quasi-stationary behavior). In the case of time-homogeneous Markov environment dynamics, the joint dynamics of environment and population is time-homogeneous and hence enters the scope of our general results for homogeneous Markov processes of [5]. The case of quenched quasi-stationary behavior is more delicate since typical realizations of the environment will include periods of growth and periods of decay of the populaton of arbitrary lengths. In particular, this requires more stringent irreducibility assumptions (see (6.1) and (6.2) below) than what one would expect in the annealed case.

6.1. General result

Let $(X_t)_{t\in\mathbb{R}_+}$ be a time inhomogeneous birth and death process reflected at 1, with measurable birth rates $b_i(t) > 0$ and death rates $d_i(t) \geq 0$ at time $t \geq 0$ from state $i \geq 1$, such that $d_1(t) = 0$ for all $t \geq 0$ and $d_i(t) > 0$ for $i \geq 2$. We also consider the penalization defined by

$$Z_{s,t} = e^{\int_s^t \kappa(u,X_u)du},$$

where $\kappa : \mathbb{R}_+ \times \{1, 2, \dots\} \to \mathbb{R}$ is a bounded measurable function. Note that the study of the distribution of a birth and death process Y on $\mathbb{Z}_+$ absorbed at 0 (with the same coefficients except $d_1(t) > 0$) and conditioned not to hit 0 (*i.e.* penalized by $\mathbb{1}_{Y_t\neq 0}$) enters this setting since

$$\mathbb{E}_{x,s}(f(Y_t) \mid Y_t \neq 0) = \frac{\mathbb{E}_{x,s}\left(f(X_t)e^{-\int_s^t d_1(u)\mathbb{1}_{X_u=1}du}\right)}{\mathbb{E}_{x,s}\left(e^{-\int_s^t d_1(u)\mathbb{1}_{X_u=1}du}\right)}.$$

Similarly, the case of birth and death processes with catastrophe (*i.e.* with killing) occurring at bounded rate depending on the position of the process (see Sect. 4.1, [5]) also enters this setting.

We will need irreducibility and stability assumptions:

$$\gamma_F := \inf_{s\geq 0,\ x,y\in F} \mathbb{P}_{s,x}(X_{s+1} = y) > 0, \quad \text{for all finite } F \subset \mathbb{N} \tag{6.1}$$

and

$$\rho_x := \inf_{s\geq 0,\ u\in[s,s+1]} \mathbb{P}_{s,x}(X_u = x) > 0, \quad \forall x \in \mathbb{N}. \tag{6.2}$$

These two conditions are satisfied for example if, for each $n \in \mathbb{N}$, the functions $b_n(t)$ and $d_n(t)$ are uniformly bounded and bounded away from 0.

Theorem 6.1. *Assume that (6.1) and (6.2) hold true and that, for some $\lambda > \|\kappa\|_\infty + \log(\gamma_{\{1\}}^{-1})$, there exists a finite $F \subset \mathbb{N}$ and an unbounded $\mathcal{T} \subset \mathbb{R}_+$ such that*

$$A := \sup_{t\in\mathcal{T}} \sup_{x\in\mathbb{N}} \mathbb{E}_{t,x}\left(e^{\lambda(T_F^X - t)}\right) < \infty, \tag{6.3}$$

where T_F^X is the first hitting time of the set F by X. We also assume that there exists $b \geq 2$ such that the set

$$\mathcal{T}_b := \left\{ s_1 \in \mathcal{T},\ \exists s_2 \in \mathcal{T} \text{ s.t. } \quad t_0 + 2 \leq s_2 - s_1 \leq t_0 + b \right\} \tag{6.4}$$

is unbounded, where

$$t_0 = \left\lceil \frac{\log A}{\log(\gamma_{\{1\}}^{-1})} \right\rceil. \tag{6.5}$$

Then, there exist $\gamma > 0$ such that, for all probability measures μ_1, μ_2 on $\mathbb{N}$ and for all $s \in \mathbb{N}$ and $t \geq s$,

$$\left\|\mu_1 K_{s,t}^T - \mu_2 K_{s,t}^T\right\|_{TV} \leq \exp\left(-\gamma\ N_{b,s,t}\right)\ \|\mu_1 - \mu_2\|_{TV}$$

and

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV} \leq 2\exp\left(-\gamma\ N_{b,s,t}\right),$$

where $N_{b,s,t} := \operatorname{Card}\left\{k \in \mathbb{N} \cap [s, t - t_0 - 2] : \mathcal{T}_b \cap [k, k+1) \neq \emptyset\right\}$. Moreover, the conclusions of Proposition 3.1 and Theorem 3.4 are satisfied, except for (3.1) and (3.8), which have to be replaced respectively by

$$\left| \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})} - \frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})} \right| \leq C_{s,y} \exp\left(-\gamma\ N_{b,s,t}\right), \quad \forall x, y \in E,\ \forall s \leq t \leq u,$$

for some constant $C_{s,y}$ only depending on s and y, and

$$\|\mathbb{Q}_{s,x}(X_t \in \cdot) - \mathbb{Q}_{s,y}(X_t \in \cdot)\|_{TV} \leq 2\exp\left(-\gamma\ N_{b,s,t}\right), \quad \forall x, y \in \mathbb{N}.$$

Since $\mathcal{T}_b$ is unbounded, we obtain in particular convergence in total variation in Theorem 6.1. Moreover, the exponential speed of convergence is governed by the asymptotic density of the set $\mathcal{T}_b$. In subsection 6.2, we apply Theorem 6.1 to the case of a birth and death process evolving in a quenched random environment.

Proof. We first notice that replacing κ by $\kappa - \|\kappa\|_\infty$ does not change the operators Φ and K in (2.2) and (2.7), and hence the measures ν_s and the constants d'_s are not modified. Therefore, we can assume without loss of generality that κ is non-positive. As observed before Theorem 2.1, the penalized process can then be interpreted as a time-inhomogeneous birth and death process Y with killing. More precisely, let Y be the time inhomogeneous birth and death process on $\mathbb{Z}_+$ with birth and death rates $b_n(t)$ and $d_n(t)$ at time t from state $n \geq 1$, with additional jump rate $-\kappa(t,n)$ at time t from $n \geq 1$ to 0, which is assumed to be an absorbing point. Then

$$\Phi_{s,t}(\mu)(f) = \mathbb{E}_{\mu,s}(f(Y_t) \mid Y_t \neq 0).$$

The process Y can be constructed from the paths of X with an additional killing rate, in which case $T_F \wedge T_0 \leq T_F^X$, where T_F is the first hitting time of the set F by Y, and $T_0 = T_{\{0\}}$. Therefore, assumption (6.3) implies that, for some constant $A < \infty$, for all $s \in \mathcal{T}$,

$$\sup_{x\in\mathbb{N}} \mathbb{E}_{s,x}\left(e^{\lambda(T_F\wedge T_0 - s)}\right) \leq A. \tag{6.6}$$

Step 1: Preliminary computations.

Let $s < s+1 \leq t$ and $u \in [s+1,t]$. For all $x \in F$, by Markov's property, (6.1) and (6.2),

$$\begin{aligned}\left(e^{-\|\kappa\|_\infty}\gamma_{\{1\}}\right)^{\lfloor u-s\rfloor -1} & e^{-\|\kappa\|_\infty}\rho_1\, e^{-\|\kappa\|_\infty}\gamma_F\, \mathbb{P}_{u,x}(t<T_0) \\ &\leq \mathbb{P}_{s,1}\left(Y_s = Y_{s+1} = \cdots = Y_{s+\lfloor u-s\rfloor-1} = Y_{u-1} = 1\right) \mathbb{P}_{u-1,1}(Y_u = x)\, \mathbb{P}_{u,x}(t<T_0) \\ &\leq \mathbb{P}_{s,1}(t<T_0).\end{aligned}$$

Thus, for $C = e^{\|\kappa\|_\infty}\gamma_{\{1\}}/(\rho_1\gamma_F)$ and for all $u \in [s+1,t]$,

$$e^{-\lambda(u-s)} \sup_{x\in F} \mathbb{P}_{u,x}(t<T_0) \leq C\mathbb{P}_{s,1}(t<T_0). \tag{6.7}$$

Now, for $u \in [s,s+1]$, by (6.2),

$$e^{-\|\kappa\|_\infty}\rho_x\, \mathbb{P}_{u,x}(t<T_0) \leq \mathbb{P}_{s,x}(t<T_0),$$

and hence, increasing C if necessary, we obtain that for all $u \in [s,t]$,

$$e^{-\lambda(u-s)} \sup_{x\in F} \mathbb{P}_{u,x}(t<T_0) \leq C \sup_{x\in F} \mathbb{P}_{s,x}(t<T_0). \tag{6.8}$$

Step 2: Dobrushin coefficient.

For this step and the next one, we fix $s_1 \in \mathcal{T}_b$ and let $s_2 \in \mathcal{T}$ such that $t_0 + 2 \leq s_2 - s_1 \leq t_0 + b$. Using (6.6), for all $t \geq s_1$ and $x \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}_{s_1,x}(T_F < t) = \mathbb{P}_{s_1,x}(T_F < t\wedge T_0) &\geq \mathbb{P}_{s_1,x}(t<T_0) - \mathbb{P}_{s_1,x}(t<T_F\wedge T_0) \\ &\geq e^{-\|\kappa\|_\infty(t-s_1)} - Ae^{-\lambda(t-s_1)}.\end{aligned}$$

Hence, it follows from the definition of t_0 in (6.5) that there exists a constant $c_0 > 0$ such that $\mathbb{P}_{s_1,x}(T_F < t) \geq c_0 > 0$ for all $t \geq s_1 + t_0$.

By assumption (6.1), $\inf_{s\geq 0,\ y\in F} \mathbb{P}_{s,y}(Y_{s+1} = 1) \geq \gamma_F > 0$, thus the Markov property entails

$$\mathbb{P}_{s_1,x}(Y_{s_1+t_0+1} = 1) \geq \mathbb{E}_{s_1,x}\left[\mathbb{1}_{T_F < s_1+t_0} \inf_{u\geq 0,\ y\in F} \mathbb{P}_{u,y}(Y_{u+1}=1)\rho_1^{\lceil t_0\rceil} e^{-\|\kappa\|_\infty t_0}\right] \geq c_1,$$

where the constant c_1 does not depend on $s_1 \in \mathcal{T}_b$ and $x \in \mathbb{N}$. Since for all $x \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}_+$,

$$\Phi_{s_1,s_1+t_0+1}(\delta_x)(f) \geq \mathbb{E}_{s_1,x}[f(Y_{s_1+t_0+1})\mathbb{1}_{s_1+t_0+1<T_0}] \geq f(1)\mathbb{P}_{s_1,x}(Y_{s_1+t_0+1} = 1),$$

we deduce that

$$\nu_{s_1,s_1+t_0+1} := \min_{x\in\mathbb{N}} \Phi_{s_1,s_1+t_0+1}(\delta_x) \geq c_1\delta_1.$$

Step 3: Comparison of survival probabilities.

Given any $s \in \mathcal{T}$, using (6.6), Markov's property and inequality (6.8) twice (first with $u = t$ and second for all $u \in [s,t]$), we have for all $t \geq s$ and $x \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{P}_{s,x}(t < T_0) &\leq \mathbb{P}_{s,x}\left(t < T_F \wedge T_0\right) + \mathbb{P}_x\left(T_F \wedge T_0 \leq t < T_0\right) \\
&\leq Ae^{-\lambda(t-s)} + \int_s^t \sup_{y\in F\cup\{0\}} \mathbb{P}_{u,y}(t < T_0)\mathbb{P}_{s,x}(T_F \wedge T_0 \in du) \\
&\leq AC \sup_{y\in F} \mathbb{P}_{s,y}(t < T_0) + C \sup_{y\in F} \mathbb{P}_{s,y}(t < T_0) \int_s^t e^{\lambda(u-s)}\, \mathbb{P}_{s,x}(T_F \wedge T_0 \in du) \\
&\leq 2AC \sup_{y\in F} \mathbb{P}_{s,y}(t < T_0).
\end{aligned} \tag{6.9}$$

Recall that we fixed $s_1 \in \mathcal{T}_b$ and $s_2 \in \mathcal{T}$ such that $t_0 + 2 \leq s_2 - s_1 \leq t_0 + b$. For all $x \in \mathbb{N}$, if $t \geq s_2$, (6.9) and (6.7) entail

$$\begin{aligned}
\mathbb{P}_{s_1+t_0+1,x}(t < T_0) &= \sum_{y\in\mathbb{N}} \mathbb{P}_{s_1+t_0+1,x}(Y_{s_2} = y)\mathbb{P}_{s_2,y}(t < T_0) \\
&\leq 2AC \sum_{y\in\mathbb{N}} \mathbb{P}_{s_1+t_0+1,x}(Y_{s_2} = y) \sup_{z\in F} \mathbb{P}_{s_2,z}(t < T_0) \\
&\leq 2AC \sup_{z\in F} \mathbb{P}_{s_2,z}(t < T_0) \\
&\leq 2AC^2\, e^{\lambda(s_2-(s_1+t_0+1))}\, \mathbb{P}_{s_1+t_0+1,1}(t < T_0) \\
&\leq 2AC^2\, e^{\lambda(b-1)}\, \mathbb{P}_{s_1+t_0+1,1}(t < T_0).
\end{aligned}$$

Since we assumed that the catastrophe rate $-\kappa$ is uniformly bounded, the last inequality extends to any $t \in [s_1 + t_0 + 1, s_2]$ (increasing the constant if necessary).

Step 4: Conclusion

Combining Steps 2 and 3, there exists $c' > 0$ such that, for all $s_1 \in \mathcal{T}_b$,

$$\begin{aligned}
d'_{s_1,s_1+t_0+1} &:= \inf_{t\geq s_1+t_0+1} \frac{\mathbb{P}_{s_1+t_0+1,\nu_{s_1,s_1+t_0+1}}(t < T_0)}{\sup_{x\in\mathbb{N}} \mathbb{P}_{s_1+t_0+1,x}(t < T_0)} \\
&\geq c_1 \inf_{t\geq s_1+t_0+1} \frac{\mathbb{P}_{s_1+t_0+1,1}(t < T_0)}{\sup_{x\in\mathbb{N}} \mathbb{P}_{s_1+t_0+1,x}(t < T_0)} \geq c'.
\end{aligned}$$

Theorem 2.1 and Remark 2.3 then imply that there exists $\gamma_0 > 0$ such that

$$\left\|\mu_1 K^T_{s,t} - \mu_2 K^T_{s,t}\right\|_{TV} \leq \exp\left(-\gamma_0\ C_{b,s,t}\right)\ \|\mu_1 - \mu_2\|_{TV}$$

and

$$\|\Phi_{s,t}(\mu_1) - \Phi_{s,t}(\mu_2)\|_{TV} \leq 2\exp\left(-\gamma_0\ C_{b,s,t}\right),$$

where

$$C_{b,s,t} := \sup\Big\{k \geq 1 : \exists s \leq t_1 < t_2 < \ldots < t_k \leq t - t_0 - 1,\ t_i \in \mathcal{T}_b,\ \forall i = 1, 2, \ldots, k,\\ t_{i+1} - t_i \geq t_0 + 1,\ \forall i = 1, 2, \ldots, k-1\Big\}.$$

Since $N_{b,s,t} \leq (t_0+1)C_{b,s,t}$, this concludes the proof of 6.1 with $\gamma = \gamma_0/(t_0+1)$. □

6.2. An example with alternating favorable and unfavorable periods in a quenched random environment

To illustrate how the assumptions of Theorem 6.1 can be checked in practice, we consider the case of alternating phases of favorable and unfavorable birth and death rates. By favorable, we mean a process which comes down fast from infinity (see assumption (6.10) below), a criterion which is known to be related to uniform convergence to quasi-stationary distributions for time-homogeneous birth and death processes [5, 16]. We study the problem of *quenched* stationary behavior of the birth and death process: we assume that the time length of the favorable and unfavorable periods are the realizations of a random environment and we study properties that hold almost surely with respect to the environment.

More precisely, we consider two sequences $(u_j, j \geq 0)$ and $(v_j, j \geq 0)$ of positive real numbers and a family of sequence of pairs of nonnegative real numbers $\{(b^j_n, d^j_n)_{n\geq 1}, j \geq 0\}$ such that, for all $j \geq 0$, $d^j_1 = 0$, $b^j_n > 0$ for all $n \geq 1$ and $d^j_n > 0$ for all $n \geq 2$. The sequence $(u_j, j \geq 0)$ (resp. $(v_j, j \geq 0)$) represents the lengths of successive unfavorable (resp. favorable) time intervals. Without loss of generality, we assume that the first phase is unfavorable. Therefore, if we set $s_0 = 0$

$$\sigma_j = s_j + u_j \quad \text{and} \quad s_{j+1} = \sigma_j + v_j, \quad \forall j \geq 0,$$

then the unfavorable time intervals are $[s_j, \sigma_j)$, $j \geq 0$ and the favorable time intervals are $[\sigma_j, s_{j+1})$, $j \geq 0$. During each favorable time interval, we assume that the birth and death rates satisfy

$$b_n(t) \leq b^j_n \quad \text{and} \quad d_n(t) \geq d^j_n, \quad \forall t \in [\sigma_j, s_{j+1}).$$

The fact that the process comes down from infinity during favorable time intervals is expressed in the following condition, assumed throughout this section:

$$\sup_{j\geq 0} S^j_n \xrightarrow[n\to+\infty]{} 0, \tag{6.10}$$

where

$$S^j_n := \sum_{m\geq n} \frac{1}{d^j_m \alpha^j_m} \sum_{\ell \geq m} \alpha^j_\ell < \infty, \tag{6.11}$$

with $\alpha^j_\ell = \left(\prod_{i=1}^{\ell-1} b^j_i\right) / \left(\prod_{i=1}^{\ell} d^j_i\right)$. For example, easy computations allow to check that (6.10) is true if, for all $j \geq 0$, $d^j_n \geq a_1(n-1)^{1+\delta}$ and $b^j_n \leq a_2 n$ for some $a_1, \delta > 0$ and $a_2 < \infty$.

We recall that, if S_1^j is finite for some j, then the time-homogeneous birth and death process Y^j with birth rates b_i^j and death rates d_i^j from state i, comes down from infinity (see for instance [20]). In addition, the distribution of Y^j starting from ∞ is well-defined and, for all $n \geq 1$,

$$S_n^j = \mathbb{E}_\infty(T_n^j) = \sum_{\ell \geq n} \mathbb{E}_{\ell+1}(T_\ell^j), \tag{6.12}$$

where T_ℓ^j is the first hitting time of i by the process Y^j.

In particular, assumption (6.10) means that on each time interval $[\sigma_j, s_{j+1})$ with $j \geq 0$, the process X comes down from infinity. Note that we make no assumption on the unfavorable time intervals, except that the process is not explosive.

If we think of the time lengths u_j and v_j as modeling the influence of a random environment on the previous birth and death process, the next result shows that the conditions of Theorem 6.1 are almost surely true for quenched random environments under very general conditions.

Theorem 6.2. *Assume that the times (u_j, v_j) are drawn as i.i.d. realizations of a random couple (U, V), where U and V are positive and $\mathbb{E}(U) < \infty$. Then, for any $\lambda > 0$, there exists a finite $F \subset \mathbb{N}$ and an infinite $J \subset \mathbb{N}$ such that, for almost all realization of the random variables $(u_j, v_j)_{j \geq 0}$,*

$$A_\lambda := \sup_{j \in J} \sup_{x \in \mathbb{N}} \mathbb{E}_{\sigma_j, x}\left(e^{\lambda(T_F^X - \sigma_j)}\right) < \infty \tag{6.13}$$

and for all $t_0 > 0$, there exists $b \geq 2$ such that the set

$$J_b := \left\{ j \in J, \exists k \in J \text{ s.t. } t_0 + 2 \leq \sigma_k - \sigma_j \leq t_0 + b \right\} \tag{6.14}$$

is infinite. If in addition assumptions (6.1) and (6.2) are satisfied, then the conclusions of Theorem 6.1 hold true for almost all realization of the random variables $(u_j, v_j)_{j \geq 0}$.

Remark 6.3. Note that, since the random variables (u_j, v_j) are i.i.d. and because of the renewal argument of the proof of Lemma 6.5 below, one can check that the set J_b has a positive asymptotic density, in the sense that, for almost all realization of the random variables $(u_j, v_j)_{j \geq 0}$,

$$\liminf_{T \to +\infty} \frac{1}{T} \mathrm{Card}\{\sigma_j \leq T : j \in J_b\} > 0.$$

Therefore, under the assumptions of the last theorem, all the convergences in Theorem 6.1 are exponential. More precisely, $\exp(-\gamma\ N_{b,s,t})$ can be replaced everywhere in Theorem 6.1 by $C \exp(-\gamma'(t-s))$ for some constants $C, \gamma' > 0$ *a priori* dependent on the realization of $(u_j, v_j)_{j \geq 0}$.

We now come to the proof of Theorem 6.2. This result actually holds true under the following more general assumptions. We will divide the proof in two steps, first proving this more general result (Lem. 6.4) and second, checking that its assumptions are implied by those of Theorem 6.2 (Lem. 6.5).

Given fixed positive numbers $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$, we set for all $j \geq 0$ and $\lambda > 0$

$$C_{\lambda,j} = \sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{\ell=1}^{n} \left(\lambda u_{j+\ell} - \frac{\log v_{j+\ell-1}}{2} \right). \tag{6.15}$$

We will need the next two assumptions: there exists $\lambda > 0$ such that

$$\exists J_\lambda \subset \mathbb{N} \text{ infinite such that } (C_{\lambda,j}, j \in J_\lambda) \text{ is bounded} \tag{6.16}$$

and

$$\forall t_0 > 0, \quad \liminf_{j \in J_\lambda,\ j \to +\infty} \inf \{\sigma_k - \sigma_j :\ k \in J_\lambda,\ k > j,\ \sigma_k - \sigma_j > t_0\} < \infty. \tag{6.17}$$

Lemma 6.4. *Assume that there exists $\lambda > 0$ such that (6.16) is satisfied. Then there exists a finite $F \subset \mathbb{N}$ such that*

$$\sup_{j \in J_\lambda} \sup_{x \in \mathbb{N}} \mathbb{E}_{\sigma_j, x} \left(e^{\lambda (T_F - \sigma_j)} \right) < \infty. \tag{6.18}$$

If in addition (6.17) is satisfied for the same $\lambda > 0$, then conditions (6.3) and (6.4) of Theorem 6.1 are true for this value of λ. In particular, if assumptions (6.1) and (6.2) are satisfied and $\lambda > \|\kappa\|_\infty + \log(\gamma_{\{1\}}^{-1})$, then the conclusions of Theorem 6.1 hold true.

The next lemma shows that the conditions of Lemma 6.4 are satisfied almost surely under the conditions of Theorem 6.2. In particular, Theorem 6.2 is a straightforward consequence of Lemmata 6.4 and 6.5.

Lemma 6.5. *Assume that the times (u_j, v_j) are drawn as i.i.d. realizations of a random variable (U, V), where U and V are positive and $\mathbb{E}(U) < \infty$. Then, for all $\lambda > 0$, (6.16) and (6.17) are satisfied.*

Remark 6.6. The conditions of Lemma 6.4 can be checked in different situations. For example, if for all $j \geq 0$, $v_j \geq \varepsilon$ for some $\varepsilon > 0$, then

$$C_{\lambda, j} \leq -\frac{\log \varepsilon}{2} + \lambda \sup_{n \geq 1} \frac{1}{n} \sum_{\ell=1}^{n} u_{j+\ell}.$$

As a consequence (6.16) holds true for any sequence $(u_j, j \geq 0)$ (not necessarily drawn as an independent sequence) such that

$$\liminf_{j \to +\infty} \sup_{n \geq 1} \frac{1}{n} \sum_{\ell=1}^{n} u_{j+\ell} < \infty.$$

Proof of Lemma 6.4. For all $s, t \geq 0$, we define

$$\alpha(s, t) = \sup_{x \in \mathbb{N}} \mathbb{E}_{s, x} \left(e^{\lambda (T_F - s) \wedge t} \right).$$

For all $j \geq 0$, we have

$$\alpha(s_j, t) \leq e^{\lambda u_j} \alpha(\sigma_j, t) \tag{6.19}$$

and, by Markov's property,

$$\begin{aligned} \alpha(\sigma_j, t) &\leq \sup_{x \in \mathbb{N}} \mathbb{E}_{\sigma_j, x} \left(e^{\lambda (T_F - \sigma_j) \wedge t} \mathbb{1}_{T_F \leq s_{j+1}} \right) + \sup_{x \in \mathbb{N}} \mathbb{E}_{\sigma_j, x} \left(e^{\lambda v_j} \mathbb{1}_{T_F > s_{j+1}} \right) \alpha(s_{j+1}, t) \\ &\leq \sup_{x \in \mathbb{N}} \mathbb{E}_x \left(e^{\lambda T_F^j} \right) + \sup_{x \in \mathbb{N}} \mathbb{E}_x \left(e^{\lambda T_F^j} \mathbb{1}_{T_F^j > v_j} \right) \alpha(s_{j+1}, t), \end{aligned}$$

where T^j_F is the first hitting time of the set F by the time homogeneous process Y^j defined above (6.12). Using Cauchy–Schwartz and Markov's inequalities,

$$\begin{aligned}\alpha(\sigma_j,t) &\leq \sup_{x\in\mathbb{N}}\ \mathbb{E}_x\left(e^{\lambda T^j_F}\right)+\sup_{x\in\mathbb{N}}\left(\mathbb{E}_x\left(e^{2\lambda T^j_F}\right)\mathbb{P}_x(T^j_F>v_j)\right)^{1/2}\alpha(s_{j+1},t)\\ &\leq \sup_{x\in\mathbb{N}}\ \mathbb{E}_x\left(e^{\lambda T^j_F}\right)+\sup_{x\in\mathbb{N}}\left(\mathbb{E}_x\left(e^{2\lambda T^j_F}\right)\mathbb{E}_x(T^j_F)\right)^{1/2}\frac{\alpha(s_{j+1},t)}{\sqrt{v_j}}\end{aligned}$$

It is standard (*cf. e.g.* [5]) to deduce from (6.10) that, given $\lambda>0$, there exists a finite $F_0=\{1,2,\dots,\max F_0\}\subset\mathbb{N}$ such that

$$A_{F_0}:=\sup_{j\geq 0,\ x\in\mathbb{N}}\mathbb{E}_x\left(e^{2\lambda T^j_{F_0}}\right)<\infty.$$

Since $\mathbb{E}_x\left(e^{2\lambda T^j_F}\right)$ is non-increasing in F, $A_F\leq A_{F_0}$ for all $F\supset F_0$. Given $F\supset F_0$ such that $F=\{1,2,\dots,\max F\}$, we deduce from (6.12) that

$$\alpha(\sigma_j,t)\leq A_{F_0}+\sqrt{A_{F_0}}\sup_{k\geq 0}(S^k_{\max F})^{1/2}\frac{\alpha(s_{j+1},t)}{\sqrt{v_j}}.$$

We set $\varepsilon=\exp(-C^*-1)$ with $C^*=\sup_{j\in J_\lambda}C_{\lambda,j}<\infty$. We then deduce from (6.10) that there exists a finite $F\subset\mathbb{N}$ such that

$$\alpha(\sigma_j,t)\leq A_{F_0}+\varepsilon\,\frac{\alpha(s_{j+1},t)}{\sqrt{v_j}}. \tag{6.20}$$

Combining (6.19) and (6.20), for all $j\geq 0$,

$$\alpha(\sigma_j,t)\leq A_{F_0}+\frac{\varepsilon}{\sqrt{v_j}}e^{\lambda u_{j+1}}\alpha(\sigma_{j+1},t).$$

A straightforward induction then implies that, for all $n\geq 0$,

$$\alpha(\sigma_j,t)\leq A_{F_0}\left[1+\sum_{k=1}^{n}e^{\lambda(u_{j+1}+\dots+u_{j+k})}\frac{\varepsilon^k}{\sqrt{v_j\dots v_{j+k-1}}}\right]+e^{\lambda(u_{j+1}+\dots+u_{j+n+1})}\frac{\varepsilon^{n+1}\ \alpha(\sigma_{j+n+1},t)}{\sqrt{v_j\dots v_{j+n}}},$$

and hence, since $\alpha(s,t)\leq e^{\lambda t}$ for all $s\geq 0$,

$$\begin{aligned}\alpha(\sigma_j,t)\leq &A_{F_0}\left[1+\sum_{k=1}^{+\infty}\varepsilon^k\ \exp\left(\sum_{\ell=1}^{k}\lambda u_{j+\ell}-\frac{\log v_{j+\ell-1}}{2}\right)\right]\\ &+\liminf_{n\to+\infty}\ \varepsilon^{n+1}\ \exp\left(\lambda t+\sum_{\ell=1}^{n}\lambda u_{j+\ell}-\frac{\log v_{j+\ell-1}}{2}\right).\end{aligned}$$

Assuming that j belongs to the set J_λ of assumption (6.16), by definition of ε and C^*, we deduce that

$$\alpha(\sigma_j,t)\leq A_{F_0}\sum_{k=0}^{+\infty}e^{-k}+\liminf_{n\to+\infty}\ e^{\lambda t-n-1}=\frac{A_{F_0}}{1-1/e}.$$

Letting $t \to +\infty$, we finally obtain

$$\sup_{j\in J_\lambda} \sup_{x\in\mathbb{N}} \mathbb{E}_{\sigma_j,x}\left[e^{\lambda(T_F-\sigma_j)}\right] \le \frac{A_{F_0}}{1-1/e}. \qquad \square$$

Proof of Lemma 6.5. Given $\varepsilon > 0$ such that $\mathbb{P}(V \ge \varepsilon) > 0$, we can assume without loss of generality that $V \ge \varepsilon > 0$ almost surely since, otherwise, we may modify the sequences $(u_j, j \ge 0)$ and $(v_j, j \ge 0)$ by removing all the favorable time intervals such that $v_j < \varepsilon$ and concatenating them with the surrounding unfavorable intervals. It is easy to check that this modifies the sequence $(u_j, v_j)_{j\ge 0}$ as an i.i.d. sample of a new random couple (U', V') such that $V' \ge \varepsilon$ almost surely, and $\mathbb{E}U' = \mathbb{E}(U \mid V \ge \varepsilon) + \frac{\mathbb{E}(U+V|V<\varepsilon)}{\mathbb{P}(V\ge\varepsilon)} < \infty$.

For all $i < j$, we introduce

$$S_{i,j} = \frac{1}{j-i}(u_{i+1} + \cdots + u_j).$$

Since $\mathbb{E}U < \infty$, the strong law of large numbers implies that $S_{i,j}$ converges to $\mathbb{E}U$ when $j \to +\infty$ for all $i \ge 0$ and hence $\sup_{j>i} S_{i,j} < \infty$ almost surely. Therefore, there exists $A > 0$ such that

$$\mathbb{P}\left(\sup_{j>i} S_{i,j} \le A\right) > \frac{1}{2}, \quad \forall i \ge 0.$$

Then, for all $k_0 \ge 1$,

$$\mathbb{P}\left(\sup_{j>i} S_{i,j} \le A \text{ and } \sup_{j>i+k_0} S_{i+k_0,j} \le A\right) > 0, \quad \forall i \ge 0.$$

For any given $t_0 > 0$, we choose $k_0 \in \mathbb{N}$ such that $k_0\varepsilon \ge t_0$. There exists a finite constant C such that

$$p := \mathbb{P}\left(\sup_{j>i} S_{i,j} \le A,\ \sup_{j>i+k_0} S_{i+k_0,j} \le A \text{ and } v_i + \cdots + v_{i+k_0-1} \le C\right) > 0, \quad \forall i \ge 0.$$

Now, for all $i \ge 0$ and $n \ge 1$, define

$$\Gamma_{i,n} := \begin{cases} \{S_{i,i+n} > A \text{ or } v_i + \cdots + v_{i+n-1} > C\} & \text{if } n \le k_0, \\ \{S_{i,i+n} > A \text{ or } S_{i+k_0,i+n} > A \text{ or } v_i + \cdots + v_{i+k_0-1} > C\} & \text{if } n \ge k_0+1 \end{cases}$$

and consider the following random sequence

$$I_0 = 0 \quad \text{and} \quad I_{k+1} = \begin{cases} I_k + \inf\{n \ge 1 \text{ s.t. } \Gamma_{I_k,n} \text{ is satisfied}\} & \text{if } I_k < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Since $\Gamma_{i,n}$ is measurable with respect to $\sigma(u_{i+1}, \dots, u_{i+n}, v_i, \dots, v_{i+n-1})$, the sequence $(I_k, k \ge 0)$ is a Markov chain in $\mathbb{N} \cup \{+\infty\}$ absorbed at $+\infty$, with independent increments up to absorption. Moreover, at each step, the probability of absorption is equal to $p > 0$. We deduce that

$$\mathbb{P}\left(\forall i \ge 0,\ \sup_{j>i} S_{i,j} > A \text{ or } \sup_{j>i+k_0} S_{i+k_0,j} > A \text{ or } v_i + \cdots + v_{i+k_0-1} > C\right)$$

$$\leq \mathbb{P}((I_k, k \geq 0) \text{ is never absorbed at } +\infty) = 0.$$

As a consequence, for any fixed $i_0 \geq 0$,

$$\mathbb{P}\left(\forall i \geq i_0,\ \sup_{j>i} S_{i,j} > A \text{ or } \sup_{j>i+k_0} S_{i+k_0,j} > A \text{ or } v_i + \cdots + v_{i+k_0-1} > C\right) = 0,$$

from which we deduce that

$$\mathbb{P}\left(\sup_{j>i} S_{i,j} \leq A,\ \sup_{j>i+k_0} S_{i+k_0,j} \leq A,\ v_i + \cdots + v_{i+k_0-1} \leq C \text{ for infinitely many } i \geq 0\right) = 1.$$

Since,

$$\sigma_{i+k_0} - \sigma_i = v_i + u_{i+1} + v_{i+1} + \cdots + v_{i+k_0-1} + u_{i+k_0},$$

since $\sup_{j>i} S_{i,j} \leq A$ implies that $u_{i+1} + \cdots + u_{i+k_0} \leq k_0 A$ and since $V \geq \varepsilon$ almost surely, we deduce that

$$\begin{aligned}\mathbb{P}\Bigg(& C_{\lambda,i} \leq \lambda A - \frac{\log \varepsilon}{2},\ C_{\lambda,i+k_0} \leq \lambda A - \frac{\log \varepsilon}{2}, \\ & \text{and } k_0\varepsilon \leq \sigma_{i+k_0} - \sigma_i \leq k_0 A + C \text{ for infinitely many } i \geq 0\Bigg) = 1.\end{aligned}$$

In other words, we proved that there exists $A > 0$ such that

$$J_\lambda := \{j \geq 0 : C_{\lambda,j} \leq A\}$$

is infinite, and that, for all $t_0 > 0$, setting $k_0 = \lceil t_0/\varepsilon \rceil$,

$$\liminf_{j \in J_\lambda,\ j \to +\infty} \inf \{\sigma_k - \sigma_j :\ k \in J_\lambda,\ k > j,\ \sigma_k - \sigma_j > t_0\} \leq k_0 A + C.$$

This concludes the proof of (6.16) and (6.17) and hence of Lemma 6.5. □

7. Proof of Theorem 2.1

Step 1: Control of the normalized distribution after a time 1
Let us show that, for all $s \geq 0$, $T \geq s+1$ and $x_1, x_2 \in E$, there exists a measure $\nu^{s,T}_{x_1,x_2}$ with mass greater than d_{s+1} such that, for all non-negative measurable function $f : E \to \mathbb{R}_+$,

$$\delta_{x_i} K^T_{s,s+1} f \geq \nu^{s,T}_{x_1,x_2}(f), \text{ for } i = 1, 2. \tag{7.1}$$

Fix $x_1, x_2 \in E$, $i \in \{1, 2\}$, $t \geq 1$ and a measurable non-negative function $f : E \to \mathbb{R}_+$. Using the Markov property, we have

$$\begin{aligned}\mathbb{E}_{s,x_i}(f(X_{s+1}) Z_{s,T}) &= \mathbb{E}_{s,x_i}(f(X_{s+1}) Z_{s,s+1} \mathbb{E}_{s+1,X_{s+1}}(Z_{s+1,T})) \\ &\geq \nu_{s+1,x_1,x_2}\left(f(\cdot)\mathbb{E}_{s+1,\cdot}(Z_{s+1,T})\right) \mathbb{E}_{s,x_i}(Z_{s,s+1}),\end{aligned}$$

by definition of ν_{s+1,x_1,x_2}. Dividing both sides by $\mathbb{E}_{s,x_i}(Z_{s,T})$, we deduce that

$$\delta_{x_i} K^T_{s,s+1}(f) \geq \nu_{s+1,x_1,x_2}\left(f(\cdot)\mathbb{E}_{s+1,\cdot}(Z_{s+1,T})\right)\frac{\mathbb{E}_{s,x_i}(Z_{s,s+1})}{\mathbb{E}_{s,x_i}(Z_{s,T})}.$$

But we have

$$\mathbb{E}_{s,x_i}(Z_{s,T}) \leq \mathbb{E}_{s,x_i}(Z_{s,s+1})\sup_{y\in E}\mathbb{E}_{s+1,y}(Z_{s+1,T}),$$

so that

$$\delta_{x_i} K^T_{s,s+1} f \geq \frac{\nu_{s+1,x_1,x_2}\left(f(\cdot)\mathbb{E}_{s+1,\cdot}(Z_{s+1,T})\right)}{\sup_{y\in E}\mathbb{E}_{s+1,y}(Z_{s+1,T})}.$$

Now, by definition of d_{s+1}, the non-negative measure

$$\nu^{s,T}_{x_1,x_2} : f \mapsto \frac{\nu_{s,x_1,x_2}\left(f(\cdot)\mathbb{E}_{s+1,\cdot}(Z_{s+1,T})\right)}{\sup_{y\in E}\mathbb{E}_{s+1,y}(Z_{s+1,T})}$$

has a total mass greater than d_{s+1}. Therefore (7.1) holds.

Step 2: Exponential contraction for Dirac initial distributions and proof of (2.8)
We now prove that, for all $x, y \in E$ and $0 \leq s \leq s+1 \leq t \leq T$

$$\left\|\delta_x K^T_{s,t} - \delta_y K^T_{s,t}\right\|_{TV} \leq 2\prod_{k=0}^{\lfloor t-s\rfloor -1}\left(1-d_{t-k}\right). \tag{7.2}$$

We deduce from (7.1) that, for all $x_1, x_2 \in E$,

$$\begin{aligned}\left\|\delta_{x_1} K^T_{s,s+1} - \delta_{x_2} K^T_{s,s+1}\right\|_{TV} &\leq \left\|\delta_{x_1} K^T_{s,s+1} - \nu^{s,T}_{x_1,x_2}\right\|_{TV} + \left\|\delta_{x_2} K^T_{s,s+1} - \nu^{s,T}_{x_1,x_2}\right\|_{TV}\\ &\leq 2(1-d_s).\end{aligned}$$

It is then standard (see *e.g.* [5]) to deduce that, for any probability measures μ_1 and μ_2 on E,

$$\left\|\mu_1 K^T_{s,s+1} - \mu_2 K^T_{s,s+1}\right\|_{TV} \leq (1-d_s)\|\mu_1-\mu_2\|_{TV}.$$

Using the semi-group property of $(K^T_{s,t})_{s,t}$, we deduce that, for any $x, y \in E$,

$$\begin{aligned}\left\|\delta_x K^T_{s,t} - \delta_y K^T_{s,t}\right\|_{TV} &= \left\|\delta_x K^T_{s,t-1}K^T_{t-1,t} - \delta_y K^T_{0,t-1}K^T_{t-1,t}\right\|_{TV}\\ &\leq (1-d_t)\left\|\delta_x K^T_{s,t-1} - \delta_y K^T_{s,t-1}\right\|_{TV}\\ &\leq \ \dots \ \leq \prod_{k=0}^{\lfloor t-s\rfloor -1}\left(1-d_{t-k}\right)\left\|\delta_x K^T_{s,t-\lfloor t-s\rfloor} - \delta_y K^T_{s,t-\lfloor t-s\rfloor}\right\|_{TV}\\ &\leq 2\prod_{k=0}^{\lfloor t-s\rfloor -1}\left(1-d_{t-k}\right).\end{aligned}$$

One deduces (2.8) with standard arguments as above.

Step 3: Exponential contraction for general initial distributions
We prove now that for any pair of initial probability measures μ_1, μ_2 on E, for all $0 \leq s \leq s+1 \leq t \leq T \geq 0$,

$$\left\| \frac{\mathbb{E}_{s,\mu_1}(\mathbb{1}_{X_t \in \cdot} Z_{s,T})}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} - \frac{\mathbb{E}_{s,\mu_2}(\mathbb{1}_{X_t \in \cdot} Z_{s,T})}{\mathbb{E}_{s,\mu_2}(Z_{s,T})} \right\|_{TV} \leq 2 \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}) . \tag{7.3}$$

Taking $t = T$ then entails (2.9) and ends the proof of Theorem 2.1.

Let μ_1 be a probability measure on E and $x \in E$. We have

$$\begin{aligned}
&\left\| \frac{\mathbb{E}_{s,\mu_1}(\mathbb{1}_{X_t \in \cdot} Z_{s,T})}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} - \frac{\mathbb{E}_{s,x}(\mathbb{1}_{X_t \in \cdot} Z_{s,T})}{\mathbb{E}_{s,x}(Z_{s,T})} \right\|_{TV} \\
&= \frac{1}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} \left\| \mathbb{E}_{s,\mu_1}(\mathbb{1}_{X_t \in \cdot} Z_{s,T}) - \mathbb{E}_{s,\mu_1}(Z_{s,T}) \delta_x K^T_{s,t} \right\|_{TV} \\
&\leq \frac{1}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} \int_{y \in E} \left\| \mathbb{E}_{s,y}(\mathbb{1}_{X_t \in \cdot} Z_{s,T}) - \mathbb{E}_{s,y}(Z_{s,T}) \delta_x K^T_{s,t} \right\|_{TV} d\mu_1(y) \\
&\leq \frac{1}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} \int_{y \in E} \mathbb{E}_{s,y}(Z_{s,T}) \left\| \delta_y K^T_{s,t} - \delta_x K^T_{s,t} \right\|_{TV} d\mu_1(y) \\
&\leq \frac{1}{\mathbb{E}_{s,\mu_1}(Z_{s,T})} \int_{y \in E} \mathbb{E}_{s,y}(Z_{s,T})\, 2 \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k})\, d\mu_1(y) \\
&\leq 2 \prod_{k=0}^{\lfloor t-s \rfloor -1} (1 - d_{t-k}) .
\end{aligned}$$

The same computation, replacing δ_x by any probability measure, leads to (7.3).

8. Proof of Proposition 3.1 and Corollary 3.2

8.1. Proof of Proposition 3.1

Fix $s \geq 0$. Let us first prove (3.1). Note that, if $d'_v = 0$ for all $v \geq s+1$, there is nothing to prove, so let us assume the converse. Fix $t \geq s+1$ such that $d'_t > 0$. Then the measure ν_t is positive and we define for all $x \in E$ and $u \geq t$

$$\eta_{t,u}(x) = \frac{\mathbb{E}_{t,x}\left(Z_{t,u}\right)}{\mathbb{E}_{t,\nu_t}(Z_{t,u})}.$$

For all $u \geq t$ and $x, y \in E$, we have

$$\begin{aligned}
\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})} &= \frac{\mathbb{E}_{s,x}\left(Z_{s,t} \mathbb{E}_{t,X_t}(Z_{t,u})\right)}{\mathbb{E}_{s,y}(Z_{s,t} \mathbb{E}_{t,X_t}(Z_{t,u}))} \\
&= \frac{\Phi_{s,t}(\delta_x)(\mathbb{E}_{t,\cdot}(Z_{t,u}))}{\Phi_{s,t}(\delta_y)(\mathbb{E}_{t,\cdot}(Z_{t,u}))} \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})} \\
&= \frac{\Phi_{s,t}(\delta_x)(\eta_{t,u})}{\Phi_{s,t}(\delta_y)(\eta_{t,u})} \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}.
\end{aligned}$$

Therefore

$$\left|\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})}-\frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\right| = \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\,\frac{|\Phi_{s,t}(\delta_x)(\eta_{t,u})-\Phi_{s,t}(\delta_y)(\eta_{t,u})|}{\Phi_{s,t}(\delta_y)(\eta_{t,u})}$$
$$\leq \frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\,\frac{\|\eta_{t,u}\|_\infty}{\Phi_{s,t}(\delta_y)(\eta_{t,u})}\prod_{k=0}^{\lfloor t-s\rfloor-1}(1-d_{t-k})\,, \tag{8.1}$$

where we used the bound (2.9) of Theorem 2.1 in the last inequality.

Let us first prove that $\eta_{t,u}$ is uniformly bounded and that we have $\Phi_{s,t}(\mu)(\eta_{t,u})\geq 1$ for all positive measure μ on E. First, by definition of d'_t, we have

$$\eta_{t,u}(x)=\frac{\mathbb{E}_{t,x}\left(Z_{t,u}\right)}{\mathbb{E}_{t,\nu_t}(Z_{t,u})}\leq 1/d'_t. \tag{8.2}$$

Second, by Markov's property,

$$\Phi_{s,t}(\mu)(\eta_{t,u})=\frac{\mathbb{E}_{s,\mu}(Z_{s,t}\eta_{t,u}(X_t))}{\mathbb{E}_{s,\mu}(Z_{s,t})}$$
$$=\frac{\mathbb{E}_{s,\mu}(Z_{s,u})}{\mathbb{E}_{s,\mu}(Z_{s,t})\mathbb{E}_{t,\nu_t}(Z_{t,u})},$$

where, using the definition of ν_t,

$$\mathbb{E}_{s,\mu}(Z_{s,u})=\mathbb{E}_{s,\mu}\left[Z_{s,t-1}\mathbb{E}_{t-1,X_{t-1}}(Z_{t-1,t}\mathbb{E}_{t,X_t}(Z_{t,u}))\right]$$
$$=\mathbb{E}_{s,\mu}\Big\{Z_{s,t-1}\Phi_{t-1,t}(\delta_{X_{t-1}})\left[\mathbb{E}_{t,\cdot}(Z_{t,u})\right]\mathbb{E}_{t-1,X_{t-1}}(Z_{t-1,t})\Big\}$$
$$\geq \mathbb{E}_{s,\mu}\left[Z_{s,t-1}\mathbb{E}_{t,\nu_t}(Z_{t,u})\mathbb{E}_{t-1,X_{t-1}}(Z_{t-1,t})\right]$$
$$=\mathbb{E}_{s,\mu}(Z_{s,t})\mathbb{E}_{t,\nu_t}(Z_{t,u}).$$

Hence,

$$\Phi_{s,t}(\mu)(\eta_{t,u})\geq 1. \tag{8.3}$$

Now, let t_1 be the smallest $v\geq s+1$ such that $d'_{v_1}>0$. Using a similar computation as in the proof of (8.3) above, we have

$$\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})}=\frac{\mathbb{E}_{s,x}[Z_{s,t_1-1}\Phi_{t_1-1,t_1}(\delta_{X_{t_1-1}})(\mathbb{E}_{t_1,\cdot}(Z_{t_1,u}))\mathbb{E}_{t_1-1,X_{t_1-1}}(Z_{t_1-1,t_1})]}{\mathbb{E}_{s,y}[Z_{s,t_1}\mathbb{E}_{t_1,X_{t_1}}(Z_{t_1,u})]}$$
$$\geq \frac{\mathbb{E}_{t_1,\nu_{t_1}}(Z_{t_1,u}))}{\sup_{z\in E}\mathbb{E}_{t_1,z}(Z_{t_1,u})}\,\frac{\mathbb{E}_{s,x}(Z_{s,t_1})}{\mathbb{E}_{s,y}(Z_{s,t_1})}$$
$$\geq d'_{t_1}\frac{\mathbb{E}_{s,x}(Z_{s,t_1})}{\sup_{z\in E}\mathbb{E}_{s,z}(Z_{s,t_1})}, \tag{8.4}$$

where we used the definition of d'_{t_1} in the last inequality. Note that the right-hand side of (8.4) does not depend on u and y and is positive by (2.1).

Inserting the inequalities (8.2), (8.3) and (8.4) in (8.1), we obtain

$$\left|\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})}-\frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\right|\leq\frac{\sup_{z\in E}\mathbb{E}_{s,z}(Z_{s,t_1})}{d'_{t_1}\mathbb{E}_{s,y}(Z_{s,t_1})}\,\frac{1}{d'_t}\prod_{k=0}^{\lfloor t-s\rfloor-1}(1-d_{t-k})$$
$$=C_{s,y}\frac{1}{d'_t}\prod_{k=0}^{\lfloor t-s\rfloor-1}(1-d_{t-k})$$

where $C_{s,y}$ only depends on s and y.

To complete the proof of (3.1), it remains to observe that, for any $u\geq t\geq s+1$ (not necessarily such that $d'_t>0$) and for all $v\in[s+1,t]$ such that $d'_v>0$, we have $t_1\leq v$ and hence we can apply the last inequality to obtain

$$\left|\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})}-\frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\right|\leq\left|\frac{\mathbb{E}_{s,x}(Z_{s,v})}{\mathbb{E}_{s,y}(Z_{s,v})}-\frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,y}(Z_{s,t})}\right|+\left|\frac{\mathbb{E}_{s,x}(Z_{s,v})}{\mathbb{E}_{s,y}(Z_{s,v})}-\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,y}(Z_{s,u})}\right|$$
$$\leq 2C_{s,y}\frac{1}{d'_v}\prod_{k=0}^{\lfloor v-s\rfloor-1}(1-d_{v-k})\,.$$

Now, we assume that (3.2) holds true. We fix $x_0\in E$. It follows from (3.1) that $x\mapsto\frac{\mathbb{E}_{s,x}(Z_{s,t})}{\mathbb{E}_{s,x_0}(Z_{s,t})}$ converges uniformly when $t\to+\infty$ to some function η_s, which is positive because of (8.4).

Moreover, for all $s\leq t\leq u$,

$$\mathbb{E}_{s,x}\left(Z_{s,t}\frac{\mathbb{E}_{t,X_t}(Z_{t,u})}{\mathbb{E}_{t,x_0}(Z_{t,u})}\right)=\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{t,x_0}(Z_{t,u})}$$
$$=\frac{\mathbb{E}_{s,x}(Z_{s,u})}{\mathbb{E}_{s,x_0}(Z_{s,u})}\;\mathbb{E}_{s,x_0}\left(Z_{s,t}\frac{\mathbb{E}_{t,X_t}(Z_{t,u})}{\mathbb{E}_{t,x_0}(Z_{t,u})}\right).\tag{8.5}$$

For all probability measure μ on E, integrating both sides of the equation with respect to μ, letting $u\to\infty$ and using Lebesgue's theorem, we deduce that, for all $s\leq t\in I$, there exists a positive constant $c_{s,t}$ which does not depend on μ such that

$$c_{s,t}=\frac{\mathbb{E}_{s,\mu}(Z_{s,t}\eta_t(X_t))}{\mu(\eta_s)}.$$

In addition, for all $s\leq t\leq u\in I$,

$$c_{s,t}c_{t,u}=\frac{\mathbb{E}_{s,x}(Z_{s,t}\eta_t(X_t))}{\eta_s(x)}\,\frac{\mathbb{E}_{t,\mu}(Z_{t,u}\eta_u(X_u))}{\mu(\eta_t)}.$$

Choosing the probability measure μ defined by $\mu(f)=\frac{\mathbb{E}_{s,x}(Z_{s,t}f(X_t))}{\mathbb{E}_{s,x}(Z_{s,t})}$ for all bounded measurable f and using Markov's property, we obtain

$$c_{s,t}c_{t,u}=\frac{\mathbb{E}_{s,x}(Z_{s,t})\,\mathbb{E}_{t,\mu}(Z_{t,u}\eta_u(X_u))}{\eta_s(x)}=\frac{\mathbb{E}_{s,x}(Z_{s,u}\eta_u(X_u))}{\eta_s(x)}=c_{s,u}.$$

Because of the last equality, replacing for all $s\geq 0$ the function $\eta_s(x)$ by $\eta_s(x)/c_{0,s}$ entails (3.4).

8.2. Proof of Corollary 3.2

Let $(f_s)_{s\geq 0}$ be a solution of (3.5) satisfying (3.6). Fix $x_0 \in E$ and for all $s \geq 0$, let $\nu_s = \mathbb{E}_{s,x_0}(Z_{0,s})$. Using (3.6) and applying (2.9) with $\mu_1 = \delta_x$ and $\mu_2 = \nu_s$, we have for all $s \geq 0$, $x \in E$ and for $t \to +\infty$,

$$f_s(x) = \mathbb{E}_{s,x}(Z_{s,t} f_t(X_t)) \sim \mathbb{E}_{s,x}(Z_{s,t}) \frac{\mathbb{E}_{0,x_0}(Z_{0,t} f_t(X_t))}{\mathbb{E}_{s,\nu_s}(Z_{s,t})}.$$

Using (3.3) (integrated with respect to $\nu_s(\mathrm{d}x)$), we deduce

$$\begin{aligned} f_s(x) &\sim \frac{\eta_s(x)}{\nu_s(\eta_s)} \, \mathbb{E}_{s,\nu_s}(Z_{s,t}) \, \frac{f_0(x_0)}{\mathbb{E}_{s,\nu_s}(Z_{s,t})} \\ &\sim \frac{\eta_s(x)}{\eta_0(x_0)} \, f_0(x_0). \end{aligned}$$

Since both sides are independent of t, we obtain

$$f_s = \eta_s \, \frac{f_0(x_0)}{\eta_0(x_0)}.$$

9. Proof of Theorem 3.4

Let us define the probability measure $Q^t_{s,x}$ by

$$dQ^t_{s,x} = \frac{Z_{s,t}}{\mathbb{E}_{s,x}(Z_{s,t})} d\mathbb{P}_{s,x}, \quad \text{on } \mathcal{F}_{s,t}$$

We have, for all $0 \leq s \leq u \leq t$,

$$\frac{\mathbb{E}_{s,x}(Z_{s,t} \mid \mathcal{F}_{s,u})}{\mathbb{E}_{s,x}(Z_{s,t})} = \frac{Z_{s,u}\mathbb{E}_{u,X_u}(Z_{u,t})}{\mathbb{E}_{s,x}[Z_{s,u}\mathbb{E}_{u,X_u}(Z_{u,t})]} = \frac{Z_{s,u}\eta_{u,t}(X_u)}{\mathbb{E}_{s,x}[Z_{s,u}\eta_{u,t}(X_u)]}.$$

By Proposition 3.1, this converges almost surely when $t \to \infty$ to

$$M_{s,u} := \frac{Z_{s,u}\eta_u(X_u)}{\mathbb{E}_{s,x}[Z_{s,u}\eta_u(X_u)]},$$

where $\mathbb{E}_{s,x}(M_{s,u}) = 1$.

By the penalization's theorem of Roynette, Vallois and Yor ([19], Thm. 2.1), these two conditions (almost sure convergence and $\mathbb{E}_{s,x}(M_{s,u}) = 1$) imply that $(M_{s,t}, t \geq s)$ is a martingale under $\mathbb{P}_{s,x}$ and that $Q^t_{s,x}(\Lambda_{s,u})$ converges to $\mathbb{E}_{s,x}\left(M_{s,u}\mathbb{1}_{\Lambda_{s,u}}\right)$ for all $\Lambda_{s,u} \in \mathcal{F}_{s,u}$ when $t \to \infty$. This means that $\mathbb{Q}_{s,x}$ is well defined and

$$\left.\frac{d\mathbb{Q}_{s,x}}{d\mathbb{P}_{s,x}}\right|_{\mathcal{F}_{s,u}} = M_{s,u}.$$

Let us now prove that the family $(\mathbb{Q}_{s,x})_{s\in I, x\in E}$ defines a time inhomogeneous Markov process, that is for all $s \leq u \leq t$, all $x \in E$ and all positive measurable function f,

$$\mathbb{E}_{\mathbb{Q}_{s,x}}(f(X_t) \mid \mathcal{F}_{s,u}) = \mathbb{E}_{\mathbb{Q}_{u,X_u}}(f(X_t)).$$

We easily check from the definition of the conditional expectation that

$$\begin{aligned}
M_{s,u}\mathbb{E}_{\mathbb{Q}_{s,x}}(f(X_t)\mid\mathcal{F}_{s,u}) &= \mathbb{E}_{s,x}\left(M_{s,t}f(X_t)\mid\mathcal{F}_{s,u}\right)\\
&= \frac{\mathbb{E}_{s,x}[Z_{s,t}\eta_t(X_t)f(X_t)\mid\mathcal{F}_{s,u}]}{\mathbb{E}_{s,x}(Z_{s,t}\eta_t(X_t))}\\
&= \frac{Z_{s,u}\mathbb{E}_{u,X_u}(Z_{u,t}\eta_t(X_t))}{\mathbb{E}_{s,x}(Z_{s,t}\eta_t(X_t))}\mathbb{E}_{u,X_u}\left(\frac{Z_{u,t}\eta_t(X_t)}{\mathbb{E}_{u,X_u}(Z_{u,t}\eta_t(X_t))}f(X_t)\right)\\
&= \frac{Z_{s,u}\mathbb{E}_{u,X_u}(Z_{u,t}\eta_t(X_t))}{\mathbb{E}_{s,x}(Z_{s,t}\eta_t(X_t))}\mathbb{E}_{\mathbb{Q}_{u,X_u}}\left(f(X_t)\right),
\end{aligned}$$

where we used the Markov property of X under $\mathbb{P}_{s,x}$, the fact that $Z_{s,t}=Z_{s,u}Z_{u,t}$ and the definition of $\mathbb{Q}_{u,X_u}$. Using the above equality with $f=1$, we conclude that

$$\frac{Z_{s,u}\mathbb{E}_{u,X_u}(Z_{u,t}\eta_t(X_t))}{\mathbb{E}_{s,x}(Z_{s,t}\eta_t(X_t))}=M_{s,u}$$

(we could also use (8.5)). Hence, the Markov property holds for $(\mathbb{Q}_{s,x})_{s\in I,x\in E}$.

The inequality (3.8) is a direct consequence of (2.8) in Theorem 2.1.

10. Proof of Theorem 4.1

We define for all $0\leq s\leq t$

$$A(s,t):=\frac{\sup_{x\in E}\mathbb{E}_{s,x}(Z_{s,t})}{\inf_{y\in E}\mathbb{E}_{s,y}(Z_{s,t})}.$$

Our first goal is to prove that $A(s,t)$ is uniformly bounded. For all $s\leq t\leq T$ and all $x\in E$, it follows from the definition (4.2) of $\varepsilon_{s,t}$ that, for all $y\in E$,

$$\begin{aligned}
\mathbb{E}_{s,x}(Z_{s,T}) &= \mathbb{E}_{s,x}(Z_{s,t})\varPhi_{s,t}(\delta_x)\left(\mathbb{E}_{t,\cdot}(Z_{t,T})\right)\\
&\leq \mathbb{E}_{s,x}(Z_{s,t})\left[\varPhi_{s,t}(\delta_y)\left(\mathbb{E}_{t,\cdot}(Z_{t,T})\right)+\varepsilon_{s,t}\sup_{z\in E}\mathbb{E}_{t,z}(Z_{t,T})\right]\\
&\leq \mathbb{E}_{s,x}(Z_{s,t})\frac{\mathbb{E}_{s,y}(Z_{s,T})}{\mathbb{E}_{s,y}(Z_{s,t})}\left[1+\varepsilon_{s,t}\sup_{z\in E}\frac{\mathbb{E}_{t,z}(Z_{t,T})}{\mathbb{E}_{t,\varPhi_{s,t}(\delta_y)}(Z_{t,T})}\right]\\
&\leq A(s,t)\mathbb{E}_{s,y}(Z_{s,T})\left[1+\varepsilon_{s,t}A(t,T)\right].
\end{aligned}$$

Therefore,

$$A(s,T)\leq A(s,t)\left(1+\varepsilon_{s,t}A(t,T)\right).$$

We deduce that, for all $k\in\mathbb{N}$,

$$\begin{aligned}
A(s,s+k) &\leq A(s,s+1)\prod_{\ell=1}^{k-1}\left(1+\varepsilon_{s,s+\ell}A(s+\ell,s+\ell+1)\right)\\
&\leq e^{2\|\kappa\|_\infty}\exp\left(e^{2\|\kappa\|_\infty}\sum_{\ell=1}^{k-1}\varepsilon_{s,s+\ell}\right).
\end{aligned}$$

It follows from (4.4) that $A(s,s+k)$ is uniformly bounded for $s \geq 0$ and $k \in \mathbb{N}$. Since κ is bounded, we deduce that $A(s,t)$ is bounded by some constant $\bar{A}$ for all $0 \leq s \leq t$.

Next, we compute a lower bound for the measure $\bar{\nu}_s$. We define for some fixed $x_0 \in E$ and all $s \geq 0$

$$\alpha_s := \frac{\mathbb{E}_{0,x_0}(Z_{0,s}\mathbb{1}_{X_s\in\cdot})}{\mathbb{E}_{0,x_0}(Z_{0,s})} = \phi_{0,s}(\delta_{x_0}).$$

Note that it follows from the semigroup property (2.3) that, for all $0 \leq s \leq t$, $\Phi_{s,t}(\alpha_s) = \alpha_t$. Fix a Borel subset B of E, a probability measure μ on E and an integer $k \geq 1$. Then,

$$\begin{aligned}\Phi_{s,s+2k}(\mu)(B) &= \Phi_{s+k,s+2k}(\Phi_{s,s+k}(\mu))(B)\\ &= \frac{\mathbb{E}_{s+k,\Phi_{s,s+k}(\mu)}\left[Z_{s+k,s+2k}\mathbb{1}_{X_s+2k\in B}\right]}{\mathbb{E}_{s+k,\Phi_{s,s+k}(\mu)}\left[Z_{s+k,s+2k}\right]}\\ &\geq \frac{1}{\bar{A}}\,\frac{\mathbb{E}_{s+k,\Phi_{s,s+k}(\mu)}\left[Z_{s+k,s+2k}\mathbb{1}_{X_s+2k\in B}\right]}{\mathbb{E}_{s+k,\alpha_{s+k}}(Z_{s+k,s+2k})}.\end{aligned}$$

Now, $\Phi_{s,s+k}(\mu) \geq \alpha_{s+k} - \nu^+_{s,\mu}$ where $\nu^+_{s,\mu} = [\alpha_{s+k} - \Phi_{s,s+k}(\mu)]_+$, where $[\cdot]_+$ (resp. $[\cdot]_-$) denotes the positive (resp. negative) part in the sense of measures. Hence,

$$\begin{aligned}\Phi_{s,s+2k}(\mu)(B) &\geq \frac{\alpha_{s+2k}(B)}{\bar{A}} - \frac{\int_E \mathbb{E}_{s+k,y}\left[Z_{s+k,s+2k}\mathbb{1}_{X_s+2k\in B}\right]\nu^+_{s,\mu}(\mathrm{d}y)}{\bar{A}\mathbb{E}_{s+k,\alpha_{s+k}}(Z_{s+k,s+2k})}\\ &\geq \frac{\alpha_{s+2k}(B)}{\bar{A}} - \frac{\int_E \Phi_{s+k,s+2k}(\delta_y)(B)\mathbb{E}_{s+k,y}(Z_{s+k,s+2k})\nu^+_{s,\mu}(\mathrm{d}y)}{\bar{A}\mathbb{E}_{s+k,\alpha_{s+k}}(Z_{s+k,s+2k})}\\ &\geq \frac{\alpha_{s+2k}(B)}{\bar{A}} - \int_E \Phi_{s+k,s+2k}(\delta_y)(B)\nu^+_{s,\mu}(\mathrm{d}y)\\ &= \alpha_{s+2k}(B)\left(\frac{1}{\bar{A}} - \nu^+_{s,\mu}(E)\right) - \int_E \left[\Phi_{s+k,s+2k}(\delta_y)(B) - \alpha_{s+2k}(B)\right]\nu^+_{s,\mu}(\mathrm{d}y)\\ &\geq \alpha_{s+2k}(B)\left(\frac{1}{\bar{A}} - \nu^+_{s,\mu}(E)\right) - \int_E \left(\Phi_{s+k,s+2k}(\delta_y)(B) - \alpha_{s+2k}(B)\right)_+\nu^+_{s,\mu}(\mathrm{d}y),\end{aligned}$$

where $(\cdot)_+$ is the positive part (in the sense of real numbers). Since $\nu^+_{s,\mu} \leq \alpha_{s+k}$ in the sense of measures we obtain

$$\Phi_{s,s+2k}(\mu)(B) \geq \alpha_{s+2k}(B)\left(\frac{1}{\bar{A}} - \nu^+_{s,\mu}(E)\right) - \int_E \left(\Phi_{s+k,s+2k}(\delta_y)(B) - \alpha_{s+2k}(B)\right)_+\alpha_{s+k}(\mathrm{d}y).$$

Now, using the fact that $\alpha_{s+k} = \Phi_{s,s+k}(\alpha_s)$, we deduce from the definition (4.2) of $\varepsilon_{s,t}$ that

$$\Phi_{s,s+2k}(\mu)(B) \geq \alpha_{s+2k}(B)\left(\frac{1}{\bar{A}} - \varepsilon_{s,s+k}\right) - \int_E \left(\Phi_{s+k,s+2k}(\delta_y)(B) - \alpha_{s+2k}(B)\right)_+\alpha_{s+k}(\mathrm{d}y).$$

Hence, for all $B_1,\dots,B_n$ measurable partition of E and all $\mu_1,\dots,\mu_n$ probability measures on E,

$$\begin{aligned}\sum_{i=1}^n \Phi_{s,s+2k}(\mu_i)(B_i) &\geq \alpha_{s+2k}(E)\left(\frac{1}{\bar{A}} - \varepsilon_{s,s+k}\right) - \int_E \sum_{i=1}^n \left(\Phi_{s+k,s+2k}(\delta_y)(B_i) - \alpha_{s+2k}(B_i)\right)_+\alpha_{s+k}(\mathrm{d}y)\\ &\geq \frac{1}{\bar{A}} - \varepsilon_{s,s+k} - \int_E \left\|\Phi_{s+k,s+2k}(\delta_y) - \alpha_{s+2k}\right\|_{TV}\alpha_{s+k}(\mathrm{d}y).\end{aligned}$$

Since $\alpha_{s+2k}=\Phi_{s+k,s+2k}(\alpha_{s+k})$, we finally obtain

$$\sum_{i=1}^n \Phi_{s,s+2k}(\mu_i)(B_i)\geq \frac{1}{\bar{A}}-\varepsilon_{s,s+k}-\varepsilon_{s+k,s+2k}.$$

Since this inequality is true for any partition (B_i) of E and any choices of (μ_i), we deduce that the infimum measure $\bar{\nu}^k_{s+2k}:=\min_{x\in E}\Phi_{s,s+2k}(\delta_x)$ satisfies

$$\bar{\nu}^k_{s+2k}(E)\geq \frac{1}{\bar{A}}-\varepsilon_{s,s+k}-\varepsilon_{s+k,s+2k}.$$

In particular, by (4.3), this is a larger than $1/(2\bar{A})$ if k is large enough. Given such an integer k, we set $n_0=2k$ and hence $\bar{\nu}_s=\bar{\nu}^k_s$ for all $s\geq n_0$. Combining the last estimate with the fact that $A(s,t)$ is bounded, we deduce that, for all $s\geq n_0$, $\bar{d}'_s\geq\frac{1}{2\bar{A}^2}$. This concludes the proof of Theorem 4.1.

11. Proof of Theorem 4.4

Fix $x_0\in E$ and set $c_0:=\inf_{s\geq 0,\,u\in[1,2]}\mathbb{P}_{s,x_0}(X_{s+u}=x_0)>0$. We define $\lambda=\sup\kappa-\inf\kappa-\ln c_0$ and consider the finite set $K\subset E$ (obtained from assumption (4.8)) such that

$$A:=\sup_{s\geq 0,\,x\in E}\ \mathbb{E}_x\left(e^{\lambda(T_K-s)}\right)<\infty.$$

Setting $t_0=2+(\ln 2+\ln A)/\lambda$, we immediately obtain by Markov inequality that

$$\inf_{s\geq 0,\ x\in E}\mathbb{P}_{s,x}(T_K<s+t_0)\geq 1/2.$$

In particular, using Markov property, the irreducibility assumption (4.7) and the finiteness of K,

$$c_1:=\inf_{s\geq 0,\ x\in E}\mathbb{P}_{s,x}(X_{s+2t_0}=x_0)\geq \inf_{s\geq 0,\ x\in E}\mathbb{P}_{s,x}(T_K<s+t_0)\inf_{u\in[0,t_0],\ y\in K}\mathbb{P}_{s+u,y}(X_{s+2t_0-u}=x_0)>0.$$

We deduce that, for all $s\geq 2t_0$,

$$\Phi_{s-2t_0,s}(\delta_x)\geq e^{-2t_0(\sup\kappa-\inf\kappa)}c_1\delta_{x_0},$$

hence

$$\varphi^{(2t_0)}_s:=\inf_{x\in E}\Phi_{s-2t_0,s}(\delta_x)\geq e^{-2t_0(\sup\kappa-\inf\kappa)}c_1\delta_{x_0}. \tag{11.1}$$

Now, for all $0\leq s\leq s+1\leq u\leq t$, we have for all $x\in K$,

$$\begin{aligned}\mathbb{E}_{s,x_0}(Z_{s,t})&\geq \mathbb{E}_{s,x_0}(Z_{u,t})e^{-(u-s)\sup\kappa}\\&\geq \mathbb{E}_{s,x_0}\left(\mathbb{1}_{X_{\lceil s\rceil+1}=x_0,\dots,X_{\lfloor u-1\rfloor-1}=x_0,X_{u-1}=x_0,X_u=x}Z_{u,t}\right)\ e^{-(u-s)\sup\kappa}\\&\geq c_0^{\lfloor u-s-1\rfloor}\inf_{u\geq 1,y\in K}\mathbb{P}_{u-1,x_0}(X_u=y)\,\mathbb{E}_{u,x}(Z_{u,t})e^{-(u-s)\sup\kappa},\end{aligned}$$

where $\inf_{u\geq 1,y\in K}\mathbb{P}_{u-1,x_0}(X_u=y)>0$ by finiteness of K and by (4.7). Applying this inequality to the argmax of $\mathbb{E}_{u,x}(Z_{u,t})$ over $x\in K$, we deduce that there exists a constant $c_2>0$ such that, for all $s\leq u\leq t$ (the inequality is immediate for $u \in [s, s+1)$),

$$\mathbb{E}_{s,x_0}\left(Z_{s,t}\right) \geq c_2\, e^{-(u-s)(\sup\kappa - \ln c_0)} \max_{x\in K} \mathbb{E}_{u,x}(Z_{u,t}). \tag{11.2}$$

For all $x \in E$, we deduce, using Markov inequality and the strong Markov property at time T_K and (11.2), that

$$\begin{aligned}
\mathbb{E}_{s,x}(Z_{s,t}) &= \mathbb{E}_{s,x}(Z_{s,t}\mathbb{1}_{t\leq T_K}) + \mathbb{E}_{s,x}(Z_{s,t}\mathbb{1}_{T_K<t})\\
&\leq e^{-(t-s)\inf\kappa} A\, e^{-\lambda(t-s)} + \mathbb{E}_{s,x}\left(Z_{s,T_K}\mathbb{E}_{T_K,X_{T_K}}(Z_{T_K,t})\mathbb{1}_{T_K<t}\right)\\
&\leq A\, e^{-(t-s)(\sup\kappa-\ln c_0)} + \frac{\mathbb{E}_{s,x_0}(Z_{s,t})}{c_2}\,\mathbb{E}_{s,x}\left(Z_{s,T_K}e^{(T_K-s)(\sup\kappa-\ln c_0)}\right)\\
&\leq \frac{\mathbb{E}_{s,x_0}(Z_{s,t})}{c_2}\left(A + \mathbb{E}_{s,x}\left(e^{(T_K-s)(\sup\kappa-\inf\kappa-\ln c_0)}\right)\right) \leq \frac{2A}{c_2}\,\mathbb{E}_{s,x_0}(Z_{s,t}).
\end{aligned}$$

We finally obtain using (11.2)

$$\inf_{t\geq s}\frac{\mathbb{E}_{s,\nu_s^{(2t_0)}}\left(Z_{s,t}\right)}{\sup_{x\in E}\mathbb{E}_{s,x}\left(Z_{s,t}\right)} \geq c_1\, e^{-2t_0(\sup\kappa-\inf\kappa)}\frac{2A}{c_2}.$$

This is a uniform lower bound for d'_s (with a time increment of $-2t_0$ instead of -1, see Rem. 2.3), hence Theorem 2.1 allows us to conclude the proof.

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