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STOCHASTIC FORMULATIONS OF THE PARAMETRIX METHOD[☆]

Arturo Kohatsu-Higa[*] and Gô Yûki

Abstract. In this manuscript, we consider stochastic expressions of the parametrix method for solutions of d-dimensional stochastic differential equations (SDEs) with drift coefficients which belong to $L^p(\mathbf{R}^d)$, $p > d$. We prove the existence and Hölder continuity of probability density functions for distributions of solutions at fixed points and obtain an explicit expansion *via* (stochastic) parametrix methods. We also obtain Gaussian type upper and lower bounds for these probability density functions.

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1. Introduction

In this paper, we consider multi-dimensional stochastic differential equations (SDEs) of the form

$$X_t = x_0 + B_t + \int_0^t b(X_s)\mathrm{d}s, \tag{1.1}$$

where $x_0 \in \mathbf{R}^d$, $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion and $b : \mathbf{R}^d \to \mathbf{R}^d$. We will show that if b belongs to $L^p(\mathbf{R}^d)$ with $p > d$ then X_t admits a continuous density and it satisfies two-sided Gaussian bounds.

The regularity of transition densities for solutions to SDEs has been studied by many researchers and it is well known that a key property to solve this problem is the regularity of the coefficients of the SDE. For hypoelliptic SDEs, if the coefficients are sufficiently smooth and have bounded derivatives then Malliavin calculus arguments work well and the solution admits a smooth density (see *e.g.* [20, 22]).

In recent years, one of the directions in this area of research is to develop tools to deal with the case of non-smooth coefficients. In this case, it is not easy to use Malliavin calculus directly. However, in [12], the authors prove Hölder continuity of densities for bounded non-Lipschitz drift coefficient b by applying Malliavin calculus in combination with some approximation techniques. This regularity property is obtained via. an estimation of the integrability order of the Fourier transform of X_t. This result was extended in [13] for drift coefficients in Sobolev type spaces.

Other results by applying Malliavin calculus for SDEs with non-Lipschitz coefficients are available. See for example: [16, 17]. In these articles, the main tool used is a combination of Girsanov's theorem and Malliavin calculus.

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Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577, Japan.

[*] Corresponding author: arturokohatsu@gmail.com

One of the main purposes of the present paper is to present the link between this technique with the classical parametrix method in analysis as exposed in [11] and to show that the interaction between these two techniques may be fruitful for various models in stochastic analysis.

As a possible application, we obtain upper and lower bounds and regularity properties for the density of X_t when the drift coefficient satisfies that $b \in L^p(\mathbf{R}^d)$ with $p > d$. In particular, unbounded drifts are amenable to this condition and we are interested in finding conditions so that Gaussian bounds can be obtained even if the drift function, b, may explode at some points.

Historically, this problem is important in applications and one of the first works in the direction of this article is the book of Portenko [24].

It is well known that Markov type SDEs are associated with second order parabolic type partial differential equations (PDEs). Especially, transition densities of solutions to SDEs are associated with fundamental solutions to parabolic PDEs. The parametrix method is a standard approach to fundamental solutions (see Ref. [11]). By using the parametrix method and its extensions, Gaussian upper and lower bounds for fundamental solutions have been obtained in a variety of settings.

Certainly, the amount of results in this topic is very large. Other results that fall into this class are [15, 28, 29] which use Dirichlet forms techniques and give upper and/or lower bounds for the density. Similarly, we refer the reader to [7, 8, 27] for other results that also show applications of these estimates in order to obtain strong existence for stochastic equations. For other related results we also refer to [1].

The Hölder regularity of densities for solutions to (1.1) in the case of bounded path-dependent drift coefficients and constant diffusion coefficient is studied in [3, 4]. The authors show that there exists a density which is α-Hölder continuous for any $\alpha \in (0,1)$. Also they obtain Gaussian type bounds for these densities.

For path-dependent SDEs with bounded Dini continuous coefficients and non-constant diffusion coefficient [19] extends a method introduced in [21], showing that the solution of the stochastic equation admits a Hölder continuous density and it satisfies Gaussian type bounds.

In the above probabilistic results, the Girsanov theorem plays an important role in the proofs. On the other hand, a probabilistic interpretation of the parametrix method for SDEs with Hölder and bounded continuous diffusion and drift coefficients is given in [2]. Its application to the Monte-Carlo simulation is also introduced in [2]. Still, no discussion of its relation with Girsanov's theorem is introduced in that article.

Our first goal is to find the relation between the above Girsanov method and the parametrix expansion technique which may be fruitful for other applications. Secondly, we apply the result in order to analyze situations under which one can still obtain upper and lower Gaussian bounds although the drift coefficient may blow up at certain points. Thirdly, we would like to understand the possible effects of the irregularity of the drift coefficient at certain points on the regularity of the density of X_t at these points.

Our arguments will first consider a solution to (1.1) with bounded b and obtain a infinite series expression of $\mathbf{E}[f(X_t)]$ with bounded Borel measurable function f via. the Girsanov theorem (to remove b from (1.1)). Roughly speaking, we apply the Itô formula for the Girsanov density repeatedly and obtain infinite series expression for $\mathbf{E}[f(X_t)]$ with bounded Borel measurable function f. This probabilistic proof corresponds to the classical parametrix expansion method.

Next we consider an approximative solution, say $X^{(n)}$, to (1.1) with drift coefficient $b^{(n)} \in L^p(\mathbf{R}^d)$, $p > d$. That is, for each $n \in \mathbf{N}$, $X^{(n)}$ is given as the solution to SDE:

$$X_t^{(n)} = x_0 + B_t + \int_0^t b^{(n)}(X_s^{(n)})\mathrm{d}s,$$

where $b^{(n)}$ is a bounded function from $\mathbf{R}^d$ into $\mathbf{R}^d$ and converges to b in $L^p(\mathbf{R}^d)$. Then we prove that $\mathbf{E}[f(X_t)]$ is given by the same infinite series expression for $\mathbf{E}[f(X_t^{(n)})]$ with $b^{(n)}$ replaced by b. This will give an expression for the density function of the law of X_t.

As applications, we obtain the upper and lower Gaussian bounds for the density and the Hölder continuity of the density. We also remark that our infinite series expression for $\mathbf{E}[f(X_t)]$ can be used for an exact Monte-Carlo simulation like the one introduced in [2] which is related to an analytical approximation given in [10]. In [6], the authors discuss generalizations to integrated diffusions.

The article is divided as follows. In Section 2, we give our main notations. In Section 3, we give our probabilistic interpretation of the parametrix method for bounded drift coefficients.

In Section 4, we start considering the limit arguments needed in order to consider the density function of SDEs with non bounded drifts. Finally, in Section 5, we consider the properties of the densities deduced in Section 4. In particular, we discuss an important one-dimensional example which clarifies the effect of the irregularity of the drift on the Hölder continuity of the density function. Some conclusions and future work are given in Section 6 and we close with some auxiliary results in an Appendix A.

2. Notations

For $n, m \in \mathbf{N}$, $\mathcal{M}_b(\mathbf{R}^n; \mathbf{R}^m)$ denotes the space of bounded measurable functions defined on $\mathbf{R}^n$ into $\mathbf{R}^m$. For a bounded function $f \in \mathcal{M}_b(\mathbf{R}^n; \mathbf{R}^m)$, the uniform norm is denoted by $\|f\|_\infty := \operatorname{ess\,sup}_{x \in \mathbf{R}^n} |f(x)|$, where $|\cdot|$ stands for the Euclidean norm on $\mathbf{R}^m$. The inner product between two vectors x, y may be denoted by $x \cdot y$ or $\langle x, y \rangle$.

For $n, m, k \in \mathbf{N}$, $C_b^k(\mathbf{R}^n; \mathbf{R}^m)$ denotes the space of bounded functions defined on $\mathbf{R}^n$ into $\mathbf{R}^m$ with bounded derivatives up to order k. Similarly, $C_c^k(\mathbf{R}^n; \mathbf{R}^m)$ denotes the subspace of $C_b^k(\mathbf{R}^n; \mathbf{R}^m)$ with compact support.

For $n, m \in \mathbf{N}$ and $p \geq 1$, $L^p(\mathbf{R}^n; \mathbf{R}^m)$ denotes the space of measurable functions $f : \mathbf{R}^n \to \mathbf{R}^m$ such that $|f|^p$ is integrable, and as usual, $\|f\|_p$ denotes its corresponding norm. If $n = m$ then we will use $\mathcal{M}_b(\mathbf{R}^n)$ (and also for C_b^k and L^p) instead of $\mathcal{M}_b(\mathbf{R}^n; \mathbf{R}^n)$.

Given $t > 0$ and $n \in \mathbf{N}$, we define the set

$$\Delta_{n,t} := \{(s_1, \ldots, s_n) \in (0,t)^n; s_{m+1} < s_m, \text{for any } 1 \leq m \leq n-1\}.$$

For $(s_1, \ldots, s_n) \in \Delta_{n,t}$, we sometimes use the simplifying notation $\mathbf{s_1^n} := (s_1, \ldots, s_n)$. Given a region $A \subset \mathbf{R}^n$ and $f \in L^1(\mathbf{R}^n; \mathbf{R})$, we define

$$\int_A f(\mathbf{x_1^n}) \mathbf{dx_1^n} := \int_A f(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

For each $t > 0$, $x_0 \in \mathbf{R}^n$ and $n \in \mathbf{N}$, the function $g_t(x_0, \cdot) : \mathbf{R}^n \to \mathbf{R}$ denotes the density function of Gaussian distribution with mean vector x_0 and the covariance matrix tI, where I is the $n \times n$-unit matrix. If $x_0 = 0$, we use $g_t(\cdot)$ instead of $g_t(0, y)$.

Let $\{a_n\}_{n \in \mathbf{N}}$ be any sequence of real numbers. For $n > m$, we define

$$\prod_{j=n}^{m} a_j := 1.$$

3. The parametrix method *via* the Itô–Taylor expansion

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbf{P})$ be a complete filtered probability space and $\{B_t\}_{t \geq 0}$ be a d-dimensional $\{\mathcal{F}_t\}_{t \geq 0}$-Brownian motion. We consider the following SDE;

$$X_t = x_0 + B_t + \int_0^t b(X_s) \mathrm{d}s, \tag{3.1}$$

where $x_0 \in \mathbf{R}^d$ and $b \in \mathcal{M}_b(\mathbf{R}^d)$. For sufficient conditions for existence and uniqueness for the equation (3.1), we refer the reader to the traditional results in Section 1.2 in [5].

In this section, we introduce a parametrix representation for $\mathbf{E}[f(X_t)]$ with $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$ by using the Itô–Taylor expansion for the Girsanov density.

3.1. Expansion for the Girsanov density

In this section, we will provide the expansion of the density for the solution of (3.1). The idea is based on the application of the Girsanov formula, Itô–Taylor expansion and partial Malliavin calculus. This expansion has allowed the theoretical study of the qualitative properties of the density of the solution to (3.1). This idea already appeared in [13], where the following discussion is done in the particular case that the test function f is the complex exponential function. Other applications were also exposed in [16, 17].

Although the calculations at the beginning of this section follow in a similar manner as in Section 5 of [13] we describe them here in detail for the sake of completeness. This formulation has some connections with the PDE approach formulated in [30]. The main difference being that the expressions in [30] correspond to the kernels of the Itô chaos expansion of X_t while ours correspond to the density of X_t.

Fix $t \in (0,T]$ and define the probability measure $\mathbf{Q}$ as

$$\left.\frac{\mathrm{d}\,\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathscr{F}_u} = \exp\left(-\int_0^u b(X_s)\mathrm{d}B_s - \frac{1}{2}\int_0^u |b(X_s)|^2\mathrm{d}s\right),\ u \in [0,t]. \tag{3.2}$$

Then by Girsanov's theorem,

$$W_u := X_u - x_0,\ u \in [0,t],$$

is a $\{\mathscr{F}_t\}_{t\geq 0}$-Brownian motion under the measure $\mathbf{Q}$. From now on, we denote by $\mathbf{E}$ and $\mathbb{E}$ the expectations under $\mathbf{P}$ and $\mathbf{Q}$, respectively. Define the following processes for $u \in [0,t]$,

$$Z_u := \exp\left(\int_0^u b(W_s + x_0)\mathrm{d}W_s - \frac{1}{2}\int_0^u |b(W_s + x_0)|^2\mathrm{d}s\right). \tag{3.3}$$

Then we have the following.

Lemma 3.1. *Assume that $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$. Then we have*

$$\mathbf{E}\left[f(X_t)\right] = \mathbb{E}\left[f(W_t + x_0)Z_t\right].$$

Proof. Let $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$. From the definitions of $\mathbf{Q}$ and W, we have

$$\begin{aligned}
\mathbf{E}\left[f(X_t)\right] &= \mathbf{E}\left[f(X_t - x_0 + x_0)\right] \\
&= \mathbb{E}\left[f(W_t + x_0)\exp\left(\int_0^t b(X_s)\mathrm{d}B_s + \frac{1}{2}\int_0^t |b(X_s)|^2\mathrm{d}s\right)\right] \\
&= \mathbb{E}\left[f(W_t + x_0)\exp\left(\int_0^t b(W_s + x_0)\mathrm{d}W_s - \frac{1}{2}\int_0^t |b(W_s + x_0)|^2\mathrm{d}s\right)\right].
\end{aligned}$$

Now the definition of Z implies that the last term equals to $\mathbb{E}[f(W_t + x_0)Z_t]$. This completes the proof. $\square$

Itô's formula applied to (3.3) implies that Z satisfies the following linear SDE;

$$Z_u = 1 + \int_0^u Z_s\mathrm{d}M_s, \tag{3.4}$$

where M is a square integrable martingale defined by

$$M_u := \int_0^u b(W_s + x_0)\mathrm{d}W_s,\ u \in [0,t].$$

Now we define recursively $Z_u^{(0)} := 1$ and

$$Z_u^{(n)} := \int_0^u Z_s^{(n-1)}\mathrm{d}M_s,\ u \in [0,t], \quad n \in \mathbf{N}.$$

Then for any $N \in \mathbf{Z}_+$, using (3.4), $N+1$ times we have

$$\mathbb{E}\left[f(W_t + x_0)Z_t\right] = \sum_{n=0}^{N} I_n(t,x_0) + R_N(t,x_0), \tag{3.5}$$

where for $n, N \in \mathbf{Z}_+$, I_n and R_N is given by

$$I_n(u,x_0) := \mathbb{E}\left[f(W_u + x_0)Z_u^{(n)}\right],\ u \in [0,t],$$

and

$$R_N(t,x_0) := \mathbb{E}\left[f(W_t + x_0)\int_0^t \cdots \int_0^{s_N} Z_{s_{N+1}}\mathrm{d}M_{s_{N+1}} \cdots \mathrm{d}M_{s_1}\right]. \tag{3.6}$$

3.1.1. Estimate for R_N

Now we prove that the series in the right hand side of (3.5) uniformly converges in t and x_0. To do this, we first give an estimate for R_N.

Proposition 3.1. *Let $t \in (0,T]$, $x_0 \in \mathbf{R}^d$ and $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$. Then for each $N \in \mathbf{Z}_+$, we have*

$$\begin{aligned}|I_N(t,x_0)| \leq& \|f\|_\infty \frac{\|b\|_\infty^N T^{N/2}}{\sqrt{N!}},\\ |R_N(t,x_0)| \leq& C_N\|f\|_\infty,\end{aligned}$$

where $C_N := \sqrt{\frac{T^{N+1}e^{T\|b\|_\infty^2}}{(N+1)!}}$.

Proof. Let $t \in (0,T]$ and $x_0 \in \mathbf{R}^d$. We will prove the second inequality as the first follows similarly.

Since $\mathbb{E}[Z_u^2] \leq e^{u\|b\|_\infty^2}$ holds for any $u \in [0,t]$, we have

$$\int_{\Delta_{N+1,t}} \mathbb{E}[Z_{s_{N+1}}^2]\mathbf{ds_1^{N+1}} \leq \int_{\Delta_{N+1,t}} e^{s_{N+1}\|b\|_\infty^2}\mathbf{ds_1^{N+1}}. \tag{3.7}$$

We remark that the right hand side of (3.7) is the $N+1$th remainder term of the Maclaurin expansion of the function $e^{u\|b\|_\infty^2}$. Therefore, we obtain that

$$\int_{\Delta_{N+1,t}} \mathbb{E}[Z_{s_{N+1}}^2]\mathbf{ds_1^{N+1}} \leq \frac{t^{N+1}e^{t\|b\|_\infty^2}}{(N+1)!\|b\|_\infty^{2(N+1)}}.$$

Now applying the Hölder inequality and the L^2-isometry of the stochastic integral to (3.6), we have

$$|R_N(t,x_0)| \le \|f\|_\infty \sqrt{\frac{t^{N+1}e^{t\|b\|_\infty^2}}{(N+1)!}}. \tag{3.8}$$

Since $0 < t \le T$, we have that

$$\frac{t^{N+1}e^{t\|b\|_\infty^2}}{(N+1)!} \le \frac{T^{N+1}e^{T\|b\|_\infty^2}}{(N+1)!}. \tag{3.9}$$

Substitute (3.9) in (3.8) and define $C_N := \sqrt{\frac{T^{N+1}e^{T\|b\|_\infty^2}}{(N+1)!}}$. From here, the result follows. □

Since C_N tends to zero as $N \to +\infty$ and $\sum_n \frac{\|b\|_\infty^n u^{n/2}}{\sqrt{n!}} < \infty$, we have the following expansion for $\mathbb{E}\left[f(W_t + x_0)Z_t\right]$.

Proposition 3.2. *Let $t \in (0,T]$, $x_0 \in \mathbf{R}^d$ and $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$. Then the following equality is satisfied.*

$$\mathbf{E}\left[f(X_t)\right] = \mathbb{E}\left[f(W_t + x_0)Z_t\right] = \sum_{n=0}^{+\infty} I_n(t,x_0). \tag{3.10}$$

The convergence holds uniformly in $t \in (0,T]$ and $x_0 \in \mathbf{R}$.

3.2. Stochastic parametrix representation

The goal of this section is to rewrite each term I_n in the expansion (3.10) without the use of stochastic integrals. This will lead to expressions using instead Lebesgue integrals.

In this sense, the result to follow is linked to the classical parametric method of PDEs (see Ref. [11] for the analytic argument in the case of PDEs) although the deduction here is completely probabilistic. This section is different from the one in [13] where a Fourier analysis approach is taken as the interest in that paper was to study theoretical properties of the density of X_t.

In order to introduce the probabilistic form of the parametrix expansion, we define

$$H(u,v,x_0) := \frac{\langle W_v - W_u, b(W_u + x_0)\rangle}{v-u}, \tag{3.11}$$

for $x_0 \in \mathbf{R}^d$ and $u, v \in [0,t]$ with $u < v$.

Theorem 3.1. *For any $f \in \mathcal{M}_b(\mathbf{R}^d;\mathbf{R})$, we have*

$$\mathbf{E}\left[f(X_t)\right] = \mathbb{E}\left[f(W_t + x_0)\right] + \sum_{n=1}^{+\infty} \int_{\Delta_{n,t}} \mathbb{E}\left[f(W_t + x_0)\prod_{j=0}^{n-1} H(s_j, s_{j+1}, x_0)\right] \mathbf{ds_1^n}. \tag{3.12}$$

Proof. The proof can be carried out using Malliavin calculus which was used in [17] and [16]. That argument required to assume first that $f \in C_b^1(\mathbf{R}^d;\mathbf{R})$ and $b \in C_b^\infty(\mathbf{R}^d)$ and then take limits in order to assure that they are valid in general.

The same argument can also be achieved using the Feynman–Kac formulas as in [21] and [19] which use this argument up to the first order expansion only.

We will use the second option here. Define recursively for any sequence $t = s_0 > s_1 > s_2 > \cdots > 0$:

$$\begin{aligned} u^1(s_1, x) &= \mathbb{E}\left[f(W_t + x_0) / W_{s_1} = x \right], \; s_1 \in [0, t], \\ u^n(s_n, x) &= \mathbb{E}\left[\nabla u^{n-1}(s_{n-1}, W_{s_{n-1}}) \cdot b(W_{s_{n-1}} + x_0) / W_{s_n} = x \right], \; s_n \in [0, s_{n-1}], \; n \geq 2. \end{aligned}$$

Note that the differentiability of u^n follows easily as the transition density of the Wiener process is explicit. Furthermore u^n satisfies the heat equation $(\partial_{s_n} + \frac{1}{2}\partial_x^2)u^n(s_n, x) = 0$. In fact, one has

$$\begin{aligned} \nabla u^1(s_1, x) &= \mathbb{E}\left[f(W_t + x_0) \frac{W_t - W_{s_1}}{t - s_1} \Big/ W_{s_1} = x \right], \\ \nabla u^n(s_n, x) &= \mathbb{E}\left[\nabla u^{n-1}(s_{n-1}, W_{s_{n-1}}) \cdot b(W_{s_{n-1}} + x_0) \frac{W_{s_{n-1}} - W_{s_n}}{s_{n-1} - s_n} \Big/ W_{s_n} = x \right]. \end{aligned}$$

In order to obtain an alternative expression for $I_n(t, x_0)$, one uses the above elements and the fact that $Z^{(n)}$ is a square integrable martingale with mean zero, the Itô's formula for u^i, $i = 1, \ldots, n$ and the fact u^i satisfies the heat equation to obtain that

$$\begin{aligned} I_n(t, x_0) &= \mathbb{E}\left[u^1(t, W_t) Z_t^{(n)} \right] \\ &= \mathbb{E}\left[\int_0^t \nabla u^1(s_1, W_{s_1}) dW_{s_1} Z_t^{(n)} \right]. \end{aligned}$$

By calculating the quadratic variation and again use the Itô formula, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^t \nabla u^1(s_1, W_{s_1}) \mathrm{d}W_{s_1} Z_t^{(n)} \right] &= \mathbb{E}\left[\int_0^t \nabla u^1(s_1, W_{s_1}) \cdot b(W_{s_1} + x_0) Z_{s_1}^{(n-1)} \mathrm{d}s_1 \right] \\ &= \mathbb{E}\left[\int_0^t u^2(s_1, W_{s_1}) Z_{s_1}^{(n-1)} \mathrm{d}s_1 \right]. \end{aligned}$$

Finally, one proves inductively that

$$I_n(t, x_0) = \int_{\Delta_{n,t}} \mathbb{E}\left[f(W_t + x_0) \prod_{j=0}^{n-1} H(s_j, s_{j+1}, x_0) \right] \mathbf{ds_1^n}.$$

From the above, one achieves (3.12). □

Further, we can also obtain the following probabilistic representation which may be of independent interest.

Theorem 3.2. *Let $f \in \mathcal{M}_b(\mathbf{R}^d; \mathbf{R})$ then*

$$\mathbf{E}[f(X_t(x_0))] = e^{\lambda t} \mathbb{E}\left[f(W_t + x_0) \prod_{j=0}^{N_t - 1} H(\tau_j, \tau_{j+1}, x_0) \right].$$

Here, N is a Poisson process with mean rate $\lambda > 0$ independent of W. The random times $\tau_i := t - \eta_i$ are such that η_i is the time of the ith jump of the Poisson process N, where we define $\eta_0 = 0$.

Proof. We only need to remark that given that a Poisson process of parameter $\lambda > 0$ has n jumps in the interval $[0,t]$ then the jumps times are distributed as the order statistics of n independent uniform random variables in the interval $[0,t]$. This density corresponds to

$$e^{\lambda t}\int_{\Delta_{n,t}} f(\mathbf{s_1^n})\frac{n!}{t^n}\mathrm{d}\mathbf{s_1^n}\mathbf{P}(N_t=n)=\mathbf{E}[f(\tau_\mathbf{1}^\mathbf{n})1(N_t=n)].$$

From here the result follows. □

Remark 3.1. In the case that $f\in L^p(\mathbf{R}^d;\mathbf{R})$ with $\mathbb{E}[|f(W_t)|^p]<+\infty$ for some $p>1$, the above discussion remains essentially valid, although the estimates in Proposition 3.1 would slightly change. In particular, Theorems 3.1 and 3.2 hold for a such f.

Remark 3.2. A similar argument may be achieved using Brownian bridges. In fact, if we condition on W_t, we have

$$\mathbb{E}\left[Z_t/W_t=y-x_0\right]=\mathbb{E}\left[\exp\left(\int_0^t b(W_s+x_0)\mathrm{d}W_s-\frac{1}{2}\int_0^t|b(W_s+x_0)|^2\mathrm{d}s\right)\Big/\, W_t=y-x_0\right].$$

Under the enlarged filtration $\mathcal{F}_s\vee\sigma(W_t)$, $s\le t$, W is a semimartingale with decomposition

$$W_s=V_s+\int_0^s\frac{W_t-W_u}{t-u}\mathrm{d}u.$$

Here V is a Brownian motion in the enlarged filtration. Then one has that

$$\mathbb{E}\left[Z_t/W_t=y-x_0\right]=\mathbb{E}\left[\exp\left(\int_0^t b(W_s+x_0)\frac{W_t-W_s}{t-s}\mathrm{d}s\right)\Big/\, W_t=y-x_0\right].$$

From here expressions similar to the ones of Theorem 3.1 can be obtained. This idea has been used in [26] and [25] without obtaining the full expansion as in (3.12).

Another method to obtain the above results is to use Malliavin Calculus in the spirit of [17]. In that framework, one may also generalize the above result to the case of a drift which is time dependent.

4. A study of non-bounded drifts: the L^p-case.

In this section, we consider the case that $b\in L^p(\mathbf{R}^d)$.

Let $\{b^{(k)}\}_{k\in\mathbf{N}}$ be a sequence of $L^p(\mathbf{R}^d)\cap\mathcal{M}_b(\mathbf{R}^d)$ which converges to b in $L^p(\mathbf{R}^d)$ and almost everywhere. Now we consider the following SDE:

$$X_t^{(k)}=x_0+B_t+\int_0^t b^{(k)}(X_s^{(k)})\mathrm{d}s. \tag{4.1}$$

Note that since $b^{(k)}$ is bounded for each $k\in\mathbf{N}$, all results in previous section are applicable. We first prove that the limit $\lim_{k\to+\infty}\mathbf{E}[f(X_t^{(k)})]$ exists for any measurable function $f:\mathbf{R}^d\to\mathbf{R}$ which is at most polynomial growth or belongs to $L^p(\mathbf{R}^d;\mathbf{R})$. After that we prove the sequence $\{X^{(k)}\}_{k\in\mathbf{N}}$ is tight and its weak convergent limit, say Y, is uniquely determined and is a solution to the SDE:

$$Y_t=x_0+B_t+\int_0^t b(Y_s)\mathrm{d}s.$$

4.1. Proof of the existence of $\lim_{k\to+\infty}\mathbf{E}[f(X_t^{(k)})]$

Let $f \in L^p(\mathbf{R}^d;\mathbf{R})$. Then from Theorem 3.1, we have that

$$\mathbf{E}\left[f(X_t^{(k)})\right] = \mathbb{E}_k\left[f(W_t+x_0)\right] + \sum_{n=1}^{+\infty}\mathbb{E}_k\left[f(W_t+x_0)\int_{\Delta_{n,t}}\prod_{j=0}^{n-1}H_k(s_j,s_{j+1},x_0)\mathbf{ds_1^n}\right], \tag{4.2}$$

where H_k is defined by (3.11) with $b=b^{(k)}$ and $\mathbb{E}_k[\cdot]$ denotes the expectation under the measure $\mathbf{Q}_k$ defined by (3.2) with $b=b^{(k)}$. By using the Markov property, we have for any $n\in\mathbf{N}$,

$$\begin{aligned}&\mathbb{E}_k\left[f(W_t+x_0)\prod_{j=0}^{n-1}H_k(s_j,s_{j+1},x_0)\right]\\&=\int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n)f\left(\sum_{l=0}^n y_l+x_0\right)\prod_{j=0}^{n-1}\frac{\left\langle y_j g_{s_j-s_{j+1}}(y_j), b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right\rangle}{s_j-s_{j+1}}\mathbf{dy_0^n}.\end{aligned}$$

Remark 4.1. The above equality corresponds to the $n-1$th term in the parametrix expansion for solutions of parabolic PDEs associated to (3.1) as suggested in Remark 2.2 of [6]. In this sense, the expression given in Theorem 3.2 is the probabilistic equivalent of the parametrix expansion.

The following estimate will be used later.

Lemma 4.1. *For any $\beta\geq 0$ and $u>0$, the equation:*

$$\sup_{y\in\mathbf{R}^d}|y|^\beta\exp\left(-\frac{|y|^2}{4u}\right)=\left(\frac{2\beta u}{e}\right)^{\frac{\beta}{2}},$$

holds. In particular, for any $d\in\mathbf{N}$, $y\in\mathbf{R}^d$, $u,v>0$ with $u<v$ and $\beta\geq 0$, we have

$$\frac{|y|^\beta g_{v-u}(y)}{v-u}\leq C_\beta\frac{g_{2(v-u)}(y)}{(v-u)^{1-\frac{\beta}{2}}}, \tag{4.3}$$

where $C_\beta := 2^{\frac{d+\beta}{2}}(\frac{\beta}{e})^{\frac{\beta}{2}}$.

This proof follows by computing $\sup_{x>0}x^\beta e^{-\frac{x^2}{4u}}$ for $u>0$. The proof is omitted and follows by basic computations of maximum of functions. Here, we remark that we interpret $\beta^{\frac{\beta}{2}}=1$ in the case that $\beta=0$.

The constants C_β with $\beta=0,1$ will be used frequently in what follows. We will also use for p_*, the Hölder conjugate of $p>1$, the constant $C_g(p_*) := \|g_2\|_{p_*} = (4\pi)^{-\frac{d}{2p}}p_*^{-\frac{d}{2p_*}}$

We will also use the following explicit expression for a Beta type iterated integral:

$$\int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}}\prod_{j=0}^{n-1}(s_j-s_{j+1})^{-\frac{p+d}{2p}}\mathbf{ds_1^n} = t^{\frac{n(p-d)-d}{2p}}\frac{\Gamma(\frac{p-d}{2p})^n\Gamma(\frac{2p-d}{2p})}{\Gamma(\frac{n(p-d)+2p-d}{2p})},\ (t>0,\quad d\in\mathbf{N},\ p>d). \tag{4.4}$$

The proof is given in Section A.1 in Appendix A. In the following Lemma we give the basic estimate of a general term of the parametrix expansion based on the L^p-norms of f and b.

Lemma 4.2. *Let $t>0$, $p\in[1,+\infty]$ and $\{b^{(k)}\}_{k\in\mathbf{N}}$ be a sequence of $\mathcal{M}_b(\mathbf{R}^d)$. If $\sup_{k\in\mathbf{N}}\|b^{(k)}\|_p<+\infty$ then for any $f\in L^p(\mathbf{R}^d;\mathbf{R})$, $n\in\mathbf{N}$ and $\mathbf{s_1^n}\in\Delta_{n,t}$, we have*

$$\begin{aligned}&\int_{\mathbf{R}^{(n+1)d}}\left|g_{s_n}(y_n)f\left(\sum_{l=0}^n y_l+x_0\right)\prod_{j=0}^{n-1}\frac{\left\langle y_jg_{s_j-s_{j+1}}(y_j),b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right\rangle}{s_j-s_{j+1}}\right|\mathbf{dy_0^n}\\&\le C_{n,b,p}\|f\|_p s_n^{-\frac{d}{2p}}\prod_{j=0}^{n-1}(s_j-s_{j+1})^{-\frac{d+p}{2p}},\end{aligned}$$

where $s_0:=t$ and

$$C_{n,b,p}:=C_1^nC_0C_g(p_*)^{n+1}\left(\sup_{k\in\mathbf{N}}\|b^{(k)}\|_p\right)^n.$$

In particular, if $p\in(d,+\infty)$ then we obtain the following estimate:

$$\mathbf{E}\left[\left|f(X_t^{(k)})\right|\right]\le\frac{\|f\|_p}{t^{\frac{d}{2p}}}\sum_{n=0}^{+\infty}C_{n,b,p}t^{\frac{(p-d)n}{2p}}\frac{\Gamma\left(\frac{p-d}{2p}\right)^n\Gamma\left(\frac{2p-d}{2p}\right)}{\Gamma\left(\frac{(2+n)p-(n+1)d}{2p}\right)}<+\infty,$$

where Γ is the Gamma function.

Proof. From the Schwartz inequality and (4.3) in Lemma 4.1, we obtain that

$$\begin{aligned}&\int_{\mathbf{R}^{(n+1)d}}\left|g_{s_n}(y_n)f\left(\sum_{l=0}^n y_l+x_0\right)\prod_{j=0}^{n-1}\frac{\left\langle y_jg_{s_j-s_{j+1}}(y_j),b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right\rangle}{s_j-s_{j+1}}\right|\mathbf{dy_0^n}\\&\le\int_{\mathbf{R}^{(n+1)d}}g_{s_n}(y_n)\left|f\left(\sum_{l=0}^n y_l+x_0\right)\right|\prod_{j=0}^{n-1}\frac{|y_j|g_{s_j-s_{j+1}}(y_j)\left|b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right|}{s_j-s_{j+1}}\mathbf{dy_0^n}\\&\le\Lambda(\mathbf{s_1^n})C_1^nC_0\int_{\mathbf{R}^{(n+1)d}}g_{2s_n}(y_n)\left|f\left(\sum_{l=0}^n y_l+x_0\right)\right|\prod_{j=0}^{n-1}g_{2(s_j-s_{j+1})}(y_j)\left|b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right|\mathbf{dy_0^n}.\end{aligned}$$

In order to obtain the last inequality, we have also used the convention

$$\Lambda(\mathbf{s_1^n}):=\prod_{j=0}^{n-1}\frac{1}{\sqrt{s_j-s_{j+1}}}.$$

Now, the Hölder inequality applied to $|f(\sum_{l=0}^n y_l+x_0)|\prod_{j=0}^{n-1}\left|b^{(k)}\left(\sum_{l=j+1}^n y_l+x_0\right)\right|$ and $\prod_{j=0}^n g_{2(s_j-s_{j+1})}(y_j)$ implies that

$$\int_{\mathbf{R}^{(n+1)d}} g_{2s_n}(y_n)\left|f\left(\sum_{l=0}^{n} y_l + x_0\right)\right| \prod_{j=0}^{n-1} g_{2(s_j-s_{j+1})}(y_j)\left|b^{(k)}\left(\sum_{l=j+1}^{n} y_l + x_0\right)\right| \mathbf{dy_0^n}$$
$$\leq \|f\|_p \sup_{k\in\mathbf{N}} \|b^{(k)}\|_p^n \|g_{2s_n}\|_{p_*} \prod_{j=0}^{n-1} \|g_{2(s_j-s_{j+1})}\|_{p_*}, \tag{4.5}$$

where p_* is a Hölder conjugate for p. Furthermore, using the convention that $s_{n+1} = 0$, we obtain that

$$\prod_{j=0}^{n} \|g_{2(s_j-s_{j+1})}\|_{p_*} = \prod_{j=0}^{n} (4\pi(s_j - s_{j+1}))^{-\frac{d}{2p}} p_*^{-\frac{d}{2p_*}} = s_n^{-\frac{d}{2p}} \Lambda(\mathbf{s_1^n})^{\frac{d}{p}} C_g(p_*)^{n+1}. \tag{4.6}$$

Then we obtain that

$$\Lambda(\mathbf{s_1^n}) C_1^n C_0 \|f\|_p \sup_{k\in\mathbf{N}} \|b^{(k)}\|_p^n \|g_{2s_n}\|_{p_*} \prod_{j=0}^{n-1} \|g_{2(s_j-s_{j+1})}\|_{p_*}$$
$$\leq C_1^n C_0 C_g(p_*)^{n+1} \sup_{k\in\mathbf{N}} \|b^{(k)}\|_p^n \|f\|_p s_n^{-\frac{d}{2p}} \Lambda(\mathbf{s_1^n})^{\frac{d+p}{p}}.$$

Finally, using the definition of $C_{n,b,p}$, the first result follows.

Furthermore, if $p \in (d, +\infty)$ then from the first result, (4.2) and (4.4), we have

$$\mathbf{E}\left[\left|f(X_t^{(k)})\right|\right] \leq \mathbb{E}\left[|f(W_t + x_0)|\right] + \|f\|_p \sum_{n=1}^{+\infty} C_{n,b,p} \int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{p+d}{2p}} \mathbf{ds_1^n}$$
$$\leq \frac{\|f\|_p}{t^{\frac{d}{2p}}} \sum_{n=0}^{+\infty} C_{n,b,p} t^{\frac{(p-d)n}{2p}} \frac{\Gamma\left(\frac{p-d}{2p}\right)^n \Gamma\left(\frac{2p-d}{2p}\right)}{\Gamma\left(\frac{n(p-d)+2p-d}{2p}\right)},$$

where Γ is the Gamma function. Now from the Stirling's approximation and d'Alembert ratio test, we see that the last series converges absolutely. Therefore we obtain the second stated result. □

Corollary 4.1. *Under same assumptions in Lemma 4.2, for any $f, h \in L^p(\mathbf{R}^d; \mathbf{R})$ and $t > 0$, we have*

$$\sup_{k\in\mathbf{N}} \mathbf{E}\left[\left|f(X_t^{(k)}) - h(X_t^{(k)})\right|\right] \leq \frac{\|f-h\|_p}{t^{\frac{d}{2p}}} \sum_{n=0}^{+\infty} C_{n,b,p} t^{\frac{(p-d)n}{2p}} \frac{\Gamma\left(\frac{p-d}{2p}\right)^n \Gamma\left(\frac{2p-d}{2p}\right)}{\Gamma\left(\frac{n(p-d)+2p-d}{2p}\right)},$$

where $C_{n,b,p}$ is the same constant which appeared in Lemma 4.2.

From Lemma 4.2, we obtain a representation formula for $\lim_{k\to+\infty} \mathbf{E}\left[f(X_t^{(k)})\right]$.

Proposition 4.1. *Let $p \in (d, +\infty]$, $b \in L^p(\mathbf{R}^d)$ and $\{b^{(k)}\}_{k\in\mathbf{N}}$ be a sequence of $\mathcal{M}_b(\mathbf{R}^d)$ which converges to b in $L^p(\mathbf{R}^d)$. Then for any $f \in L^p(\mathbf{R}^d;\mathbf{R})$ and $t>0$, we have*

$$\lim_{k\to+\infty} \mathbf{E}\left[f(X_t^{(k)})\right] = \int_{\mathbf{R}^d} f(y+x_0)g_t(y)dy \\ +\sum_{n=1}^{+\infty}\int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) f\left(\sum_{l=0}^{n} y_l + x_0\right)\prod_{j=0}^{n-1}\frac{\left\langle y_j g_{s_j - s_{j+1}}(y_j), b\left(\sum_{l=j+1}^{n} y_l + x_0\right)\right\rangle}{s_j - s_{j+1}}\mathbf{dy_0^n}.$$

Remark 4.2. If we consider some suitable probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbf{P}})$ and a d-dimensional Brownian motion $\tilde{W}$ on this space, the equation in Proposition 4.1 can be written as

$$\lim_{k\to+\infty} \mathbf{E}\left[f(X_t^{(k)})\right] = \tilde{\mathbf{E}}\left[f(\tilde{W}_t + x_0)\right] + \sum_{n=1}^{+\infty}\tilde{\mathbf{E}}\left[f(\tilde{W}_t + x_0)\int_{\Delta_{n,t}}\prod_{j=0}^{n-1} H(s_j, s_{j+1}, x_0)\mathbf{ds_1^n}\right],$$

where H is defined by (3.11) with replacing W into $\tilde{W}$ and we put $s_0 := t$. When $p = +\infty$, it corresponds to Theorem 3.1.

The following result is useful in proving the tightness of the sequence $X^{(k)}$.

Proposition 4.2. *Let $\alpha > 1$ and $p \in (d, \infty)$. Under same assumptions in Proposition 4.1, for any $t > s > 0$, we have*

$$\sup_{k\in\mathbf{N}} \mathbf{E}\left[\left|X_t^{(k)} - X_s^{(k)}\right|^{\alpha}\right] \le \sum_{n=0}^{+\infty} C_{n,b,p,\alpha}(t-s)^{\frac{\alpha}{2}+\frac{(p-d)n}{2p}} \frac{\Gamma\left(\frac{p-d}{2p}\right)^n \Gamma\left(\frac{2p-d}{2p}\right)}{\Gamma\left(\frac{n(p-d)+2p-d}{2p}\right)},$$

where

$$C_{n,b,p,\alpha} := (n+1)^{\alpha-1} C_g(p_*)^{n+1} C_1^{n-1}\left(nC_{1+\alpha}C_0 + C_{\alpha/2}C_1\right)\sup_{k\in\mathbf{N}}\|b^{(k)}\|_p^n.$$

Proof. It is enough to estimate the following term:

$$\int_{\Delta_{n,t-s}}\int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n)\left|\sum_{l=0}^{n} y_l\right|^{\alpha}\prod_{j=0}^{n-1}\left|\frac{\left\langle y_j g_{s_j-s_{j+1}}(y_j), b^{(k)}\left(\sum_{m=j+1}^{n} y_m\right)\right\rangle}{s_j - s_{j+1}}\right|\mathbf{dy_0^n ds_1^n}.$$

Since $\alpha > 1$, for each $n \in \mathbf{N}$, we have

$$\left|\sum_{l=0}^{n} y_l\right|^{\alpha} \le (n+1)^{\alpha-1}\sum_{l=0}^{n}|y_l|^{\alpha}.$$

From the above and Schwartz inequality, we obtain that

$$\int_{\mathbf{R}^{(n+1)d}} \left| g_{s_n}(y_n) \left| \sum_{l=0}^{n} y_l \right|^{\alpha} \prod_{j=0}^{n-1} \frac{\left\langle y_j g_{s_j - s_{j+1}}(y_j), b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right\rangle}{s_j - s_{j+1}} \right| \mathbf{dy_0^n}$$
$$\leq (n+1)^{\alpha-1} \sum_{l=0}^{n} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) \, |y_l|^{\alpha} \prod_{j=0}^{n-1} \frac{|y_j| g_{s_j - s_{j+1}}(y_j)}{s_j - s_{j+1}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}. \tag{4.7}$$

Now (4.3) implies that

$$\sum_{l=0}^{n} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) \, |y_l|^{\alpha} \prod_{j=0}^{n-1} \frac{|y_j| g_{s_j - s_{j+1}}(y_j)}{s_j - s_{j+1}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}$$
$$\leq \sum_{l=0}^{n} C_{n,\alpha,l} (s_l - s_{l+1})^{\frac{\alpha}{2}} \int_{\mathbf{R}^{(n+1)d}} g_{2s_n}(y_n) \prod_{j=0}^{n-1} \frac{g_{2(s_j - s_{j+1})}(y_j)}{\sqrt{s_j - s_{j+1}}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}, \tag{4.8}$$

where

$$C_{n,\alpha,l} := \begin{cases} C_\alpha C_1^n, & \text{if } l = n, \\ C_{1+\alpha} C_0 C_1^{n-1}, & \text{if } l < n. \end{cases}$$

Since $s_l - s_{l+1} \leq t - s$ for any $0 \leq l \leq n$, we obtain that for $C_{n,\alpha} = C_1^{n-1}(nC_{1+\alpha}C_0 + C_\alpha C_1)$,

$$\sum_{l=0}^{n} C_{n,\alpha,l} (s_l - s_{l+1})^{\frac{\alpha}{2}} \int_{\mathbf{R}^{(n+1)d}} g_{2s_n}(y_n) \prod_{j=0}^{n-1} \frac{g_{2(s_j - s_{j+1})}(y_j)}{\sqrt{s_j - s_{j+1}}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}$$
$$\leq (t-s)^{\frac{\alpha}{2}} C_{n,\alpha} \int_{\mathbf{R}^{(n+1)d}} g_{2s_n}(y_n) \prod_{j=0}^{n-1} \frac{g_{2(s_j - s_{j+1})}(y_j)}{\sqrt{s_j - s_{j+1}}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}. \tag{4.9}$$

Now (4.5) and (4.6) imply that

$$(t-s)^{\frac{\alpha}{2}} C_{n,\alpha} \int_{\mathbf{R}^{(n+1)d}} g_{2s_n}(y_n) \prod_{j=0}^{n-1} \frac{g_{2(s_j - s_{j+1})}(y_j)}{\sqrt{s_j - s_{j+1}}} \left| b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right| \mathbf{dy_0^n}$$
$$\leq (t-s)^{\frac{\alpha}{2}} C_{n,\alpha} C_g(p_*)^{n+1} \sup_{k \in \mathbf{N}} \|b^{(k)}\|_p^n s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{d+p}{2p}}. \tag{4.10}$$

Therefore, from (4.7) to (4.10), we obtain that

$$\int_{\Delta_{n,t-s}} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) \left| \sum_{l=0}^{n} y_l \right|^{\alpha} \prod_{j=0}^{n-1} \left| \frac{\left\langle y_j g_{s_j - s_{j+1}}(y_j), b^{(k)} \left(\sum_{m=j+1}^{n} y_m \right) \right\rangle}{s_j - s_{j+1}} \right| \mathbf{dy_0^n} \mathbf{ds_1^n}$$
$$\leq (n+1)^{\alpha-1} C_{n,\alpha} C_g(p_*)^{n+1} \sup_{k \in \mathbf{N}} \|b^{(k)}\|_p^n (t-s)^{\frac{\alpha}{2}} \int_{\Delta_{n,t-s}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{d+p}{2p}} \mathbf{ds_1^n}.$$

Now, as in the proof of Lemma 4.2, we obtain the result. $\square$

4.2. The limit of $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$

We now show that $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ converges weakly and that its limit process, say $\tilde{X}_\cdot$, is a solution to the SDE:

$$\tilde{X}_t = x_0 + B_t + \int_0^t b(\tilde{X}_s)\mathrm{d}s\,. \tag{4.11}$$

Note that the strong existence and path-wise uniqueness of the solution to (4.11) with $b \in L^p(\mathbf{R}^d)$ follow from Theorem 2.1 in [18].

To show this convergence, we first prove the tightness of $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ in $C[0,T]$.

For this purpose, it is enough (see for example Problem 4.11 in Chap. 2 of [14]) to show that there exist some positive constants C, α and β such that

$$\sup_{k\in\mathbf{N}} \mathbf{E}\left[\left|X_t^{(k)} - X_s^{(k)}\right|^\alpha\right] \leq C(t-s)^{1+\beta}.$$

From Proposition 4.2, we see that this estimate holds with $\alpha = 4$ and $\beta = 1$, for example. Hence, $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ is tight in $C[0,T]$ for any $T > 0$.

Now we assume that $\{X_\cdot^{(k_m)}\}_{m\in\mathbf{N}}$ is any convergent subsequence of $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ and $\tilde{X}_\cdot$ be its limit. Then using Markov's property, we have for any $0 < t_1 < \cdots < t_l < T$ and $f \in C_c(\mathbf{R}^{d\times m})$

$$\lim_{m\to+\infty} \mathbf{E}\left[f(X_{t_1}^{(k_m)}, \ldots, X_{t_l}^{(k_m)})\right] = \mathbf{E}\left[f(\tilde{X}_t)\right].$$

On the other hand, from Proposition 4.1, the left hand side of the above equality does not depend on a choice of sub sequence $\{X_\cdot^{(k_m)}\}_{m\in\mathbf{N}}$. Therefore $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ weakly converges to a continuous process $\tilde{X}_\cdot$.

Lemma 4.3. *Let $T > 0$. Then the sequence of processes $\{(X_\cdot^{(k)}, \int_0^\cdot b(X_s^{(k)})\mathrm{d}s)\}_{k\in\mathbf{N}}$ weakly converges to the process $(\tilde{X}_\cdot, \int_0^\cdot b(\tilde{X}_s)\mathrm{d}s)$.*

Proof. Let us choose any $n, k \in \mathbf{N}$ and $\varepsilon > 0$ and assume that $f : C([0,T];\mathbf{R}^d) \times C([0,T];\mathbf{R}^d) \to \mathbf{R}$ is Lipschitz continuous. Then we have

$$\begin{aligned}
&\left|\mathbf{E}\left[f\left(X_\cdot^{(k)}, \int_0^\cdot b^{(k)}(X_s^{(k)})\mathrm{d}s\right)\right] - \mathbf{E}\left[f\left(\tilde{X}_\cdot, \int_0^\cdot b(\tilde{X}_s)\mathrm{d}s\right)\right]\right| \\
&\leq \left|\mathbf{E}\left[f\left(X_\cdot^{(k)}, \int_0^\cdot b^{(k)}(X_s^{(k)})\mathrm{d}s\right)\right] - \mathbf{E}\left[f\left(X_\cdot^{(k)}, \int_0^\cdot b^{(n)}(X_s^{(k)})\mathrm{d}s\right)\right]\right| \\
&\quad + \left|\mathbf{E}\left[f\left(X_\cdot^{(k)}, \int_0^\cdot b^{(n)}(X_s^{(k)})\mathrm{d}s\right)\right] - \mathbf{E}\left[f\left(\tilde{X}_\cdot, \int_0^\cdot b^{(n)}(\tilde{X}_s)\mathrm{d}s\right)\right]\right| \\
&\quad + \left|\mathbf{E}\left[f\left(\tilde{X}_\cdot, \int_0^\cdot b^{(n)}(\tilde{X}_s)\mathrm{d}s\right)\right] - \mathbf{E}\left[f\left(\tilde{X}_\cdot, \int_0^\cdot b(\tilde{X}_s)\mathrm{d}s\right)\right]\right| \\
&=: I_{n,k} + J_{n,k} + L_n.
\end{aligned}$$

Since f is Lipschitz continuous, there exists some positive constant C_f such that

$$I_{n,k} \leq C_f \int_0^T \mathbf{E}\left[\left|b^{(k)}(X_s^{(k)}) - b^{(n)}(X_s^{(k)})\right|\right] \mathrm{d}s.$$

Now by applying Corollary 4.1, we have

$$C_f \int_0^T \mathbf{E}\left[\left|b^{(k)}(X_s^{(k)}) - b^{(n)}(X_s^{(k)})\right|\right] \mathrm{d}s \leq \|b^{(k)} - b^{(n)}\|_p \sum_{m=0}^{+\infty} C_{m,b,p} \frac{T^{\frac{(p-d)m}{2p}+1}}{\frac{(p-d)m}{2p}+1} \frac{\Gamma\left(\frac{p-d}{2p}\right)^m \Gamma\left(\frac{2p-d}{2p}\right)}{\Gamma\left(\frac{m(p-d)+2p-d}{2p}\right)}.$$

Note that since $p > d$ and $C_{m,b,p} \leq C^m$ for some positive constant C, we can conclude that there exists some positive constant $C_{f,b,p,T}$ which depends only on f, b, p and T such that

$$I_{n,k} \leq C_{f,b,p,T} \|b^{(k)} - b^{(n)}\|_p \,.$$

Similarly, we also obtain that

$$L_n \leq C_{f,b,p,T} \|b^{(n)} - b\|_p \,.$$

Since $\{b_m\}_{m \in \mathbf{N}}$ converges to b, there exists $n_0 \in \mathbf{N}$ such that for any $n, k \geq n_0$,

$$C_{f,b,p,T}\left(\|b^{(n)} - b\|_p + \|b^{(n)} - b^{(k)}\|_p\right) < \frac{2\varepsilon}{3}.$$

Now fix such n_0 and $n \geq n_0$. Then since $\{X_\cdot^{(k)}\}_{k\in\mathbf{N}}$ weakly converges to $\tilde{X}_\cdot$, there exists $n_1 \in \mathbf{N}$ such that for any $k \geq n_1$ $J_{n,k} < \frac{\varepsilon}{3}$. Consequently, we obtain that

$$\left|\mathbf{E}\left[f\left(X_\cdot^{(k)}, \int_0^\cdot b^{(k)}(X_s^{(k)})\mathrm{d}s\right)\right] - \mathbf{E}\left[f\left(\tilde{X}_\cdot, \int_0^\cdot b(\tilde{X}_s)\mathrm{d}s\right)\right]\right| < \varepsilon,$$

for sufficiently large k. This conclude the proof. □

Now we turn to prove that the above limit process $\tilde{X}_\cdot$ is a solution to (4.11). Let us define $Y_0 := x_0$ and

$$Y_t := \tilde{X}_t - \int_0^t b(\tilde{X}_s)\mathrm{d}s,\ t \in (0,T],$$
$$Y_t^{(k)} := X_t^{(k)} - \int_0^t b^{(k)}(X_s^{(k)})\mathrm{d}s,\ t \in (0,T],\ k \in \mathbf{N}.$$

Then to prove that $\tilde{X}_\cdot$ is a solution to (4.11), it is enough to show that $Y_\cdot$ is a continuous martingale and for each $t > 0$, the quadratic variation of $Y_t - Y_0$ equals to t. We first prove that $Y_\cdot - Y_0$ is a martingale null at zero. Let $0 \leq s \leq t \leq T$, $n \in \mathbf{N}$, $h \in C_b(\mathbf{R}^{nd}; \mathbf{R})$. Choose any sequence $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq s$. Then from Lemma 4.3, we have

$$\mathbf{E}[Y_t h(Y_{s_1}, \ldots, Y_{s_n})] = \lim_{k\to+\infty} \mathbf{E}[Y_t^{(k)} h(Y_{s_1}^{(k)}, \ldots, Y_{s_n}^{(k)})].$$

Since $X_\cdot^{(k)}$ is a solution to SDE (4.1), $Y_\cdot^{(k)}$ is a martingale and hence

$$\mathbf{E}[Y_t^{(k)} h(Y_{s_1}^{(k)}, \ldots, Y_{s_n}^{(k)})] = \mathbf{E}[Y_s^{(k)} h(Y_{s_1}^{(k)}, \ldots, Y_{s_n}^{(k)})],$$

holds for any k. Now by taking a limit for k, we obtain that

$$\mathbf{E}[Y_t h(Y_{s_1}, \ldots, Y_{s_n})] = \mathbf{E}[Y_s h(Y_{s_1}, \ldots, Y_{s_n})].$$

This concludes that Y is a martingale null at zero with respect to the filtration $\{\mathscr{F}_t^Y := \sigma(Y_s; 0 \le s \le t)\}_{0\le t\le T}$.

Now, we show that the quadratic variation of $Y_t - Y_0 = t$. Let $N \in \mathbf{N}$ and define the function $f_N : \mathbf{R}^d \to \mathbf{R}$ as the following:

$$f_N(x) := \begin{cases} |x|^2, & \text{if } |x| \le N, \\ N^2, & \text{if } |x| > N. \end{cases}$$

Then from Lemma 4.3, for any $N \in \mathbf{N}$ and $s, t \in (0, T]$ with $s < t$, we have

$$\mathbf{E}\left[f_N(Y_t - Y_0)\right] = \lim_{k\to+\infty} \mathbf{E}\left[f_N\left(X_t^{(k)} - X_0^{(k)} + \int_0^t b^{(k)}(X_u^{(k)})\mathrm{d}u\right)\right].$$

However, since $X_\cdot^{(k)}$ is a solution to SDE (4.1), we see that

$$\mathbf{E}\left[f_N\left(X_t^{(k)} - X_0^{(k)} + \int_0^t b^{(k)}(X_u^{(k)})\mathrm{d}u\right)\right] = \mathbf{E}\left[f_N\left(B_t\right)\right].$$

Now letting $N \to +\infty$, we obtain that

$$\mathbf{E}\left[|Y_t - Y_0|^2\right] = \mathbf{E}\left[|B_t|^2\right] = t.$$

This implies that the quadratic variation of $Y_t - Y_0$ equals to t. Summarizing the above, we obtain the following result.

Theorem 4.1. *Let $d \in \mathbf{N}$, $p > d$, $b \in L^p(\mathbf{R}^d)$, $x_0 \in \mathbf{R}^d$ and X_t be the solution to (1.1). Then for any $f \in L^p(\mathbf{R}^d; \mathbf{R})$, we have*

$$\mathbf{E}[f(X_t)] = \sum_{n=0}^{+\infty} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) f\left(\sum_{l=0}^n y_l + x_0\right) \prod_{j=0}^{n-1} \frac{\left\langle y_j g_{s_j - s_{j+1}}(y_j), b\left(\sum_{l=j+1}^n y_l + x_0\right)\right\rangle}{s_j - s_{j+1}} \mathbf{dy_0^n}.$$

Remark 4.3. Assume that $b = b_1 + b_2$, where $b_1 \in L^p(\mathbf{R}^d)$ and $b_2 \in L^\infty(\mathbf{R}^d)$. Then by using

$$\left(\|f\|_p \|g_{2s_n}\|_{p_*} + \|b_2\|_\infty\right) \prod_{j=0}^{n-1} \left(\|g_{2(s_j - s_{j+1})}\|_{p_*} \sup_{k\in\mathbf{N}} \|b_1^{(k)}\|_p + \|b_2\|_\infty\right),$$

instead of the upper bound in (4.5), one can show that Theorem 4.1 still holds in this case.

5. Properties of the density

Throughout this section and the rest of the article we always assume that $b \in L^p(\mathbf{R}^d)$ for $p > d$. Then Theorems 3.2 and 4.1 give the following expressions for the density of X_t.

Theorem 5.1. *Let us define*

$$\theta_{u,v}(x, y) := \frac{\langle x - y, b\,(y)\rangle}{v - u},\ \ u, v \in [0, t] \text{ with } u < v,\ \ x, y \in \mathbf{R}^d,$$

and

$$\tilde{p}_{n,t}(x_0,y) := \int_{\Delta_{n,t}} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(x_0,z_n) \prod_{j=0}^{n-1} g_{s_j-s_{j+1}}(z_{j+1},z_j)\theta_{s_{j+1},s_j}(z_j,z_{j+1})\mathbf{dz_1^n ds_1^n},$$

where $s_0 = t$, $z_0 := y$. *Then the density of* X_t, *denoted by* $p_t(x_0,\cdot)$, *exists and it has the following equivalent expressions;*

$$p_t(x_0,y) = e^{\lambda t}\mathbb{E}\left[p_{\tau_1,s_0}(y-x_0-W_{\tau_1})\frac{\langle y-x_0-W_{\tau_1}, b(Y_{\tau_1}+x_0)\rangle}{\tau_1-\tau_2}\prod_{j=1}^{N_t-1} H(\tau_j,\tau_{j+1},x_0)\right], \tag{5.1}$$

and

$$p_t(x_0,x) = g_t(x_0,x) + \sum_{n=1}^{+\infty} \tilde{p}_{n,t}(x_0,x). \tag{5.2}$$

Proof. If the density p_t exists then (5.1) immediately follows from Theorem 3.2. Therefore, in the following, we only prove (5.2). Let $f \in C_c(\mathbf{R}^d;\mathbf{R})$. Note that Theorem 4.1 implies that

$$\mathbf{E}[f(X_t)] = \sum_{n=0}^{+\infty} \int_{\Delta_{n,t}} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) f\left(\sum_{l=0}^{n} y_l + x_0\right) \prod_{j=0}^{n-1} \frac{\left\langle y_j g_{s_j-s_{j+1}}(y_j), b\left(\sum_{l=j+1}^n y_l + x_0\right)\right\rangle}{s_j - s_{j+1}} \mathbf{dy_0^n ds_1^n}.$$

We now turn to calculate the n-th term of the above sum. Apply the change of variables $z_j := \sum_{l=j}^n y_l + x_0$ for $0 \le j \le n$, in order to obtain

$$\begin{aligned}
&\int_{\Delta_{n,t}} \int_{\mathbf{R}^{(n+1)d}} g_{s_n}(y_n) f\left(\sum_{l=0}^{n} y_l + x_0\right) \prod_{j=0}^{n-1} \frac{\left\langle y_j g_{s_j-s_{j+1}}(y_j), b\left(\sum_{l=j+1}^n y_l + x_0\right)\right\rangle}{s_j - s_{j+1}} \mathbf{dy_0^n ds_1^n} \\
&= \int_{\Delta_{n,t}} \int_{\mathbf{R}^{(n+1)d}} g_t(x_0,z_n) f(z_0) \prod_{j=0}^{n-1} \frac{\left\langle (z_j - z_{j+1}) g_{s_j-s_{j+1}}(z_{j+1},z_j), b(z_{j+1})\right\rangle}{s_j - s_{j+1}} \mathbf{dz_0^n ds_1^n}.
\end{aligned}$$

Now, we define

$$p_t(x_0,y) := g_t(x_0,y) + \sum_{n=1}^{+\infty} \tilde{p}_{n,t}(x_0,y),$$

where for each $n \in \mathbf{N}$,

$$\begin{aligned}
\tilde{p}_{n,t}(x_0,y) &:= \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} g_{s_n}(x_0,z_n) \prod_{j=0}^{n-1} \frac{\left\langle (z_j - z_{j+1}) g_{s_j-s_{j+1}}(z_{j+1},z_j), b(z_{j+1})\right\rangle}{s_j - s_{j+1}} \mathbf{dz_1^n ds_1^n} \\
&= \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} g_{s_n}(x_0,z_n) \prod_{j=0}^{n-1} g_{s_j-s_{j+1}}(z_{j+1},z_j)\theta_{s_{j+1},s_j}(z_j,z_{j+1})\mathbf{dz_1^n ds_1^n}.
\end{aligned}$$

Note that repeated application of Fubini's theorem implies that $\tilde{p}_{n,t}(x_0,y)$ and $p_t(x_0,y)$ are well defined a.e.

Then from Theorem 4.1, for any continuous positive function f with compact support, we have

$$\mathbf{E}\left[f(X_t)\right] = \int_{\mathbf{R}^d} f(y)p_t(x_0, y)\mathrm{d}y.$$

From here, the non-negativity of p_t follows and this completes the proof. □

We first prove that the density defined in Theorem 5.1 has a Gaussian type upper bound.

Proposition 5.1. *Let $t > 0$, $x_0 \in \mathbf{R}^d$ and X_t is a solution to (4.11). Assume that p_t is a density function for X_t given by (5.2). Then there exists some positive constant $C_{b,p}(t)$ such that for any $y \in \mathbf{R}^d$,*

$$p_t(x_0, y) \leq g_{2t}(x_0, y)C_{b,p}(t).$$

Proof. Fix $t > 0$ and $x_0 \in \mathbf{R}^d$. Recall that from the previous theorem, the density function for X_t is given by

$$p_t(x_0, y) = g_t(x_0, y) + \sum_{n=1}^{+\infty} \tilde{p}_{n,t}(x_0, y),$$

where

$$\tilde{p}_{n,t}(x_0, y) := \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} g_{s_n}(x_0, z_n) \prod_{j=0}^{n-1} \frac{\left\langle (z_j - z_{j+1}) g_{s_j - s_{j+1}}(z_{j+1}, z_j), b\left(z_{j+1}\right)\right\rangle}{s_j - s_{j+1}} \mathbf{dz_1^n ds_1^n},$$

and we put $z_0 := y$. Note that for any $s > 0$ and $z, w \in \mathbf{R}^d$, it holds that

$$(g_s(z, w))^{p_*} = (2\pi s)^{-\frac{d(p_ast - 1)}{2}} p_*^{-\frac{d}{2}} g_{\frac{s}{p_*}}(z, w).$$

Then as in the proof of Lemma 4.2, we have

$$\begin{aligned}
\tilde{p}_{n,t}(x_0, y) &\leq C_1^n C_0 \|b\|_p^n \int_{\Delta_{n,t}} \prod_{j=0}^{n-1} \frac{1}{\sqrt{s_j - s_{j+1}}} \left(\int_{\mathbf{R}^{nd}} \left| g_{2s_n}(x_0, z_n) \prod_{j=0}^{n-1} g_{2(s_j - s_{j+1})}(z_{j+1}, z_j) \right|^{p_*} \mathbf{dz_1^n} \right)^{\frac{1}{p_*}} \mathbf{ds_1^n} \\
&= \tilde{C}_{n,b,p} \int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{p+d}{2p}} \left(\int_{\mathbf{R}^d} g_{\frac{2s_n}{p_*}}(x_0, z_n) g_{\frac{2(t-s_n)}{p_*}}(z_n, y) dz_n \right)^{\frac{1}{p_*}} \mathbf{ds_1^n},
\end{aligned}$$

where $s_{n+1} := 0$, p_* is the Hölder conjugate for p and

$$\tilde{C}_{n,b,p} := C_g(p_*)^{n-1} C_1^n C_0 \|b\|_p^n.$$

Therefore, from the Chapman–Kolmogorov property, we have

$$\begin{aligned}
&\tilde{C}_{n,b,p} \int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{p+d}{2p}} \left(\int_{\mathbf{R}^d} g_{\frac{2s_n}{p_*}}(x_0, z_n) g_{\frac{2(t-s_n)}{p_*}}(z_n, y) dz_n \right)^{\frac{1}{p_*}} \mathbf{ds_1^n} \\
&= g_{\frac{2t}{p_*}}(x_0, y)^{\frac{1}{p_*}} \tilde{C}_{n,b,p} \int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1} (s_j - s_{j+1})^{-\frac{p+d}{2p}} \mathbf{ds_1^n}.
\end{aligned}$$

Since

$$g_{\frac{2t}{p_*}}(x_0,y)^{\frac{1}{p_*}} = g_{2t}(x_0,y)C_g(p_*),$$

(4.4) implies that

$$g_{\frac{2t}{p_*}}(x_0,y)^{\frac{1}{p_*}}\tilde{C}_{n,b,p}\int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}}\prod_{j=0}^{n-1}(s_j-s_{j+1})^{-\frac{p+d}{2p}}\mathbf{ds_1^n} = g_{2t}(x_0,y)C_g(p_*)\tilde{C}_{n,b,p}t^{\frac{n(p-d)-d}{2p}}\frac{\Gamma(\frac{p-d}{2p})^n\Gamma(\frac{2p-d}{2p})}{\Gamma(\frac{n(p-d)+2p-d}{2p})}.$$

Now, define

$$C_{n,b,p} := \tilde{C}_{n,b,p}C_g(p_*)\frac{\Gamma(\frac{p-d}{2p})^n\Gamma(\frac{2p-d}{2p})}{\Gamma(\frac{n(p-d)+2p-d}{2p})},$$

and then we have

$$\begin{aligned}p_t(x_0,y) &\le g_t(x_0,y) + g_{2t}(x_0,y)\sum_{n=1}^{+\infty}C_{n,b,p}t^{\frac{n(p-d)}{2p}}\\ &\le g_{2t}(x_0,y)\left(C_0+\sum_{n=1}^{+\infty}C_{n,b,p}t^{\frac{n(p-d)}{2p}}\right).\end{aligned}$$

Hence we obtain the result by setting

$$C_{b,p}(t) := C_0 + \sum_{n=1}^{+\infty}C_{n,b,p}t^{\frac{n(p-d)}{2p}}.$$

□

This Gaussian type upper bound implies the following lower Gaussian type bound for p_t.

Proposition 5.2. *Let $t>0$, $x_0\in\mathbf{R}^d$ and X_t is a solution to (4.11). Let p_t be the density function for X_t which is given in (5.2). Then there exists some positive constants L and C such that for any $x_0,y\in\mathbf{R}^d$,*

$$p_t(x_0,y) \ge Lg_{Ct}(y-x_0).$$

Proof. Let $t>0$. From the triangle inequality and (5.2), we have

$$p_t(x_0,y) \ge g_t(x_0,y) - \sum_{n=1}^{+\infty}|\tilde{p}_{n,t}(x_0,y)|,$$

for any $x_0,y\in\mathbf{R}^d$. Then from the proof of Proposition 5.1, we see that

$$p_t(x_0,y) \ge g_t(x_0,y) - g_{2t}(x_0,y)\sum_{n=1}^{+\infty}C_{n,b,p}t^{\frac{n(p-d)}{2p}},$$

for any $x_0, y \in \mathbf{R}^d$. In particular, if $|y - x_0| \leq \sqrt{2td \log 2}$ then we have $g_t(x_0, y) \geq g_{2t}(x_0, y)$. Therefore we have

$$g_t(x_0, y) - g_{2t}(x_0, y) \sum_{n=1}^{+\infty} C_{n,b,p} t^{\frac{n(p-d)}{2p}} \geq g_{2t}(x_0, y) \left(1 - \sum_{n=1}^{+\infty} C_{n,b,p} t^{\frac{n(p-d)}{2p}} \right),$$

if $|y - x_0| \leq \sqrt{2td \log 2}$ holds. Furthermore, as $p > d$, there exists some positive constant t_* such that for any $t \leq t_*$,

$$1 - \sum_{n=1}^{+\infty} C_{n,b,p} t^{\frac{n(p-d)}{2p}} \geq \frac{1}{2}.$$

As a result, for any $t \in (0, t_*]$ and $|y - x_0| \leq \sqrt{2td \log 2}$, we have the following local lower bound for p_t:

$$p_t(x_0, y) \geq \frac{1}{2} g_{2t}(x_0, y). \tag{5.3}$$

Now we prove a global lower bound by chaining the local lower bounds using the Chapman–Kolmogorov formula for p_t. Let $t > 0$, $R(t) := \sqrt{2td \log 2}$ and $x_0, y \in \mathbf{R}^d$. We let M be defined as

$$M := \left\lfloor \max \left\{ \frac{2|y - x_0|^2}{td \log 2}, \ \frac{t}{4t_*} \right\} \right\rfloor.$$

Here $\lfloor \cdot \rfloor$ denotes the floor function. If $M = 0$ then (5.3) holds. Assume that $M \geq 1$. We now define

$$z_i := x_0 + \frac{i}{4M}(y - x_0), \ (0 \leq i \leq 4M).$$

Then from the Chapman–Kolmogorov equation, we have

$$\begin{aligned} p_t(x_0, y) &= \int_{\mathbf{R}^{(4M-1)d}} \prod_{i=0}^{4M-1} p_{\frac{t}{4M}}(w_i, w_{i+1}) \mathbf{dw_1^{4M-1}} \\ &\geq \int_{\mathbf{R}^{(4M-1)d}} \prod_{i=0}^{4M-1} p_{\frac{t}{4M}}(w_i, w_{i+1}) 1_{B_i}(w_{i+1}) \mathbf{dw_1^{4M-1}}, \end{aligned} \tag{5.4}$$

where $w_0 := x_0$, $w_{4M} := y$, $c := \frac{3M-1}{8Md}$,

$$B_i := \{x \in \mathbf{R}^d ; |x - z_i| < cR(t/4M)\}, \ (1 \leq i \leq 4M),$$

and 1_{B_i} denotes the indicator function for B_i. Note that $1_{B_{4M}}(w_{4M}) = 1$ because $z_{4M} = w_{4M} = y$ and that although the definition of c depends on M, we have that $\frac{1}{8d} \leq c \leq \frac{3}{8d}$. From the definitions of M, z_i, B_i and c, we see that if $w_i \in B_i$, $w_{i+1} \in B_{i+1}$, $0 \leq i \leq 4M - 1$ then

$$\begin{aligned} |w_i - w_{i+1}| &\leq |w_i - z_i| + |z_i - z_{i+1}| + |z_{i+1} - w_{i+1}| \\ &\leq 2cR(t/4M) + \frac{|y - x_0|}{4M} \\ &\leq \frac{3}{4d} R(t/4M) + \frac{\sqrt{td \log 2}}{4\sqrt{2M}} \leq R(t/4M). \end{aligned}$$

Hence, from (5.4) to (5.3) and the fact that $g_{\frac{t}{4M}}(x)$ is a decreasing function on $|x|$, we obtain that

$$\begin{aligned} p_t(x_0,y) &\geq 2^{-4M} \int_{\mathbf{R}^{(4M-1)d}} \prod_{i=0}^{4M-1} g_{\frac{t}{4M}}(w_{i+1},w_i) 1_{B_i}(w_{i+1}) \mathbf{dw_1^{4M-1}} \\ &\geq 2^{-4M} \left(cR(t/4M)\right)^{(4M-1)d} V_d^{4M-1} G_{d,M,t}^{4M}, \end{aligned} \tag{5.5}$$

where $V_d := \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ is the volume of the unit ball in $\mathbf{R}^d$ and

$$G_{d,M,t} := \left(\frac{2M}{\pi t}\right)^{\frac{d}{2}} \exp\left(-\frac{2MR(t/4M)^2}{t}\right).$$

Furthermore, as $\frac{R(t/4M)^2 4M}{2t} = d\log 2$ then

$$G_{d,M,t}^{4M} = (\pi t)^{-2Md}(2M)^{2Md} e^{-4Md\log 2}.$$

Substitute this in (5.5) to obtain that for some appropriate strictly positive constants C and L which depend on the dimension d, we have

$$\begin{aligned} p_t(x_0,y) &\geq 2^{-4M} \left(cR(t/4M)\right)^{(4M-1)d} V_d^{4M-1} (\pi t)^{-2Md}(2M)^{2Md} e^{-4Md\log 2} \\ &= 2^{-4M}(c\sqrt{d\log 2})^{(4M-1)d} t^{-d/2} \pi^{-2Md}(2M)^{d/2} e^{-4Md\log(2)} \\ &= C_d t^{-d/2} \exp\left(4Md\log(2^{-1}c\sqrt{d\pi\log 2}) + \frac{d}{2}\log(2M) - 4Md\log 2\right) \\ &\geq L g_{Ct}(x_0,y). \end{aligned}$$

In the last inequality, we have used that $\log(2^{-1}c\sqrt{d\pi\log 2}) < 0$, $\log(x) \geq -x$ for $x \geq 2$ and $M \leq \frac{2|x_0-y|^2}{td\log 2} + 1$. $\square$

Remark 5.1. The method of proof is essentially due to Aronson and based on some crucial estimate by Nash. We replaced Nash's estimate with a similar estimate derived directly from the parametrix method. This idea can be applied to the study of stochastic PDEs (see Ref. [23]). We would like to thank a referee for pointing this and a mistake in the proof of a preliminary version.

5.1. L^p modulus of continuity of the density of X_t

Now we discuss about the L^p modulus of continuity of the density p_t. For $p > 1$ and $b \in L^p(\mathbf{R}^d)$, we define a function h from $\mathbf{R}^d$ into $[0,+\infty)$ which measures the modulus of continuity of b. That is,

$$h(r) := \left(\int_{\mathbf{R}^d} |b(z) - b(z+r)|^p \, dz\right)^{\frac{1}{p}}.$$

Then we obtain the following estimate.

Theorem 5.2. *Let $p > d$, $t > 0$ and $b \in L^p(\mathbf{R}^d)$. Then there exists a positive constant $C_{b,p,d,t}$ such that*

$$|p_t(x_0,x) - p_t(x_0,y)| \leq C_{b,p,d,t} \max\left\{|x-y|, h(x-y)\right\}.$$

Proof. From the equation (5.2) in Theorem 5.1, we have

$$|p_t(x_0,x)-p_t(x_0,y)| \le |g_t(x_0,x)-g_t(x_0,y)| + \sum_{n=1}^{+\infty} |\tilde{p}_{n,t}(x_0,x)-\tilde{p}_{n,t}(x_0,y)|.$$

Since g_t is Lipschitz continuous, there exists positive constant $C_{t,d}$ such that $|g_t(x_0,x)-g_t(x_0,y)| \le C_{t,d}|x-y|$ holds for any $x,y \in \mathbf{R}^d$.

Now we turn to estimate for $\tilde{p}_{n,t}$. To do this, we first prepare notations. For each fixed $n \in \mathbf{N}$, $t>0$ and $x_0 \in \mathbf{R}^d$, we define functions A_j and B_j on $\mathbf{R}^d \times \mathbf{R}^{nd}$ as follows:

$$A_j(y;\mathbf{z_1^n}) := \begin{cases} \frac{\left\langle (y-z_1)g_{t-s_1}(y-z_1), b(z_1)\right\rangle}{t-s_1}, & j=0,\\ \frac{\left\langle (z_j-z_{j+1})g_{s_j-s_{j+1}}(z_{j+1},z_j), b(z_{j+1})\right\rangle}{s_j-s_{j+1}}, & 1\le j\le n-1,\\ g_{s_n}(z_n-x_0), & j=n. \end{cases}$$

$$B_j(y;\mathbf{z_1^n}) := \begin{cases} \frac{\left\langle -z_1 g_{t-s_1}(-z_1), b(z_1+y)\right\rangle}{t-s_1}, \ j=0,\\ \frac{\left\langle (z_j-z_{j+1})g_{s_j-s_{j+1}}(z_{j+1},z_j), b(z_{j+1}+y)\right\rangle}{s_j-s_{j+1}}, \ 1\le j\le n-1,\\ g_{s_n}(z_n+y-x_0), \ j=n. \end{cases}$$

Then from the definition of $\tilde{p}_{n,t}$, we have

$$|\tilde{p}_{n,t}(x_0,x)-\tilde{p}_{n,t}(x_0,y)| = \left| \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n A_j(x;\mathbf{z_1^n}) \right) \mathbf{dz_1^n ds_1^n} - \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n A_j(y;\mathbf{z_1^n}) \right) \mathbf{dz_1^n ds_1^n} \right|.$$

Now applying the change of variables with $w_j = z_j - x$ for the former integral and $w_j = z_j - y$ for the latter one, we have that

$$\begin{aligned} &\left| \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n A_j(x;\mathbf{z_1^n}) \right) \mathbf{dz_1^n ds_1^n} - \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n A_j(y;\mathbf{z_1^n}) \right) \mathbf{dz_1^n ds_1^n} \right| \\ &= \left| \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n B_j(x;\mathbf{w_1^n}) \right) \mathbf{dw_1^n ds_1^n} - \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} \left(\prod_{j=0}^n B_j(y;\mathbf{w_1^n}) \right) \mathbf{dw_1^n ds_1^n} \right| \\ &\le \sum_{k=0}^n \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} |B_k(x;\mathbf{w_1^n}) - B_k(y;\mathbf{w_1^n})| \prod_{j=0}^{k-1} |B_j(x;\mathbf{w_1^n})| \prod_{j=k+1}^{n} |B_j(y;\mathbf{w_1^n})| \, \mathbf{dw_1^n ds_1^n}. \end{aligned}$$

Here we use Jensen's inequality and put $w_0 := 0$ to obtain the last inequality. Furthermore, from the definition of B_j, we see that the following estimates hold for any $x,y \in \mathbf{R}^d$ and $\mathbf{w_1^n} \in \mathbf{R}^{nd}$.

$$|B_j(x;\mathbf{w_1^n}) - B_j(y;\mathbf{w_1^n})| \le \begin{cases} C_1 \dfrac{g_{2(t-s_1)}(-w_1)|b(w_1+x)-b(w_1+y)|}{\sqrt{t-s_1}}, & j=0,\\ C_1 \dfrac{g_{2(s_j-s_{j+1})}(w_j-w_{j+1})|b(w_{j+1}+x)-b(w_{j+1}+y)|}{\sqrt{s_j-s_{j+1}}}, & 1\le j\le n-1,\\ \left| g_{s_n}(w_n+x-x_0) - g_{s_n}(w_n+y-x_0) \right|, & j=n, \end{cases}$$

and

$$|B_j(x;\mathbf{w_1^n})| \leq \begin{cases} C_1 \dfrac{g_{2(t-s_1)}(-w_1)|b(w_1+x)|}{\sqrt{t-s_1}}, & j=0, \\ C_1 \dfrac{g_{2(s_j-s_{j+1})}(w_j-w_{j+1})|b(w_{j+1}+x)|}{\sqrt{s_j-s_{j+1}}}, & 1 \leq j \leq n-1, \\ g_{s_n}(w_n+x-x_0), & j=n. \end{cases}$$

Now from the Hölder inequality and the above estimations, we obtain as in the proof of Theorem 4.2 that for any $x, y \in \mathbf{R}^d$ and with $s_{n+1}=0$,

$$\begin{aligned} &\sum_{k=0}^{n} \int_{\Delta_{n,t}} \int_{\mathbf{R}^{nd}} |B_k(x;\mathbf{w_1^n}) - B_k(y;\mathbf{w_1^n})| \prod_{j=0}^{k-1} |B_j(x;\mathbf{w_1^n})| \prod_{j=k+1}^{n} |B_j(y;\mathbf{w_1^n})| \, \mathbf{dw_1^n ds_1^n} \\ &\quad \leq nh(x-y)\|b\|_p^{n-1} C_g(p_*)^{n+1} g_{2t}(x_0,y) \int_{\Delta_{n,t}} \prod_{j=0}^{n} (s_j-s_{j+1})^{-\frac{d}{2p}} \mathbf{ds_1^n} \\ &\quad + |x-y| C_1 \|b\|_p^n C_g(p_*)^{n+1} \left(g_{2t}(x_0,x)+g_{2t}(x_0,y)\right) \int_{\Delta_{n,t}} s_n^{-\frac{1}{2}} \prod_{j=0}^{n} (s_j-s_{j+1})^{-\frac{d}{2p}} \mathbf{ds_1^n}. \end{aligned}$$

Since $p > d$, we can apply Lemma A.1 for the above integrals and then we have

$$\begin{aligned} &nh(x-y)\|b\|_p^{n-1} C_g(p_*)^{n+1} g_{2t}(x_0,y) \int_{\Delta_{n,t}} \prod_{j=0}^{n} (s_j-s_{j+1})^{-\frac{d}{2p}} \mathbf{ds_1^n} \\ &\quad + |x-y| C_1 \|b\|_p^n C_g(p_*)^{n+1} \left(g_{2t}(x_0,x)+g_{2t}(x_0,y)\right) \int_{\Delta_{n,t}} s_n^{-\frac{1}{2}} \prod_{j=0}^{n} (s_j-s_{j+1})^{-\frac{d}{2p}} \mathbf{ds_1^n} \\ &\quad \leq C_{n,b,p,d,t} \left(h(x-y) t^{-\frac{(p+1)d}{2p}} + |x-y| t^{-\frac{d+1}{2}} \right), \end{aligned}$$

where

$$C_{n,b,p,d,t} := t^{n(1-\frac{d}{2p})} C_g(p_*)^{n+1} \max \left\{ n\|b\|_p^{n-1} \frac{\Gamma\left(\frac{2p-d}{2p}\right)^{n+1}}{\Gamma\left((n+1)\left(\frac{2p-d}{2p}\right)\right)}, 2C_1\|b\|_p^n \frac{\Gamma\left(\frac{2p-d}{2p}\right)^{n} \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+n\left(\frac{2p-d}{2p}\right)\right)} \right\}.$$

Note that $C_{n,b,p,d,t}$ is summable in n and hence we can conclude that

$$|p_t(x_0,x) - p_t(x_0,y)| \leq C_{b,p,d,t} \max\left\{h(x-y), |x-y|\right\},$$

where

$$C_{b,p,d,t} := t^{-\frac{d+1}{2}} \sum_{n=1}^{+\infty} C_{n,b,p,d,t}.$$

□

5.2. Local Hölder continuity in the one-dimensional case

If $d=1$, we can obtain another estimate about the Hölder continuity of the density of X_t. In this case, Lemma A.2 in Appendix A is a key lemma to obtain our results.

5.2.1. Some definitions and basic inequalities

Before the discussion, we set up some notations. Let $x, x_0 \in \mathbf{R}$, $t>0$ and $n \geq 1$. In the following, we define $z_0 := x$ and $z_{n+1} := x_0$. We define the following three functions.

$$\begin{aligned}
\varphi_{n,t,x_0}(x, \mathbf{z_1^n}) &:= \frac{1}{\sqrt{2\pi t}} \exp\left(- \frac{\left(\sum_{k=1}^{n+1} |z_{k-1} - z_k| \right)^2}{2t} \right), \\
G_{n,t,x_0}(x, \mathbf{z_1^n}) &:= \varphi_{n,t,x_0}(x, \mathbf{z_1^n}) \prod_{k=1}^{n} \operatorname{sgn}(z_{k-1} - z_k), \\
h_{n,t,x_0}(x, \mathbf{z_1^n}) &:= \begin{cases} G_{1,t,x_0}(x, z_1), & n=1, \\ \varphi_{n,t,x_0}(x, \mathbf{z_1^n}) \operatorname{sgn}(x - z_1) \operatorname{sgn}(z_1 - z_2), & n \geq 2. \end{cases}
\end{aligned}$$

Then from Lemma A.2, we have

$$G_{n,t,x_0}(x, \mathbf{z_1^n}) = \int_{\Delta_{k,t}} g_{s_k}(z_k) \prod_{j=0}^{k-1} \frac{g_{s_j - s_{j+1}}(z_j - z_{j+1})}{s_j - s_{j+1}} \mathbf{ds_1^k} \prod_{k=1}^{n} (z_{k-1} - z_k),$$

for almost every $\mathbf{z_1^n} \in \mathbf{R}^n$. Notice that for any $x \in \mathbf{R}$ and almost every $\mathbf{z_1^n} \in \mathbf{R}^n$, it holds that

$$|h_{n,t,x_0}(x, \mathbf{z_1^n})| = |G_{n,t,x_0}(x, \mathbf{z_1^n})| = \varphi_{n,t,x_0}(x, \mathbf{z_1^n}).$$

We also remark that for any $x \in \mathbf{R}$ and $\mathbf{z_1^n} \in \mathbf{R}^n$, the inequality

$$\begin{aligned}
\varphi_{n,t,x_0}(x, \mathbf{z_1^n}) &= \sqrt{2} \exp\left(- \frac{\left(|x - z_1| + |z_1 - z_2| + \sum_{k=3}^{n+1} |z_{k-1} - z_k| \right)^2}{4t} \right) \varphi_{n,2t,x_0}(x, \mathbf{z_1^n}) \\
&\leq \sqrt{2} \exp\left(- \frac{\left(|x - z_2| + \sum_{k=3}^{n+1} |z_{k-1} - z_k| \right)^2}{4t} \right) \varphi_{n,2t,x_0}(x, \mathbf{z_1^n}) \\
&= 2\sqrt{2\pi t} \varphi_{n-1,2t,x_0}(x, \mathbf{z_2^n}) \varphi_{n,2t,x_0}(x, \mathbf{z_1^n}).
\end{aligned} \tag{5.6}$$

Similarly, it is also holds that

$$\varphi_{n,t,x_0}(x, \mathbf{z_1^n}) \leq \varphi_{n-1,t,x_0}(x, \mathbf{z_2^n}). \tag{5.7}$$

5.2.2. Hölder continuity of $p_t(x_0, \cdot)$

Let $x \in \mathbf{R}$. In this section, we assume that there exist positive constants $\alpha, \beta \in (0,1)$ such that

$$\lim_{\delta \downarrow 0} \delta^{-\alpha} \int_x^{x+\delta} |b(z)| \mathrm{d}z < +\infty, \tag{5.8}$$

$$\lim_{\delta\downarrow 0}\delta^{-\beta}\int_{x-\delta}^{x}|b(z)|\mathrm{d}z<+\infty. \tag{5.9}$$

Then the following quantities $R(x,\alpha)$ and $L(x,\alpha)$ determine the right or left continuity of the density $p_t(x_0,x)$.

$$R(x,\alpha):=\lim_{\delta\downarrow 0}\delta^{-\alpha}\int_{x}^{x+\delta}b(z)\mathrm{d}z, \tag{5.10}$$

$$L(x,\beta):=\lim_{\delta\downarrow 0}\delta^{-\beta}\int_{x-\delta}^{x}b(z)\mathrm{d}z. \tag{5.11}$$

Now we have the following result about the continuity of the density p_t.

Theorem 5.3. *Let $d=1$, $p>1$, $t>0$, $x\in\mathbf{R}$, $b\in L^p(\mathbf{R})$. Assume that (5.8) holds for some $\alpha\in(0,1)$. Then for any $x_0\in\mathbf{R}$, we have*

$$\lim_{\delta\downarrow 0}\frac{p_t(x_0,x+\delta)-p_t(x_0,x)}{\delta^{\alpha}}=2R(x,\alpha)p_t(x_0,x). \tag{5.12}$$

Similarly, if (5.9) holds for some $\beta\in(0,1)$. Then for any $x_0\in\mathbf{R}$, we have

$$\lim_{\delta\downarrow 0}\frac{p_t(x_0,x)-p_t(x_0,x-\delta)}{\delta^{\beta}}=2L(x,\beta)p_t(x_0,x). \tag{5.13}$$

Proof. We only prove (5.12) because (5.13) can be proved in the similar discussion.

Let $\delta\in(0,1)$ and fix $x,x_0\in\mathbf{R}$. From the equation (5.2) in Theorem 5.1, we have

$$p_t(x_0,x+\delta)-p_t(x_0,x)=g_t(x_0,x+\delta)-g_t(x_0,x)+\sum_{n=1}^{+\infty}(\tilde{p}_{n,t}(x_0,x+\delta)-\tilde{p}_{n,t}(x_0,x)).$$

From the Lipschitz continuity of $g_t(x_0,\cdot)$ and Lemma A.3, we see that

$$\lim_{\delta\downarrow 0}\delta^{-\alpha}\left(g_t(x_0,x+\delta)-g_t(x_0,x)+\tilde{p}_{1,t}(x_0,x+\delta)-\tilde{p}_{1,t}(x_0,x)\right)=2R(x,\alpha)g_t(x_0,x).$$

We turn to the case $n\geq 2$. From Lemma A.2, we have

$$\tilde{p}_{n,t}(x_0,x+\delta)-\tilde{p}_{n,t}(x_0,x)=\int_{\mathbf{R}^n}(G_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-G_{n,t,x_0}(x,\mathbf{z_1^n}))\prod_{k=1}^{n}b(z_k)\mathbf{dz_1^n}.$$

We define

$$f(\mathbf{z_2^n}):=\prod_{k=2}^{n}b(z_k)\prod_{k=3}^{n}\mathrm{sgn}(z_{k-1}-z_k),$$

where $z_{n+1} := x_0$ Then we can rewrite the previous integral as follows.

$$\begin{aligned}&\int_{\mathbf{R}^n} (G_{n,t,x_0}(x+\delta, \mathbf{z_1^n}) - G_{n,t,x_0}(x, \mathbf{z_1^n})) \prod_{k=1}^{n} b(z_k) \mathbf{dz_1^n} \\ &= \int_{\mathbf{R}^{n-1}} f(\mathbf{z_2^n}) \left(\int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x+\delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n}) \right) \mathrm{d}z_1 \right) \mathbf{dz_2^n}.\end{aligned}$$

Lemma A.3 implies that for almost every $\mathbf{z_2^n} \in \mathbf{R}^{n-1}$, we have

$$\begin{aligned}&\lim_{\delta \downarrow 0} \delta^{-\alpha} f(\mathbf{z_2^n}) \int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x+\delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n}) \right) \mathrm{d}z_1 \\ &= 2R(x,\alpha) f(\mathbf{z_2^n}) \operatorname{sgn}(x - z_2) \varphi_{n-1,t,x_0}(x, \mathbf{z_2^n}) \\ &= 2R(x,\alpha) G_{n-1,t,x_0}(x, \mathbf{z_2^n}) \prod_{k=2}^{n} b(z_k).\end{aligned}$$

On the other hand, from inequalities (A.2) and (5.7), we have

$$\begin{aligned}&\left| \delta^{-\alpha} \int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x+\delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n}) \right) \mathrm{d}z_1 \right| \\ &\leq \delta^{1-\alpha} 4\sqrt{\pi} \|b\|_p \|g_{4t}\|_{p_*} C_1 (\varphi_{n,4t,x_0}(x, \mathbf{z_2^n}) + \varphi_{n,4t,x_0}(x+\delta, \mathbf{z_2^n})) \\ &\quad + (\varphi_{n,t,x_0}(x, \mathbf{z_2^n}) + \varphi_{n,t,x_0}(x+\delta, \mathbf{z_2^n})) \delta^{-\alpha} \int_x^{x+\delta} |b(z)| \mathrm{d}z.\end{aligned}$$

From (5.8), the above inequality implies that there exists positive constant C which is independent of δ, n and $\mathbf{z_2^n}$ such that

$$\begin{aligned}\left| \delta^{-\alpha} \int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x+\delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n}) \right) \mathrm{d}z_1 \right| &\leq C (\varphi_{n,4t,x_0}(x+\delta, \mathbf{z_2^n}) + \varphi_{n,4t,x_0}(x, \mathbf{z_2^n})) \\ &= C \left(|G_{n-1,4t}(x_0, x, \mathbf{z_2^n})| + |G_{n-1,4t}(x_0, x+\delta, \mathbf{z_2^n})| \right).\end{aligned}$$

Now from Lemma A.2, we see that

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}} \left(|G_{n-1,4t,x_0}(x, \mathbf{z_2^n})| + |G_{n-1,4t,x_0}(x+\delta, \mathbf{z_2^n})| \right) \prod_{k=2}^{n} |b(z_k)| \mathbf{dz_2^n} \\ &= \int_{\Delta_{n-1,2t}} \int_{\mathbf{R}^{n-1}} \prod_{k=2}^{n} |b(z_k)| \left(g_{2s_n}(x_0, z_n) \prod_{k=1}^{n-1} \frac{|z_k - z_{k+1}| g_{2(s_k - s_{k+1})}(z_k - z_{k+1})}{2(s_k - s_{k+1})} \right) \mathbf{dz_2^n ds_2^n} \\ &\quad + \int_{\Delta_{n-1,2t}} \int_{\mathbf{R}^{n-1}} \prod_{k=2}^{n} |b(z_k)| \left(g_{2s_n}(x_0, z_n) \prod_{k=1}^{n-1} \frac{|z_k - z_{k+1}| g_{2(s_k - s_{k+1})}(z_k - z_{k+1})}{2(s_k - s_{k+1})} \right) \mathbf{dz_2^n ds_2^n},\end{aligned}$$

where we put $z_1 := x$ and $s_1 := 2t$ in the former integral and $z_1 := x + \delta$ and $s_1 := 2t$ in the latter integral. In the discussion of Section 4, we have already seen that these integrals are finite. Therefore, by applying the dominated convergence theorem, we obtain that

$$\begin{aligned}&\lim_{\delta\downarrow 0}\delta^{-\alpha}\int_{\mathbf{R}^{n-1}} f(\mathbf{z_2^n})\left(\int_{\mathbf{R}} b(z_1)\left(h_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-h_{n,t,x_0}(x,\mathbf{z_1^n})\right)\mathrm{d}z_1\right)\mathbf{dz_2^n}\\&=2R(x,\alpha)\int_{\mathbf{R}^{n-1}} G_{n-1,t}(x_0,x,\mathbf{z_2^n})\prod_{k=2}^{n} b(z_k)\mathbf{dz_2^n}\\&=2R(x,\alpha)\tilde{p}_{n-1,t}(x_0,x),\end{aligned}$$

for all $n \geq 2$. This completes the proof. □

Example 5.1. Let $\delta > 0$, $\alpha \in (0,1)$ and $b(z) := \frac{1}{z^\alpha}1_{(0,1)}(z)$. In this case, we see that

$$R(x,1-\alpha)=\begin{cases}\dfrac{1}{1-\alpha} & x=0,\\ 0 & x\neq 0.\end{cases}$$

Therefore, by applying Proposition 5.3, we have

$$\lim_{\delta\downarrow 0}\frac{p_t(x_0,x+\delta)-p_t(x_0,x)}{\delta^{1-\alpha}}=\begin{cases}\dfrac{1}{1-\alpha}p_t(x_0,0) & x=0,\\ 0 & x\neq 0,\end{cases}$$

for all $x_0 \in \mathbf{R}$ and $t > 0$. Furthermore, since the density $p_t(x_0,\cdot)$ has a Gaussian lower bound (see Prop. 5.2), $\frac{1}{1-\alpha}p_t(x_0,0) > 0$.

On the other hand, it also holds that $L(x,\beta) = 0$ for all $x \in \mathbf{R}$ and $\beta \in (0,1)$. Therefore, again from Proposition 5.3, we see that

$$\lim_{\delta\downarrow 0}\frac{p_t(x_0,x)-p_t(x_0,x-\delta)}{\delta^{\beta}}=0,$$

holds for all $x, x_0 \in \mathbf{R}$, $\beta \in (0,1)$ and $t > 0$.

These facts tell us that the right Hölder regularity of the density $p_t(x_0,0)$ is different from the its left Hölder regularity. Furthermore, the irregularity point, $x = 0$, is related to the singular point of b and independent of the initial value x_0 and t.

Remark 5.2. Note that this example and Theorem 5.3 rely on Lemma A.2 which is a one-dimensional result. Therefore multi-dimensional extensions of this example are non-trivial.

6. Some conclusions

In this article, we have given a presentation which hopefully clarifies the link between the mathematical expressions for the density or fundamental solution using the classical parametrix method in PDEs and its stochastic counterpart. We have used this methodology to give an application for the study of densities of SDEs with drift coefficient which may explode mildly in the sense that $b \in L^p(\mathbf{R}^d)$.

In particular, we tried to give a first hint towards the question of the existence of upper and lower Gaussian bounds and the Hölder regularity of the density for such drift coefficients. Note that due to results in [9], one has that if the drift is smooth at a neighborhood then the density will also be smooth in that neighborhood.

In Section 6.5 of [14], we can find an example of non-differentiable density with discontinuous bounded drift coefficient and see that the discontinuous point of the drift corresponds with non-differentiable point of the density. Moreover, one can check that although the density is not differentiable, but Lipschitz continuous.

On the other hand, as we have seen in Example 5.1, if the drift explodes at $x = 0$ then the density is no longer Lipschitz at this point, but Hölder continuous. Hence, intuitively, the important factor in this example

TABLE 1. Regularity of the density at $x=0$ with discontinuous drift at $x=0$.

Singularity of the drift at $x=0$	Regularity of density	Example of drift
(Locally) bounded	(Locally) Lipschitz	$\mathrm{sgn}(x)$
Locally unbounded and integrable	$\min\{1-\alpha, 1-\beta\}$-Hölder	$x^{-\alpha}1_{(0,1)}(x)+\lvert x\rvert^{-\beta}1_{(-1,0)}(x)$
Locally unbounded and not integrable	Discontinuous	$(2x)^{-1}$
Locally unbounded and not integrable	$2^{-1}(c-2^{-1})$-Hölder	cx^{-1}, $c>2^{-1}$

which determines the regularity of the density is the explosion speed of the drift at $x=0$ (described in (5.10) and (5.11)). As it is well known, cases such as $b(x)=\frac{c}{x}$ lead to Bessel processes whose densities have different properties according to the value of c. In Table 1, heuristic conclusions are expressed. Table 1 also shows the large amount of problems that need to be tackled in the future.

Still, we believe that there are many possible extensions of this methodology that need to be explored. For example, by extending it to a larger class of driving processes or by studying fine properties of the density functions associated to the solution of the corresponding SDE when diffusion coefficients are not elliptic. Another application is the parametrix expansion of the density in the sense of the Taylor expansions in powers of time.

Furthermore, there is the need to understand better the structure presented in formulas such as (5.1).

APPENDIX A

A.1 Representations of integrals via Gamma functions

In this manuscript, we often use the equation:

$$\int_{\Delta_{n,t}} s_n^{-\frac{d}{2p}} \prod_{j=0}^{n-1}(s_j-s_{j+1})^{-\frac{p+d}{2p}} \mathbf{ds_1^n} = t^{\frac{n(p-d)-d}{2p}} \frac{\Gamma(\frac{p-d}{2p})^n \Gamma(\frac{2p-d}{2p})}{\Gamma(\frac{n(p-d)+2p-d}{2p})},\ (t>0,\quad d\in\mathbf{N},\ p>d).$$

In this section, we prove this equality. We first recall the well known equation between Gamma and Beta functions. Let $B(\cdot,\cdot)$ be the Beta function and Γ be the Gamma function. Then for any $x,y>0$, we have

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Lemma A.1. *Let $\alpha,\beta\in(-\infty,1)$ and $n\in\mathbf{N}$. Then for any $t>0$, we have*

$$\begin{aligned}\int_{\Delta_{n,t}} s_n^{-\alpha}\prod_{j=0}^{n-1}(s_j-s_{j+1})^{-\beta}\mathbf{ds_1^n} &= t^{-\alpha+n(1-\beta)}\prod_{j=0}^{n-1}B(1-\alpha+j(1-\beta),1-\beta)\\ &= t^{-\alpha+n(1-\beta)}\frac{\Gamma(1-\beta)^n\Gamma(1-\alpha)}{\Gamma(1-\alpha+n(1-\beta))}.\end{aligned} \tag{A.1}$$

Proof. Choose and fix $t>0$ arbitrary. We prove by the induction. For $n=1$, (A.1) follows from definition of the Beta function. Now we assume that (A.1) holds for $n=m$. Then we have

$$\int_{\Delta_{n,t}} s_n^{-\alpha}\prod_{j=0}^{m}(s_j-s_{j+1})^{-\beta}\mathbf{ds_1^m} = \int_0^t s_1^{-\alpha+m(1-\beta)}(t-s_1)^{-\beta}\mathrm{d}s_1\prod_{j=0}^{m-1}B(1-\alpha+j(1-\beta),1-\beta).$$

Therefore, the proof follows from the case $n = 1$ replacing α by $\alpha + m(\beta - 1) < 1$. The second formula follows by using the relation between the Beta and Gamma functions. $\square$

A.2 The Laplace transform of the time convolution of Gaussian densities

Let $f : (0, +\infty) \to \mathbf{R}$ and $\int_0^{+\infty} f(t)e^{-\theta t}\mathrm{d}t < +\infty$. We define the Laplace transform for f by

$$(\mathscr{L}f)(\theta) := \int_0^{+\infty} f(t)e^{-\theta t}\mathrm{d}t,$$

where $\theta > 0$. We also define following two functions.

$$\phi_y(s) := \frac{1}{s\sqrt{2\pi s}}e^{-\frac{y^2}{s}}1_{(0,+\infty)}(s),\ \ s \in \mathbf{R}, y \in \mathbf{R},$$
$$\psi_y(s) := \frac{1}{\sqrt{2\pi s}}e^{-\frac{y^2}{s}}1_{(0,+\infty)}(s),\ \ s \in \mathbf{R}, y \in \mathbf{R}.$$

Then one can show that

$$(\mathscr{L}\phi_y)(\theta) = \frac{1}{|y|}e^{-\sqrt{2\theta|y|^2}},\ \ y \neq 0,$$
$$(\mathscr{L}\psi_y)(\theta) = \frac{1}{\sqrt{2\theta}}e^{-\sqrt{2\theta|y|^2}},\ \ y \neq 0.$$

Then we have the following lemma.

Lemma A.2. *Let $k \in \mathbf{N}$. If $|y_j| > 0$ for any $0 \le j \le k$ then we have*

$$\int_{\Delta_{k,t}} g_{s_k}(y_k)\prod_{j=0}^{k-1}\frac{g_{s_j - s_{j+1}}(y_j)}{s_j - s_{j+1}}\mathbf{ds_1^k} = \psi_{\sum_{j=0}^k |y_j|}(t)\prod_{j=0}^{k-1}\frac{1}{|y_j|}.$$

Proof. We only show that $k = 1$ case, that is, for any $y_0, y_1 \in \mathbf{R} \setminus \{0\}$, we have

$$\int_0^t g_s(y_1)\frac{g_{t-s}(y_0)}{t-s}\mathrm{d}s = \psi_{|y_0|+|y_1|}(t)\frac{1}{|y_0|}.$$

In fact, from the definitions of ϕ and ψ, we have

$$\int_0^t g_s(y_1)\frac{g_{t-s}(y_0)}{t-s}\mathrm{d}s = \int_{\mathbf{R}} \psi_{y_1}(s)\phi_{y_0}(t-s)\mathrm{d}s = \psi_{y_1} * \phi_{y_0}(t)$$

for any $t > 0$. Since $y_1, y_2 \in \mathbf{R} \setminus \{0\}$, we have that

$$\begin{aligned}(\mathscr{L}\psi_{y_1} * \phi_{y_0})(\theta) &= (\mathscr{L}\psi_{y_1})(\theta)(\mathscr{L}\phi_{y_0})(\theta)\\ &= \frac{1}{|y_1|\sqrt{2\theta}}e^{-\sqrt{2\theta(|y_0|+|y_1|)^2}}\\ &= \frac{1}{|y_1|}(\mathscr{L}\psi_{|y_0|+|y_1|})(\theta).\end{aligned}$$

Therefore, the uniqueness of the Laplace transform implies that

$$\psi_{y_1} * \phi_{y_0}(t) = \psi_{|y_0|+|y_1|}(t)\frac{1}{|y_1|}$$

for any $t > 0$ and $y_0, y_1 \in \mathbf{R} \setminus \{0\}$. □

Let $\varphi_{n,t,x_0}(x, \mathbf{z_1^n})$, $G_{n,t,x_0}(x, \mathbf{z_1^n})$ and $h_{n,t,x_0}(x, \mathbf{z_1^n})$ be functions defined in Section 5.2.

Lemma A.3. *Let $d = 1$, $x, x_0 \in \mathbf{R}$, $t > 0$. Assume that $b \in L^p(\mathbf{R})$ and (5.8) holds for some $\alpha \in (0, 1)$. Then for almost all $\mathbf{z_2^n} \in \mathbf{R}^{n-1}$, we have*

$$\lim_{\delta \downarrow 0} \delta^{-\alpha} \int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n})\right) \mathrm{d}z_1 = 2R(x, \alpha) \operatorname{sgn}(x - z_2)\varphi_{n-1,t,x_0}(x, \mathbf{z_2^n}).$$

Similarly, if (5.9) holds for $\beta \in (0, 1)$ then we have

$$\lim_{\delta \downarrow 0} \delta^{-\beta} \int_{\mathbf{R}} b(z_1) \left(h_{n,t,x_0}(x, \mathbf{z_1^n}) - h_{n,t,x_0}(x - \delta, \mathbf{z_1^n})\right) \mathrm{d}z_1 = 2L(x, \alpha) \operatorname{sgn}(x - z_2)\varphi_{n-1,t,x_0}(x, \mathbf{z_2^n}).$$

Proof. We only prove for the case that (5.8) holds. We first prove that for all sufficiently small δ,

$$\begin{aligned}
&\left| \int_{-\infty}^{x} b(z_1) \left(h_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n})\right) \mathrm{d}z_1 + \int_{x+\delta}^{\infty} b(z_1) \left(h_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n})\right) \mathrm{d}z_1 \right| \\
&\leq 16\sqrt{\pi}\|b\|_p \|g_{4t}\|_{p_*} C_1 \delta(\varphi_{n,4t,x_0}(x, \mathbf{z_2^n}) + \varphi_{n,4t,x_0}(x + \delta, \mathbf{z_2^n})),
\end{aligned} \tag{A.2}$$

holds. From the definition of h_{n,t,x_0}, as long as $z_1 \notin [x, x + \delta]$, we have

$$|h_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - h_{n,t,x_0}(x, \mathbf{z_1^n})| = |\varphi_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - \varphi_{n,t,x_0}(x, \mathbf{z_1^n})|.$$

Now the mean value theorem yields that

$$\begin{aligned}
|\varphi_{n,t,x_0}(x + \delta, \mathbf{z_1^n}) - \varphi_{n,t,x_0}(x, \mathbf{z_1^n})| &= \frac{1}{\sqrt{2\pi t}} \left| \int_{|x-z_1|}^{|x+\delta-z_1|} \frac{\mathrm{d}}{\mathrm{d}y} \exp\left(-\frac{\left(y + \sum_{k=2}^{n+1} |z_{k-1} - z_k|\right)^2}{2t} \right) \mathrm{d}y \right| \\
&\leq \frac{\sqrt{2}C_1\delta}{\sqrt{2\pi t}\sqrt{t}} \exp\left(-\frac{\left(\min\{|x+\delta-z_1|, |x-z_1|\} + \sum_{k=2}^{n+1} |z_{k-1}-z_k|\right)^2}{4t} \right) \\
&= \frac{2C_1\delta}{\sqrt{t}} \left(\varphi_{n,2t,x_0}(x+\delta, \mathbf{z_1^n}) 1_{(x+\delta,+\infty)}(z_1) + \varphi_{n,2t,x_0}(x, \mathbf{z_1^n}) 1_{(-\infty,x)}(z_1) \right).
\end{aligned}$$

Furthermore, from the inequality (5.6), we have that

$$\varphi_{n,2t,x_0}(x, \mathbf{z_1^n}) \leq 4\sqrt{\pi t}\varphi_{n-1,4t,x_0}(x, \mathbf{z_2^n})\varphi_{n,4t,x_0}(x, \mathbf{z_1^n}).$$

Hence by applying the Jensen inequality, we obtain that

$$
\begin{aligned}
&\left|\int_{-\infty}^{x} b(z_1)\left(h_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-h_{n,t,x_0}(x,\mathbf{z_1^n})\right)\mathrm{d}z_1+\int_{x+\delta}^{\infty} b(z_1)\left(h_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-h_{n,t,x_0}(x,\mathbf{z_1^n})\right)\mathrm{d}z_1\right|\\
&\le \frac{2C_1\delta}{\sqrt{t}}\left(\int_{-\infty}^{x}|b(z_1)|\varphi_{n,2t,x_0}(x,\mathbf{z_1^n})\mathrm{d}z_1+\int_{x+\delta}^{+\infty}|b(z_1)|\varphi_{n,2t,x_0}(x+\delta,\mathbf{z_1^n})\mathrm{d}z_1\right)\\
&\le 8\sqrt{\pi}C_1\delta(\varphi_{n-1,4t,x_0}(x,\mathbf{z_2^n})+\varphi_{n-1,4t,x_0}(x+\delta,\mathbf{z_2^n}))\left(\int_{\mathbf{R}}|b(z_1)|(\varphi_{n,4t,x_0}(x,\mathbf{z_1^n})+\varphi_{n,4t,x_0}(x+\delta,\mathbf{z_1^n}))\mathrm{d}z_1\right).
\end{aligned}
$$

Now it is clear that the above integral is bounded by $2\|b\|_p\|g_{4t}\|_{p_*}$. Therefore we have (A.2). From this upper bound, we see that

$$
\lim_{\delta\downarrow 0}\delta^{-\alpha}\left|\int_{[x,x+\delta]^c} b(z_1)\left(h_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-h_{n,t,x_0}(x,\mathbf{z_1^n})\right)\mathrm{d}z_1\right|=0.
$$

On the other hand, when z_1 moves in $[x,x+\delta]$, it holds that $\operatorname{sgn}(x+\delta-z_1)=1$ and $\operatorname{sgn}(x-z_1)=-1$. Therefore we have

$$
\begin{aligned}
&\int_{x}^{x+\delta} b(z_1)\left(h_{n,t,x_0}(x+\delta,\mathbf{z_1^n})-h_{n,t,x_0}(x,\mathbf{z_1^n})\right)\mathrm{d}z_1\\
&=\int_{x}^{x+\delta} b(z_1)\operatorname{sgn}(z_1-z_2)(\varphi_{n,t,x_0}(x+\delta,\mathbf{z_1^n})+\varphi_{n,t,x_0}(x,\mathbf{z_1^n}))\mathrm{d}z_1,
\end{aligned}
$$

Except for $z_2=x$, by taking δ sufficiently small, we see that $\operatorname{sgn}(z_1-z_2)=\operatorname{sgn}(x-z_2)$. Therefore we have

$$
\begin{aligned}
&\int_{x}^{x+\delta} b(z_1)\operatorname{sgn}(z_1-z_2)(\varphi_{n,t,x_0}(x+\delta,\mathbf{z_1^n})+\varphi_{n,t,x_0}(x,\mathbf{z_1^n}))\mathrm{d}z_1\\
&=\operatorname{sgn}(x-z_2)\int_{x}^{x+\delta} b(z_1)(\varphi_{n,t,x_0}(x+\delta,\mathbf{z_1^n})+\varphi_{n,t,x_0}(x,\mathbf{z_1^n}))\mathrm{d}z_1.
\end{aligned}
$$

Since φ_{n,t,x_0} is continuous, for any $\varepsilon>0$, there exists $\delta_0>0$ such that for any $z_1\in[x,x+\delta_0]$,

$$
|(\varphi_{n,t,x_0}(x+\delta_0,\mathbf{z_1^n})+\varphi_{n,t,x_0}(x,\mathbf{z_1^n}))-\varphi_{n-1,t,x_0}(x,\mathbf{z_2^n})|<\varepsilon.
$$

This implies that

$$
\begin{aligned}
&\lim_{\delta\downarrow 0}\delta^{-\alpha}\operatorname{sgn}(x-z_2)\int_{x}^{x+\delta} b(z_1)(\varphi_{n,t,x_0}(x+\delta,\mathbf{z_1^n})+\varphi_{n,t,x_0}(x,\mathbf{z_1^n}))\mathrm{d}z_1\\
&=2R(x,\alpha)\operatorname{sgn}(x-z_2)\varphi_{n-1,t,x_0}(x,\mathbf{z_2^n}).
\end{aligned}
$$

This completes the proof. □

References

[1] W. Arendt, G. Metafune and D. Pallara, Gaussian estimates for elliptic operators with unbounded drift. *J. Math. Anal. Appl.* **338** (2008) 505–517.

[2] V. Bally and A. Kohatsu-Higa, A probabilistic interpretation of the parametrix method. *Ann. Appl. Probab.* **25** (2015) 3095–3138.

[3] D. Baños and P. Krühner, Optimal density bounds for marginals of Itô processes. *Commun. Stoch. Anal.* **10** (2016) 1.

[4] D. Baños and P. Krühner, Hölder continuous densities of solutions of SDEs with measurable and path dependent drift coefficients. *Stoch. Proc. Appl.* **127** (2017) 1785–1799.

[5] A.S. Cherny and H.-J. Engelbert, Singular Stochastic Differential Equations, Vol. 1858. Springer Science & Business Media, Berlin (2005).

[6] F. Corielli, P. Foschi and A. Pascucci. Parametrix approximation of diffusion transition densities. *SIAM J. Finan. Math.* **1** (2010) 833–867.

[7] G. Da Prato, F. Flandoli, E. Priola and M. Röckner, Strong uniqueness for stochastic evolution equations in hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** (2013) 3306–3344.

[8] G. Da Prato, F. Flandoli, E. Priola and M. Röckner, Strong uniqueness for stochastic evolution equations with unbounded measurable drift term. *J. Theor. Probab.* **28** (2015) 1571–1600.

[9] S. De Marco, Smoothness and asymptotic estimates of densities for sdes with locally smooth coefficients and applications to square root-type diffusions. *Ann. Appl. Probab.* **21** (2011) 1282–1321.

[10] M.D. Francesco and A. Pascucci, On a class of degenerate parabolic equations of kolmogorov type. *AMRX Appl. Math. Res. Express* **3** (2005) 77–116.

[11] A. Friedman, Partial Differential Equations of Parabolic Type. Prentice-Hall, Inc., Englewood Cliffs, NJ (1964).

[12] M. Hayashi, A. Kohatsu-Higa and G. Yûki, Local Hölder continuity property of the densities of solutions of SDEs with singular coefficients. *J. Theoret. Probab.* **26** (2013) 1117–1134.

[13] M. Hayashi, A. Kohatsu-Higa and G. Yûki, Hölder continuity property of the densities of SDEs with singular drift coefficients. *Electron. J. Probab.* **19** (2014) 1–22.

[14] I. Karatzas and S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd Edn. Vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (1991).

[15] P. Kim and R. Song, Two-sided estimates on the density of brownian motion with singular drift. *Illinois J. Math.* **50** (2006) 635–688.

[16] A. Kohatsu-Higa and A. Makhlouf, Estimates for the density of functionals of SDEs with irregular drift. *Stoch. Process. Appl.* **123** (2013) 1716–1728.

[17] A. Kohatsu-Higa and A. Tanaka, A Malliavin calculus method to study densities of additive functionals of SDE's with irregular drifts. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (2012) 871–883.

[18] N. V. Krylov and M. Röckner, Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Relat. Fields* **131** (2005) 154–196.

[19] S. Kusuoka, Continuity and gaussian two-sided bounds of the density functions of the solutions to path-dependent stochastic differential equations via perturbation. *Stoch. Process. Appl*. (2016) 359–384.

[20] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32** (1985) 1–76.

[21] A. Makhlouf, Existence and Gaussian bounds for the density of Brownian motion with random drift. *Commun. Stoch. Anal.* **10** (2016) 151–162.

[22] D. Nualart, The Malliavin Calculus and Related Topics, 2nd Edn. *Probability and its Applications (New York)*. Springer-Verlag, Berlin (2006).

[23] A. Pascucci and A. Pesce, The Parametrix Method for Parabolic Spdes. Preprint arXiv:1803.06543 (2018).

[24] N. Portenko, Generalized Diffusion Processes. Translations of Mathematical Monographs. American Mathematical Society, Providence, RI (1990).

[25] Z. Qian and W. Zheng, A representation formula for transition probability densities of diffusions and applications. *Stoch. Process. Appl.* **111** (2004) 57–76.

[26] Z. Qian, F. Russo and W. Zheng, Comparison theorem and estimates for transition probability densities of diffusion processes. *Probab. Theory Relat. Fields* **127** (2003) 388–406.

[27] M. Romito, A Simple Method for the Existence of a Density for Stochastic Evolutions with Rough Coefficients. Preprint arXiv:1707.05042v2 (2017).

[28] J. Shin and G. Trutnau, On singular stochastic differential equations and dirichlet forms. *Rev. Roumaine Math. Pures Appl.* **62** (2017a) 217–258.

[29] J. Shin and G. Trutnau, On the stochastic regularity of distorted brownian motions. *Trans. Am. Math. Soc.* **369** (2017b) 7883–7915.

[30] A.J. Veretennikov and N.V. Krylov, Explicit formulae for the solutions of stochastic equations. *Mat. Sb. (N.S.)* **100** (1976) 266–284.