ON THE BOUNDED LAWS OF ITERATED LOGARITHM IN BANACH SPACE

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Abstract. In the present paper, by using the inequality due to Talagrand's isoperimetric method, several versions of the bounded law of iterated logarithm for a sequence of independent Banach space valued random variables are developed and the upper limits for the non-random constant are given.

1. Introduction

Let B be a real separable Banach space with $||\cdot||$ and topological dual B^*. For a B-valued random variable X and some $p \in [1, 2]$, we write $X \in WM_0^p$ if for all $f \in B^*$, we have $Ef(X) = 0$ and $E|f(X)|^p < \infty$. Throughout $\{X_n, n \geq 1\}$ is a sequence of independent B-valued random variables defined on a probability space (Ω, F, P) and $\{\epsilon_n, n \geq 1\}$ is an independent Rademacher series supported on the same probability space (Ω, F, P) and independent of $\{X_n, n \geq 1\}$. For each $n \geq 1$, put $S_n = \sum_{i=1}^n X_i, W_{n,p} = \sup_{f \in B_1^*}(\sum_{i=1}^n |f(X_i)|^p)^{1/p}, s_{n,p} = \sup_{f \in B_1^*}(\sum_{i=1}^n E|f(X_i)|^p)^{1/p}$ where B_1^* is the unit ball of B^*. As usual, $L_2 x$ denotes the function $\log \log \max\{e^e, x\}$.

Griffin and Kuelbs [10, 11] established some extensions of the law of iterated logarithm (LIL) *via* self-normalizations for independent real valued random variables both in the symmetric and non-symmetric cases. However, many of these results require a symmetry assumption. Afterwards, Godbole [9] discussed the self-normalized bounded law of iterated logarithm (SNBLIL) for B-valued random variables and gave the following definition for SNBLIL: $\{X_n, n \geq 1\}$ will be said to satisfy the SNBLIL ($\{X_n\} \in SNBLIL$) if there exists a non-random constant $0 < M < \infty$ such that for some $p \in [1, 2]$ and $r > 0$,

$$\limsup_{n \to \infty} \frac{||\sum_{i=1}^n X_i||}{(\sum_{i=1}^n ||X_i||^p)^{1/p}(L_2 \sum_{i=1}^n ||X_i||^p)^r} = M \qquad \text{a.s.} \qquad (1.1)$$

Godbole [9] proved that under some conditions, $\{X_n, n \geq 1\} \in SNBLIL$ for $r = \frac{1}{2}$ and $r = \frac{p-1}{p}$. The advantage of SNBLIL is to drop the standard bounded assumption for the random variables $\{X_n, n \geq 1\}$. But these theorems do not give the accurate value for the non-random constant M and deeply depend on the type of Banach spaces. In the case of real symmetric random variables, Marcinkiewicz proved that $M \leq 1$ for $p = 2$

and $r = \frac{1}{2}$ (see [11] for a simple proof of Marcinkiewicz's result). Therefore it was of interest to ask whether $M \leq 1$ for B-valued symmetric random variables or to give the accurate estimate for M. By replacing the self-normalizer $(\sum_{i=1}^{n} ||X_i||^p)^{1/p}(L_2 \sum_{i=1}^{n} ||X_i||^p)^r$ by $(\sup_{f \in B_1^*} \sum_{i=1}^{n} |f(X_i)|^p)^{1/p}(L_2 \sup_{f \in B_1^*} \sum_{i=1}^{n} |f(X_i)|^p)^r$, Deng [6] proved the following results.

Theorem 1.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent symmetric* B*-valued random variables. Suppose that for some* $p \in [1, 2]$*, the following conditions hold:*

$$\lim_{n \to \infty} W_{n,p}^p \to +\infty \ \ a.s. \tag{1.2}$$

$$\frac{\sup_{f \in D} |f(X_n)|(L_2 W_{n,p}^p)^{1/2}}{W_{n,p}} \to 0 \ \ a.s. \tag{1.3}$$

$$S_n/(2W_{n,p}^2 L_2 W_{n,p}^p)^{1/2} \to 0 \ \ in \ probability. \tag{1.4}$$

Then

$$\limsup_{n \to \infty} \frac{||S_n||}{(2W_{n,p}^2 L_2 W_{n,p}^p)^{1/2}} \leq 1 \ \ a.s.$$

In particular, if (1.2), (1.3) and (1.4) hold for $p = 2$*, then*

$$\limsup_{n \to \infty} \frac{||S_n||}{(2W_{n,2}^2 L_2 W_{n,2}^2)^{1/2}} = 1 \ \ a.s.$$

Theorem 1.2. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent symmetric* B*-valued random variables. Suppose that for some* $p \in [1, 2]$*, the following conditions hold:*

$$\lim_{n \to \infty} W_{n,p}^p = +\infty \ \ a.s.$$

$$\frac{\sup_{f \in D} |f(X_n)|(L_2 W_{n,p}^p)^{\frac{p-1}{p}}}{W_{n,p}} \to 0 \ \ a.s.$$

$$S_n/W_{n,p}(L_2 W_{n,p}^p)^{\frac{p-1}{p}} \to 0 \ \ in \ probability.$$

Then

$$\limsup_{n \to \infty} \frac{||S_n||}{2W_{n,p}(L_2 W_{n,p}^p)^{\frac{p-1}{p}}} \leq \begin{cases} \frac{1}{2} & p = 1 \\ (\frac{p}{p-1})^{\frac{p-1}{p}} & 1 < p < 2 \ \ a.s. \\ \frac{\sqrt{2}}{2} & p = 2. \end{cases}$$

Now it is of concern that the equation (1.1) holds if the self-normalizers $(\sum_{i=1}^{n} ||X_i||^p)^{1/p}(L_2 \sum_{i=1}^{n} ||X_i||^p)^r$ and $(\sum_{i=1}^{n} |f(X_i)|^p)^{1/p}(L_2 \sum_{i=1}^{n} |f(X_i)|^p)^r$ are replaced by the nonrandom normalizers $(\sum_{i=1}^{n} E||X_i||^p)^{1/p}(L_2 \sum_{i=1}^{n} E||X_i||^p)^r$ and $(\sum_{i=1}^{n} E|f(X_i)|^p)^{1/p}(L_2 \sum_{i=1}^{n} E|f(X_i)|^p)^r$. Moreover, if (1.1) holds for non-random normalizers, what the nonrandom constant M equals? or can the estimate of upper limit for M be given?

The purpose of the present paper is to solve the above questions. In some sense, we generalize the bounded law of iterated logarithm for B-valued random variables. In fact, the classical bounded law of iterated logarithm can be obtained for $p = 2$. Our approach is mainly due to Ledoux and Talagrand [13]. In Ledoux and Talagrand [13],

they developed the isoperimetric inequality and the finite dimensional approximation argument *via* some entropy estimate which is based on the Sudakov type minoration. This approach is very effective for dealing with the questions of strong limit theorems of B-valued random variables and plays an important role in the accurate estimates of upper limits of BLIL in Banach space. The paper is organized as follows. In Section 2, we state the main results. In Section 3, we develop a new inequality. The proofs of theorems are obtained by using an entropy approach and the new inequality.

2. Main results

We start with introducing some notations. For a sequence of independent B-valued random variables $\{X_n, n \geq 1\}$, put $S_n = \sum_{i=1}^{n} X_i, s_{n,p} = (\sup_{f \in B_1^*} \sum_{i=1}^{n} E|f(X_i)|^p)^{1/p}$ for some $p \in [1, 2]$, where B_1^* is the unit ball of B^*. Note that if D is a countable subset of the unit ball B_1^* such that $||x|| = \sup_{f \in D} |f(x)|$ for every $x \in B, s_{n,p} = (\sup_{f \in D} \sum_{i=1}^{n} E|f(X_i)|^p)^{1/p}$. By replacing the self-normalizer with $s_{n,p}(L_2 s_{n,p}^p)^r$, we may obtain the results with respect to the BLIL. At first we discuss the case that $r = \frac{1}{2}$ and state our main theorems.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables with $X_n \in WM_0^p$ for some $p \in [1, 2]$. Assuming that the following statements hold:*

$$\lim_{n \to \infty} s_{n,p}^p = +\infty, \tag{2.1}$$

$$||X_n|| \leq (\eta_n s_{n,p})/(L_2 s_{n,p}^p)^{1/2} \ a.s. \tag{2.2}$$

for some sequence of real number $\{\eta_n\}$ with $1 \geq \eta_n \to 0$, and

$$S_n/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2} \to 0 \ in \ probability. \tag{2.3}$$

Then

$$\limsup_{n \to \infty} \frac{||S_n||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \leq \begin{cases} 2^{1+1/p} & \text{for } 1 \leq p < 2 \\ 1 & \text{for } p = 2. \end{cases} \ a.s.$$

Note that the Wittmann's LIL holds under the condition that $\sum_{n=1}^{\infty} \frac{E||X_n||^r}{(s_n^2 L_2 s_n^2)^{r/2}} < +\infty$ for $r > 2$ where $s_n^2 = s_{n,2}^2 = \sup_{f \in B_1^*} \sum_{i=1}^{n} Ef^2(X_i)$ (see [3], Th. 2.1). Now one would like to know whether there is a similar result if $s_{n,2}$ is replaced by $s_{n,p}$ for $1 \leq p \leq 2$. Actually we have the following result.

Theorem 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables with $X_n \in WM_0^p$ for some $p \in [1, 2]$. If the following statements hold:*

$$\sum_{n=1}^{\infty} (s_{n,p}^2 L_2 s_{n,p}^p)^{-\alpha/2} E||X_n||^\alpha < +\infty \qquad \text{for some } \alpha > p; \tag{2.4}$$

$$s_{n,p} \to \infty. \tag{2.5}$$

Then

$$\limsup_{n \to \infty} \frac{||S_n||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \leq \Gamma \qquad a.s. \ for \ some \ \Gamma > 0$$

holds provided that $S_n/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}$ is bounded in probability. In addition, if $S_n/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2} \to 0$ in probability,

$$\limsup_{n \to \infty} \frac{||S_n||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \leq \begin{cases} 2^{1+1/p} & \text{for } 1 \leq p < 2 \\ 1 & \text{for } p = 2 \end{cases} \ a.s.$$

Now we discuss the result for i.i.d. B-valued random variables.

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. B-valued random variables with $X \in WM_0^p$ for some $p \in [1, 2]$. Let $\{c_n, n \geq 1\}$ be a sequence of positive numbers satisfying*

$$c_n \geq n^{1/p}(L_2 n)^{1/2}. \tag{2.6}$$

Then for some $\Gamma > 0$,

$$\limsup_{n \to \infty} \frac{||S_n||}{c_n} \leq \Gamma \qquad a.s.$$

holds provided that

$$S_n/c_n \to 0 \qquad in \ probability, \tag{2.7}$$

and

$$\sum_{n=1}^{\infty} P(||X_1|| \geq c_n) < +\infty. \tag{2.8}$$

In particular,

$$\limsup_{n \to \infty} \frac{||S_n||}{(2n^{2/p}L_2 n)^{1/2}} \leq \begin{cases} 2^{1+1/p}\sigma_p & for \ 1 \leq p < 2 \\ \sigma_2 & for \ p = 2 \end{cases} \quad a.s.$$

holds provided that

$$S_n/(2n^{2/p}L_2 n)^{1/2} \to 0 \qquad in \ probability, \tag{2.9}$$

and

$$E \frac{||X_1||^p}{(L_2||X_1||)^{p/2}} < +\infty, \tag{2.10}$$

where $\sigma_p = \sup_{f \in B_1^*}(E|f(X)|^p)^{1/p}$.

Next we state the results for $r = \frac{1}{q} = \frac{p}{p-1}$.

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables with $X_n \in WM_0^p$ for some $p \in [1, 2]$. Assuming the following statements hold:*

$$\lim_{n \to \infty} s_{n,p}^p = +\infty, \tag{2.11}$$

$$||X_n|| \leq \eta_n s_{n,p}/(L_2 s_{n,p}^p)^{1/p} \ a.s., \tag{2.12}$$

for some sequence of real number $\{\eta_n\}$ with $1 \geq \eta_n \to 0$, and

$$S_n/(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q} \to 0 \ in \ probability. \tag{2.13}$$

Then

$$\limsup_{n \to \infty} \frac{||S_n||}{(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q}} \leq \Lambda_p \ a.s.$$

where

$$\Lambda_p = \begin{cases} 0 & for \ p = 1 \\ (16K_0)^{1/p}q^{1/q} & for \ 1 < p < 2 \\ 1 & for \ p = 2 \end{cases}$$

and K_0 is a universal constant in isoperimetric inequality in Talagrand [12].

Theorem 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables with $X_n \in WM_0^p$ for some $p \in [1, 2]$. Suppose that the following statements hold:*

$$\sum_{n=1}^{\infty} (s_{n,p}^q L_2 s_{n,p}^p)^{-\alpha/q} E||X_n||^\alpha < +\infty \qquad \text{for some } \alpha > p; \qquad (2.14)$$

$$s_{n,p} \to \infty. \qquad (2.15)$$

Then

$$\limsup_{n\to\infty} \frac{||S_n||}{(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q}} \leq \Gamma \qquad \text{a.s. for some } \Gamma > 0$$

holds provided that $S_n/(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q}$ is bounded in probability. In addition, if $S_n/(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q} \to 0$ in probability,

$$\limsup_{n\to\infty} \frac{||S_n||}{(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q}} \leq \Lambda_p \text{ a.s.}$$

where Λ_p is the same as that in Theorem 2.4.

Similar to Theorem 2.3, we have the following result.

Theorem 2.6. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. B-valued random variables with $X \in WM_0^p$ for some $p \in [1, 2]$. Let $\{c_n, n \geq 1\}$ be a sequence of positive numbers satisfying*

$$c_n \geq n^{1/p}(L_2 n)^{1/q}.$$

Then for some $\Gamma > 0$,

$$\limsup_{n\to\infty} \frac{||S_n||}{c_n} \leq \Gamma \qquad \text{a.s.}$$

holds provided that (2.7) and (2.8) hold. In particular,

$$\limsup_{n\to\infty} \frac{||S_n||}{(2n^{q/p}L_2 n)^{1/q}} \leq \Lambda_p \sigma_p \text{ a.s.}$$

holds provided that

$$S_n/(2n^{q/p}L_2 n)^{1/q} \to 0 \qquad \text{in probability,}$$

and

$$E\frac{||X_1||^p}{(L_2||X_1||)^{p-1}} < +\infty,$$

where $\sigma_p = \sup_{f \in B_1^*} (E|f(X)|^p)^{1/p}$ and Λ_p is the same as that in Theorem 2.4.

Remarks. (1) The above theorems answer the questions proposed in Section 1 for normalized constants $s_{n,p}(L_2 s_{n,p}^p)^{1/2}$ and $s_{n,p}(L_2 s_{n,p}^p)^{\frac{p-1}{p}}$ and, give an accurate estimate of the upper limit for the nonrandom constant. In particular, for $p = 2$, Theorems 2.1 and 2.4 are versions of Kolmogorov's BLIL, Theorems 2.2 and 2.5 are the versions of Wittmann's BLIL for B-valued random variables and Theorems 2.3 and 2.6 is

the versions of BLIL for i.i.d. B-valued random variables. In that sense, our results are the extensions of the BLIL for B-valued random variables. Also, some results are new even for real valued random variables. Since $s_{n,p} \leq (\sum_{i=1}^{n} E||X_i||^p)^{1/p}$, the similar results may be obtained for the normalizer constant $(\sum_{i=1}^{n} E||X_i||^p)^{1/p}(L_2 \sum_{i=1}^{n} E||X_i||^p)^{1/2}$.

(2) In Theorems 2.2 and 2.5, the assumptions that Conditions (2.4) and (2.14) holds "for some $\alpha > p$" can not be strengthened to say "for some $\alpha \geq p$". In fact (2.4) and (2.14) fail for $s_{n,p}$ when $\alpha = p$. Indeed, if (2.4) holds with $\alpha = p$, that is,

$$\sum_{n=1}^{\infty} (s_{n,p}^2 L_2 s_{n,p}^p)^{-p/2} E||X_n||^p < +\infty,$$

then defining n_k to be the smallest positive integer satisfying that $s_{n,p}^p > \rho^k$, $k \geq 1$, where $\rho > 1$, we have that

$$(s_{n_{k+1},p}^2 L_2 s_{n_{k+1},p}^p)^{p/2}(s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{-p/2} \sim \rho, \qquad \text{and} \qquad s_{n_k,p}^p \sim \rho^k.$$

Hence, setting $\gamma_i = \sup\{k : i \geq n_k\}$,

$$\sum_{k=1}^{\infty} \frac{1}{(s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{p/2}} \sum_{i=1}^{n_k} E||X_i||^p \leq \sum_{j=1}^{\infty} E||X_j||^p \sum_{k=\gamma_j}^{\infty} \frac{1}{(s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{p/2}}$$

$$\leq \sum_{j=1}^{\infty} E||X_j||^p \sum_{k=\gamma_j}^{\infty} \frac{1}{(s_{n_{\gamma_i},p}^2 L_2 s_{n_{\gamma_i},p}^p)^{p/2} \rho^{(k-\gamma_j)}}$$

$$\leq \sum_{j=1}^{\infty} \frac{E||X_j||^r}{(s_{j,p}^2 L_2 s_{j,p}^p)^{p/2}} \sum_{k=0}^{\infty} \frac{1}{\rho^k}$$

$$\leq (1 - \frac{1}{\rho})^{-1} \sum_{j=1}^{\infty} \frac{E||X_j||^r}{(s_{j,p}^2 L_2 s_{j,p}^p)^{p/2}} < +\infty.$$

On the other hand,

$$\sum_{k=1}^{\infty} \frac{1}{(s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{p/2}} \sum_{i=1}^{n_k} E||X_i||^p \geq$$

$$\sum_{k=1}^{\infty} \frac{1}{s_{n_k,p}^p (L_2 s_{n_k,p}^p)^{p/2}} \sum_{i=1}^{n_k} E||X_i||^p \geq \sum_{k=1}^{\infty} (L_2 s_{n_k,p}^p)^{-1} \geq \sum_{k=1}^{\infty} (\log(k \log \rho))^{-1} = +\infty,$$

which results in a contradiction.

(3) Obviously, for $1 < p < 2$, the upper limits in the above theorems are not best and perhaps can be improved. On the other hand, the lower limits for the nonrandom constants do not be given and it is unknown whether the lower limits are greater than zero or not.

(4) Furthermore, from Theorem 2.6, the law of large number is obtained for $p = 1$ and the LIL is obtained for $p = 2$. The conjecture is that there exists an continuous increasing function $\Lambda(p)$ such that $\Lambda(0) = 0$, $\Lambda(2) = 1$ and for i.i.d. B-valued random variables such that

$$\limsup_{n \to \infty} ||S_n||/(2n^{q/p} L_2 n)^{1/q} = \Lambda(p) \text{ a.s.}$$

3. The proofs of theorems

Now we first prove Theorem 2.1.

The proof of Theorem 2.1.

By the Hoffmann-Jørgensen inequality and the standard method of symmetrization, it is easy to prove that (2.2) and (2.3) imply that

$$\lim_{n\to\infty} \frac{E||S_n||}{s_{n,p}t_n} = \lim_{n\to\infty} \frac{E||\sum_{i=1}^n \epsilon_i X_i||}{s_{n,p}t_n} = 0, \tag{3.1}$$

where $t_n = (2L_2 s_{n,p}^p)^{1/2}$ and $\{\epsilon_n, n \geq 1\}$ is a Rademacher sequence which is independent of $\{X_n, n \geq 1\}$.

For $\rho > 1$, let $n_k = \min\{n; s_{n,p} > \rho^k\}$. Then, $s_{n_k,p} \sim \rho^k$, and $s_{n_{k+1},p}/s_{n_k,p} \sim \rho$. To the claim of Theorem 2.1, it will be sufficient to show that for every $\epsilon > 0$ and some $\rho > 1$,

$$\sum_{k=1}^\infty P\{||S_{n_k}|| > (2^{1+1/p} + \epsilon)s_{n_k,p}t_{n_k}\} < +\infty. \tag{3.2}$$

Let now $\epsilon > 0$ and $\rho > 1$ be fixed. For $f, g \in D$ and every k, set

$$d_2^k(f,g) = \frac{1}{s_{n_k,p}} \left(\sum_{i=1}^{n_k} E(f-g)^2(X_i) \right)^{1/2}$$

and

$$\beta_k = E||\sum_{i=1}^{n_k} \epsilon_i X_i||/(s_{n_k,p}t_{n_k}).$$

By (3.1), $\beta_k \to 0$ as $k \to \infty$.

Let $N(D, d_2^k, \epsilon)$ denote the entropy, that is, $N(D, d_2^k, \epsilon)$ is the minimal number of elements g in D such that for every $f \in D$ there exists such a g with $d_2^k(f,g) < \epsilon$. By Theorem 2 in Chow and Teicher [5] and von Bahr-Esseen's inequality (see von Bahr and Esseen [2], Th. 4), we have that for every $f \in B_1^*$,

$$\left[\sum_{i=1}^n Ef^2(X_i) \right]^{1/2} = \left[E \left| \sum_{i=1}^n f(X_i) \right|^2 \right]^{1/2} \leq A_1^{-1} E \left| \sum_{i=1}^n f(X_i) \right|$$

$$\leq A_1^{-1} \left(2^{p-1} \sum_{i=1}^n E|f(X_i)|^p \right)^{1/p} \leq 2^{1+1/p} \left(\sum_{i=1}^n E|f(X_i)|^p \right)^{1/p}$$

and thus,

$$\sigma_n = \left(\sup_{f \in B_1^*} \sum_{i=1}^n Ef^2(X_i) \right)^{1/2} \leq 2^{1+1/p} s_{n,p}. \tag{3.3}$$

By using Proposition 4.13 in Ledoux and Talagrand [14], the argument like the proof of Lemma 8.3 in Ledoux and Talagrand [14] and (3.3), one may prove that

$$N(D, d_2^k, \epsilon) \leq \exp(\beta_k t_{n_k}^2). \tag{3.4}$$

Now we establish (3.2). According to (3.4), we denote, for each k (large enough) and f in D, by $g_k(f)$ an element of D such that $d_2^k(f,g) < \epsilon$ in such a way that set D_k of all $g_k(f)$'s has a cardinality less than $\exp(\beta_k t_{n_k}^2)$.

Then we have that

$$\left|\left|\sum_{i=1}^{n_k} X_i\right|\right| \leq \sup_{g \in D_k} \left|\sum_{i=1}^{n_k} g(X_i)\right| + \sup_{h \in D_k'} \left|\sum_{i=1}^{n_k} h(X_i)\right|$$

where $D_k' = \{f - g_k(f) : f \in D\}$. The main observation concerning D_k' is that

$$\sup_{h \in D_k'} [\sum_{i=1}^{n_k} E|h(X_i)|^2)^{1/2} \leq \epsilon s_{n_k,p}.$$

In order that (3.2) holds, it is sufficient to show that for some constant

$$\sum_k P\left\{\sup_{h \in D_k'} \left|\sum_{i=1}^{n_k} h(X_i)\right| > M\epsilon s_{n_k,p} t_{n_k}\right\} < \infty \tag{3.5}$$

and

$$\sum_k P\left\{\sup_{g \in D_k} \left|\sum_{i=1}^{n_k} g(X_i)\right| > (2^{1+1/p} + \epsilon) s_{n_k,p} t_{n_k}\right\} < \infty. \tag{3.6}$$

To prove (3.5), replacing $\{X_i\}$ by $\{X_i - X_i'\}$ where $\{X_i'\}$ is an independent copy of the sequence $\{X_i\}$ and noting that

$$s_{n,p} \leq \sup_{f \in D} \left(\sum_{i=1}^{n} E|f(X_i - X_i')|^p\right)^{1/p} \leq 2^{\frac{p-1}{p}} s_{n,p},$$

we can assume the symmetrization of random variables. By taking $q = 2K_0, k = [t_{n_k}^2] + 1, s = 2\epsilon s_{n_k,p} t_{n_k}, t = M'\epsilon s_{n_k,p} t_{n_k}$ and using proposition 1.1 in Ledoux and Talagrand [13], we have that

$$P\left\{\sup_{h \in D_k'} \left|\sum_{i=1}^{n_k} h(X_i)\right| > M\epsilon s_{n_k,p} t_{n_k}\right\} \leq \left(\frac{1}{2}\right)^{t_{n_k}^2} + P\left\{\sum_{i=1}^{[t_{n_k}^2]+1} ||X_i||^* > 2\epsilon s_{n_k,p} t_{n_k}\right\}$$

$$+ 4\exp\left\{-\frac{(M'\epsilon s_{n_k,p} t_{n_k})^2}{128K_0\sigma_k^2}\right\} + 4\exp\left\{-\frac{(M'\epsilon s_{n_k,p} t_{n_k})^2 t_{n_k}^2}{768M_k 2 s_{n_k,p} t_{n_k}}\right\}$$

where

$$\sigma_k = \sup_{h \in D_k'} \left(\sum_{i=1}^{n_k} Eh^2(X_i I\{||X_i|| \leq s/k\})\right)^{1/2} \leq \sup_{h \in D_k'} \left(\sum_{i=1}^{n_k} Eh^2(X_i)\right)^{1/2} \leq \epsilon s_{n_k,p}$$

and

$$\frac{M_k}{s_{n_k,p} t_{n_k}} = \frac{E||\sum_{i=1}^{n_k} X_i I\{||X_i|| \leq s/k\}||}{s_{n_k,p} t_{n_k}} \leq \frac{E||\sum_{i=1}^{n_k}||}{s_{n_k,p} t_{n_k}} \to 0.$$

Noting that for large k,

$$\sum_{i=1}^{[t_{n_k}^2]+1} ||X_i||^* \leq ([t_{n_k}^2] + 1) \max_{i \leq n_k} ||X_i|| \leq ([t_{n_k}^2] + 1)\epsilon s_{n_k,p}/t_{n_k} \leq 2\epsilon s_{n_k,p} t_{n_k},$$

we have that

$$P\left\{\sum_{i=1}^{[t_{n_k}^2]+1}||X_i||^* > 2\epsilon s_{n_k,p}t_{n_k}\right\} = 0.$$

Thus, we have that for $M(M')$ large enough,

$$P\left\{\sup_{h\in D_k'}|\sum_{i=1}^{n_k}h(X_i)| > M\epsilon s_{n_k,p}t_{n_k}\right\} \leq 2^{-t_{n_k}^2} + 4\exp\left\{-\frac{M'^2\epsilon^2 s_{n_k,p}^2}{128K_0\epsilon^2 s_{n_k,p}^2}t_{n_k}^2\right\} + 4\exp\left\{-\frac{M'^2\epsilon^2}{1536\frac{M_k}{s_{n_k,p}t_{n_k}}}t_{n_k}^2\right\}$$

$$\leq 2^{-t_{n_k}^2} + 4\exp\left\{-\frac{M'^2}{128K_0}t_{n_k}^2\right\} + 4\exp\left\{-\frac{M'^2}{1536}t_{n_k}^2\right\}$$

$$\leq 2^{-2L_2\rho^{kp}} + 8\exp\left\{-2\log_2\rho^{kp}\right\}.$$

This completes the proof of (3.5). In order to prove (3.6), by using Lemma 1.6 in Ledoux and Talagrand [14] and noting $\sigma_n^2 \leq 2^{2+2/p}s_{n,p}^2$, for all large enough k's

$$P\left\{\left|\sum_{i=1}^{n_k}g(X_i)\right| > (2^{1+1/p}+\epsilon)s_{n_k,p}t_{n_k}\right\}$$

$$\leq 2\exp\left\{-\frac{(2^{1+1/p}+\epsilon)^2 s_{n_k,p}^2 t_{n_k}^2}{2\sigma_{n_k}^2}\left(2-\exp\left[\frac{(2^{1+1/p}+\epsilon)s_{n_k,p}t_{n_k}\frac{\eta_{n_k}s_{n_k,p}}{t_{n_k}}}{\sigma_{n_k}^2}\right]\right)\right\}$$

$$\leq 2\exp\left\{-\frac{(2^{1+1/p}+\epsilon)^2 s_{n_k,p}^2 2L_2 s_{n_k,p}^p}{2^{3+2/p}s_{n_k,p}^2}\left(2-\exp\left[\frac{(2^{1+1/p}+\epsilon)\eta_{n_k}s_{n_k,p}^2}{2^{2+2/p}s_{n_k,p}^2}\right]\right)\right\}$$

$$\leq 2\exp\left\{-\left(1+\frac{\epsilon}{4}\right)\log_2 s_{n_k,p}^p\right\}.$$

Hence for k_0 large enough,

$$\sum_{k\geq k_0}P\{\sup_{g\in D_k}|\sum_{i=1}^{n_k}g(X_i)| > (2^{1+1/p}+\epsilon)s_{n_k,p}t_{n_k}\} \leq \sum_{k\geq k_0}2\,\mathrm{card}(D_k)\exp\{-(1+\frac{\epsilon}{4})\log_2 s_{n_k,p}^p\}$$

$$\leq \sum_{k\geq k_0}2\exp\{2\beta_k\log_2 s_{n_k,p}^p\}\exp\{-(1+\frac{\epsilon}{4})\log_2 s_{n_k,p}^p\}$$

$$\leq 2\sum_{k\geq k_0}\exp\{-(1+\frac{\epsilon}{8})\log_2 s_{n_k,p}^p\} < +\infty.$$

The proof of Theorem 2.1 is complete. □

Now we are in the position to prove Theorem 2.2. In order to establish this theorem, we need the following proposition.

Proposition 3.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent B-valued random variables. Suppose that for a positive sequence of real numbers $\{a_n, n \geq 1\}$ with $a_n \uparrow +\infty$ and a real number $\alpha \geq 1$, the following statements hold*

$$\sum_{n=1}^{\infty}a_n^{-\alpha}E||X_n||^{\alpha} < \infty, \tag{3.7}$$

$$\limsup_{n\to\infty}a_{n+1}/a_n < +\infty. \tag{3.8}$$

(1) For $1 \leq \alpha \leq 2$ and $X_n \in WM_0^\alpha$,

$$\limsup_{n \to \infty} ||S_n||/a_n = \begin{cases} 0 \\ \Gamma & for\ some\ \Gamma > 0, \end{cases} \quad a.s. \tag{3.9}$$

if and only if

$$\limsup_{n \to \infty} ||S_n||/a_n = \begin{cases} 0 \\ M & for\ some\ M > 0, \end{cases} \quad in\ probability \tag{3.10}$$

respectively.

(2) Assume, moreover, that there is some $\beta > 0$ such that for any $n \geq 1$

$$1 \leq a_{n+1}/a_n \leq (s_{n+1,p}/s_{n,p})^\beta \tag{3.11}$$

and for some $1 \leq p \leq 2, X_n \in WM_0^p$ and

$$\limsup_{n \to \infty} (s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}/a_n < +\infty. \tag{3.12}$$

Then for $\alpha > 2$,

$$\limsup_{n \to \infty} ||S_n||/a_n < +\infty\ a.s. \tag{3.13}$$

if and only if

$$\limsup_{n \to \infty} ||S_n||/a_n < +\infty\ in\ probability. \tag{3.14}$$

In addition, assume that

$$\limsup_{n \to \infty} (s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}/a_n = 0. \tag{3.15}$$

Then,

$$\limsup_{n \to \infty} ||S_n||/a_n = 0\ a.s. \tag{3.16}$$

if and only if

$$\limsup_{n \to \infty} ||S_n||/a_n = 0\ in\ probability. \tag{3.17}$$

Proof. We only prove that under the Conditions (3.7) and (3.8), (3.10) implies (3.9). As in the proof of Theorem 2.1, it can be shown that (3.10) implies that

$$\limsup_{n \to \infty} E||S_n||/a_n \begin{cases} = 0 \\ < +\infty. \end{cases} \tag{3.18}$$

Set $\rho > D = \limsup_{n \to \infty} a_{n+1}/a_n$. For $k \geq 1$, let n_k be the smallest positive integer satisfying that $a_n > \rho^k$. it is easy to see that

$$a_{n_k} \sim \rho^k \qquad and \qquad a_{n_{k+1}}/a_{n_k} \sim \rho.$$

By the standard argument, in order to prove that $\limsup_{n \to \infty} ||\sum_{i=1}^n X_i||/a_n = 0 (= \Gamma)$ a.s., it is sufficient to prove that for $\forall \epsilon > 0$ (or for some $\epsilon > 0$),

$$\sum_{k=1}^\infty P \left\{ \left|\left| \sum_{i=n_k+1}^{n_{k+1}} X_i \right|\right| \geq \epsilon a_{n_{k+1}} \right\} < +\infty.$$

By (3.18), for k large enough, $E||\sum_{i=n_k+1}^{n_{k+1}} X_i||/a_{n_{k+1}} \le \epsilon/2$. Thus, by de Acosta's inequality (see de Acosta [1]), for sufficiently large k,

$$P\left\{\left\|\sum_{i=n_k+1}^{n_{k+1}} X_i\right\| \ge \epsilon a_{n_{k+1}}\right\} \le P\left\{\left|\left\|\sum_{i=n_k+1}^{n_{k+1}} X_i\right\| - E\left\|\sum_{i=n_k+1}^{n_{k+1}} X_i\right\|\right| \ge \frac{\epsilon}{2}a_{n_{k+1}}\right\}$$

$$\le \frac{2^\alpha}{\epsilon^\alpha a_{n_{k+1}}^\alpha} \sum_{i=n_k+1}^{n_{k+1}} E||X_i||^\alpha \le \frac{2^\alpha}{\epsilon^\alpha} \sum_{i=n_k+1}^{n_{k+1}} \frac{E||X_i||^\alpha}{a_i^\alpha}.$$

Therefore,

$$\sum_{k=1}^\infty P\left\{\left\|\sum_{i=1}^{n_k} X_i\right\| \ge \epsilon a_{n_{k+1}}\right\} < +\infty.$$

The proof of Part (1) is complete.

Now we are in the position to prove Part (2). We only prove that under the Conditions (3.7), (3.11) and (3.15), (3.17) implies (3.16). Now, by Lemma 3.3 in Wittmann [15], for any $M > 1$, there exists a strictly increasing subsequence of positive integers such

$$Ma_{n_k} \le a_{n_{k+1}} \le M^3 a_{n_k+1}. \tag{3.19}$$

From Proposition 2 of Chen [3], it is sufficient to show that

$$\sum_{k=1}^\infty \exp\left\{-\frac{\delta a_{n_k}^2}{\sigma_k^2}\right\} < +\infty \qquad \text{for every } \delta > 0 \tag{3.20}$$

where $\sigma_k^2 = \sup_{f \in S_1^*} \sum_{i=1}^{n_k} Ef^2(X_i)(k \ge 1)$. Now from (3.11), $s_{n_{k+1},p}/s_{n_k,p} \sim M^{1/\beta}$ and from which, (3.15) and (3.3), it is easy to see that for sufficiently large k,

$$\frac{\delta a_{n_k}}{\sigma_k^2} \ge \frac{\delta' s_{n_k,p}^2 L_2 s_{n_k,p}^p}{\sigma_k^2} \ge \delta'' L_2 s_{n_k,p}^p \ge 2\log(k\log M^{1/\beta}).$$

Hence (3.9) holds. This completes the proof of Proposition 3.1. □

The proof of Theorem 2.2

The theorem follows from Proposition 3.1 for $1 \le p < \alpha \le 2$ and from Wittmann's LIL (see Chen [3]) for $\alpha > p = 2$, respectively. We only prove that for $p < 2 < \alpha$, under the Conditions (2.4) and (2.5), $S_n/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2} \to 0$ in probability implies that

$$\limsup_{n\to\infty} \frac{||S_n||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \le 2^{1+1/p} \text{ a.s.} \tag{3.21}$$

For this purpose, define

$$X_n' = X_n I\{||X_n|| < (2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}\},$$
$$X_n'' = X_n I\{||X_n|| \ge (2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}\} = X_n - X_n'.$$

By (2.4),

$$\sum_{n=1}^{\infty} P(||X_n|| \geq (2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}) \leq \sum_{n=1}^{\infty} (2s_{n,p}^2 L_2 s_{n,p}^p)^{-\alpha/2} E||X_n||^{\alpha} < +\infty.$$

By the Borel-Cantelli Lemma,

$$\lim_{n\to\infty} (2s_{n,p}^2 L_2 s_{n,p}^p)^{-1/2} \sum_{i=1}^{n} X_i'' = 0 \qquad \text{a.s.} \qquad (3.22)$$

and for $n \geq 1$,

$$\left\| \sum_{i=1}^{n} EX_i'' \right\| = \sup_{f \in B_1^*} \sum_{i=1}^{n} Ef(X_i'')$$

$$\leq \sup_{f \in B_1^*} \left(\sum_{i=1}^{n} E|f(X_i)|^p \right)^{1/p} \left(\sum_{i=1}^{n} P(||X_i|| \geq (2s_{i,p}^2 L_2 s_{i,p}^p)^{1/2}) \right)^{\frac{p-1}{p}} \leq K s_{n,p} \to \infty \quad (n \to \infty)$$

where $K = (\sum_{i=1}^{\infty} P(||X_i|| \geq (2s_{i,p}^2 L_2 s_{i,p}^p)^{1/2}))^{\frac{p-1}{p}}$. Thus,

$$\frac{|| \sum_{i=1}^{n} EX_i'' ||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \to 0 \quad (n \to \infty). \qquad (3.23)$$

Since $\sum_{i=1}^{n} X_i/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2} \to 0$ in probability, by (3.22) and (3.23), we may obtain that

$$\sum_{i=1}^{n} (X_i' - EX_i')/(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2} \to 0 \quad \text{in probability.} \qquad (3.24)$$

Again by (3.22) and (3.23), in order to prove the theorem, it is sufficient to show that

$$\limsup_{n\to\infty} \frac{|| \sum_{i=1}^{n} (X_i' - EX_i') ||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} \leq \Lambda_1 \quad \text{a.s.} \qquad (3.25)$$

By (2.4) and Kronecker Lemma, we have that

$$(2s_{n,p}^2 L_2 s_{n,p}^p)^{-\alpha/2} \sum_{i=1}^{n} E||X_i||^{\alpha} \to 0 \quad (n \to \infty). \qquad (3.26)$$

Hence by the boundedness of $\{X_i' - EX_i'\}$ and (3.24), applying the standard method of symmetrization and Lemma 7.2 of Ledoux and Talagrand [14], we may obtain that

$$\lim_{n\to\infty} \frac{E|| \sum_{i=1}^{n} (X_i' - EX_i') ||}{(2s_{n,p}^2 L_2 s_{n,p}^p)^{1/2}} = 0. \qquad (3.27)$$

By Lemma 3.3 of Wittmann [15], for any $M > 1$, there exists a subsequence of positive integers $\{n_k\}$ with $M s_{n_k,p} \leq s_{n_{k+1},p} \leq M^3 s_{n_k+1,p}$. Thus, by the standard argument, to prove (3.25), it is enough to show that for $M > 1$,

$$\sum_{k=1}^{\infty} P \left\{ \left\| \sum_{i=1}^{n_k} (X_i' - EX_i') \right\| > (2^{1+1/p} + 4\epsilon)(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2} \right\} < +\infty. \qquad (3.28)$$

Now for α satisfying (2.4) and $\gamma_j = \inf\{k : j \le n_k\}$, we have that

$$\sum_{k=1}^{\infty} \frac{1}{(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{\alpha/2}} \sum_{j=1}^{n_k} E\|X_j\|^{\alpha} \le \sum_{j=1}^{\infty} E\|X_j\|^{\alpha} \sum_{k=\gamma_j}^{\infty} \frac{1}{(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{\alpha/2}}$$

$$\le \sum_{j=1}^{\infty} E\|X_j\|^{\alpha} \sum_{k=\gamma_j}^{\infty} \frac{C}{(2s_{n_{\gamma_j},p}^2 L_2 s_{n_{\gamma_j},p})^{\alpha/2} M^{(k-\gamma_j)r}}$$

$$\le \sum_{j=1}^{\infty} \frac{E\|X_j\|^{\alpha}}{(2s_{j,p}^2 L_2 s_{j,p}^p)^{\alpha/2}} \sum_{k=0}^{\infty} \frac{C}{M^{kr}}$$

$$\le C(1 - \frac{1}{M^r})^{-1} \sum_{j=1}^{\infty} \frac{E\|X_j\|^{\alpha}}{(2s_{j,p}^p L_2 s_{j,p}^p)^{\alpha/2}}.$$

It is obvious that (2.4) implies the finiteness of the right side. Thus we have proved that

$$\sum_{k=1}^{\infty} \frac{1}{(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{\alpha/2}} \sum_{j=1}^{n_k} E\|X_j\|^{\alpha} < +\infty. \tag{3.29}$$

By (3.29), one can find a sequence $\{\eta_k\}$ of positive numbers tending to zero such that

$$\sum_{k=1}^{\infty} (\eta_k (2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2})^{-\alpha} \sum_{j=1}^{n_k} E\|X_j\|^{\alpha} < +\infty. \tag{3.30}$$

For $i \le n_k$, define

$$Y_{ik} = X_i' I\left\{\|X_i'\| < \frac{\eta_k s_{n_k,p}}{(2L_2 s_{n_k,p}^p)^{1/2}}\right\} = X_i I\left\{\|X_i\| < \frac{\eta_k s_{n_k,p}}{(2L_2 s_{n_k,p}^p)^{1/2}}\right\}$$

$$Z_{ik} = X_i' I\left\{\|X_i'\| \ge \frac{\eta_k s_{n_k,p}}{(2L_2 s_{n_k,p}^p)^{1/2}}\right\} = X_i I\left\{\frac{\eta_k s_{n_k,p}}{(2L_2 s_{n_k,p}^p)^{1/2}} \le \|X_i\| < (2s_{i,p}^2 L_2 s_{i,p}^p)^{1/2}\right\}.$$

Thus to prove (3.28), it sufficient to show that

$$\sum_{k=1}^{\infty} P\left\{\left\|\sum_{i=1}^{n_k} (Y_{ik} - EY_{ik})\right\| \ge (2^{1+1/p} + 3\epsilon)(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\} < +\infty \tag{3.31}$$

and

$$\sum_{k=1}^{\infty} P\left\{\left\|\sum_{i=1}^{n_k} (Z_{ik} - EZ_{ik})\right\| \ge \epsilon(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\} < +\infty. \tag{3.32}$$

Now we first prove (3.32). By Lemma 2.5 of Gine and Zinn [8], it is enough to show that for each $\epsilon > 0$,

$$\sum_{k=1}^{\infty} P\left\{\left\|\sum_{i=1}^{n_k} \epsilon_i Z_{ik}\right\| > \epsilon(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\} < +\infty. \tag{3.33}$$

This is equivalent to showing, by the Hoffmann-Jørgensen's inequality (Ledoux and Talagrand [12] Prop. 2.1), that for $\forall \epsilon > 0$,

$$\sum_{k=1}^{\infty} P\left\{\max_{i \le n_k} \|Z_{ik}\| > \epsilon(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\} < +\infty \tag{3.34}$$

and

$$\sum_{k=1}^{\infty}\left(P\left\{\left\|\sum_{i=1}^{n_k}\epsilon_i Z_{ik}\right\| > \epsilon(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\}\right)^2 < +\infty. \tag{3.35}$$

It is easy to see that (3.29) implies (3.34). Thus, to prove (3.35), by taking $q = [2K_0] + 1, m_k = [\delta_k^{-\frac{1}{2}}]$ (where $\delta_k = (\eta_k)^{-\alpha}(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{-\alpha/2}\sum_{i=1}^{n_k} E\|X_i\|^\alpha$, $s = \eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}$ and $t = \frac{\epsilon}{3}(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}$ and applying Proposition 1.1 in Ledoux and Talagrand [13], we have that for some $\delta > 0$,

$$P\left\{\left\|\sum_{i=1}^{n_k}\epsilon_i Z_{ik}\right\| > (\frac{\epsilon}{3} + 2\eta_k)(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2} + 8qM_k\right\}$$

$$\leq \left(\frac{1}{2}\right)^{m_k} + P\left\{\sum_{i=1}^{m_k}\|Z_{ik}\|^* > \eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right\}$$

$$+ 4\exp\left\{-\frac{(\frac{\epsilon}{3})^2 2s_{n_k,p}^2 L_2 s_{n_k,p}^p}{64([2K_0]+1)\sigma_k^2}\right\} + 4\exp\left\{-\frac{(\frac{\epsilon}{3})^2 2m_k s_{n_k,p}^2 L_2 s_{n_k,p}^p}{768 M_k \eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}}\right\}$$

$$\leq \left(\frac{1}{2}\right)^{m_k} + \frac{m_k E \max_{i \leq n_k}\|Z_{ik}\|^\alpha}{\eta_k^\alpha(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{\alpha/2}} + 4\exp\left\{-\frac{\delta^2(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)}{\sigma_k^2}\right\}$$

$$+ 4\exp\left\{-\frac{\delta^2 m_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}}{M_k \eta_k}\right\},$$

where

$$M_k = E\left\|\sum_{i=1}^{n_k}\epsilon_i Z_{ik} I\left\{\|Z_{ik}\| \leq \eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}/m_k\right\}\right\|,$$

$$\sigma_k^2 = \sup_{f \in B_1^*}\sum_{i=1}^{n_k} E[f^2(Z_{ik})I\{\|Z_{ik}\| \leq \eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}/m_k\}].$$

By (3.27) and Levy's inequality, we have

$$\lim_{k\to\infty}\frac{M_k}{2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}} = 0.$$

Thus for k large enough,

$$\left[P\left(\left\|\sum_{i=1}^{n_k}\epsilon_i Z_{ik}\right\| > \epsilon 2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}\right)\right]^2 \leq C\left\{\left(\frac{1}{2}\right)^{2m_k} + \delta_k + \exp\left(-\delta^2 m_k\right) + \exp\left(-\frac{2\delta^2 a_{n_k}^2}{\sigma_k^2}\right)\right\}$$

$$\leq C\left\{\left(\frac{1}{2}\right)^{2m_k} + \delta_k + \frac{1}{m_k^2} + \exp\left(-\frac{2\delta^2 a_{n_k}^2}{\sigma_k^2}\right)\right\} \leq C\left\{\delta_k + \exp\left(-\frac{2\delta^2 a_{n_k}^2}{\sigma_k^2}\right)\right\}$$

where C is the constant independent of k and varies from line to line, and the second inequality follows from the fact that

$$\exp(-x) \leq C_2 \frac{1}{x^2} \qquad \forall x \geq 1 \text{ and for some constant } C_2.$$

By (3.30), in order to show (3.35), it is sufficient to prove that

$$\sum_{k=1}^{\infty} \exp(-2\delta^2(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)/\sigma_k^2) < +\infty.$$

Let $N_1 = \{k \in N, 2L_2 s_{n_k,p}^p < m_k = [\delta_k^{-\frac{1}{2}}]\}$. For any $k \in N_i = 1$, we have $\frac{\eta_k(s_{n_k,p})}{(2L_2 s_{n_k,p}^p)^{1/2}} > \frac{\eta_k(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)^{1/2}}{m_k}$. Thus by the definition of $\{Z_{ik}\}$, $M_k = 0$ and $\sigma_k = 0$. Hence,

$$\sum_{k \in N} \exp\left\{-\frac{2\delta^2(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)}{\sigma_k^2}\right\} = \sum_{k \in N \setminus N_1} \exp\left\{-\frac{2\delta^2(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)}{\sigma_k^2}\right\}.$$

By the argument like the proof of Theorem 2.1, we have that $\sigma_k \leq 2^{1+\frac{1}{p}} s_{n_k,p}$. Therefore,

$$\sum_{k \in N} \exp\left\{-\frac{2\delta^2(2s_{n_k,p}^2 L_2 s_{n_k,p}^p)}{\sigma_k^2}\right\} = \sum_{k \in N \setminus N_1} \exp\left\{-\frac{2\delta^2}{2^{1+1/p}} 2L_2 s_{n_k,p}^p\right\}$$

$$\leq C \sum_{k \in N \setminus N_1} \frac{1}{(2L_2 s_{n_k,p}^p)^2} \leq C \sum_{k \in N \setminus N_1} \frac{1}{m_k^2} \leq C \sum_{k=1}^{\infty} \delta_k < +\infty.$$

(3.35) is proved.

Similar to the proof of Theorem 2.1, we easily show that for $\forall \epsilon > 0$,

$$\sum_{k=1}^{\infty} P\left\{\left\|\sum_{i=1}^{n_k}(Y_{ik} - EY_{ik})\right\| > (2^{1+1/p} + 3\epsilon)a_{n_k}\right\} < +\infty.$$

Therefore we have actually proved that

$$\limsup_{n \to \infty} \frac{\|\sum_{i=1}^{n}(X_i' - EX_i')\|}{a_n} \leq 2^{1+1/p} \qquad \text{a.s.}$$

This completes the proof of Theorem 2.2. □

In order to prove Theorem 2.3, we need the following lemma (see Lem. 5 in Einmahl [7]).

Lemma 3.2. *Let* $\xi : \Omega \to [0, \infty)$ *be a random variable such that*

$$\sum_{n=1}^{\infty} P\{\xi > c_n\} < +\infty,$$

where $\{c_n\}$ *is a sequence satisfying for some* $\alpha > 0$,

$$c_n/n^{\alpha} \text{ is nondecreasing.}$$

Then we have that

$$(a) \sum_{n=1}^{\infty} E\xi^r I\{\xi \leq c_n\}/c_n^r < +\infty \qquad \text{provided that } r > \alpha^{-1},$$

$$(b) \sum_{n=1}^{\infty} P(\xi > \epsilon c_n) < +\infty \qquad \text{for any } \epsilon > 0.$$

The proof of Theorem 2.3.

For each $i \geq 1$, define

$$X_i' = X_i I\{||X_i|| \leq c_n\}, \quad X_i'' I\{||X_i|| > c_i\} = X_i - X_i'.$$

Then similar to the proof of Theorem 2.2, we easily obtain that

$$\lim_{n \to \infty} \sum_{i=1}^n (X_i'' - EX_i'')/c_n = 0 \quad \text{a.s.} \tag{3.36}$$

and thus, by (2.7) we also have that

$$\sum_{i=1}^n (X_i' - EX_i')/c_n \to 0 \quad \text{in probability.} \tag{3.37}$$

Now by using Lemma 3.2, (2.6) and (2.8) imply that for $r > p = \alpha^{-1}$,

$$\sum_{n=1}^\infty \frac{E||X'||^r}{c_n^r} = \sum_{n=1}^\infty E||X_n||^r I\{||X_n|| \leq c_n\}/c_n^r = \sum_{n=1}^\infty E||X||^r I\{||X|| \leq c_n\}/c_n^r < +\infty. \tag{3.38}$$

Hence the conditions in Proposition 3.1 are obtained from (3.37) and (3.38). Therefore, there exists some $\Gamma > 0$ such that

$$\limsup_{n \to \infty} \frac{||\sum_{i=1}^n (X_i' - EX_i')||}{c_n} \leq \Gamma.$$

Again by (3.36), $\limsup_{n \to \infty} ||S_n||/c_n < \Gamma$. In particular, if $c_n = \sigma_p n^{1/p}(2L_2 n)^{1/2}$, the conclusion follows from Theorem 2.2.

Finally we give the proof of Theorem 2.3. We first give a proposition which seems to be an extension of Proposition 1.1 in Ledoux and Talagrand [13]. By using Proposition 3.1 in Deng [6] and Theorem 4.12 in Ledoux and Talagrand [14], the proof may follow from the similar argument like the proof of Proposition 1.1 in Ledoux and Talagrand [13] or Theorem 6.17 in Ledoux and Talagrand [14] and thus is omitted.

Proposition 3.3. *Let $\{X_i, n \geq 1\}$ be a sequence of independent symmetric B-valued random variables. For any integer $k \geq \omega$ and positive numbers s and t, we have that,*

$$P \left\{ \left|\left| \sum_{i=1}^N X_i \right|\right| > t + 2s + 8\omega M \right\}$$

$$\leq \left(\frac{K_0}{\omega} \right)^k + P \left\{ \sum_{i=1}^k ||X_i||^* > s \right\} + 4 \exp \left\{ -\frac{t^q}{2^q q (4\omega \sigma^p + 24 p M (s/k)^{p-1})^{q/p}} \right\}$$

where $p \in (1, 2], q = \frac{p}{p-1}$,

$$M = E \left|\left| \sum_{i=1}^N u_i \right|\right| \qquad \sigma^p = \sup_{f \in B_1^*} \left(\sum_{i=1}^N E|f(X_i)|^p \right),$$

$u_i = X_i I\{||X_i|| \leq s/k\}(i \leq N)$, $\{||X_i||^*, i \leq N\}$ is the nondecreasing rearrangement of the sequence $\{||X_i||, i \leq N\}$, and K_0 is a universal constant.

The proof of Theorem 2.4.

We first prove Theorem 2.4 for $p = 1$. In view of (2.11), for $\rho > 1$, and each k, let $n_k = \min\{n : s_{n,1} > \rho^k\}$. Then,

$$s_{n_k,1} \sim \rho^k, \qquad\qquad s_{n_{k+1},1}/s_{n_k,1} \sim \rho.$$

Set $I(k) = \{n_k + 1, ..., n_{k+1}\}$. From Corollary 7.3 in Ledoux and Talagrand [14], it is sufficient to prove that $\limsup_{k\to\infty} \|\sum_{i\in I(k)} X_i\|/s_{n_k,1} = 0$. Now, set $q = [2K_0] + 1, s = [2L_2 s_{n_k,1}]$ and for integer r, set $X_{I(k)}^{(r)} = \|X_i\|$ whenever $\|X_i\|$ is the r-th maximum of the sample $\{\|X_i\|, i \in I(k)\}$. On account of (2.12), for all $\epsilon > 0$ and for large k,

$$\sum_{r=1}^{[2L_2 s_{n_k,1}]} X_{I(k)}^{(r)} \le [2L_2 s_{n_k,1}]\eta_{n_k} s_{n_k,1}/L_2 s_{n_k,1} < \epsilon s_{n_k,1},$$

$$M_k = E\|\sum_{i\in I(k)} X_i I_{\{\|X_i\|\le \epsilon s_{n_k,1}/[2L_2 s_{n_k,1}]\}}\| = E\|\sum_{i\in I(k)} X_i\|,$$

$$\sigma_k^2 = \sup_{f\in B_1^*}\left(\sum_{i\in I(k)} Ef^2(X_i I_{\{\|X_i\|\le \epsilon s_{n_k,1}/[2L_2 s_{n_k,1}]\}})\right) = \sup_{f\in B_1^*}\left(\sum_{i\in I(k)} Ef^2(X_i)\right)$$

$$\le \sup_{f\in B_1^*}\left(\sum_{i\in I(k)} E|f(X_i)|\eta_i s_{i,1}/L_2 s_{i,1}\right) \le s_{n_k,1}^2 \eta_{n_k}/L_2 s_{n_k,1}$$

and thus for a large k_0,

$$\sum_{k\ge k_)} \exp(-\epsilon^2 s_{n_k,1}^2/\sigma_k^2) \le \sum_{k\ge k_0} \exp(-\epsilon L_2 s_{n_k,1}/\eta_{n_k}) \le C \sum_{k\ge k_o} \exp\{-2(\log k \log \rho)\} < +\infty.$$

From Theorem 7.5 in Ledoux and Talagrand [14], it follows that $\limsup_{n\to\infty} S_n/s_{n,1} = 0$.

Next by Borel-Cantelli Lemma and Levy's inequality, in order to establish the theorem for $p \in (1, 2)$, it suffices to show that for all $\epsilon > 0$,

$$\sum_{k=1}^{\infty} P\{\|S_{n_k}\| > (\Lambda_p + 3\epsilon)(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}\} < +\infty \tag{3.39}$$

where $\{n_k\}$ is the same as above. Taking $\omega = 2K_0, k = [2L_2 s_{n_k,p}^p], s = \epsilon(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}, t = (\Lambda_p + \epsilon)(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}$ and applying Proposition 3.3, it is easy to see that

$$P\{\|S_{n_k}\| > t + 2s + 8\omega M_k\}$$

$$\le \left(\frac{1}{2}\right)^{[2L_2 s_{n_k,p}^p]} + P\left(\sum_{i=1}^{[2L_2 s_{n_k,p}^p]} \|X_i\|^* > s\right)$$

$$+ 4\exp\left(-\frac{[(\Lambda + \epsilon)(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}]^q}{2^q q(4\omega\sigma_k^2 + 24pM_k[\epsilon(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}/([2L_2 s_{n_k,p}^p])]^{p-1})^{q/p}}\right)$$

where

$$M_k = E \left\| \sum_{i=1}^{n_k} X_i I_{\{||X_i|| \le s/k\}} \right\|,$$

$$\sigma_k^p = \sup_{f \in B_1^*} \left(\sum_{i=1}^{n_k} E|f(X_i)I_{\{||X_i|| \le s/k\}}|^p \right)$$

and thus $M_k = ||S_{n_k}||$, $\sigma_k^p = s_{n_k,p}^p$ for sufficiently large k since $||X_i|| \le \eta_i s_{i,p}/(L_2 s_{i,p}^p)^{1/p} \le \epsilon(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}/$
$(2L_2 s_{n_k,p}^p) \le s/k$. Again, by using the standard method, it is easy to see that (2.13) implies that $\lim_{n \to \infty} \alpha_k \equiv \lim_{n \to \infty} E||S_n||/(2s_{n,p}^q L_2 s_{n,p}^p)^{1/q} = 0$. Now note that for sufficiently large k,

$$\sum_{i=1}^{[2L_2 s_{n_k,p}^p]} ||X_i||^* \le [2L_2 s_{n_k,p}^p]\eta_{n_k} s_{n_k,p}/(L_2 s_{n_k,p}^p)^{1/p} \le s,$$

$$24pM_k[\epsilon(2s_{n_k,p}^q L_2 s_{n_k,p}^p)^{1/q}/([2L_2 s_{n_k,p}^p])]^{p-1}$$

$$\le 24p\epsilon^{p-1} M_k s_{n_k,p}^{p-1}/(2L_2 s_{n_k,p}^p)^{1/q}$$

$$\le 24p\epsilon^{p-1} s_{n_k,p}^p ||S_{n_k}||/(2s_{n_k,p}^q L_2 s_{n_k,p})^{1/q} \le C\alpha_k s_{n_k,p}^p.$$

Therefore, for k large enough,

$$P\{||S_{n_k}|| > (\Lambda_p + 3\epsilon)2s_{n_k,p}(L_2 s_{n_k,p}^p)^{1/q}\}$$

$$\le P\{||S_{n_k,p}|| > t + 2s + 8\omega M_k\}$$

$$\le (\frac{1}{2})^{2\log_2 \rho^{pk}} + 4\exp\left\{ -\frac{[(16K_0)^{1/p}q^{1/q} + \epsilon]^q 2s_{n_k,p}^q L_2 s_{n_k,p}^p}{2^q q(4\omega + \alpha_k)s_{n_k,p}^q} \right\}$$

$$\le 2^{-2\log(pk\log\rho)} + 4\exp\left\{ -\frac{[(8K_0)^{1/p}q^{1/q} + \epsilon/2^{1/p}]^q}{q(8K_0 + \alpha_k)^{q/p}} \log_2 s_{n_k,p}^p \right\}$$

$$\le 2^{-2\log(pk\log\rho)} + 4\exp\left\{ -(1 + \frac{\epsilon}{2})\log_2 s_{n_k,p}^p \right\},$$

from which, (3.10) follows. For $p = 2$ the result follows from Kolmogorov's LIL (see Chen [4]). The proof of Theorem 2.4 is complete. $\square$

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