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# COMPARISON OF ORDER STATISTICS IN A RANDOM SEQUENCE TO THE SAME STATISTICS WITH I.I.D. VARIABLES

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**Abstract.** The paper is motivated by the stochastic comparison of the reliability of non-repairable $k$-out-of-$n$ systems. The lifetime of such a system with nonidentical components is compared with the lifetime of a system with identical components. Formally the problem is as follows. Let $U_i$, $i = 1, ..., n$, be positive independent random variables with common distribution $F$. For $\lambda_i > 0$ and $\mu > 0$, let consider $X_i = U_i/\lambda_i$ and $Y_i = U_i/\mu$, $i = 1, ..., n$. Remark that this is no more than a change of scale for each term. For $k \in \{1, 2, ..., n\}$, let us define $X_{k:n}$ to be the $k$th order statistics of the random variables $X_1, ..., X_n$, and similarly $Y_{k:n}$ to be the $k$th order statistics of $Y_1, ..., Y_n$. If $X_i$, $i = 1, ..., n$, are the lifetimes of the components of a $n$+1-$k$-out-of-$n$ non-repairable system, then $X_{k:n}$ is the lifetime of the system. In this paper, we give for a fixed $k$ a sufficient condition for $X_{k:n} \geq_{st} Y_{k:n}$ where $st$ is the usual ordering for distributions. In the Markovian case (all components have an exponential lifetime), we give a necessary and sufficient condition. We prove that $X_{k:n}$ is greater that $Y_{k:n}$ according to the usual *stochastic ordering* if and only if

$$\binom{n}{k} \mu^k \geq \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1} \lambda_{i_2} ... \lambda_{i_k}.$$

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## 1. Introduction

In reliability theory, a system of $n$ identical independent components is usually said $k$-out-of-$n$ when it is functioning if and only if at least $k$ of the components are functioning. In this case, there are $n-k$ failures. Such systems are getting more and more frequent in industrial processes. For example, a given parameter (presence or not of a train, temperature, ...) might be controlled by several devices and the decision rule used to fix the value of this parameter is of type $k$-out-of-$n$. This reliability notion is, in fact, the same as the order statistics notion. If the lifetimes of the components are independent identically distributed (i.i.d.) random variables $X_1, X_2, ..., X_n$, then the lifetime of the $n$+1-$k$-out-of-$n$ system is exactly the $k$th order statistics (denoted $X_{k:n}$) of the random variables $X_i$, $i = 1, ..., n$.

*Keywords and phrases.* Stochastic ordering, Markov system, order statistics, $k$-out-of-$n$.

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Let us now describe the practical problem which motivated this paper. We consider a $n$+1-$k$-out-of-$n$ system where the $n$ components have independent exponential lifetimes $X_1, ..., X_n$, but not necessarily identically distributed. We denote $\lambda_1, ..., \lambda_n$ the respective parameters of these exponential lifetimes. The problem is to compare this system with an equivalent system with i.i.d. components. Practically, when we have to replace many different components (from different factories) by identical components (from the same factory), we need to guarantee the same quality.

Let us denote by $Y_1, ..., Y_n$, the random lifetimes of the identical components with the common parameter $\mu$. The problem of comparison is equivalent to the following one. What are the values of $\mu$ which characterize the stochastic inequality $X_{k:n} \geq_{st} Y_{k:n}$, *i.e.* $\mathbb{P}(X_{k:n} > t) \geq \mathbb{P}(Y_{k:n} > t)$, $\forall t > 0$ ?

A general result on the stochastic comparison of order statistics was obtained by Pledger and Proschan [6], in connection with the *Schur's majorization* (see Marshall and Olkin [4]). In our context, this result provides a sufficient condition which can be written as follows:

$$\mu = \frac{\lambda_1 + \lambda_2 + ... + \lambda_n}{n} \quad \Rightarrow \quad X_{k:n} \geq_{st} Y_{k:n}.$$

More recently, Khaledi and Kochar [3] studied the case $k = n$ and proved that

$$\mu = \sqrt[n]{\lambda_1 \lambda_2 ... \lambda_n} \quad \Rightarrow \quad X_{n:n} \geq_{hr} Y_{n:n},$$

where $hr$ denotes the *hazard rate ordering.* See Shaked and Shantikhumar [7] for an overview of the different notions of ordering.

Here we extend this results referring to *stochastic ordering* which gives the comparison of survival functions. With the exponential assumption, we propose a necessary and sufficient condition on the parameters for the inequality $X_{k:n} \geq_{st} Y_{k:n},\ k = 1, 2, ..., n$. More generally, we give sufficient conditions for the stochastic comparison in the case where the distribution $F$ is not exponential.

## 2. Elementary symmetrical functions

First, let us introduce some notations and recall some results about elementary symmetrical functions.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$, $n > 1$ a vector with positive components. For $j \in \{1, 2, ..., n\}$,

$$S_j(\mathbf{x}) = \sum_{1 \leq i_1 < i_2 < ... < i_j \leq n} x_{i_1} x_{i_2} ... x_{i_j} \tag{1}$$

is the $j$th elementary symmetrical function of the positive $x_1, x_2, ..., x_n$.
As usual $S_0(\mathbf{x}) = 1$, and $S_j(\mathbf{x}) = 0$, for $j > n$.
For $p, q \in \{1, 2, ..., n\}$, $p \neq q$, we denote $\mathbf{x}^p = (..., x_{p-1}, x_{p+1}, ...) \in (0, \infty)^{n-1}$ and $\mathbf{x}^{p,q} = (\mathbf{x}^p)^q \in (0, \infty)^{n-2}$. Therefore

$$S_j(\mathbf{x}^p) = \sum_{\substack{i_1, ..., i_j \in \{1, ..., n\} \setminus \{p\} \\ i_1 < i_2 < ... < i_j}} x_{i_1} x_{i_2} ... x_{i_j} \tag{2}$$

is the $j$th elementary symmetrical function obtained without the component of number $p$ and

$$S_j(\mathbf{x}^{p,q}) = \sum_{\substack{i_1, ..., i_j \in \{1, ..., n\} \setminus \{p, q\} \\ i_1 < i_2 < ... < i_j}} x_{i_1} x_{i_2} ... x_{i_j} \tag{3}$$

is the $j$th elementary symmetrical function made without $p$ and $q$. In the case where all coordinates of $\mathbf{x}$ are equal to $m$, we have

$$S_k(\mathbf{x}) = \binom{n}{k} m^k. \tag{4}$$

These functions satisfy the elementary relations:

$$S_j(\mathbf{x}) = x_p S_{j-1}(\mathbf{x}^p) + S_j(\mathbf{x}^p) \tag{5}$$

and

$$S_j(\mathbf{x}) = x_p x_q S_{j-2}(\mathbf{x}^{p,q}) + (x_p + x_q) S_{j-1}(\mathbf{x}^{p,q}) + S_j(\mathbf{x}^{p,q}). \tag{6}$$

Moreover we define

$$m_j(\mathbf{x}) = \left( \binom{n}{j}^{-1} S_j(\mathbf{x}) \right)^{\frac{1}{j}} = \left( \binom{n}{j}^{-1} \sum_{|J|=j} \prod_{i \in J} x_i \right)^{\frac{1}{j}} \tag{7}$$

the $j$th symmetrical mean, $j = 1, 2, ..., n$. These different averages are classical and satisfy the well-known Mac Laurin's inequalities (see Hardy, Littlewood and Pólya [2]):

$$m_1(\mathbf{x}) \geq m_2(\mathbf{x}) \geq ... \geq m_n(\mathbf{x}).$$

Two special cases are of interest. If $j = 1$ and $j = n$ then $m_1(\mathbf{x}) = \frac{x_1 + x_2 + ... + x_n}{n}$ and $m_n(\mathbf{x}) = \sqrt[n]{x_1 x_2 ... x_n}$ are the arithmetical and geometrical means. In order to compare the elementary symmetrical functions of two different vectors, we shall use the following lemma.

**Lemma 1.** *Let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ be two vectors such that $0 < x_i \leq y_i$ for all $i = 1, ..., m$. If $r \in \{0, 1, ..., m-1\}$ then*

$$\frac{S_{r+1}(\mathbf{x})}{S_r(\mathbf{x})} \leq \frac{S_{r+1}(\mathbf{y})}{S_r(\mathbf{y})}. \tag{8}$$

*Moreover, if there exists $i_0$ so that $x_{i_0} < y_{i_0}$ then the inequality (8) is strict.*

*Proof.* Let $r$ be an integer, $0 \leq r < m$. It is sufficient to prove that the symmetrical function $\gamma_r : (0, \infty)^m \to (0, \infty)$, $\gamma_r(\mathbf{x}) = \frac{S_{r+1}(\mathbf{x})}{S_r(\mathbf{x})}$ is strictly increasing in $x_1$.

The property is clear for $r = 0$. Let us consider $r > 0$. Since $S_{r+1}(\mathbf{x}) = x_1 S_r(\mathbf{x}^1) + S_{r+1}(\mathbf{x}^1)$ and $S_r(\mathbf{x}) = x_1 S_{r-1}(\mathbf{x}^1) + S_r(\mathbf{x}^1)$ we have

$$\frac{\partial \gamma_r}{\partial x_1}(\mathbf{x}) = \frac{\left(S_r(\mathbf{x}^1)\right)^2 - S_{r-1}(\mathbf{x}^1) S_{r+1}(\mathbf{x}^1)}{\left(S_r(\mathbf{x})\right)^2}.$$

The relation $\frac{\partial \gamma_r}{\partial x_1}(\mathbf{x}) > 0$ is equivalent to:

$$\left(S_r(\mathbf{x}^1)\right)^2 > S_{r-1}(\mathbf{x}^1) S_{r+1}(\mathbf{x}^1),$$

for $\mathbf{x}^1 = (x_2, ..., x_n)$. But this is an immediate consequence of the Newton inequalities (see Hardy, Littlewood and Pólya [2]). □

In the sequel, we shall use the following consequence of Lemma 1.

**Corollary 1.** *Let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ be two vectors with $m$ positive components such that $\alpha \leq \frac{y_i}{x_i} \leq \beta$, $i = 1, 2, ..., m$, where $0 < \alpha < \beta$. Then, for $r \in \{0, 1, ..., m-1\}$, the next inequalities are true:*

$$\alpha \frac{S_r(\mathbf{y})}{S_r(\mathbf{x})} \leq \frac{S_{r+1}(\mathbf{y})}{S_{r+1}(\mathbf{x})} \leq \beta \frac{S_r(\mathbf{y})}{S_r(\mathbf{x})}, \tag{9}$$

*and at least one of these inequalities is a strict inequality.*

The assertion follows easily by replacing the pair $(\mathbf{x}, \mathbf{y})$ by $(\alpha\mathbf{x}, \mathbf{y})$ and $(\mathbf{y}, \beta\mathbf{x})$ in the above lemma.

**Remark.** We give also an elementary result about the comparison of two fractions. The proof is omitted. If $a, b, c, d, e, f$ are positive with $\frac{a}{b} \leq \frac{c}{d} \leq \frac{e}{f}$ and $\frac{a}{b} < \frac{e}{f}$ then

$$\frac{a+c}{b+d} < \frac{c+e}{d+f}. \tag{10}$$

In order to prove our main results, we present a sufficient condition to recognize the minimum value of a symmetrical function. This result is interesting by itself and, to the best of our knowledge, it is new.

**Lemma 2.** *For $n > 1$, let $\psi : (0, \infty)^n \to (0, \infty)$ be a symmetrical and continuously differentiable mapping, and $k$ an integer, $1 \leq k \leq n$. Let us assume that, for any vector $\mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n$, with $x_p = \min x_i$ and $x_q = \max x_i$ we have:*

$$x_p < x_q \quad \Rightarrow \quad \frac{\frac{\partial \psi}{\partial x_p}(\mathbf{x})}{\frac{\partial S_k}{\partial x_p}(\mathbf{x})} < \frac{\frac{\partial \psi}{\partial x_q}(\mathbf{x})}{\frac{\partial S_k}{\partial x_q}(\mathbf{x})}. \tag{11}$$

*Then, for any $\mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n$, the following inequality holds:*

$$\psi(x_1, ..., x_n) \geq \psi(\underbrace{m_k(\mathbf{x}), ..., m_k(\mathbf{x})}_{n \ times}). \tag{12}$$

*Proof.* The case $k = 1$ is well-known (see, for example, Marshall and Olkin [4]).
Now we suppose $k > 1$. For a fixed vector $\mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n$ let consider $a = \min x_i$, $b = \max x_i$, $m = m_k(\mathbf{x})$ and $\mathbf{m} = (\underbrace{m, ..., m}_{n \ times})$. Inequality (12) is an equality for $a = b$.
Let us assume $a < b$. Then $m \in (a, b)$. We consider the compact subset $K$ of $(0, \infty)^n$:

$$K = \{\mathbf{t} = (t_1, t_2, ..., t_n) \in [a, b]^n \mid m_k(\mathbf{t}) = m\}.$$

Clearly, $\mathbf{x}$ and $\mathbf{m}$ belong to $K$. From Weierstrass's theorem it follows that the continuous mapping $\psi$ reaches an absolute minimum on the compact $K$ on some point $\mathbf{u} = (u_1, u_2, ..., u_n) \in K$.

Now let us assume $\mathbf{u} \neq \mathbf{m}$. In this case, there exists $p, q \in \{1, 2, ..., n\}$ such that $a \leq u_p = \min u_i < \max u_i = u_q \leq b$. Rewriting condition $m_k(\mathbf{u}) = m$ from relation (6), yields to

$$u_p u_q S_{k-2}(\mathbf{u}^{p,q}) + (u_p + u_q) S_{k-1}(\mathbf{u}^{p,q}) + S_k(\mathbf{u}^{p,q}) = \binom{n}{k} m^k.$$

The equation $S_{k-2}(\mathbf{u}^{p,q}) z^2 + 2 S_{k-1}(\mathbf{u}^{p,q}) z + S_k(\mathbf{u}^{p,q}) = \binom{n}{k} m^k$ has a positive solution in $z$ which is denoted $z_1$. Clearly $u_p < z_1 < u_q$. For $t \in [u_p, z_1)$, let us consider the function $g(t)$ defined on $[u_p, z_1)$ by the relation:

$$t g(t) S_{k-2}(\mathbf{u}^{p,q}) + (t + g(t)) S_{k-1}(\mathbf{u}^{p,q}) + S_k(\mathbf{u}^{p,q}) = \binom{n}{k} m^k.$$

We have $g(u_p) = u_q$ and more generally:

$$g(t) = \frac{\binom{n}{k} m^k - t S_{k-1}(\mathbf{u}^{p,q}) - S_k(\mathbf{u}^{p,q})}{t S_{k-2}(\mathbf{u}^{p,q}) + S_{k-1}(\mathbf{u}^{p,q})} \in (z_1, u_q], \quad \forall t \in [u_p, z_1).$$

Let us denote by $\mathbf{u}(t)$ the vector with the components $u_p(t) = t$, $u_q(t) = g(t)$ and $u_i(t) = u_i$ for $i \in \{1, 2, ..., n\} \setminus \{p, q\}$. We have $\mathbf{u}(t) \in K$ and

$$S_k(\mathbf{u}(t)) = \binom{n}{k} m^k, \quad \forall t \in [u_p, z_1).$$

The continuously differentiable decreasing function $g$ has the following derivative:

$$g'(t) = -\frac{\frac{\partial S_k}{\partial x_p}(\mathbf{u}(t))}{\frac{\partial S_k}{\partial x_q}(\mathbf{u}(t))}. \tag{13}$$

Now let us consider the continuously differentiable function $\varphi : [u_p, z_1) \to I\!R, \quad \varphi(t) = \psi(\mathbf{u}(t))$.
From relation (13) we obtain:

$$\varphi'(t) = \frac{\partial \psi}{\partial x_p}(\mathbf{u}(t)) + \frac{\partial \psi}{\partial x_q}(\mathbf{u}(t)) \cdot g'(t) = \frac{\partial S_k}{\partial x_p}(\mathbf{u}(t)) \left( \frac{\frac{\partial \psi}{\partial x_p}(\mathbf{u}(t))}{\frac{\partial S_k}{\partial x_p}(\mathbf{u}(t))} - \frac{\frac{\partial \psi}{\partial x_q}(\mathbf{u}(t))}{\frac{\partial S_k}{\partial x_q}(\mathbf{u}(t))} \right).$$

But, from assumption (11), it follows that $\varphi'(u_p) < 0$. Hence, there exists $\varepsilon > 0$ such that $u_p + \varepsilon < z_1$ and $\varphi'(t) < 0, \quad \forall t \in [u_p, u_p + \varepsilon)$. Therefore, $\psi(\mathbf{u}(t)) < \psi(\mathbf{u}(u_p)) = \psi(\mathbf{u})$, for any $t \in (u_p, u_p + \varepsilon)$. This gives the contradiction. Then the unique minimum point of $\psi$ on $K$ is $\mathbf{m}$ and the relation (12) follows. $\square$

**Remark.** The assumption (11) can be replaced by the following more restrictive assumption:

$$\forall \mathbf{x} \in (0,\infty)^n \quad \forall i, j \in \{1, ..., n\} \qquad x_i \neq x_j \;\Rightarrow\; (x_i - x_j) \left( \frac{\frac{\partial \psi}{\partial x_i}(\mathbf{x})}{\frac{\partial S_k}{\partial x_i}(\mathbf{x})} - \frac{\frac{\partial \psi}{\partial x_j}(\mathbf{x})}{\frac{\partial S_k}{\partial x_j}(\mathbf{x})} \right) > 0.$$

In the case $k = 1$, one obtains again a well-known sufficient condition of Schur convexity (see Marshall and Olkin [4]).

## 3. The main results

This section is concerned with the characterizations of the comparison between a system with different lifetime components $(X_i)$ and a system with i.i.d. lifetime components $(Y_i)$.
Formally, let $X$ and $Y$ be two random variables with support $I\!R^+$, having the survival functions $\overline{F}_X = 1 - F_X$ and $\overline{F}_Y$, respectively. $F_X$ and $F_Y$ are assumed to be continuously differentiable.
The variable $X$ is said *stochastically larger* than $Y$ (denoted $X \geq_{st} Y$) when $\overline{F}_X(t) \geq \overline{F}_Y(t), \ \forall t \geq 0$.

The following theorem provides sufficient conditions for the stochastic comparison between the same order statistics in two sequences of independent random variables.

**Theorem 1.** *Let $U_1, U_2, ..., U_n$ be i.i.d. positive random variables with the common distribution function $F$ having a positive non-increasing hazard rate $h(x) = \frac{F'(x)}{1-F(x)}, \ x \in (0,\infty)$.*
*For a vector $\lambda = (\lambda_1, ..., \lambda_n)$ with positive components and $k \in \{1, ..., n\}$,*

$$m_k(\lambda) = \left( \binom{n}{k}^{-1} \sum_{|J|=k} \prod_{i \in J} \lambda_i \right)^{\frac{1}{k}}$$

*is the kth symmetrical mean of* $\lambda$. *Let us define* $X_i = U_i/\lambda_i$ *and* $Y_i = U_i/m_k(\lambda)$ *for all* $i \in \{1, ..., n\}$. *Let us denote by* $X_{k:n}$ *(resp.* $Y_{k:n}$*) the kth order statistics of* $(X_1, ..., X_n)$ *(resp.* $(Y_1, ..., Y_n)$*). If* $\frac{F(x)}{x(1-F(x))}$ *is an increasing function on* $(0, \infty)$*, then*

$$X_{k:n} \ \geq_{st} \ Y_{k:n}.$$

*Proof.* From the definition, the random variable $X_i$ has the distribution function $F_{X_i}(t) = F(\lambda_i t), \ \ t \geq 0, \ \ i = 1, 2, ..., n$.

First, let us consider the case $k = 1$. The random variables $X_{1:n}$ and $Y_{1:n}$ have the survival functions $\overline{F}_{X_{1:n}}(t) = \prod_{i=1}^{n} \overline{F}(\lambda_i t)$ and $\overline{F}_{Y_{1:n}}(t) = \left(\overline{F}(\frac{\lambda_1 + ... + \lambda_n}{n} t)\right)^n$ respectively. From $\left(\log \overline{F}\right)' = -h$ with $h$ a non-increasing function, it follows that $\log \overline{F}$ is a convex function. Thus, from Jensen's inequality we get $\overline{F}_{X_{1:n}}(t) \geq \overline{F}_{Y_{1:n}}(t), \ \forall t \geq 0$, and the conclusion is proved for $k = 1$.

Let us assume now $k > 1$. The survival function of $X_{k:n}$ is:

$$\overline{F}_{X_{k:n}}(t) = \sum_{j=0}^{k-1} \sum_{|J|=j} \left( \prod_{i \in J} F(\lambda_i t) \right) \left( \prod_{i' \notin J} \overline{F}(\lambda_{i'} t) \right) = \prod_{i=1}^{n} \overline{F}(\lambda_i t) \sum_{j=0}^{k-1} \sum_{|J|=j} \left( \prod_{i \in J} \frac{F(\lambda_i t)}{\overline{F}(\lambda_i t)} \right). \tag{14}$$

Let us consider the functions $y : (0, \infty) \to (0, \infty), \ y(x) = \frac{F(x)}{\overline{F}(x)}$ and $\psi : (0, \infty)^n \to (0, 1)$,

$$\psi(x_1, ..., x_n)) = \frac{\sum_{j=0}^{k-1} S_j(y(x_1), ..., y(x_n))}{\prod_{i=1}^{n} (1 + y(x_i))}.$$

For $t > 0$ we have $\overline{F}_{X_{k:n}}(t) = \psi(\lambda_1 t, ..., \lambda_n t)$.

Similarly, $\overline{F}_{Y_{k:n}}(t) = \psi(\underbrace{m_k(\lambda t), ..., m_k(\lambda t)}_{n \ times})$. To obtain the conclusion $X_{k:n} \geq_{st} Y_{k:n}$, it is sufficient to prove the following property of the symmetrical and continuously differentiable mapping $\psi$:

$$\psi(x_1, ..., x_n) \geq \psi(\underbrace{m_k(\mathbf{x}), ..., m_k(\mathbf{x})}_{n \ times}), \ \forall \ \mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n. \tag{15}$$

For $\mathbf{x} = (x_1, ..., x_n) \in (0, \infty)^n$, we use the notation $\mathbf{y} = \mathbf{y}(\mathbf{x}) = (y(x_1), ..., y(x_n))$ and $y_i = y(x_i)$. The function $\psi$ has the following partial derivatives:

$$\begin{aligned} \frac{\partial \psi}{\partial x_s}(\mathbf{x}) &= \frac{y'(x_s) \prod_{i \neq s}(1 + y_i) \left\{ (1 + y_s) \sum_{j=1}^{k-1} S_{j-1}(\mathbf{y}^s) - \left[ 1 + \sum_{j=1}^{k-1} \left( y_s S_{j-1}(\mathbf{y}^s) + S_j(\mathbf{y}^s) \right) \right] \right\}}{\prod_{i=1}^{n} (1 + y(x_i))^2} \\ &= -\frac{y'(x_s)}{1 + y(x_s)} \ \frac{S_{k-1}(\mathbf{y}^s)}{\prod_{i=1}^{n} (1 + y_i)} \\ &= -h(x_s) \ \frac{S_{k-1}(\mathbf{y}^s)}{\prod_{i=1}^{n} (1 + y_i)}, \quad s = 1, ..., n. \end{aligned} \tag{16}$$

We denote $x_p = \min x_i$ and $x_q = \max x_i$. Let us assume that $x_p < x_q$.
For $k < n$, using relation (16), we get:

$$\begin{aligned} \left( \frac{\frac{\partial \psi}{\partial x_p}}{\frac{\partial S_k}{\partial x_p}} - \frac{\frac{\partial \psi}{\partial x_q}}{\frac{\partial S_k}{\partial x_q}} \right)(\mathbf{x}) &= \frac{1}{\prod_{i=1}^{n} (1 + y_i)} \left( h(x_q) \frac{S_{k-1}(\mathbf{y}^q)}{S_{k-1}(\mathbf{x}^q)} - h(x_p) \frac{S_{k-1}(\mathbf{y}^p)}{S_{k-1}(\mathbf{x}^p)} \right) \\ &= \frac{1}{\prod_{i=1}^{n} (1 + y_i)} \left( h(x_q) \frac{y_p S_{k-2}(\mathbf{y}^{p,q}) + S_{k-1}(\mathbf{y}^{p,q})}{x_p S_{k-2}(\mathbf{x}^{p,q}) + S_{k-1}(\mathbf{x}^{p,q})} - h(x_p) \frac{y_q S_{k-2}(\mathbf{y}^{p,q}) + S_{k-1}(\mathbf{y}^{p,q})}{x_q S_{k-2}(\mathbf{x}^{p,q}) + S_{k-1}(\mathbf{x}^{p,q})} \right). \end{aligned} \tag{17}$$

Since $x \to \frac{F(x)}{x(1-F(x))}$ is assumed to be an increasing function on $(0,\infty)$, we have $\frac{y_p}{x_p} < \frac{y_q}{x_q}$ and

$$\frac{y_p}{x_p} \le \frac{y_i}{x_i} \le \frac{y_q}{x_q},\ \forall i \in \{1,...,n\} \setminus \{p,q\}.$$

Corollary 1 can be applied for $m = n-2,\ r = k-2,\ \alpha = \frac{y_p}{x_p},\ \beta = \frac{y_q}{x_q}$ and the following inequalities hold:

$$\frac{y_p}{x_p}\frac{S_{k-2}(\mathbf{y}^{p,q})}{S_{k-2}(\mathbf{x}^{p,q})} \le \frac{S_{k-1}(\mathbf{y}^{p,q})}{S_{k-1}(\mathbf{x}^{p,q})} \le \frac{y_q}{x_q}\frac{S_{k-2}(\mathbf{y}^{p,q})}{S_{k-2}(\mathbf{x}^{p,q})}.$$

Moreover, at least one of these inequalities is strict. Hence, from relation (10) the next inequality follows:

$$\frac{y_pS_{k-2}(\mathbf{y}^{p,q}) + S_{k-1}(\mathbf{y}^{p,q})}{x_pS_{k-2}(\mathbf{x}^{p,q}) + S_{k-1}(\mathbf{x}^{p,q})} < \frac{y_qS_{k-2}(\mathbf{y}^{p,q}) + S_{k-1}(\mathbf{y}^{p,q})}{x_qS_{k-2}(\mathbf{x}^{p,q}) + S_{k-1}(\mathbf{x}^{p,q})}.$$

But $h$ is a positive non-increasing function. Thus, $0 < h(x_q) \le h(x_p)$. Therefore, from (17), we get:

$$\frac{\frac{\partial\psi}{\partial x_p}(\mathbf{x})}{\frac{\partial S_k}{\partial x_p}(\mathbf{x})} < \frac{\frac{\partial\psi}{\partial x_q}(\mathbf{x})}{\frac{\partial S_k}{\partial x_q}(\mathbf{x})}.$$

Hence, inequality (15) may be deduced from Lemma 2 and the conclusion follows.
It is worth to note that equation (17) cannot be used for $k = n$. But, in this case, using equation (16), we get

$$\left(\frac{\frac{\partial\psi}{\partial x_p}}{\frac{\partial S_k}{\partial x_p}} - \frac{\frac{\partial\psi}{\partial x_q}}{\frac{\partial S_k}{\partial x_q}}\right)(\mathbf{x}) = \frac{1}{\prod_{i=1}^n(1+y_i)}\prod_{i\neq p,q}\frac{y_i}{x_i}\left(h(x_q)\frac{y_p}{x_p} - h(x_p)\frac{y_q}{x_q}\right) < 0.$$

And the conclusion follows from Lemma 2. □

One important field of application concerns the exponential distribution (see below). But the result is more general. We give an example of a distribution function which satisfies the assumptions of Theorem 1.
*Example.* For $a > 1$, let us define the distribution function $F(x) = 1 - \frac{1}{(1+x)^a},\ x \ge 0$. We have $h(x) = \frac{F'(x)}{1-F(x)} = \frac{a}{1+x},\ x \ge 0$. The function $g : (0,\infty) \to (0,\infty),\ g(x) = \frac{F(x)}{x(1-F(x))} = \frac{(1+x)^a-1}{x}$ has the derivative $g'(x) = \frac{1+(1+x)^{a-1}(ax-1)}{x^2}$. But $v(x) = 1+(1+x)^{a-1}(ax-1)$ is a positive function on $(0,\infty)$, since $v(0) = 0$ and $v'(x) = a(a-1)x(1+x)^{a-2} > 0,\ \forall\ x > 0$. Therefore $h$ is a decreasing function and $g$ is an increasing function on $(0,\infty)$.

Theorem 1 can be naturally applied to the comparisons of Markov systems in reliability. Let us consider a system which is composed of $n$ components and is considered failed when $k$ components are failed. Let us assume that the failure rates of the components are constant. With such properties, the system is a $n$+1-$k$-out-of-$n$ Markov system. If the system is starting as new, the lifetime of the system is nothing but the $k$th order statistics of the exponential lifetimes of the components.

The next theorem gives a necessary and sufficient condition for the stochastic comparison of the lifetimes of two $n$+1-$k$-out-of-$n$ Markov systems in $k$th order statistics language. This result supplements the known results on this subject.

**Theorem 2.** *Let $X_i,\ i = 1,...,n,$ be independent exponential random variables with respective parameters $\lambda_i > 0,\ i = 1,2,...,n.$ Let $Y_i,\ i = 1,2,...,n,$ be independent exponential random variables with the common parameter $\mu > 0$. For $k \in \{1,2,...,n\}$, let us denote by $X_{k:n}$ the kth order statistics of the random variables $X_1, X_2, ..., X_n,$ and similarly let us denote by $Y_{k:n}$ the kth order statistics of the random variables $Y_1, Y_2, ..., Y_n$.*

*Then*

$$X_{k:n} \geq_{st} Y_{k:n} \quad \text{if and only if} \quad \mu \geq \left( \binom{n}{k}^{-1} \sum_{|J|=k} \prod_{i \in J} \lambda_i \right)^{\frac{1}{k}}.$$

*Proof.* We shall denote by $F(x) = 1 - \mathrm{e}^{-x},\ x \geq 0$, the exponential distribution function with parameter 1.

At first, let us assume that $X_{k:n} \geq_{st} Y_{k:n}$, *i.e.* $\overline{F}_{X_{k:n}}(t) \geq \overline{F}_{Y_{k:n}}(t),\ \forall t > 0$. The survival function of the random variable $X_{k:n}$ can be written as:

$$\overline{F}_{X_{k:n}}(t) = 1 - \mathrm{e}^{-t \sum_{i=1}^{n} \lambda_i} \sum_{j=k}^{n} S_j(\mathrm{e}^{\lambda_1 t} - 1, ..., \mathrm{e}^{\lambda_n t} - 1).$$

Using the Taylor's expansion about 0, we obtain:

$$\overline{F}_{X_{k:n}}(t) = 1 - S_k(\lambda_1, ..., \lambda_n) t^k + o(t^k), \quad t \to 0.$$

In the same way, we have:

$$\overline{F}_{Y_{k:n}}(t) = 1 - \binom{n}{k} \mu^k t^k + o(t^k), \quad t \to 0.$$

Therefore, $S_k(\lambda_1, ..., \lambda_n) \leq \binom{n}{k} \mu^k$. This can be rewritten as $\mu \geq m_k(\lambda_1, ..., \lambda_n)$.

Conversely, suppose that $\mu \geq m_k(\lambda_1, ..., \lambda_n)$. Since the survival function of $Y_{k:n}$ is clearly decreasing in $\mu$, it suffices to prove the assertion $\overline{F}_{X_{k:n}}(t) \geq \overline{F}_{Y_{k:n}}(t),\ \forall\ t > 0$, for $\mu = m_k(\lambda_1, ..., \lambda_n)$.

The exponential distribution function $F$ has a constant hazard rate $h(x) = 1,\ \forall\ x \geq 0$. Moreover the function $\frac{F(x)}{x(1-F(x))} = \frac{e^x - 1}{x}$ is increasing on $(0, \infty)$. Hence, the exponential distribution $F$ satisfies the assumptions of Theorem 1. Clearly, the distribution function of the random variable $X_i$ is $F_{X_i}(t) = F(\lambda_i t),\ t \geq 0,\ i = 1, ..., n$. Similarly, we have $F_{Y_i}(t) = F(\mu t),\ t \geq 0$. Then, applying Theorem 1 we get the conclusion $X_{k:n} \geq_{st} Y_{k:n}$. □

The practical interest of this result is to give precise production constraints on the components of a $k$-out-of-$n$ system. For example, in the case of replacing several components with well-known failure rates $(\lambda_i)_i$ by identical components, the previous theorem gives an exact value $m_{n+1-k}(\lambda)$ for the characteristic of the new components in order to preserve the reliability. This result was known for $k = 2$, it has been proved by Păltănea [5]. In the same spirit, Bon and Păltănea [1] have obtained necessary and sufficient conditions about comparisons of convolutions of exponential variables.

## 4. Numerical examples

The previous results can be illustrated as follows. Let us consider $n$ exponential independent random variables $X_i$ with parameter $\lambda_i$ and $X_{k:n}(\lambda)$ the $k$th order statistics. Let us denote by $Y_{k:n}(m_j(\lambda))$ the $k$th order statistics of $n$ exponential independent random variables $Y_i$ with common parameter $m_j(\lambda),\ j = 1, ..., n$.

The survival functions of $X_{k:n}(\lambda)$ and $Y_{k:n}(m_j(\lambda)),\ j = 1, ..., n$ are plotted in Figure 1. It can be clearly seen that

$$\overline{F}_{X_{k:n}(\lambda)} \geq \overline{F}_{Y_{k:n}(m_j(\lambda))} \quad \Leftrightarrow \quad j \leq k. \tag{18}$$

In Figure 2, we give an example of a distribution which satisfies the assumptions of Theorem 1 such that this ordering is true again:

$$F(x) = 1 - \frac{1}{(x+1)^2}, \quad x \geq 0.$$

If we consider a distribution function with an strict increasing hazard rate (usually named IFR) then the assumptions of Theorem 1 are not satisfied. Figure 3 refers to an IFR Weibull distribution. It can be seen that inequality (18) does not hold.

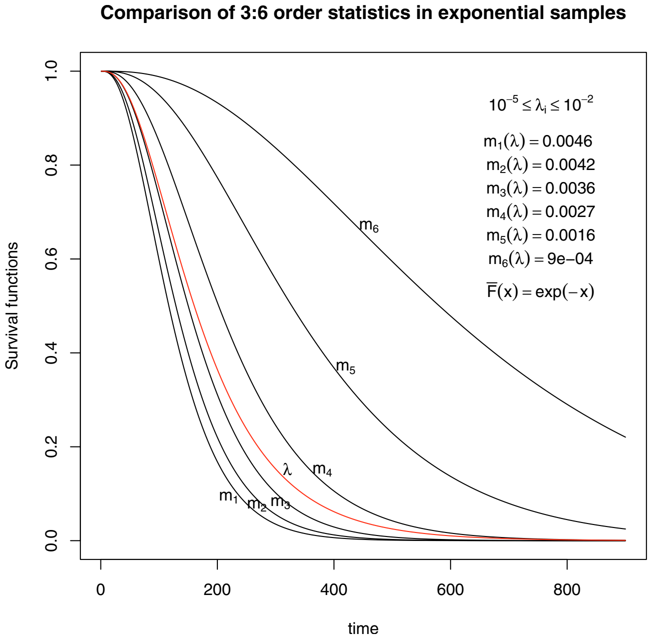


FIGURE 1. The exponential case for $n = 6$ and $k = 3$.

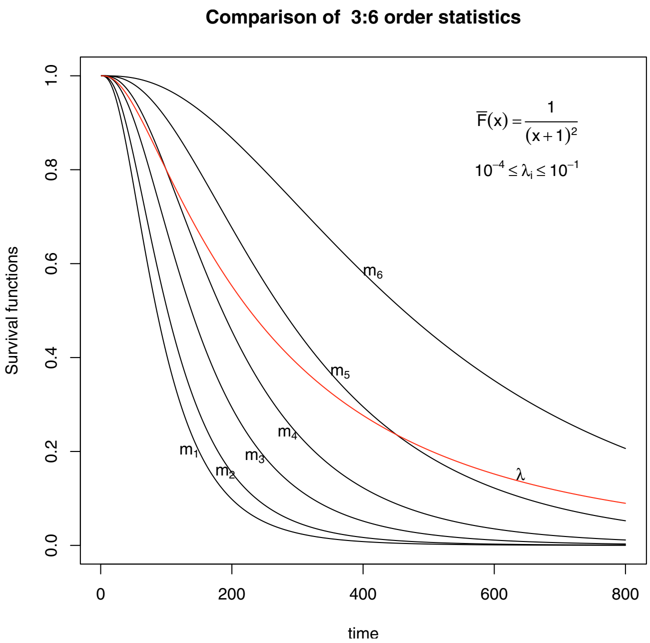


FIGURE 2. The case of the distribution function $F(x) = 1 - (x+1)^{-2}$ for $n = 6$ and $k = 3$.

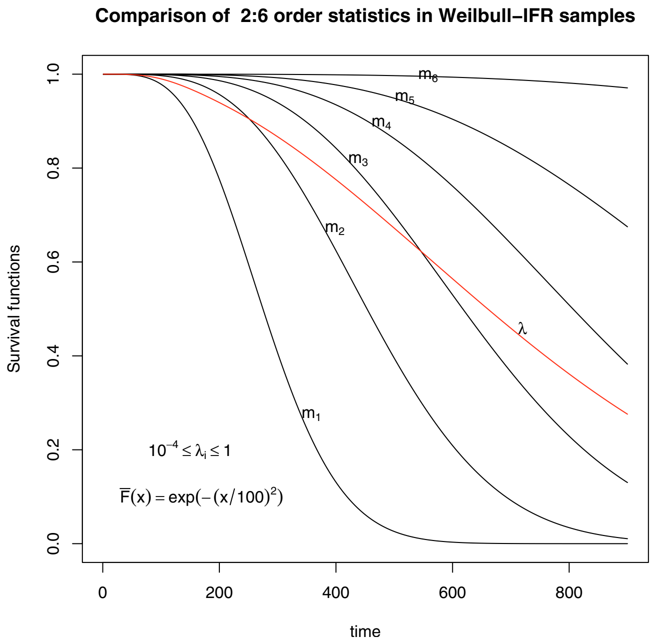


FIGURE 3. The case of an IFR Weilbull distribution for $n = 6$ and $k = 2$.

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