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STABILITY OF SOLUTIONS OF BSDES WITH RANDOM TERMINAL TIME

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Abstract. In this paper, we study the stability of the solutions of Backward Stochastic Differential Equations (BSDE for short) with an almost surely finite random terminal time. More precisely, we are going to show that if (W^n) is a sequence of scaled random walks or a sequence of martingales that converges to a Brownian motion W and if (τ^n) is a sequence of stopping times that converges to a stopping time τ, then the solution of the BSDE driven by W^n with random terminal time τ^n converges to the solution of the BSDE driven by W with random terminal time τ.

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Introduction

We want to make an approximation of the solutions of a backward stochastic differential equation (BSDE for short) with an almost surely finite random terminal time τ like

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s,\ t \geqslant 0.$$

The robustness of numerical methods to approximate solutions of BSDEs had already been studied in the case of a deterministic terminal time. Antonelli and Kohatsu-Higa in [1] and also Coquet, Mackevičius and Mémin in [6] and [7] proposed approximation schemes that use discretization of filtrations and convergence of filtrations. Briand, Delyon and Mémin in [3] and Ma, Protter, San Martín and Torres in [13] have approximated the Brownian motion by a scaled random walk. Then, in [4], Briand, Delyon and Mémin have studied another case: they approach the Brownian motion by a sequence of martingales.

We are interested in BSDEs with random terminal time because there have strong links with Partial Differential Equations as it is explained by Peng in [14]. As for BSDEs with deterministic terminal time, we study the robustness of numerical methods to approximate the solutions of those BSDEs. In this paper, we shall approximate the Brownian motion either by a scaled random walk, either by a sequence of martingales. In this study, we need moreover to approximate the random almost surely finite terminal time τ by a sequence of stopping times $(\tau^n)_n$.

In Section 1, we approximate the Brownian motion by a scaled random walk. First, we shall state the problem and study the properties of existence and uniqueness of the solutions of the BSDEs. Then, we will deal with the convergence of the solutions. To end this part, we give an example of the convergence result for hitting times.

Keywords and phrases. Backward Stochastic Differential Equations (BSDE), stability of BSDEs, weak convergence of filtrations, stopping times.

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In Section 2, we approximate the Brownian motion by a sequence of martingales. We shall see some generalizations of the results of Section 1: existence and uniqueness of the solutions of the BSDEs under study, convergence of the solutions. Moreover, we will illustrate these results by the case of discretizations of a Brownian motion and hitting times. For technical reasons, we need some results about convergence of stopped filtrations. So, in Appendix A, we will deal with stopped filtrations and stopped processes. We are going to establish a link between the convergence of a sequence of stopped processes and the convergence of the associated stopped filtrations.

In what follows, we are given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Unless otherwise specified, every σ-field will be supposed to be included in $\mathcal{A}$, every process will be indexed by $\mathbb{R}^+$ and taking values in $\mathbb{R}$, every filtration will be indexed by $\mathbb{R}^+$. $\mathbb{D} = \mathbb{D}(\mathbb{R}^+)$ denotes the space of càdlàg functions from $\mathbb{R}^+$ to $\mathbb{R}$. We endow $\mathbb{D}$ with the Skorokhod topology. If X is a process and τ a stopping time, we denote by X^τ the corresponding stopped process, *i.e.* for every t, $X_t^\tau = X_{t\wedge\tau}$.

Definition 0.1. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be a filtration and τ a $\mathcal{F}$-stopping time. We define the σ-field $\mathcal{F}_\tau$ by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau \leqslant s\} \in \mathcal{F}_s, \forall s\} \text{ where } \mathcal{F}_\infty = \bigvee_t \mathcal{F}_t$$

and the stopped filtration $\mathcal{F}^\tau$ by

$$\mathcal{F}_t^\tau = \mathcal{F}_{\tau\wedge t} = \{A \in \mathcal{F}_t : A \cap \{\tau \leqslant s\} \in \mathcal{F}_s, \forall s \leqslant t\},$$

for every t.

For technical background about Skorokhod topology, the reader may refer to Billingsley [2] or Jacod and Shiryaev [11].

1. Stability of BSDEs when the Brownian motion is approximated by a scaled random walk

1.1. Statement of the problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function: there exists $K \in \mathbb{R}^+$ such that, for every $(y, z), (y', z') \in \mathbb{R}^2$, we have:

$$|f(y,z) - f(y',z')| \leqslant K[|y-y'| + |z-z'|].$$

For clarity's sake, we consider a time-independent generator f but the results of Section 1 remain true if f is time-dependent as it is explained in Remark 1.3.

We also suppose that f is bounded and that f fills the following property of monotonicity w.r.t. y: there exists $\mu > 0$ such that

$$\forall (y,z),(y',z) \in \mathbb{R}^+ \times \mathbb{R}^2, (y-y')(f(y,z) - f(y',z)) \leqslant -\mu(y-y')^2.$$

Let W be a Brownian motion and $\mathcal{F}$ its natural filtration. Let τ be a $\mathcal{F}$-stopping time almost surely finite. We consider the following stochastic differential equation:

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s,\ t \geqslant 0, \tag{1}$$

where ξ is a bounded $\mathcal{F}_\tau$-mesurable random variable.

Definition 1.1. We call a solution of the BSDE (1) a pair (Y, Z) of progressively measurable processes verifying the equation (1) such that $Y_t = \xi$ and $Z_t = 0$ on the set $\{t > \tau\}$ and:

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}^+} e^{-2\mu t}|Y_t|^2\right] < +\infty \text{ and } \forall t,\ \mathbb{E}\left[\int_0^{t\wedge\tau} |Z_t|^2 \mathrm{d}t\right] < +\infty.$$

According to Theorem 2.1 of Royer in [15], the BSDE (1) has a unique pair solution (Y, Z) in the set of processes such that Y is continuous and uniformly bounded.

In this section, we approximate equation (1) on the following way. We consider the sequence of scaled random walks $(W^n)_{n\geqslant 1}$ defined by:

$$W_t^n = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]} \varepsilon_k^n,\ t \geqslant 0$$

where $(\varepsilon_k^n)_{k\in\mathbb{N}^*}$ is a sequence of i.i.d. symmetric Bernoulli variables. Let $\mathcal{F}^n$ be the natural filtrations of W^n, $n \geqslant 1$. We have $\mathcal{F}_t^n = \sigma(\varepsilon_k^n, k \leqslant [nt])$. Let $(\tau^n)_n$ be a sequence of bounded $(\mathcal{F}^n)$-stopping times. For each n, we can find $T_n \in \mathbb{N}$ such that $\tau^n \leqslant T_n$.

Then, for each n, we consider the following equation:

$$\begin{aligned} y^n_{\frac{k}{n}\wedge\tau^n} &= y^n_{\frac{k+1}{n}\wedge\tau^n} + \frac{1}{n}\mathbf{1}_{\{\tau^n \geqslant \frac{k}{n}\}} f\left(y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\right) - z^n_{\frac{k+1}{n}\wedge\tau^n}\frac{1}{\sqrt{n}}\varepsilon^n_{k+1},\ k = 0, \ldots, (n-1)T_n \\ y^n_{\tau^n} &= \xi^n \end{aligned} \tag{2}$$

where (ξ^n) is a sequence of $(\mathcal{F}^n_{\tau^n})$-measurable integrable random variables.

By a solution of equation (2), we mean a discrete process $\{y^n_{\frac{k}{n}}, z^n_{\frac{k+1}{n}}\}_{k\geqslant 0}$ that satisfies (2), such that $y^n_{\frac{k}{n}} = \xi^n$ and $z^n_{\frac{k}{n}} = 0$ on the set $\{\tau^n < \frac{k}{n}\}$ and such that $\{y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\}_{k\geqslant 0}$ is $\mathcal{F}^{n,\tau^n}$-adapted.

Proposition 1.2. *Equation (2) has a unique solution (y^n, z^n).*

Proof. We are going to build this solution on a converse iterative way.
For $k = nT_n$, let us put $y^n_{\frac{k}{n}\wedge\tau^n} = \xi^n$ and $z^n_{\frac{k+1}{n}\wedge\tau^n} = 0$.
Let us suppose that, for a given k, we have built $\left(y^n_{\frac{k+1}{n}\wedge\tau^n}, z^n_{\frac{k+2}{n}\wedge\tau^n}\right)$. Using equation (2), we are going to determinate $\left(y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\right)$.

Let us begin by giving an expression of $z^n_{\frac{k+1}{n}\wedge\tau^n}$.
Multiplying equation (2) by $\sqrt{n}\varepsilon^n_{k+1}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}$ and taking the conditional expectation with respect to $\mathcal{F}^{n,\tau^n}_{k/n}$, we have:

$$\begin{aligned} & z^n_{\frac{k+1}{n}\wedge\tau^n}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}} \\ = {} & \mathbb{E}\left[z^n_{\frac{k+1}{n}\wedge\tau^n}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}\middle|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}\right] \\ = {} & \sqrt{n}\mathbb{E}\left[y^n_{\frac{k+1}{n}\wedge\tau^n}\varepsilon^n_{k+1}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}\middle|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}\right] + \frac{1}{\sqrt{n}}\mathbb{E}\left[\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}} f\left(y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\right)\varepsilon^n_{k+1}\middle|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}\right] \\ & -\sqrt{n}\mathbb{E}\left[y^n_{\frac{k}{n}\wedge\tau^n}\varepsilon^n_{k+1}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}\middle|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}\right] \\ = {} & \sqrt{n}\mathbb{E}\left[y^n_{\frac{k+1}{n}\wedge\tau^n}\varepsilon^n_{k+1}\middle|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}\right]\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}} \end{aligned}$$

because $y^n_{\frac{k}{n}\wedge\tau^n}$ and $z^n_{\frac{k+1}{n}\wedge\tau^n}$ must be $\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}$-measurable, τ^n is a $\mathcal{F}^n$-stopping time and ε^n_{k+1} is independent from $\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}$ and is centered.
Moreover, by definition of a solution, for every k, $z^n_{\frac{k+1}{n}\wedge\tau^n} = z^n_{\frac{k+1}{n}\wedge\tau^n}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}$. So, we put:

$$z^n_{\frac{k+1}{n}\wedge\tau^n} = \sqrt{n}\mathbb{E}\left[y^n_{\frac{k+1}{n}\wedge\tau^n}\varepsilon^n_{k+1}\middle|\mathcal{F}^n_{\frac{k}{n}}\right]\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}.$$

We point out that $z^n_{\frac{k+1}{n}\wedge\tau^n}$ is $\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}$-measurable.

Now, we have to determinate $y^n_{\frac{k}{n}\wedge\tau^n}$.
On the set $\{\tau^n < \frac{k}{n}\}$, $y^n_{\frac{k}{n}\wedge\tau^n} = y^n_{\frac{k+1}{n}\wedge\tau^n} = y^n_{\tau^n} = \xi^n$.
On $\{\tau^n \geqslant \frac{k}{n}\}$, $y^n_{\frac{k}{n}\wedge\tau^n} = y^n_{\frac{k+1}{n}\wedge\tau^n} + \frac{1}{n}f\left(y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\right) - z^n_{\frac{k+1}{n}\wedge\tau^n}\frac{1}{\sqrt{n}}\varepsilon^n_{k+1}$ and we can write it

$$y^n_{\frac{k}{n}\wedge\tau^n} = \varphi\left(y^n_{\frac{k}{n}\wedge\tau^n}\right)$$

with $\varphi(y) = y^n_{\frac{k+1}{n}\wedge\tau^n} + \frac{1}{n}f\left(y, z^n_{\frac{k+1}{n}\wedge\tau^n}\right) - z^n_{\frac{k+1}{n}\wedge\tau^n}\frac{1}{\sqrt{n}}\varepsilon^n_{k+1}$. As f is K-Lipschitz in y, we have, for every y, y':

$$|\varphi(y) - \varphi(y')| \leqslant \frac{K}{n}|y - y'|.$$

So, for n large enough (notice that this range does not depend on k), $\frac{K}{n} < 1$ and φ is a contraction. Then, the equation $y^n_{\frac{k}{n}\wedge\tau^n} = \varphi\left(y^n_{\frac{k}{n}\wedge\tau^n}\right)$ has a unique solution for n large enough, according to a fix point Theorem. By construction, $y^n_{\frac{k}{n}\wedge\tau^n}$ is $\mathcal{F}^{n,\tau^n}_{\frac{k+1}{n}}$-measurable. But, using the predictable representation property, we have $y^n_{\frac{k+1}{n}\wedge\tau^n} - z^n_{\frac{k+1}{n}\wedge\tau^n}\frac{1}{\sqrt{n}}\varepsilon^n_{k+1} = \mathbb{E}[y^n_{\frac{k+1}{n}\wedge\tau^n}|\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}]$. So $y^n_{\frac{k}{n}\wedge\tau^n}$ is independant from ε^n_{k+1} and $y^n_{\frac{k}{n}\wedge\tau^n}$ is $\mathcal{F}^{n,\tau^n}_{\frac{k}{n}}$-measurable.
Hence, the equation (2) has a unique solution. □

Now, we define continuous time processes Y^n and Z^n by $Y^n_t = y^n_{\frac{[nt]}{n}\wedge\tau^n}$, $Z^n_t = z^n_{\frac{\lfloor nt\rfloor}{n}\wedge\tau^n}$, for every $t\in\mathbb{R}^+$ where $\lfloor x\rfloor = (x-1)^+$ if x is an integer, $[x]$ otherwise. The processes Y^n and Z^n are constant on the intervals $[k/n,(k+1)/n[$ and $]k/n,(k+1)/n]$ respectively and satisfy the following equation:

$$Y^n_t = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f(Y^n_{(s\wedge\tau^n)-}, Z^n_{s\wedge\tau^n})\mathrm{d}A^n_s - \int_{t\wedge\tau^n}^{\tau^n} Z^n_{s\wedge\tau^n}\mathrm{d}W^n_s, \tag{3}$$

where $A^n_s = \frac{[ns]}{n}$.

Remark 1.3. If f is time-dependent such that for every (y,z), we suppose that $\{f(t,y,z)\}_{t\geqslant 0}$ is progressively measurable, bounded, K-Lipschitz in y and z and verify the condition of monotonicity w.r.t. y given before. Under these assumptions, all the results of Section 1 remain true and the proofs are the same. In that case, Equation (2) becomes:

$$\begin{aligned} y^n_{\frac{k}{n}\wedge\tau^n} &= y^n_{\frac{k+1}{n}\wedge\tau^n} + \frac{1}{n}\mathbf{1}_{\{\tau^n\geqslant\frac{k}{n}\}}f\left(\frac{k}{n}\wedge\tau^n, y^n_{\frac{k}{n}\wedge\tau^n}, z^n_{\frac{k+1}{n}\wedge\tau^n}\right) - z^n_{\frac{k+1}{n}\wedge\tau^n}\frac{1}{\sqrt{n}}\varepsilon^n_{k+1},\ k = 0,\ldots,(n-1)T_n \\ y^n_{\tau^n} &= \xi^n. \end{aligned}$$

1.2. Convergence of the solutions

The aim of this section is to prove the following result of convergence of the solutions:

Theorem 1.4. *Let (Y, Z) be the solution of the BSDE (1) and (Y^n, Z^n) be the processes constant on the intervals $[k/n, (k+1)/n[$ and $]k/n, (k+1)/n]$ respectively solving equation (3). We suppose that there exists $\delta > 0$ such that $\sup_n \mathbb{E}[|\xi^n|^{1+\delta}]^{\frac{1}{1+\delta}} < +\infty$ and $\sup_n \mathbb{E}[|\tau^n|^{1+\delta}]^{\frac{1}{1+\delta}} < +\infty$ and that we have the convergences $\xi^n \xrightarrow{\mathbb{P}} \xi$, $\tau^n \xrightarrow{\mathbb{P}} \tau$ and $W^n \xrightarrow{\mathbb{P}} W$. Then*

$$\forall L \in \mathbb{N}, \sup_{t\in[0,L]} |Y^n_{t\wedge\tau^n} - Y_{t\wedge\tau}| + \int_0^{\tau\wedge\tau^n} |Z^n_{t\wedge\tau^n} - Z_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0$$

which we shall denote by $(Y^n, Z^n) \to (Y, Z)$ and also

$$\forall L \in \mathbb{N}, \sup_{t\in[0,L]} \left| \int_0^{t\wedge\tau^n} Z^n_s \mathrm{d}W^n_s - \int_0^{t\wedge\tau} Z_s \mathrm{d}W_s \right| \xrightarrow{\mathbb{P}} 0.$$

According to this theorem, we have the convergence in probability of the solutions under the rather strong assumption that the scaled random walks converge in probability to the Brownian motion and the random terminal times also converge. Actually, with Donsker's Theorem, we have the convergence in law of the scaled random walks to the Brownian motion. If we only have this convergence in law, we obtain the following corollary:

Corollary 1.5. *Let (Y, Z) be the solution of the BSDE (1) and (Y^n, Z^n) be the processes constant on the intervals $[k/n, (k+1)/n[$ and $]k/n, (k+1)/n]$ respectively solution of the equation (3). We suppose that $\forall n, \forall k, \varepsilon^n_k = \varepsilon_k$, $\xi = g(W)$ and $\xi^n = g(W^n)$ with g bounded continuous. We assume that there exists $\delta > 0$ such that $\sup_n \mathbb{E}[|\tau^n|^{1+\delta}]^{\frac{1}{1+\delta}} < +\infty$. We also suppose that we have the convergence $(W^n, \tau^n) \xrightarrow{\mathcal{L}} (W, \tau)$. Then $\left(Y^n_{.\wedge\tau^n}, \int_0^{.\wedge\tau^n} Z^n_s \mathrm{d}W^n_s\right)_n$ converges in law to $\left(Y_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_s \mathrm{d}W_s\right)$ for the Skorokhod topology.*

Proof. According to the Skorokhod representation theorem, in a space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$, we can find $(\tilde{W}^n, \tilde{\tau}^n)$ with the same law as (W^n, τ^n) and $(\tilde{W}, \tilde{\tau})$ with the same law as (W, τ) such that $(\tilde{W}^n, \tilde{\tau}^n) \xrightarrow{a.s.} (\tilde{W}, \tilde{\tau})$. We denote by $\tilde{\mathcal{F}}$ the natural filtration of $\tilde{W}$ and by $\tilde{\mathcal{F}}^n$ the natural filtrations of the $\tilde{W}^n$.
Let us show that $\tilde{\tau}$ is an $\tilde{\mathcal{F}}$-stopping time. Let us fix $t \geqslant 0$ and note that $\{\tau \leqslant t\} \in \mathcal{F}_t$ since τ is a $\mathcal{F}$-stopping time. Then, for every $\varepsilon > 0$, we can find $h : \mathbb{R}^k \to \mathbb{R}$ measurable and $s_1, \dots, s_k$ in $[0, t]$ such that

$$\int |\mathbf{1}_{\{\tau\leqslant t\}} - h(W_{s_1}, \dots, W_{s_k})| \mathrm{d}\mathbb{P} < \varepsilon.$$

$(W, \tau) \sim (\tilde{W}, \tilde{\tau})$ so $\mathbf{1}_{\{\tau\leqslant t\}} - h(W_{s_1}, \dots, W_{s_k}) \sim \mathbf{1}_{\{\tilde{\tau}\leqslant t\}} - h(\tilde{W}_{s_1}, \dots, \tilde{W}_{s_k})$. Then,

$$\int |\mathbf{1}_{\{\tilde{\tau}\leqslant t\}} - h(\tilde{W}_{s_1}, \dots, \tilde{W}_{s_k})| \mathrm{d}\tilde{\mathbb{P}} = \int |\mathbf{1}_{\{\tau\leqslant t\}} - h(W_{s_1}, \dots, W_{s_k})| \mathrm{d}\mathbb{P}.$$

So, for every $\varepsilon > 0$, we can find $h : \mathbb{R}^k \to \mathbb{R}$ measurable and $s_1, \dots, s_k$ in $[0, t]$ such that

$$\int |\mathbf{1}_{\{\tilde{\tau}\leqslant t\}} - h(\tilde{W}_{s_1}, \dots, \tilde{W}_{s_k})| \mathrm{d}\tilde{\mathbb{P}} < \varepsilon.$$

Hence, $\{\tilde{\tau} \leqslant t\} \in \tilde{\mathcal{F}}_t$. Then $\tilde{\tau}$ is an $\tilde{\mathcal{F}}$-stopping time. On the same way, the $\tilde{\tau}^n$'s are $\tilde{\mathcal{F}}^n$-stopping times.
Let (Y'^n, Z'^n) be the solution of

$$Y'^n_t = g(\tilde{W}^n) + \int_{t\wedge\tilde{\tau}^n}^{\tilde{\tau}^n} f(Y'^n_{(s\wedge\tilde{\tau}^n)-}, Z'^n_{s\wedge\tilde{\tau}^n}) \mathrm{d}A^n_s - \int_{t\wedge\tilde{\tau}^n}^{\tilde{\tau}^n} Z'^n_{s\wedge\tilde{\tau}^n} \mathrm{d}\tilde{W}^n_s$$

and (Y', Z') be the solution of

$$Y'_{t\wedge\tilde{\tau}} = g(\tilde{W}) + \int_{t\wedge\tilde{\tau}}^{\tilde{\tau}} f(Y'_s, Z'_s)\mathrm{d}s - \int_{t\wedge\tilde{\tau}}^{\tilde{\tau}} Z'_s\mathrm{d}\tilde{W}_s,\ t \geqslant 0.$$

All the assumptions of Theorem 1.4 are filled. So we have the convergence in probability of $\left(Y'^n_{.\wedge\tilde{\tau}^n}, \int_0^{.\wedge\tilde{\tau}^n} Z'^n_s\mathrm{d}\tilde{W}^n_s\right)_n$ to $\left(Y'_{.\wedge\tilde{\tau}}, \int_0^{.\wedge\tilde{\tau}} Z'_s\mathrm{d}\tilde{W}_s\right)$. Then, denoting by $(Y^n_{.\wedge\tau^n}, Z^n_{.\wedge\tau^n})$ the solution of (1) and by $(Y_{.\wedge\tau}, Z_{.\wedge\tau})$ the solution of (3), since $(Y^n_{.\wedge\tau^n}, Z^n_{.\wedge\tau^n}, W^n, \tau^n)$ has the same law as $(Y'^n_{.\wedge\tilde{\tau}^n}, Z'^n_{.\wedge\tilde{\tau}^n}, \tilde{W}^n, \tilde{\tau}^n)$ and $(Y_{.\wedge\tau}, Z_{.\wedge\tau}, W, \tau)$ has the same law as $(Y'_{.\wedge\tilde{\tau}}, Z'_{.\wedge\tilde{\tau}}, \tilde{W}, \tilde{\tau})$, we have the convergence of $\left(Y^n_{.\wedge\tau^n}, \int_0^{.\wedge\tau^n} Z^n_s\mathrm{d}W^n_s\right)_n$ to $\left(Y_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_s\mathrm{d}W_s\right)$ in distribution for the Skorokhod topology. □

We point out that in this corollary, it is necessary to have the joint convergence of $((W^n, \tau^n))_n$ to (W, τ). In section 1.3, we shall see an example where the stopping times are hitting times.

To prove the first convergence of Theorem 1.4, we are going to use the Picard approximations (Y^p, Z^p) and $((Y^{n,p}, Z^{n,p}))_n$ defined on the following way:

$$Y^{p+1}_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(Y^p_s, Z^p_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z^{p+1}_s\mathrm{d}W_s,\ \forall t \geqslant 0, \tag{4}$$

$$Y^{n,p+1}_t = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_{t\wedge\tau^n}^{\tau^n} Z^{n,p+1}_s\mathrm{d}W^n_s, \tag{5}$$

with $Y^0 = Z^0 = Y^{n,0} = Z^{n,0} = 0$ and $A^n_t = \frac{[nt]}{n}$.

We have existence and uniqueness of adapted pairs (Y^p, Z^p) and $((Y^{n,p}, Z^{n,p}))_n$ by induction on p, using respectively Theorem 2.1 of Royer in [15] and Proposition 1.2.

We write:

$$Y^n - Y = (Y^n - Y^{n,p}) + (Y^{n,p} - Y^p) + (Y^p - Y),$$
$$Z^n - Z = (Z^n - Z^{n,p}) + (Z^{n,p} - Z^p) + (Z^p - Z).$$

We are going to prove successively the convergence of the Picard approximations to the solutions, *i.e.* for every n, $(Y^{n,p}, Z^{n,p}) \to (Y^n, Z^n)$ and $(Y^p, Z^p) \to (Y, Z)$, and then check that the first convergence of the theorem is true for the Picard approximations, *i.e.* for every p, $(Y^{n,p}, Z^{n,p}) \to (Y^p, Z^p)$.

Most of the proof uses the same arguments as in the proof of Theorem 2.1 of Briand, Delyon and Mémin in [3] but there are some technical difficulties due to the stopping times.

The arguments of Lemma 4.1 in [3] are still true when we replace a deterministic terminal time by a bounded random terminal time. So we have:

$$\forall L,\ \sup_n \mathbb{E}\left[\sup_{t\in[0,L]} |Y^n_{\tau^n\wedge t} - Y^{n,p}_{\tau^n\wedge t}|^2 + \int_0^{+\infty} |Z^n_t - Z^{n,p}_t|^2\mathrm{d}t\right] \xrightarrow[p\to+\infty]{} 0. \tag{6}$$

With a truncation argument (using the fact that τ is almost surely finite), we deduce the convergence of (Y^p, Z^p) to (Y, Z), *i.e.*

$$\forall L,\ \mathbb{E}\left[\sup_{t\in[0,L]} |Y^p_{\tau\wedge t} - Y_{\tau\wedge t}|^2 + \int_0^{+\infty} |Z^p_t - Z_t|^2\mathrm{d}t\right] \xrightarrow[p\to+\infty]{} 0. \tag{7}$$

Now, let us show that for every p, $(Y^{n,p}, Z^{n,p}) \to (Y^p, Z^p)$ as $n \to +\infty$ that is

$$\forall L \in \mathbb{N}, \sup_{t\in[0,L]} |Y^{n,p}_{t\wedge\tau^n} - Y^p_{t\wedge\tau}| + \int_0^{\tau^n} |Z^{n,p}_{t\wedge\tau^n} - Z^p_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty. \tag{8}$$

We argue by induction on p.

- The property is true for $p = 0$ because $Y^0 = Z^0 = Y^{n,0} = Z^{n,0} = 0$.
- We suppose that, for given p, (8) holds. Let us prove that (8) is still true for $p+1$. The proof will be given through three steps.

In a first step, we introduce the sequence $(M^n)_n$ of processes defined by

$$M^n_t = Y^{n,p+1}_{t\wedge\tau^n} + \int_0^{t\wedge\tau^n} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}A^n_s$$

and the process M defined by

$$M_t = Y^{p+1}_{t\wedge\tau} + \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s.$$

In Lemma 1.6, we prove that $(M^n_{.\wedge\tau^n})_n$ is a sequence of $(\mathcal{F}^{n,\tau^n})$-martingales. Then, with Lemmas 1.8 and 1.9, we show that we have the following convergence:

$$\forall L, \sup_{t\in[0,L]} |M^n_{t\wedge\tau^n} - M_{t\wedge\tau}| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

The aim of the second step is to prove Lemma 1.11:

$$\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n} - Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

At last, in a third step (Lemma 1.12), we deal with the convergence of $Y^{n,p+1}$ to Y^{p+1}:

$$\forall L, \sup_{t\in[0,L]} |Y^{n,p+1}_{t\wedge\tau^n} - Y^{p+1}_{t\wedge\tau}| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Step 1. Let $(M^n)_n$ be the processes defined by $M^n_t = Y^{n,p+1}_{t\wedge\tau^n} + \int_0^{t\wedge\tau^n} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}A^n_s$.

Lemma 1.6. *For each n, M^n is a $\mathcal{F}^{n,\tau^n}$-martingale.*

Proof. Let us show that $M^n_t = M^n_0 + \int_0^{t\wedge\tau^n} Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s$. Using (5),

$$\begin{aligned}
M^n_0 &+ \int_0^{t\wedge\tau^n} Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s \\
&= Y^{n,p+1}_0 + \int_0^{t\wedge\tau^n} Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s \\
&= \xi^n + \int_0^{\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_0^{\tau^n} Z^{n,p+1}_s\mathrm{d}W^n_s + \int_0^{t\wedge\tau^n} Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s \\
&= Y^{n,p+1}_{t\wedge\tau^n} + \int_0^{t\wedge\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s = M^n_t.
\end{aligned}$$

Then, for every n, as the process $(M^n_0 + \int_0^t Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s)_{t\geqslant 0}$ is a $\mathcal{F}^n$-martingale, the stopped process $(M^n_0 + \int_0^{t\wedge\tau^n} Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}W^n_s)_{t\geqslant 0} = (M^n_t)_{t\geqslant 0}$ is a $\mathcal{F}^{n,\tau^n}$-martingale. □

So, we have $M_t^n = \mathbb{E}[M_{\tau^n}^n | \mathcal{F}_t^{n,\tau^n}]$ with $M_{\tau^n}^n = Y_{\tau^n}^{n,p+1} + \int_0^{\tau^n} f(Y_{s-}^{n,p}, Z_s^{n,p}) \mathrm{d}A_s^n$, for every n, for every t.

Before proving the convergence of $(M_{\tau^n}^n)_n$, let us see a lemma that links stopped integrals and integrals of stopped processes:

Lemma 1.7. *For every t, we have the following relations:*

$$\int_0^{t\wedge\tau} f(Y_s^p, Z_s^p)\mathrm{d}s = \int_0^t f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s + (\tau\wedge t - t) f(Y_\tau^p, Z_\tau^p),$$

$$\int_0^{t\wedge\tau^n} f(Y_s^{n,p}, Z_s^{n,p})\mathrm{d}s = \int_0^t f(Y_{s\wedge\tau^n}^{n,p}, Z_{s\wedge\tau^n}^{n,p})\mathrm{d}s + (\tau^n\wedge t - t) f(Y_{\tau^n}^{n,p}, Z_{\tau^n}^{n,p}).$$

Proof. Let us prove the first relation.
On $\{t \leqslant \tau\}$,

$$\int_0^t f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s = \int_0^t f(Y_s^p, Z_s^p)\mathrm{d}s = \int_0^{t\wedge\tau} f(Y_s^p, Z_s^p)\mathrm{d}s$$

and on $\{t > \tau\}$, we have:

$$\begin{aligned}\int_0^t f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s &= \int_0^{t\wedge\tau} f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s + \int_{t\wedge\tau}^t f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s \\ &= \int_0^{t\wedge\tau} f(Y_s^p, Z_s^p)\mathrm{d}s + \int_{t\wedge\tau}^t f(Y_\tau^p, Z_\tau^p)\mathrm{d}s \\ &= \int_0^{t\wedge\tau} f(Y_s^p, Z_s^p)\mathrm{d}s + (t - t\wedge\tau) f(Y_\tau^p, Z_\tau^p).\end{aligned}$$

The second equality is proved by the same arguments. □

Lemma 1.8. *$(M_{\tau^n}^n)_n$ converges in L^1 to $Y_\tau^{p+1} + \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s$.*

Proof. It suffices to show that $(M_{\tau^n}^n)_n$ converges to $Y_\tau^{p+1} + \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s$ in probability. Indeed, for every n, $\mathbb{E}[|M_{\tau^n}^n|^{1+\delta}]^{\frac{1}{1+\delta}} \leqslant \mathbb{E}[|\xi^n|^{1+\delta}]^{\frac{1}{1+\delta}} + \|f\|_\infty \mathbb{E}[|\tau^n|^{1+\delta}]^{\frac{1}{1+\delta}}$. So, when we take the sup in n, $\sup_n \mathbb{E}[|M_{\tau^n}^n|^{1+\delta}]^{\frac{1}{1+\delta}} < \infty$ according to the assumptions on $(\xi^n)_n$, $(\tau^n)_n$ and f. So, the sequence $(M_{\tau^n}^n)$ is uniformly integrable and then the convergence in probability implies the convergence in L^1.

Let us show that $(M_{\tau^n}^n)_n$ converges in probability to $Y_\tau^{p+1} + \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s$.

$$\left| M_{\tau^n}^n - Y_\tau^{p+1} + \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s \right| \leqslant |Y_{\tau^n}^{n,p+1} - Y_\tau^{p+1}| + \left| \int_0^{\tau^n} f(Y_{s-}^{n,p}, Z_s^{n,p})\mathrm{d}A_s^n - \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s \right|.$$

But,

$$|Y_{\tau^n}^{n,p+1} - Y_\tau^{p+1}| = |\xi^n - \xi| \xrightarrow{\mathbb{P}} 0. \tag{9}$$

We are going to conclude by showing that

$$\left| \int_0^{\tau^n} f(Y_{s-}^{n,p}, Z_s^{n,p})\mathrm{d}A_s^n - \int_0^\tau f(Y_s^p, Z_s^p)\mathrm{d}s \right| \xrightarrow{\mathbb{P}} 0.$$

For each n, for each ω, there exists a unique k_n such that τ^n is in the interval $[k_n/n, (k_n+1)/n[$. As the processes $Y^{n,p}$ and $Z^{n,p}$ are constant on the intervals of the form $[k/n, (k+1)/n[$ and $]k/n, (k+1)/n]$ respectively by construction, we have:

$$\int_0^{\tau^n} f(Y_{s-}^{n,p}, Z_s^{n,p})\mathrm{d}A_s^n = \int_0^{k_n/n} f(Y_s^{n,p}, Z_s^{n,p})\mathrm{d}s.$$

So,

$$\begin{aligned}\left|\int_0^{\tau^n} f(Y_{s-}^{n,p}, Z_s^{n,p})\mathrm{d}A_s^n - \int_0^{\tau} f(Y_s^p, Z_s^p)\mathrm{d}s\right| &= \left|\int_0^{k_n/n} f(Y_s^{n,p}, Z_s^{n,p})\mathrm{d}s - \int_0^{\tau} f(Y_s^p, Z_s^p)\mathrm{d}s\right| \\ &\leqslant \left|\int_0^{(k_n/n)\wedge\tau} (f(Y_s^{n,p}, Z_s^{n,p}) - f(Y_s^p, Z_s^p))\mathrm{d}s\right| \\ &\quad + \left|\int_{(k_n/n)\wedge\tau}^{(k_n/n)\vee\tau} (f(Y_s^{n,p}, Z_s^{n,p})\mathbf{1}_{\{\tau\leqslant k_n/n\}} - f(Y_s^p, Z_s^p)\mathbf{1}_{\{\tau>k_n/n\}})\mathrm{d}s\right|.\end{aligned}$$

Taking $t = (k_n/n)\wedge\tau$ in the two equalities of Lemma 1.7, we have:

$$\begin{aligned}\int_0^{(k_n/n)\wedge\tau} f(Y_s^p, Z_s^p)\mathrm{d}s &= \int_0^{(k_n/n)\wedge\tau} f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p)\mathrm{d}s, \\ \int_0^{(k_n/n)\wedge\tau} f(Y_s^{n,p}, Z_s^{n,p})\mathrm{d}s &= \int_0^{(k_n/n)\wedge\tau} f(Y_{s\wedge\tau^n}^{n,p}, Z_{s\wedge\tau^n}^{n,p})\mathrm{d}s.\end{aligned}$$

So,

$$\begin{aligned}&\left|\int_0^{(k_n/n)\wedge\tau} (f(Y_s^{n,p}, Z_s^{n,p}) - f(Y_s^p, Z_s^p))\mathrm{d}s\right| \\ &= \left|\int_0^{(k_n/n)\wedge\tau} (f(Y_{s\wedge\tau^n}^{n,p}, Z_{s\wedge\tau^n}^{n,p}) - f(Y_{s\wedge\tau}^p, Z_{s\wedge\tau}^p))\mathrm{d}s\right| \\ &\leqslant \int_0^{(k_n/n)\wedge\tau} K[|Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p| + |Z_{s\wedge\tau^n}^{n,p} - Z_{s\wedge\tau}^p|]\mathrm{d}s \quad \text{because } f \text{ is } K\text{-Lipschitz} \\ &\leqslant K\int_0^{\tau^n\wedge\tau} |Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p|\mathrm{d}s + K\int_0^{\tau\wedge\tau^n} |Z_{s\wedge\tau^n}^{n,p} - Z_{s\wedge\tau}^p|\mathrm{d}s.\end{aligned}$$

Using the induction assumption (8), the fact that $\tau < +\infty$ *a.s.* and Cauchy-Schwarz inequality, we prove that

$$\int_0^{\tau\wedge\tau^n} |Z_{s\wedge\tau^n}^{n,p} - Z_{s\wedge\tau}^p|^2\mathrm{d}s \xrightarrow{\mathbb{P}} 0. \tag{10}$$

On the other hand, let us fix $\varepsilon > 0$ and $\eta > 0$. Since $\tau < +\infty$ *a.s.*, we can find T such that $\mathbb{P}[\tau \geqslant T] \leqslant \varepsilon$. Then,

$$\begin{aligned}&\mathbb{P}\left[\int_0^{\tau^n\wedge\tau} |Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p|\mathrm{d}s \geqslant \eta\right] \\ &= \mathbb{P}\left[\int_0^{\tau^n\wedge\tau} |Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p|\mathrm{d}s\ \mathbf{1}_{\{\tau\wedge\tau^n<T\}} \geqslant \eta\right] + \mathbb{P}\left[\int_0^{\tau^n\wedge\tau} |Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p|\mathrm{d}s\ \mathbf{1}_{\{\tau\wedge\tau^n\geqslant T\}} \geqslant \eta\right] \\ &\leqslant \mathbb{P}\left[T\sup_{s\in[0,T]} |Y_{s\wedge\tau^n}^{n,p} - Y_{s\wedge\tau}^p| \geqslant \eta\right] + \mathbb{P}[\tau\wedge\tau^n \geqslant T] \\ &\leqslant 2\varepsilon \quad \text{by choice of } T \text{ and using (8).}\end{aligned}$$

So,

$$\int_0^{\tau^n \wedge \tau} |Y^{n,p}_{s\wedge\tau^n} - Y^p_{s\wedge\tau}|\mathrm{d}s \xrightarrow{\mathbb{P}} 0. \tag{11}$$

Finally, using the convergences (10) and (11), we have

$$\left|\int_0^{(k_n/n)\wedge\tau} (f(Y^{n,p}_s, Z^{n,p}_s) - f(Y^p_s, Z^p_s))\mathrm{d}s\right| \xrightarrow{\mathbb{P}} 0. \tag{12}$$

On the other hand, $|\tau^n - k_n/n| \leqslant 1/n$ and $\tau^n \xrightarrow{\mathbb{P}} \tau$, so we have $(k_n/n) \xrightarrow{\mathbb{P}} \tau$. Then,

$$\begin{aligned} &\left|\int_{(k_n/n)\wedge\tau}^{(k_n/n)\vee\tau} (f(Y^{n,p}_s, Z^{n,p}_s)\mathbf{1}_{\{\tau\leqslant k_n/n\}} - f(Y^p_s, Z^p_s)\mathbf{1}_{\{\tau > k_n/n\}})\mathrm{d}s\right| \\ \leqslant\ & |k_n/n - \tau|\ \|f\|_\infty \\ \xrightarrow{\mathbb{P}}\ & 0. \end{aligned} \tag{13}$$

According to the convergences (12) and (13), we have

$$\left|\int_0^{\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_0^{\tau} f(Y^p_s, Z^p_s)\mathrm{d}s\right| \xrightarrow{\mathbb{P}} 0. \tag{14}$$

Hence, according to the convergences (9) and (14),

$$M^n_{\tau^n} \xrightarrow{\mathbb{P}} \left(Y^{p+1}_\tau + \int_0^\tau f(Y^p_s, Z^p_s)\mathrm{d}s\right). \tag{15}$$

At last, Lemma 1.8 is proved. □

Let M be the process defined by $M_t = Y^{p+1}_{t\wedge\tau} + \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s$.

Lemma 1.9. *We have the convergence*

$$\forall L,\ \sup_{t\in[0,L]} |M^n_t - M_t| \xrightarrow{\mathbb{P}} 0. \tag{16}$$

Proof. Using the same computation as for M^n_t in Lemma 1.6, we find:

$$M_t = M_0 + \int_0^{t\wedge\tau} Z^{p+1}_s \mathrm{d}W_s = Y^{p+1}_{t\wedge\tau} + \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s.$$

So the process M is a $\mathcal{F}^\tau$-martingale.
Then, $M_{t\wedge\tau} = \mathbb{E}[M_\tau|\mathcal{F}^\tau_t]$, for every t, where $M_\tau = Y^{p+1}_\tau + \int_0^\tau f(Y^p_s, Z^p_s)\mathrm{d}s$. According to (15), $M^n_{\tau^n} \xrightarrow{L^1} M_\tau$.

If we prove that we have the convergence of the sequence of filtrations $(\mathcal{F}^{n,\tau^n})_n$ to $\mathcal{F}^\tau$, according to Remark 1.2 in Coquet, Mémin and Słomiński [8], we have $M^n_. \xrightarrow{\mathbb{P}} M_.$ for the Skorokhod topology.

$(W^n)_n$ is a sequence of processes with independant increments that converges in probability to W. So, according to Proposition 2 in [8], $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Moreover, W is continuous and $\tau^n \xrightarrow{\mathbb{P}} \tau$ so according to Corollary A.7, $\mathcal{F}^{n,\tau^n} \xrightarrow{w} \mathcal{F}^\tau$.

Finally, we have $M^n_\cdot \xrightarrow{\mathbb{P}} M_\cdot$ for the Skorokhod topology. Moreover, $\mathcal{F}$ is a Brownian filtration, so, every $\mathcal{F}$-martingale is continuous. In particular, $\mathbb{E}[M_\tau|\mathcal{F}_\cdot]$ is continuous. The stopped process is also continuous, *i.e.* $M_\cdot = \mathbb{E}[M_\tau|\mathcal{F}^\tau_\cdot]$ is a continuous process. So, the previous convergence is uniform in t on every compact set and Lemma 1.9 is proved. □

Step 2. In this step, we shall prove the convergence of $(Z^{n,p+1})$ to Z^{p+1} as $n \to \infty$.

First, let us see a lemma of convergence of quadratic variations:

Lemma 1.10. *$(W^{n,\tau^n}, M^{n,\tau^n}, [M^{n,\tau^n}, M^{n,\tau^n}], [M^{n,\tau^n}, W^{n,\tau^n}])$ converges to $(W^\tau, M^\tau, [M^\tau, M^\tau], [M^\tau, W^\tau])$ in probability.*

Proof. It suffices to follow the lines of Briand, Delyon and Mémin in the proof of Theorem 3.1 in [3]. □

Lemma 1.11. *$\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n} - Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}s \xrightarrow{\mathbb{P}} 0.$*

Proof. To prove this lemma, we prove that we have these two convergences:

$$\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t - \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0, \tag{17}$$

$$\forall g \in \mathrm{L}^2, \int_0^{\tau^n} g(s)(Z^{n,p+1}_{s\wedge\tau^n} - Z^{p+1}_{s\wedge\tau})\mathrm{d}s \xrightarrow{\mathbb{P}} 0. \tag{18}$$

Then, we conclude with the following lines:

$$\begin{aligned}\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n} - Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t &= \int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t + \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t - 2\int_0^{\tau^n} Z^{n,p+1}_{t\wedge\tau^n} Z^{p+1}_{t\wedge\tau} \mathrm{d}t \\ &= \left(\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t - \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t\right) + 2\left(\int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t - \int_0^{\tau^n} Z^{n,p+1}_{t\wedge\tau^n} Z^{p+1}_{t\wedge\tau} \mathrm{d}t\right) \\ &\xrightarrow{\mathbb{P}} 0 \text{ according to (17) and (18) with } g = Z^{p+1}.\end{aligned}$$

First, let us show that $\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t - \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0$

$$\begin{aligned}&\left|\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t - \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t\right| \\ &\qquad \leqslant \left|\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}t - \int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}A^n_t\right| + \left|\int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n}|^2 \mathrm{d}A^n_t - \int_0^{\tau^n} |Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t\right|.\end{aligned}$$

According to Lemma 1.10,

$$\forall L, \sup_{t\in[0,L]} \left|[M^{n,\tau^n}, M^{n,\tau^n}]_t - [M^\tau, M^\tau]_t\right| \xrightarrow{\mathbb{P}} 0.$$

But,

$$M^{n,\tau^n}_t = M^n_0 + \int_0^{t\wedge\tau^n} Z^{n,p+1}_s \mathrm{d}W^n_s = M^n_0 + \frac{1}{\sqrt{n}} \sum_{k=1}^{[n(t\wedge\tau^n)]} Z^{n,p+1}_{k/n} \varepsilon^n_k.$$

So,

$$[M^{n,\tau^n}, M^{n,\tau^n}]_t = \frac{1}{n} \sum_{k=1}^{[n(t\wedge\tau^n)]} |Z^{n,p+1}_{k/n}|^2 = \int_0^{t\wedge\tau^n} |Z^{n,p+1}_s|^2 \mathrm{d}A^n_s.$$

On the other hand, M^τ is a continuous martingale, so

$$[M^\tau, M^\tau]_t = \langle M^\tau, M^\tau\rangle_t = \int_0^{t\wedge\tau} |Z_s^{p+1}|^2 \mathrm{d}s.$$

Then,

$$\forall L, \sup_{s\in[0,L]} \left| \int_0^{s\wedge\tau^n} |Z_t^{n,p+1}|^2 \mathrm{d}A_t^n - \int_0^{s\wedge\tau} |Z_t^{p+1}|^2 \mathrm{d}t \right| \xrightarrow{\mathbb{P}} 0.$$

In particular, using the fact that $\sup_n \tau^n$ is almost surely finite because $(\tau^n)_n$ converges in probability to the almost surely finite stopping time τ, we have

$$\left| \int_0^{\tau^n} |Z_t^{n,p+1}|^2 \mathrm{d}A_t^n - \int_0^{\tau\wedge\tau^n} |Z_t^{p+1}|^2 \mathrm{d}t \right| \xrightarrow{\mathbb{P}} 0.$$

As $\tau^n \xrightarrow{\mathbb{P}} \tau$, by dominated convergence, we have:

$$\left| \int_0^{\tau^n} |Z_t^{p+1}|^2 \mathrm{d}t - \int_0^{\tau^n\wedge\tau} |Z_t^{p+1}|^2 \mathrm{d}t \right| \xrightarrow{\mathbb{P}} 0 \text{ as } n \to +\infty.$$

Moreover, $\int_0^{\tau^n} |Z_t^{n,p+1}|^2 \mathrm{d}A_t^n = \int_0^{\tau^n} |Z_t^{n,p+1}|^2 \mathrm{d}t$. So, the convergence (17) is proved.

It remains to prove convergence (18). Let us fix $g \in \mathrm{L}^2(\mathbb{R}^+)$ and show that $\int_0^{\tau^n} g(s)(Z_{s\wedge\tau^n}^{n,p+1} - Z_{s\wedge\tau}^{p+1})\mathrm{d}s \xrightarrow{\mathbb{P}} 0$. According to Lemma 1.10, we have the convergence

$$\forall L, \sup_{t\in[0,L]} \left| [M^{n,\tau^n}, W^{n,\tau^n}]_t - [M^\tau, W^\tau]_t \right| \xrightarrow{\mathbb{P}} 0.$$

But,

$$[M^{n,\tau^n}, W^{n,\tau^n}]_t = \frac{1}{n} \sum_{k=1}^{[n(t\wedge\tau^n)]} Z_{k/n}^{n,p+1} = \int_0^{t\wedge\tau^n} Z_s^{n,p+1} \mathrm{d}A_s^n.$$

On the other hand, M^τ and W^τ are continuous martingales, so

$$[M^\tau, W^\tau]_t = \langle M^\tau, W^\tau\rangle_t = \int_0^{t\wedge\tau} Z_s^{p+1} \mathrm{d}s.$$

Then,

$$\forall L, \sup_{t\in[0,L]} \left| \int_0^{t\wedge\tau^n} Z_s^{n,p+1} \mathrm{d}A_s^n - \int_0^{t\wedge\tau} Z_s^{p+1} \mathrm{d}s \right| \xrightarrow{\mathbb{P}} 0. \tag{19}$$

We write:

$$\begin{aligned} &\left| \int_0^{\tau\wedge\tau^n} g(s)(Z_{s\wedge\tau^n}^{n,p+1} - Z_{s\wedge\tau}^{p+1})\mathrm{d}s \right| \\ &\quad \leqslant \left| \int_0^{\tau\wedge\tau^n} g(s) Z_{s\wedge\tau^n}^{n,p+1} \mathrm{d}s - \int_0^{\tau\wedge\tau^n} g(s) Z_{s\wedge\tau^n}^{n,p+1} \mathrm{d}A_s^n \right| \\ &\qquad + \left| \int_0^{\tau\wedge\tau^n} g(s) Z_{s\wedge\tau^n}^{n,p+1} \mathrm{d}A_s^n - \int_0^{\tau\wedge\tau^n} g(s) Z_{s\wedge\tau}^{p+1} \mathrm{d}s \right|. \end{aligned}$$

As the function g is measurable and τ is almost surely finite, using a density argument and the convergence (19), we have:

$$\left|\int_0^{\tau^n} g(s)Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}A^n_s - \int_0^{\tau\wedge\tau^n} g(s)Z^{p+1}_{s\wedge\tau^n}\mathrm{d}s\right| \to 0. \tag{20}$$

By dominated convergence, we deduce that

$$\left|\int_0^{\tau\wedge\tau^n} g(s)Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}s - \int_0^{\tau\wedge\tau^n} g(s)Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}A^n_s\right| \xrightarrow{\mathbb{P}} 0.$$

Next, using the density of the set of continuous function with compact support in L^2, we show that

$$\left|\int_0^{\tau\wedge\tau^n} g(s)Z^{n,p+1}_{s\wedge\tau^n}\mathrm{d}A^n_s - \int_0^{\tau\wedge\tau^n} g(s)Z^{p+1}_{s\wedge\tau}\mathrm{d}s\right| \xrightarrow{\mathbb{P}} 0.$$

The convergence (18) is now proved.
Lemma 1.11 is proved. □

Step 3. Let us prove the convergence of $Y^{n,p+1}$ to Y^{p+1} when n goes to ∞.

Lemma 1.12. *For every L, $\sup_{t\in[0,L]} |Y^{n,p+1}_{t\wedge\tau^n} - Y^{p+1}_{t\wedge\tau}| \xrightarrow{\mathbb{P}} 0.$*

Proof. We fix L. We write:

$$\sup_{t\in[0,L]} |Y^{n,p+1}_{t\wedge\tau^n} - Y^{p+1}_{t\wedge\tau}| \leqslant \sup_{t\in[0,L]} |M^n_{t\wedge\tau^n} - M_{t\wedge\tau}| + \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s\right|.$$

Moreover,

$$\begin{aligned}
&\sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s\right| \\
&\quad\leqslant \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^{n,p}_{s-}, Z^{n,p}_s)\mathrm{d}A^n_s - \int_0^{t\wedge\tau^n} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}s\right| \\
&\qquad+ \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}s - \int_0^{t\wedge\tau^n} f(Y^p_s, Z^p_s)\mathrm{d}s\right|\mathbf{1}_{\tau^n\leqslant\tau} \\
&\qquad+ \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}s - \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s\right|\mathbf{1}_{\tau^n>\tau} \\
&\qquad+ \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^p_s, Z^p_s)\mathrm{d}s - \int_0^{t\wedge\tau} f(Y^p_s, Z^p_s)\mathrm{d}s\right|\mathbf{1}_{\tau^n\leqslant\tau} \\
&\qquad+ \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}s - \int_0^{t\wedge\tau} f(Y^{n,p}_s, Z^{n,p}_s)\mathrm{d}s\right|\mathbf{1}_{\tau^n>\tau}.
\end{aligned}$$

But, for every t, for every ω, we can find k_n such that $t\wedge\tau^n \in [k_n/n,(k_n+1)/n[$. So, we have

$$\begin{aligned}\left|\int_0^{t\wedge\tau^n} f(Y_{s-}^{n,p},Z_s^{n,p})\mathrm{d}A_s^n - \int_0^{t\wedge\tau^n} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s\right| &= \left|\int_{k_n/n}^{t\wedge\tau^n} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s\right|\\ &\leqslant |t\wedge\tau^n - k_n/n|\ \|f\|_\infty\\ &\leqslant \|f\|_\infty/n.\end{aligned}$$

Then,

$$\sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y_{s-}^{n,p},Z_s^{n,p})\mathrm{d}A_s^n - \int_0^{t\wedge\tau^n} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s\right| \leqslant \|f\|_\infty/n \xrightarrow{\mathbb{P}} 0. \tag{21}$$

Next,

$$\begin{aligned}&\sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s - \int_0^{t\wedge\tau^n} f(Y_s^p,Z_s^p)\mathrm{d}s\right|\mathbf{1}_{\{\tau^n\leqslant\tau\}}\\ &\qquad + \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s - \int_0^{t\wedge\tau} f(Y_s^p,Z_s^p)\mathrm{d}s\right|\mathbf{1}_{\{\tau^n>\tau\}}\\ &= \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y_{s\wedge\tau^n}^{n,p},Z_{s\wedge\tau^n}^{n,p})\mathrm{d}s - \int_0^{t\wedge\tau^n} f(Y_{s\wedge\tau}^p,Z_{s\wedge\tau}^p)\mathrm{d}s\right|\mathbf{1}_{\{\tau^n\leqslant\tau\}}\\ &\qquad + \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau} f(Y_{s\wedge\tau^n}^{n,p},Z_{s\wedge\tau^n}^{n,p})\mathrm{d}s - \int_0^{t\wedge\tau} f(Y_{s\wedge\tau}^p,Z_{s\wedge\tau}^p)\mathrm{d}s\right|\mathbf{1}_{\{\tau^n>\tau\}}\\ &\leqslant \int_0^{\tau\wedge\tau^n\wedge L} |f(Y_{s\wedge\tau^n}^{n,p},Z_{s\wedge\tau^n}^{n,p}) - f(Y_{s\wedge\tau}^p,Z_{s\wedge\tau}^p)|\mathrm{d}s\\ &\leqslant K(\tau\wedge\tau^n\wedge L)\sup_{t\in[0,L]}|Y_{t\wedge\tau^n}^{n,p}-Y_{t\wedge\tau}^p| + K(\tau\wedge\tau^n\wedge L)\left(\int_0^{\tau\wedge\tau^n}|Z_{s\wedge\tau^n}^{n,p}-Z_{s\wedge\tau}^p|^2\mathrm{d}s\right)^{1/2}\\ &\qquad \text{because } f \text{ is } K\text{-Lipschitz and using Cauchy-Schwarz inequality}\\ &\xrightarrow{\mathbb{P}} 0 \ \text{ by induction assumption (8).}\end{aligned} \tag{22}$$

Moreover,

$$\begin{aligned}&\sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y_s^p,Z_s^p)\mathrm{d}s - \int_0^{t\wedge\tau} f(Y_s^p,Z_s^p)\mathrm{d}s\right|\mathbf{1}_{\{\tau^n\leqslant\tau\}}\\ &\qquad + \sup_{t\in[0,L]}\left|\int_0^{t\wedge\tau^n} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s - \int_0^{t\wedge\tau} f(Y_s^{n,p},Z_s^{n,p})\mathrm{d}s\right|\mathbf{1}_{\{\tau^n>\tau\}}\\ &\leqslant \sup_{t\in[0,L]}|t\wedge\tau^n - t\wedge\tau|\ \|f\|_\infty\\ &\xrightarrow{\mathbb{P}} 0 \ \text{ because } \ \tau^n \xrightarrow{\mathbb{P}} \tau.\end{aligned} \tag{23}$$

Finally, according to the convergences (16, 21, 22) and (23), we have the convergence anounced in Lemma 1.12.

□

Then, using Lemmas 1.11 and 1.12, we have

$$\forall L,\ \sup_{t\in[0,L]} |Y^{n,p+1}_{t\wedge\tau^n} - Y^{p+1}_{t\wedge\tau}| + \int_0^{\tau^n} |Z^{n,p+1}_{t\wedge\tau^n} - Z^{p+1}_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0.$$

That concludes the proof of the induction.

We have shown that (8) is true for every p, *i.e.*

$$\forall p,\ \forall L,\ \sup_{t\in[0,L]} |Y^{n,p}_{t\wedge\tau^n} - Y^{p}_{t\wedge\tau}| + \int_0^{\tau^n} |Z^{n,p}_{t\wedge\tau^n} - Z^{p}_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0. \tag{24}$$

Thanks to convergences (6, 7) and (24), the first part of Theorem 1.4 is proved, *i.e.*

$$\forall L,\ \sup_{t\in[0,L]} |Y^{n}_{t\wedge\tau^n} - Y_{t\wedge\tau}| + \int_0^{\tau^n} |Z^{n}_{t\wedge\tau^n} - Z_{t\wedge\tau}|^2 \mathrm{d}t \xrightarrow{\mathbb{P}} 0.$$

To prove the second part of the theorem, we define the processes M^n and M by

$$M^n_t = M^n_0 + \int_0^{t\wedge\tau^n} Z^n_{s\wedge\tau^n} \mathrm{d}W^n_s = Y^n_{t\wedge\tau^n} + \int_0^{t\wedge\tau^n} f(Y^n_s, Z^n_s)\mathrm{d}A^n_s, \forall t$$

and

$$M_t = M_0 + \int_0^{t\wedge\tau} Z_{s\wedge\tau} \mathrm{d}W_s = Y_{t\wedge\tau} + \int_0^{t\wedge\tau} f(Y_s, Z_s)\mathrm{d}s, \forall t.$$

It is clear that, for each n, M^n is a $\mathcal{F}^{n,\tau^n}$-martingale and that M is a $\mathcal{F}^{\tau}$-martingale. So, we can write

$$M^n_t = \mathbb{E}[M_{\tau^n}|\mathcal{F}^{n,\tau^n}_t] \text{ and } M_t = \mathbb{E}[M_\tau|\mathcal{F}^{\tau}_t]$$

where $M_{\tau^n} = Y^n_{\tau^n} + \int_0^{\tau^n} f(Y^n_s, Z^n_s)\mathrm{d}A^n_s$ and $M_\tau = Y_\tau + \int_0^\tau f(Y_s, Z_s)\mathrm{d}s$. As in Lemma 1.9, we prove that $\forall L,\ \sup_{t\in[0,L]} |M^n_t - M_t| \xrightarrow{\mathbb{P}} 0$. At last, arguing like in Lemma 1.12, $\forall L,\ \sup_{t\in[0,L]} |Y^n_t - Y_t| \xrightarrow{\mathbb{P}} 0$. In particular, $M^n_0 = Y^n_0 \xrightarrow{\mathbb{P}} Y_0 = M_0$. Finally,

$$\forall L,\ \sup_{t\in[0,L]} \left| \int_0^{t\wedge\tau^n} Z^n_s \mathrm{d}W^n_s - \int_0^{t\wedge\tau} Z_s \mathrm{d}W_s \right| \xrightarrow{\mathbb{P}} 0.$$

Theorem 1.4 is proved.

1.3. **An example**

Before studying an example with particular stopping times, let us show a technical lemma that will be useful.

Lemma 1.13. *Let $(a^n)_n$ be a sequence of real numbers that converges to a. We consider the functions f and f^n defined from $\mathbb{D}$ to $\mathbb{R}$ by*

$$f(x) = \inf\{t > 0 : x(t) > a\} \quad \textit{and} \quad f^n(x) = \inf\{t \in]0, n] : x(t) > a^n\} \wedge n$$

with the convention $\inf\emptyset = +\infty$. Let y be a continuous process such that

$$\inf\{t > 0 : y(t) > a\} = \inf\{t > 0 : y(t) \geqslant a\}.$$

Let $(y^n)_n$ be a sequence of functions of $\mathbb{D}$ that converges to y for the uniform topology on every compact set. Then $f^n(y^n) \to f(y)$.

Proof. For every n, we denote $t_n = f^n(y^n)$. We have $y^n(t_n) \geqslant a^n$ or $t_n = n$.
Let t be the limit of a subsequence of (t_n) in $\bar{\mathbb{R}}^+$. Let us show that $t = f(y)$.
Without loss of generality, instead of extracting a subsequence, we suppose that $t_n \to t$. As y is continuous and (y^n) converges uniformly to y, $y^n(t_n) \to y(t)$. Then, when n tends to ∞ in the inequality $y^n(t_n) \geqslant a^n$, we have $y(t) \geqslant a$. So, $t \geqslant f(y)$ because we have assumed that $\inf\{t > 0 : y(t) > a\} = \inf\{t > 0 : y(t) \geqslant a\}$.
Let us suppose that $t > f(y)$. Let us fix $0 < \varepsilon < \frac{t-f(y)}{2}$. By definition of $f(y)$, we can find $t_0 \in [f(y), f(y)+\varepsilon[$ such that $y(t_0) > a$. We take $\alpha = \frac{y(t_0)-a}{2} \wedge \varepsilon$. Since $y^n \to y$, $t_n \to t$ and $a^n \to a$, there exists n_0 such that for every $n \geqslant n_0$, $\sup_t |y^n_t - y_t| < \alpha/4$, $|t_n - t| < \alpha/4$ and $|a^n - a| < \alpha$. In particular, for every $n \geqslant n_0$, $|y^n(t_0) - y(t_0)| < \alpha/4$. Then,

$$t_n - t_0 > t - \alpha/4 - (f(y) + \varepsilon + \alpha/4) > t - f(y) - \frac{t - f(y)}{2} > 0,$$

so $t_n > t_0$.
On the other hand, $|y^n(t_0) - y(t_0)| < \alpha/4$ and $|a^n - a| < \alpha$, so

$$y^n(t_0) - a^n > y(t_0) - \alpha/4 - a - \alpha > \frac{3\alpha}{4} > 0.$$

Hence, $y^n(t_0) > a^n$. This is in contradiction with the definition of t_n. So, $t \leqslant f(y)$.
Hence, $t = f(y)$ and $f(y)$ is the only possible limit for the convergent subsequences of $(f^n(y^n))_n$. So $f^n(y^n) \to f(y)$. □

Proposition 1.14. *With the notations of Theorem 1.4, we assume that* $\forall n, \forall k, \varepsilon^n_k = \varepsilon_k$, $\xi = g(W)$ *and* $\xi^n = g(W^n)$ *with g bounded and continuous. Let $(a^n)_n$ be a sequence of real numbers that converges to a. We define the stopping times $(\tau^n)_n$ and τ as follows:*

$$\tau^n = \inf\{t \in]0, n] : |W^n_t| > a^n\} \wedge n \quad \textit{and} \quad \tau = \inf\{t > 0 : |W_t| > a\}.$$

Then $\left(Y^n_{\cdot\wedge\tau^n}, \int_0^{\cdot\wedge\tau^n} Z^n_s \mathrm{d}W^n_s\right)$ *converges in law to* $\left(Y_{\cdot\wedge\tau}, \int_0^{\cdot\wedge\tau} Z_s \mathrm{d}W_s\right)$ *for the Skorokhod topology.*

Proof. According to Corollary 1.5, it is sufficient to prove that $(W^n, \tau^n) \xrightarrow{\mathcal{L}} (W, \tau)$ and that $\sup_n \left[(\tau^n)^2\right]^{1/2} < +\infty$.

According to Donsker's Theorem, we have the convergence in law of W^n to W. Using the Skorokhod representation Theorem, we can find a probabilistic space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$, processes $\tilde{W}^n$ and $\tilde{W}$ such that $\tilde{W}^n \sim W^n$, $\tilde{W} \sim W$ and $\tilde{W}^n \xrightarrow{a.s.} \tilde{W}$.
Denoting by E the set $\{\omega : \inf\{t > 0 : |\tilde{W}_t(\omega)| > a\} \neq \inf\{t > 0 : |\tilde{W}_t(\omega)| \geqslant a\}\}$, it is quickly proved that $\tilde{\mathbb{P}}[E] = 0$. Then, for every $\omega \notin E$, $t \mapsto \tilde{W}_t(\omega)$ is continuous, using Lemma 1.13, for every $\omega \notin E$, $f^n(\tilde{W}^n(\omega)) \to f(\tilde{W}(\omega))$. Then, $f^n(\tilde{W}^n) \xrightarrow{a.s.} f(\tilde{W})$. We also have

$$(\tilde{W}^n, f^n(\tilde{W}^n)) \xrightarrow{a.s.} (\tilde{W}, f(\tilde{W})).$$

By construction, $(\tilde{W}^n, f^n(\tilde{W}^n)) \sim (W^n, f^n(W^n))$. Then, we can find a process Y such that $(\tilde{W}, f(\tilde{W})) \sim (W, Y)$ and $(W^n, f^n(W^n)) \xrightarrow{\mathcal{L}} (W, Y)$. But, by construction, we also have $(\tilde{W}, f(\tilde{W})) \sim (W, f(W))$. Then $Y = f(W)$ *a.s.* So we have

$$(W^n, f^n(W^n)) \xrightarrow{\mathcal{L}} (W, f(W)), \ \textit{i.e.}\ (W^n, \tau^n) \xrightarrow{\mathcal{L}} (W, \tau).$$

On the other hand, following the lines of the proofs of Proposition 1.16 and Theorem 1.17 in Chung and Zhao [5], we have $\sup_n \left[(\tau^n)^2\right]^{1/2} < +\infty$.

The result follows using Corollary 1.5. □

2. Stability of BSDEs when the Brownian motion is approximated by a sequence of martingales

2.1. Statement of the problem

Let W be a Brownian motion and $\mathcal{F}$ its natural filtration. Let τ be a $\mathcal{F}$-stopping time almost surely finite. We consider the following BSDE:

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(r, Y_r, Z_r)\mathrm{d}r - \int_{t\wedge\tau}^{\tau} Z_r \mathrm{d}W_r,\ t \geqslant 0, \tag{25}$$

where ξ is a bounded random variable $\mathcal{F}_\tau$-measurable and for every (y,z), $\{f(t,y,z)\}_t$ is progressively measurable.

We approximate this equation on the following way. Let $(W^n)_n$ be a sequence of càdlàg processes and $(\mathcal{F}^n)_n$ the natural filtrations for these processes. We suppose that (W^n) is a sequence of square integrable $(\mathcal{F}^n)$-martingales which converges in probability to W. We don't suppose that W^n has the predictable representation property. Let $(\tau^n)_n$ be a sequence of $(\mathcal{F}^n)$-stopping times that converges almost surely to τ.
Then, we consider the following BSDE:

$$Y_t^n = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f^n(r, Y_{r-}^n, Z_r^n)\mathrm{d}\langle W^n\rangle_r - \int_{t\wedge\tau^n}^{\tau^n} Z_r^n \mathrm{d}W_r^n - (N_{\tau^n}^n - N_{t\wedge\tau^n}^n), t \geqslant 0 \tag{26}$$

where $(\xi^n)_n$ is a sequence of random variables $(\mathcal{F}_{\tau^n}^n)$-measurable, (N^n) is a sequence of $(\mathcal{F}^n)$ martingales orthogonal to (W^{n,τ^n}) and for every (y,z), $\{f^n(t,y,z)\}_t$ is progressively measurable with respect to $(\mathcal{F}^n)$.

We denote by $\mathcal{S}_L^p$ the set of càdlàg processes X indexed by $\mathbb{R}^+$ and taking values in $\mathbb{R}$ such that

$$\|X\|_{\mathcal{S}_L^p} = \mathbb{E}\left[\sup_{t\in[0,L]} |X_t|^p\right] < +\infty.$$

We put the following assumptions on the martingales and on the terminal conditions:

(H1) (i) $\forall L$, $W^n \xrightarrow{\mathcal{S}_L^2} W$,
(ii) $\langle W^n\rangle_t - \langle W^n\rangle_s \leqslant \rho(t-s) + a_n$
where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0^+) = 0$ and $(a_n) \downarrow 0$,
(iii) $\sup_n \langle W^n\rangle_{\tau^n} < +\infty$.

(H2) (i) $\xi^n \xrightarrow{L^2} \xi$,
(ii) $\|\xi\|_\infty + \sup_n \mathbb{E}[|\xi^n|] < \infty$.

2.2. Existence and uniqueness of the solutions for the studied BSDEs

Let us begin with the case of the equation (25).
Let us put some assumptions on the generator f:

(Hf) (i) f is K-Lipschitz in y and z,
(ii) f is monotone in y in the following way: there exists $\mu > 0$ such that
$\forall (t,y,z), (t,y',z) \in \mathbb{R}^+ \times \mathbb{R}^2, (y-y')(f(t,y,z) - f(t,y',z)) \leqslant -\mu(y-y')^2$.
(iii) f is bounded.

Under these assumptions, according to Theorem 2.1 of Royer in [15], the BSDE (25) has a unique solution (Y, Z) (in the sense of Def. 1.1) in the set of processes such that Y is continuous and uniformly bounded.

Now, let us deal with the equation (26).
Let us put some assumptions on the generators (f^n):

(Hfn) (i) for every n, f^n is K-Lipschitz in y and z,
(ii) $\sup_n \|f^n\| < \infty$.

Let us first introduce some notations:
$\mathcal{S}^{2,n}$ is the set of processes Y progressively measurable with respect to $\mathcal{F}^{n,\tau^n}$ such that $\mathbb{E}\left[\sup_{t\geqslant 0}|Y_{t\wedge\tau^n}|^2\right] < \infty$,
$\mathcal{M}^{2,n}$ is the set of predictable processes Z measurable with respect to $\mathcal{F}^{n,\tau^n}$ such that $\mathbb{E}\left[\int_0^{\tau^n} |Z_r|^2 \mathrm{d}\langle W^n\rangle_r\right] < +\infty$,
$\mathcal{H}_0^{2,n}$ is the set of squared integrable $\mathcal{F}^{n,\tau^n}$-martingales M such that $M_0 = 0$.

Now, a fixed point argument and an estimation *a priori* like Briand, Delyon and Mémin in the proof of Theorem 9 in [4] give the result of existence and uniqueness of the solution of the BSDE (26):

Theorem 2.1. *Under the assumptions (H1), (H2), (Hf) and (Hfn), the BSDE (26) has, for n large enough, a unique solution* $(Y^n_{.\wedge\tau^n}, Z^n_{.\wedge\tau^n}, N^n_{.\wedge\tau^n})$ *in* $\mathcal{S}^{2,n} \times \mathcal{M}^{2,n} \times \mathcal{H}_0^{2,n}$.

2.3. Convergence of the solutions

First, we show a result of stability for the decompositions of the terminal conditions ξ and ξ^n. This result will be the main argument in the proof of Theorem 2.3 about the convergence of the solutions.

Theorem 2.2. *We suppose that the conditions (H1) and (H2) are filled. We consider the orthogonal decomposition of* ξ^n *with respect to* $W^n_{.\wedge\tau^n}$, *i.e.* Z^n *is a predictable* $\mathcal{F}^{n,\tau^n}$*-measurable process,* N^n *is a* $\mathcal{F}^{n,\tau^n}$*-martingale orthogonal to* $W^n_{.\wedge\tau^n}$ *and*

$$M^n_{t\wedge\tau^n} = \mathbb{E}[\xi^n|\mathcal{F}^{n,\tau^n}_t] = M^n_0 + \int_0^{t\wedge\tau^n} Z^n_r \mathrm{d}W^n_r + N^n_t = M^n_0 + \int_0^{t\wedge\tau^n} Z^n_r \mathrm{d}W^n_r + N^n_{t\wedge\tau^n}.$$

We also consider the representation of ξ *as a stochastic integral:*

$$M_{t\wedge\tau} = \mathbb{E}[\xi|\mathcal{F}^\tau_t] = M_0 + \int_0^{t\wedge\tau} Z_r \mathrm{d}W_r.$$

Then we have the following convergences: for every L,

$$\left(M^n_{.\wedge\tau^n}, \int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}W^n_r, N^n_{.\wedge\tau^n}\right) \xrightarrow{\mathcal{S}^2_L} \left(M_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_r \mathrm{d}W_r, 0\right),$$
$$\left(\int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}\langle W^n\rangle_r, \int_0^{.\wedge\tau^n} |Z^n_r|^2 \mathrm{d}\langle W^n\rangle_r\right) \xrightarrow{\mathcal{S}^2_L\times\mathcal{S}^1_L} \left(\int_0^{.\wedge\tau} Z_r \mathrm{d}r, \int_0^{.\wedge\tau} |Z_r|^2 \mathrm{d}r\right).$$

Proof. The only noticeable difference with Briand, Delyon and Mémin's proof of Theorem 5 in [4] is that we have to prove the convergence of $M^n_{.\wedge\tau^n}$ to $M_{.\wedge\tau}$.
According to Proposition 3 in Briand, Delyon and Mémin in [4], we have the convergence of filtrations $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. As W is continuous and $\tau^n \xrightarrow{\mathbb{P}} \tau$, according to Corollary A.7, $(\mathcal{F}^{n,\tau^n})$ converges to $\mathcal{F}^\tau$. Moreover, according to (H2), (ξ^n) converges in L^2 so in L^1 to ξ. Then, $M^n_{.\wedge\tau^n} \xrightarrow{\mathbb{P}} M_{.\wedge\tau}$, according to Remark 1.2 in Coquet, Mémin and Słomiński [8].

$$\forall L,\ M^n_{.\wedge\tau^n} \xrightarrow{\mathcal{S}^2_L} M_{.\wedge\tau}.$$

□

We are now going to be interested in the convergence of the solutions of the BSDE.

Let us do some assumptions of convergence on the generators f and f^n.

$$\textbf{(H3)} \quad \forall(y,z),\ \{f^n(t,y,z)\}_t \text{ has càdlàg trajectories and } f^n(.,y,z) \xrightarrow{\mathcal{S}^2_L} f(.,y,z),\ \forall L.$$

Now, the following result of convergence of the solutions can be proven in a similar way as in Briand, Delyon and Mémin in the proof of Theorem 12 in [4].

Theorem 2.3. *We suppose that (H1), (H2), (H3), (Hf) and (Hfn) are filled. We denote by* (Y^n, Z^n, N^n) *the solution of the equation (26) and by* (Y, Z) *these of the equation (25). Then,* $(Y^n, Z^n, N^n) \to (Y, Z, 0)$*, i.e.* $\forall L$,

$$(Y^n_{.\wedge\tau^n}, \int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}W^n_r, N^n_{.\wedge\tau^n}) \xrightarrow{\mathcal{S}^2_L} (Y_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_r \mathrm{d}W_r, 0),$$
$$(\int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}\langle W^n\rangle_r, \int_0^{.\wedge\tau^n} |Z^n_r|^2 \mathrm{d}\langle W^n\rangle_r) \xrightarrow{\mathcal{S}^2_L\times\mathcal{S}^1_L} (\int_0^{.\wedge\tau} Z_r \mathrm{d}r, \int_0^{.\wedge\tau} |Z_r|^2 \mathrm{d}r).$$

2.4. **Application to discretizations**

In this section, we are interested in the case of the approximation of a Brownian motion W by its discretizations W^n. More precisely, we consider an increasing sequence $(\pi^n = \{t^n_k\})_n$ of subdivisions of $\mathbb{R}^+$ with mesh going to 0 and the discretized processes W^n are defined by $W^n_{t\wedge\tau^n} = W_{t^n_k}$ if $t^n_k \leqslant t\wedge\tau^n < t^n_{k+1}$.

Let $(a^n)_n$ be a sequence of real numbers which decreases to a real number a. Then, we consider the following random variables:

$$\tau = \inf\{t > 0 : |W_t| > a\} \quad \text{and} \quad \tau^n = \inf\{t \in]0, n] : |W^n_t| > a^n\} \wedge n.$$

As previously, we denote by $\mathcal{F}$ the natural filtration for W and $\mathcal{F}^n$ the natural filtrations of the processes W^n. It is clear that (W^n) is a sequence of $(\mathcal{F}^n)$-martingales, τ is a $\mathcal{F}$-stopping time and $(\tau^n)_n$ is a sequence of $(\mathcal{F}^n)$-stopping times.

Let us show now a result of convergence on these stopping times:

Lemma 2.4. $\tau^n \xrightarrow{a.s.} \tau$.

Proof. The proof will be done in two steps. In a first step, we prove that the limit of (τ^n) is a stopping time. In a second step, we shall identify the limit using properties of discretizations.

Note that $(\tau^n)_n$ is an almost surely nonincreasing sequence of $(\mathcal{F}^n)$-stopping times lower-bounded by 0. So, (τ^n) converges almost surely to a random variable $\tilde{\tau}$. $\tilde{\tau}$ is a $\mathcal{F}$-stopping time according to the following proposition:

Proposition 2.5. *We suppose that, for every* n, $\mathcal{F}^n \subset \mathcal{F}$ *and that the filtration* $\mathcal{F}$ *is right continuous and complete. Let* $(\tau^n)_n$ *be a sequence of* $(\mathcal{F}^n)$*-stopping times that converges almost surely to a random variable* τ*. Then* τ *is a* $\mathcal{F}$*-stopping time.*

Proof. Let us fix $t \in \{s : \mathbb{P}[\tau = s] = 0\}$.
Since $\tau^n \xrightarrow{a.s.} \tau$ and $\mathbb{P}[\tau = t] = 0$, we know that $\mathbf{1}_{\{\tau^n\leqslant t\}} \xrightarrow{L^1} \mathbf{1}_{\{\tau\leqslant t\}}$. Then, $\mathbb{E}[\mathbf{1}_{\{\tau^n\leqslant t\}}|\mathcal{F}_t] \xrightarrow{L^1} \mathbb{E}[\mathbf{1}_{\{\tau\leqslant t\}}|\mathcal{F}_t]$.
But, $\mathbf{1}_{\{\tau^n\leqslant t\}} = \mathbb{E}[\mathbf{1}_{\{\tau^n\leqslant t\}}|\mathcal{F}_t]$ because $\mathcal{F}^n \subset \mathcal{F}$ and τ^n is a $\mathcal{F}^n$-stopping time.
So, by uniqueness of the limit, $\mathbb{E}[\mathbf{1}_{\{\tau\leqslant t\}}|\mathcal{F}_t] = \mathbf{1}_{\{\tau\leqslant t\}}$ *a.s.* Then, $\{\tau \leqslant t\} \in \mathcal{F}_t$.

As τ is a random variable, $\{t : \mathbb{P}[\tau = t] \neq 0\}$ is countable. Let us fix t such that $\mathbb{P}[\tau = t] \neq 0$. We can find a sequence $(t^n)_n$ that decreases to t such that for every n, $\mathbb{P}[\tau = t^n] = 0$. Then $\{\tau \leqslant t\} = \bigcap_n \{\tau \leqslant t^n\}$. But, for every n, $\{\tau \leqslant t^n\} \in \mathcal{F}_{t^n}$. So $\{\tau \leqslant t\} \in \bigcap_n \mathcal{F}_{t^n}$. But $\bigcap_n \mathcal{F}_{t^n} = \mathcal{F}_{t^+} = \mathcal{F}_t$ because $\mathcal{F}$ is right continuous.
So, for every t, $\{\tau \leqslant t\} \in \mathcal{F}_t$, *i.e.* τ is a $\mathcal{F}$-stopping time. □

Using $W^n \xrightarrow{a.s.} W$, $\tau^n \xrightarrow{a.s.} \tilde{\tau}$ and W is continuous, we get $W^n_{\tau^n} \xrightarrow{a.s.} W_{\tilde{\tau}}$. Moreover, by construction of τ^n, either $\tau^n = n$, or $W^n_{\tau^n} \geqslant a^n$. When n tends to ∞, either $\tilde{\tau} = +\infty$, or $W_{\tilde{\tau}} \geqslant a$ *a.s.* Then, $\tilde{\tau} \geqslant \tau$ *a.s.*

Let us fix ω such that $\tau^n(\omega) \to \tilde{\tau}(\omega)$. We suppose that $\tilde{\tau}(\omega) \neq \tau(\omega)$, *i.e.* $\tilde{\tau}(\omega) > \tau(\omega)$. Then, we can find $t_0 < \tilde{\tau}(\omega)$ such that $W_{t_0}(\omega) \geqslant a$ and $W_{t_0+}(\omega) > a$. As W is right continuous, we can find $0 < \eta < \tilde{\tau}(\omega) - t_0$ such that for every $t \in]t_0, t_0 + \eta[$, $W_t(\omega) > a$. Let us fix $t_1 \in]t_0, t_0 + \eta[$. $(a^n)_n$ decreases to a, so we can find n_0 such that for every $n \geqslant n_0$, $a^n \leqslant \frac{W_{t_1}(\omega)-a}{2}$. As W is right continuous at time t_1, there exists $0 < \eta_1 < \tilde{\tau}(\omega) - t_0$ such that for every $t \in]t_1, t_1 + \eta_1[$, $W_t(\omega) > \frac{W_{t_1}(\omega)-a}{2}$. $|\pi^n| \to 0$ so there exists $n_1 \geqslant n_0$ such that for every $n \geqslant n_1$, we can find $t_n \in \pi^n$, $t_n \in]t_1, t_1 + \eta_1[$. Then, for every $n \geqslant n_1$, $\tau^n(\omega) \leqslant t_n < \tilde{\tau}(\omega)$. This is in contradiction with the fact that $(\tau^n(\omega))_n$ decreases to $\tilde{\tau}(\omega)$. So $\tilde{\tau}(\omega) = \tau(\omega)$. Finally, $\tau^n \xrightarrow{a.s.} \tau$.
Lemma 2.4 is proved. □

On the other hand, for every L, $\sup_{t\in[0,L]} |W^n_t - W_t| \xrightarrow{a.s.} 0$. Moreover, all the processes are bounded on $[0, L]$. So the convergence is in $\mathcal{S}^2_L$ (*cf* Sect. 2.1), ie $W^n \xrightarrow{\mathcal{S}^2_L} W$. Then, we remark that $\langle W^n \rangle$ is the discretized process of $\langle W \rangle$. Let us fix $s \leqslant t$. We can find $i, j \in \mathbb{N}$ such that $s \in [t^n_i, t^n_{i+1}[$ and $t \in [t^n_j, t^n_{j+1}[$. Then, $\langle W^n \rangle_t - \langle W^n \rangle_s = t^n_j - t^n_i \leqslant t - s + |\pi_n|$ where $|\pi_n|$, the mesh of the subdivision, goes to 0. At last, $\langle W^n \rangle_{\tau^n} \leqslant \tau^n + |\pi^n|$. Then,

$$\sup_n \langle W^n \rangle_{\tau^n} \leqslant \sup_n \tau^n + \sup_n |\pi^n|.$$

It is well known (see *e.g.* Th. 1.17 in Chung and Zhao [5]) that $\mathbb{E}[\tau] < +\infty$. So τ is almost surely finite. As $\tau^n \xrightarrow{a.s.} \tau$, $\sup_n \tau^n < +\infty$ *a.s.* As last, as $(|\pi^n|)_n$ decreases to 0, $\sup_n |\pi^n| < +\infty$ *a.s.* Finally, $\sup_n \langle W^n \rangle_{\tau^n} < +\infty$ *a.s.*
So, the assumption (H1) is satisfied.

Then, we consider terminal conditions ξ and (ξ^n) and some generators f and (f^n) such that the conditions (Hf), (Hfn), (H2) and (H3) are filled.

Let (Y, Z) be the solution of the BSDE

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(r, Y_r, Z_r)\mathrm{d}r - \int_{t\wedge\tau}^{\tau} Z_r \mathrm{d}W_r,\ t \geqslant 0,$$

and (Y^n, Z^n, N^n) the solution of the BSDE

$$Y^n_t = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f^n(r, Y^n_{r-}, Z^n_r)\mathrm{d}\langle W^n \rangle_r - \int_{t\wedge\tau^n}^{\tau^n} Z^n_r \mathrm{d}W^n_r - (N^n_{\tau^n} - N^n_{t\wedge\tau^n}), t \geqslant 0.$$

These solutions exist and are unique in specified spaces as it was proved in Section 2.2.

The assumptions of Theorem 2.3 are satisfied. So, we have the following convergences: for every L,

$$\left(Y^n_{.\wedge\tau^n}, \int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}W^n_r, N^n_{.\wedge\tau^n}\right) \xrightarrow{\mathcal{S}^2_L} \left(Y_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_r \mathrm{d}W_r, 0\right),$$

$$\left(\int_0^{.\wedge\tau^n} Z^n_r \mathrm{d}\langle W^n \rangle_r, \int_0^{.\wedge\tau^n} |Z^n_r|^2 \mathrm{d}\langle W^n \rangle_r\right) \xrightarrow{\mathcal{S}^2_L \times \mathcal{S}^1_L} \left(\int_0^{.\wedge\tau} Z_r \mathrm{d}r, \int_0^{.\wedge\tau} |Z_r|^2 \mathrm{d}r\right).$$

$\langle W^{n,\tau^n} \rangle = \langle W^n \rangle^{\tau^n}$ according to the following lemma:

Lemma 2.6. *Let M be a $\mathcal{F}$-martingale and τ a $\mathcal{F}$-stopping time. Then we have the following equality: $\langle M\rangle^\tau = \langle M^\tau\rangle$.*

Note that $\langle W^n\rangle$ is the discretization of $\langle W\rangle$: $\langle W^n\rangle_t = t_i^n$ if $t \in [t_i^n, t_{i+1}^n[$. So $\langle W^{n,\tau^n}\rangle$ is the discretization of $\langle W^\tau\rangle$. Then, using the same arguments as in Lemma 1.11, we have:

$$\mathbb{E}\left[\int_0^{\tau\wedge\tau^n} |Z^n_{t\wedge\tau^n} - Z_{t\wedge\tau}|^2\,\mathrm{d}r\right] \xrightarrow[n\to+\infty]{} 0.$$

Finally, we have the following convergence for the solutions:

$$\forall L,\ \mathbb{E}\left[\sup_{t\in[0,L]} |Y^n_{t\wedge\tau^n} - Y_{t\wedge\tau}|^2 + \int_0^{\tau\wedge\tau^n} |Z^n_{t\wedge\tau^n} - Z_{t\wedge\tau}|^2\mathrm{d}r + \sup_{t\in[0,L]} |N^n_{t\wedge\tau^n}|^2\right] \xrightarrow[n\to+\infty]{} 0.$$

We have just proved the following theorem:

Theorem 2.7. *Let $(\pi^n = \{t_k^n\})_n$ be an increasing sequence of subdivisions of $\mathbb{R}^+$ with mesh going to 0 and W^n the discretized associated processes of W. Let $(a^n)_n$ be a sequence of real numbers which decreases to a real number a. We consider the stopping times*

$$\tau = \inf\{t > 0 : |W_t| > a\} \quad \textit{and} \quad \tau^n = \inf\{t \in]0, n] : |W^n_t| > a^n\} \wedge n.$$

Let (Y, Z) be the solution of the BSDE

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(r, Y_r, Z_r)\mathrm{d}r - \int_{t\wedge\tau}^{s\wedge\tau} Z_r \mathrm{d}W_r,\ t \geqslant 0,$$

and (Y^n, Z^n, N^n) the solution of the BSDE

$$Y^n_t = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f^n(r, Y^n_{r-}, Z^n_r)\mathrm{d}\langle W^n\rangle_r - \int_{t\wedge\tau^n}^{\tau^n} Z^n_r \mathrm{d}W^n_r - (N^n_{\tau^n} - N^n_{t\wedge\tau^n}), t \geqslant 0.$$

We assume that the conditions (Hf), (Hfn), (H2) and (H3) are satisfied. Then, we have the following convergence for the solutions:

$$\forall L,\ \mathbb{E}\left[\sup_{t\in[0,L]} |Y^n_{t\wedge\tau^n} - Y_{t\wedge\tau}|^2 + \int_0^{\tau\wedge\tau^n} |Z^n_{t\wedge\tau^n} - Z_{t\wedge\tau}|^2\mathrm{d}r + \sup_{t\in[0,L]} |N^n_{t\wedge\tau^n}|^2\right] \xrightarrow[n\to+\infty]{} 0.$$

Appendix A. About stopped filtrations and stopped processes

In their paper [9], Haezendonck and Delbaen give the following characterization of the σ-field $\mathcal{F}_\tau$ when $\mathcal{F}$ is the natural filtration of a process X:

Proposition A.1. *Let X be a càdlàg process, $\mathcal{F}$ the natural filtration of X and τ a $\mathcal{F}$-stopping time. Then $\mathcal{F}_\tau = \sigma(\{X_{\tau\wedge s}, s \geqslant 0\})$.*

This characterization shows that, if $\mathcal{F}$ is the natural filtration of X and τ a $\mathcal{F}$-stopping time, the stopped filtration $\mathcal{F}^\tau$ is the natural filtration of the stopped process X^τ.

The notions of convergence of filtrations and of σ-fields have been firstly defined in Hoover [10] and then in a slightly different way in Coquet, Mémin and Słominski [8]. In [8], the filtrations are indexed by a finite interval time $[0, T]$. We generalise it to the case of filtrations indexed by $\mathbb{R}^+$.

Definition A.2. We say that $(\mathcal{F}^n)$ converges to $\mathcal{F}$ if for every $A \in \mathcal{F}_\infty$, the sequence of processes $(\mathbb{E}[\mathbf{1}_A|\mathcal{F}^n_\cdot])_n$ converges in probability to $\mathbb{E}[\mathbf{1}_A|\mathcal{F}_\cdot]$ for the Skorokhod topology. We denote $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$.

Definition A.3. We say that the sequence of σ-fields $(\mathcal{B}_n)$ converges to the σ-field $\mathcal{B}$ if for every $A \in \mathcal{B}$, the sequence of random variables $(\mathbb{E}[\mathbf{1}_A|\mathcal{B}^n])_n$ converges in probability to $\mathbf{1}_A$. We denote $\mathcal{B}_n \to \mathcal{B}$.

The following lemma shows that, when holds convergence of filtrations, to get the convergence of associated stopped filtrations we just have to check the convergence of the terminal σ-fields. More precisely,

Lemma A.4. *Let $(\mathcal{F}^n)$ be a sequence of filtrations that converges to the filtration $\mathcal{F}$. Let (τ^n) be a sequence of $\mathcal{F}^n$-stopping times that converges in probability to a $\mathcal{F}$-stopping time τ. If the convergence of the σ-fields $(\mathcal{F}^n_{\tau^n})_n$ to $\mathcal{F}_\tau$ holds, then also holds the convergence of the filtrations $(\mathcal{F}^{n,\tau^n})_n$ to $\mathcal{F}^\tau$.*

Proof. Let us fix $B \in \mathcal{F}_\tau$.
$\mathcal{F}^n_{\tau^n} \to \mathcal{F}_\tau$, so by definition $\mathbb{E}[\mathbf{1}_B|\mathcal{F}^n_{\tau^n}] \xrightarrow{\mathbb{P}} \mathbf{1}_B$. Moreover, this convergence holds in L^1 since the sequence is uniformly integrable. As $\mathcal{F}^n \to \mathcal{F}$, according to Remark 1.2 in Coquet, Mémin and Słominski [8], we have

$$\mathbb{E}[\mathbb{E}[\mathbf{1}_B|\mathcal{F}^n_{\tau^n}]|\mathcal{F}^n_\cdot] \xrightarrow{\mathbb{P}} \mathbb{E}[\mathbf{1}_B|\mathcal{F}_\cdot] \text{ for the Skorokhod topology.} \tag{27}$$

For every t, we have the relations $\mathbb{E}[\mathbb{E}[\mathbf{1}_B|\mathcal{F}^n_{\tau^n}]|\mathcal{F}^n_t] = \mathbb{E}[\mathbf{1}_B|\mathcal{F}^n_{\tau^n\wedge t}] = \mathbb{E}[\mathbf{1}_B|\mathcal{F}^{n,\tau^n}_t]$ and $\mathbb{E}[\mathbf{1}_B|\mathcal{F}_t] = \mathbb{E}[\mathbf{1}_B|\mathcal{F}^\tau_t]$ using Proposition 1.2.17 in Karatzas and Shreve [12]. So the convergence (27) can be written on the following way: $\mathbb{E}[\mathbf{1}_B|\mathcal{F}^{n,\tau^n}_\cdot] \xrightarrow{\mathbb{P}} \mathbb{E}[\mathbf{1}_B|\mathcal{F}^\tau_\cdot]$ for the Skorokhod topology. Lemma A.4 is proved. □

Using the same arguments as in the proof of Lemma 3 in [8], we have the following characterization of convergence of σ-fields:

Lemma A.5. *Let Y be a càdlàg process, $\mathcal{A} = \sigma(\{Y_t, t \geqslant 0\})$ and $(\mathcal{A}^n)$ be a sequence of σ-fields. The following assumptions are equivalent:*
i) $\mathcal{A}^n \to \mathcal{A}$,
ii) $\mathbb{E}[f(Y_{t_1}, \ldots, Y_{t_k})|\mathcal{A}^n] \xrightarrow{\mathbb{P}} f(Y_{t_1}, \ldots, Y_{t_k})$ *for every bounded continuous function $f : \mathbb{R}^k \to \mathbb{R}$ and $t_1, \ldots, t_k$ continuity points of Y.*

Then, with the characterization of Proposition A.1, we can show a link between convergence of stopped processes and convergence of stopped filtrations:

Theorem A.6. *Let (X^n) and X be càdlàg processes, $(\mathcal{F}^n)$ and $\mathcal{F}$ their natural filtrations. Let (τ^n) be a sequence of $(\mathcal{F}^n)$-stopping times that converges in probability to a $\mathcal{F}$-stopping time τ. We suppose that $X^{n,\tau^n} \xrightarrow{\mathbb{P}} X^\tau$ for the Skorokhod topology and that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Then $\mathcal{F}^{n,\tau^n} \xrightarrow{w} \mathcal{F}^\tau$.*

Proof. As τ and τ^n are respectively $\mathcal{F}$ and $\mathcal{F}^n$-stopping times, according to Proposition A.1, we have the equalities $\mathcal{F}_\tau = \sigma(\{X_{\tau\wedge s}, s \geqslant 0\})$ and $\mathcal{F}^n_{\tau^n} = \sigma(\{X^n_{\tau^n\wedge s}, s \geqslant 0\})$.
Let $t_1, \ldots t_k$ be points of continuity of X^τ and $f : \mathbb{R}^k \to \mathbb{R}$ be a bounded continuous function.
As $X^{n,\tau^n} \xrightarrow{\mathbb{P}} X^\tau$, we have: $(X^n_{t_1\wedge\tau^n}, \ldots, X^n_{t_k\wedge\tau^n}) \xrightarrow{\mathbb{P}} (X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})$. As f is bounded and continuous, we have:

$$f(X^n_{t_1\wedge\tau^n}, \ldots, X^n_{t_k\wedge\tau^n}) \xrightarrow{L^1} f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau}). \tag{28}$$

Finally,

$$\begin{aligned}
&\mathbb{P}[|\mathbb{E}[f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})|\mathcal{F}^n_{\tau^n}] - f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})| \geqslant \eta] \\
&\leqslant \mathbb{P}[|\mathbb{E}[f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})|\mathcal{F}^n_{\tau^n}] - \mathbb{E}[f(X^n_{t_1\wedge\tau^n}, \ldots, X^n_{t_k\wedge\tau^n})|\mathcal{F}^n_{\tau^n}]| \geqslant \eta/2] \\
&\quad + \mathbb{P}[|\mathbb{E}[f(X^n_{t_1\wedge\tau^n}, \ldots, X^n_{t_k\wedge\tau^n})|\mathcal{F}^n_{\tau^n}] - f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})| \geqslant \eta/2] \\
&\leqslant \frac{4}{\eta}\mathbb{E}[|f(X^n_{t_1\wedge\tau^n}, \ldots, X^n_{t_k\wedge\tau^n}) - f(X_{t_1\wedge\tau}, \ldots, X_{t_k\wedge\tau})|] \\
&\to 0 \text{ according to (28).}
\end{aligned}$$

Then, $\mathcal{F}^n_{\tau^n} \to \mathcal{F}_\tau$, according to Lemma A.5 and, using Lemma A.4, $\mathcal{F}^{n,\tau^n} \xrightarrow{w} \mathcal{F}^\tau$.
Theorem A.6 is proved □

Let us show a corollary when the limit is a continuous process.

Corollary A.7. *Let (X^n) be a sequence of càdlàg processes and X a continuous process, $(\mathcal{F}^n)$ and $\mathcal{F}$ the associated filtrations. Let (τ^n) be a sequence of $\mathcal{F}^n$-stopping times that converges in probability to a $\mathcal{F}$-stopping time τ. We suppose that $X^n \xrightarrow{\mathbb{P}} X$ for the Skorokhod topology and that $\mathcal{F}^n \xrightarrow{w} \mathcal{F}$. Then we have the convergence of the stopped filtrations $\mathcal{F}^{n,\tau^n} \xrightarrow{w} \mathcal{F}^\tau$.*

Proof. According to Theorem A.6, we just have to prove that $X^{n,\tau^n} \xrightarrow{\mathbb{P}} X^\tau$.
By definition of the Skorokhod topology, we have to prove that $\forall L \in \mathbb{N}$, $\sup_{t\in[0,L]} |X^{n,\tau^n}_t - X^\tau_t| \xrightarrow{\mathbb{P}} 0$.
Let us fix $L \in \mathbb{N}$ and $\eta > 0$. We have:

$$\mathbb{P}\left[\sup_{t\in[0,L]} |X^{n,\tau^n}_t - X^\tau_t| \geqslant \eta\right] \leqslant \mathbb{P}\left[\sup_{t\in[0,L]} |X^n_{t\wedge\tau^n} - X_{t\wedge\tau^n}| \geqslant \eta/3\right] + \mathbb{P}\left[\sup_{t\in[0,L]} |X_{t\wedge\tau^n} - X_{t\wedge\tau}|\mathbf{1}_{|\tau^n-\tau|<\alpha} \geqslant \eta/3\right]$$
$$+\mathbb{P}\left[\sup_{t\in[0,L]} |X_{t\wedge\tau^n} - X_{t\wedge\tau}|\mathbf{1}_{|\tau^n-\tau|\geqslant\alpha} \geqslant \eta/3\right] \leqslant \mathbb{P}\left[\sup_{t\in[0,L]} |X^n_t - X_t| \geqslant \eta/3\right] + \varepsilon + \mathbb{P}[|\tau^n - \tau| \geqslant \alpha]$$
$$\xrightarrow[n\to+\infty]{} 0$$

because $X^n \xrightarrow{\mathbb{P}} X$, X is continuous on the compact $[0,L]$ and $\tau^n \xrightarrow{\mathbb{P}} \tau$. So $X^{n,\tau^n} \xrightarrow{\mathbb{P}} X^\tau$. □

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