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CORRIGENDUM TO “STABILITY OF SOLUTIONS OF BSDES WITH RANDOM TERMINAL TIME”

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Abstract. This paper is a corrigendum to paper Toldo, *ESAIM, P&S* **10** (2006) 141–163 where we study the stability of the solutions of Backward Stochastic Differential Equations (BSDE for short) with an almost surely finite random terminal time.

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1. Introduction

The proofs of Theorems 1.4 and 2.3 of my paper [2] are not correct: there is no reason to have the convergence of the sequence $(Y^p, Z^p)_p$ of the Picard approximations when the terminal time is not bounded.

However, using another method, we can get similar results that are given in Theorems 2.1 and 3.1.

2. Stability of BSDEs when the Brownian motion is approximated by a scaled random walk

Let $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz function: there exists $\gamma \in \mathbb{R}^+$ such that, for every $(t, y, z), (t, y', z') \in \mathbb{R}^+ \times \mathbb{R}^2$, we have $|f(t, y, z) - f(t, y', z')| \leqslant \gamma[|y - y'| + |z - z'|]$. We also suppose that f is bounded and that f fills the following property of monotonicity w.r.t. y: there exists $\mu > 0$ such that $\forall (t, y, z), (t, y', z) \in \mathbb{R}^+ \times \mathbb{R}^2, (y - y')(f(t, y, z) - f(t, y', z)) \leqslant -\mu(y - y')^2$. Let W be a Brownian motion, $\mathcal{F}$ its natural filtration, τ a $\mathcal{F}$-stopping time almost surely finite and ξ a bounded $\mathcal{F}_\tau$-mesurable random variable. We consider the following stochastic differential equation:

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(s, Y_s, Z_s)\mathrm{d}s - \int_{t\wedge\tau}^{\tau} Z_s \mathrm{d}W_s,\ t \geqslant 0.$$

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In this section, we approximate the previous equation on the following way. We consider the sequence of scaled random walks $(W^n)_{n\geqslant 1}$ defined by $W_t^n = \frac{1}{\sqrt{n}}\sum_{k=1}^{[nt]}\varepsilon_k^n,\ t\geqslant 0$ where $(\varepsilon_k^n)_{k\in\mathbb{N}^*}$ is a sequence of i.i.d. symmetric Bernoulli variables. Let $\mathcal{F}^n$ be the natural filtrations of W^n, $n\geqslant 1$. We have $\mathcal{F}_t^n = \sigma(\varepsilon_k^n, k\leqslant [nt])$. Let $(\tau^n)_n$ be a sequence of bounded $(\mathcal{F}^n)$-stopping times (for each n, we can find $T_n\in\mathbb{N}$ such that $\tau^n\leqslant T_n$) and (ξ^n) a sequence of $(\mathcal{F}_{\tau^n}^n)$-measurable integrable random variables. We consider the processes Y^n and Z^n that are constant on the intervals $[k/n,(k+1)/n[$ and $]k/n,(k+1)/n]$ respectively and that satisfy the following equation:

$$Y_t^n = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f(s, Y_{(s\wedge\tau^n)-}^n, Z_{s\wedge\tau^n}^n)\mathrm{d}A_s^n - \int_{t\wedge\tau^n}^{\tau^n} Z_{s\wedge\tau^n}^n \mathrm{d}W_s^n \quad \text{where } A_s^n = \frac{[ns]}{n}\cdot$$

We also suppose that $\sup_n \|\xi^n\|_\infty < +\infty$ (it implies that (Y^n) is uniformly bounded).

Instead of Theorem 1.4 of [2], we have the following result of convergence:

Theorem 2.1. *We suppose that all the assumptions given before are filled and that we have the convergences* $\xi^n \xrightarrow{\mathbb{P}} \xi$, $\tau^n \xrightarrow{\mathbb{P}} \tau$ *and* $W^n \xrightarrow{\mathbb{P}} W$. *Then*

$$\sup_{t\in\mathbb{R}^+} \mathrm{e}^{-\mu t}|Y_t^n - Y_t| + \int_0^{+\infty} \mathrm{e}^{-2\mu t}|Z_t^n - Z_t|^2\mathrm{d}t \quad \xrightarrow{\mathbb{P}} \quad 0,$$

$$\forall L,\ \sup_{t\in[0,L]}\left|\int_0^t Z_s^n\mathrm{d}W_s^n - \int_0^t Z_s\mathrm{d}W_s\right| \quad \xrightarrow{\mathbb{P}} \quad 0,$$

$$\sup_{t\in\mathbb{R}^+}\left|\int_0^t \mathrm{e}^{-\mu s}Z_s^n\mathrm{d}W_s^n - \int_0^t \mathrm{e}^{-\mu s}Z_s\mathrm{d}W_s\right| \quad \xrightarrow{\mathbb{P}} \quad 0.$$

As in [2], we obtain a corollary under assumption of convergence in law:

Corollary 2.2. *We suppose that* $\forall n, \forall k, \varepsilon_k^n = \varepsilon_k$, $\xi = g(W)$ *and* $\xi^n = g(W^n)$ *with* g *bounded continuous. We also suppose that we have the convergence* $(W^n,\tau^n) \xrightarrow{\mathcal{L}} (W,\tau)$. *Then* $\left(Y_{.\wedge\tau^n}^n, \int_0^{.\wedge\tau^n} Z_s^n\mathrm{d}W_s^n\right)_n$ *converges in law to* $\left(Y_{.\wedge\tau}, \int_0^{.\wedge\tau} Z_s\mathrm{d}W_s\right)$ *for the Skorokhod topology.*

Here, we don't have the assumptions of uniform integrability for the terminal conditions and the stopping times but we have an assumption of uniform boundedness of the terminal conditions. Concerning the convergences, we have uniform convergence on every compact set and if we want a convergence on $\mathbb{R}^+$, we have to add an exponential factor.

We can't prove Theorem 2.1 as it was explained in [2] because when the stopping times are not bounded but only almost surely finite there is no reason to have the convergence of Picard approximation.

We just give the main steps of the proof of the first convergence of Theorem 2.1 (this is the wrong proof in [2]). We denote this convergence by $(Y^n,Z^n)\to(Y,Z)$. We prove successively the following convergences:

- for every K, $(Y^{n,K},Z^{n,K})\to(Y^K,Z^K)$ when n goes to $+\infty$;
- $(Y^K,Z^K)\to(Y,Z)$ when K goes to $+\infty$;
- $(Y^{n,K},Z^{n,K})\to(Y^n,Z^n)$ uniformly w.r.t. n when K goes to $+\infty$;

where (Y^K, Z^K) and $(Y^{n,K}, Z^{n,K})$ are solutions of:

$$\begin{aligned} Y^K_{t\wedge\tau\wedge K} &= \xi \mathbf{1}_{\tau\leqslant K} + \int_{t\wedge\tau\wedge K}^{\tau\wedge K} f(s, Y^K_s, Z^K_s)\mathrm{d}s - \int_{t\wedge\tau\wedge K}^{\tau\wedge K} Z^K_s \mathrm{d}W_s,\ \forall t \geqslant 0, \\ \text{and } Y^{n,K}_t &= \xi^n \mathbf{1}_{\tau^n\leqslant K} + \int_{t\wedge\tau^n\wedge K}^{\tau^n\wedge K} f(s, Y^{n,K}_{s-}, Z^{n,K}_s)\mathrm{d}A^n_s - \int_{t\wedge\tau^n\wedge K}^{\tau^n\wedge K} Z^{n,K}_s \mathrm{d}W^n_s,\ \forall t \geqslant 0. \end{aligned}$$

The reader can find more details in Chapter 3 Section 2 of [1].

3. Stability of BSDEs when the Brownian motion is approximated by a sequence of martingales

Let W be a Brownian motion, $\mathcal{F}$ its natural filtration, τ a $\mathcal{F}$-stopping time almost surely finite and ξ a bounded random variable $\mathcal{F}_\tau$-measurable. We consider the following BSDE:

$$Y_{t\wedge\tau} = \xi + \int_{t\wedge\tau}^{\tau} f(r, Y_r, Z_r)\mathrm{d}r - \int_{t\wedge\tau}^{\tau} Z_r \mathrm{d}W_r,\ t \geqslant 0. \tag{1}$$

We approximate this equation on the following way. Let $(W^n)_n$ be a sequence of càdlàg processes and $(\mathcal{F}^n)_n$ the natural filtrations for these processes. We suppose that (W^n) is a sequence of square integrable $(\mathcal{F}^n)$-martingales which converges in probability to W. We don't suppose that W^n has the predictable representation property. Let $(\tau^n)_n$ be a sequence of $(\mathcal{F}^n)$-stopping times that converges almost surely to τ. Then, we consider the following BSDE:

$$Y^n_t = \xi^n + \int_{t\wedge\tau^n}^{\tau^n} f^n(r, Y^n_{r-}, Z^n_r)\mathrm{d}\langle W^n\rangle_r - \int_{t\wedge\tau^n}^{\tau^n} Z^n_r \mathrm{d}W^n_r - (N^n_{\tau^n} - N^n_{t\wedge\tau^n}), t \geqslant 0 \tag{2}$$

where $(\xi^n)_n$ is a sequence of random variables $(\mathcal{F}^n_{\tau^n})$-measurable, (N^n) is a sequence of $(\mathcal{F}^n)$ martingales orthogonal to (W^{n,τ^n}).

We put the following assumptions on the martingales and on the terminal conditions:

(H1) (i) $\forall L$, $W^n \xrightarrow{\mathcal{S}^2_L} W$,
(ii) $| < W^n >_t - t| \leqslant a_n$ where $(a_n) \downarrow 0$.

(H2) (i) $\xi^n \xrightarrow{L^2} \xi$,
(ii) $\|\xi\|_\infty + \sup_n \mathbb{E}[|\xi^n|] < \infty$.

Let us put some assumptions on the generator f:

(Hf) (i) f is γ-Lipschitz in y and z,
(ii) f is monotone in y in the following way with constant $\mu > 0$,
(iii) f is bounded,
(iv) for every (y, z), $\{f(t, y, z)\}_t$ is progressively measurable w.r.t. $\mathcal{F}$.

Let us put some assumptions on the generators (f^n):

(Hfn) (i) for every n, f^n is γ-Lipschitz in y and z,
$(ii$ for every n, f^n is monotone w.r.t. y with constant μ,
(iii) $\sup_n \|f^n\| < \infty$,
(iv) for every (y, z), $\{f^n(t, y, z)\}_t$ is progressively measurable w.r.t. $(\mathcal{F}^n)_n$,
(v) $\forall (y, z)$, $\{f^n(t, y, z)\}_t$ has càdlàg trajectories and $f^n(., y, z) \xrightarrow{\mathcal{S}^2_L} f(., y, z)$, $\forall L$.

The only difference with [2] is the assumption **(H1)** that has been replaced by a weaker one.

Instead of Theorem 2.3 of [2], we have the following result of convergence:

Theorem 3.1. *We suppose that all the assumptions given before are filled. Then,*

$$\mathbb{E}\left[\sup_{t\in\mathbb{R}^+} \mathrm{e}^{-2\mu t}|Y_t^n - Y_t|^2 + \sup_{t\in\mathbb{R}^+} \mathrm{e}^{-2\mu t}|N_t^n|^2\right] \xrightarrow[n\to+\infty]{} 0,$$

$$\forall L,\ \mathbb{E}\left[\sup_{t\in[0,L]}\left|\int_0^t Z_s^n \mathrm{d}W_s^n - \int_0^t Z_s \mathrm{d}W_s\right|\right] \xrightarrow[n\to+\infty]{} 0.$$

Here again, we can't argue as in [2] to prove the theorem because there is no reason to have the convergence of Picard approximations. To prove the theorem, we argue using the same steps as in the proof of Theorem 2.1.

The reader can find more details in Chapter 3 Section 3 of [1].

References

[1] S. Toldo, Convergence de filtrations ; application à la discrétisation de processus et à la stabilité de temps d'arrêt. Ph.D. Thesis, Université de Rennes 1 (Novembre 2005).

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