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FROM L. EULER TO D. KÖNIG *

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Abstract. Starting from the famous Königsberg bridge problem which Euler described in 1736, we intend to show that some results obtained 180 years later by König are very close to Euler's discoveries.

Keywords. Eulerian graph, edge coloring, parity, König theorem.

Mathematics Subject Classification. 05C15.

1. Introduction

After the celebration of the 300th anniversary of Euler's birth (1707–1783) it seems appropriate to recall a famous contribution of this scholar and to show how it contained in an implicit form the essence of results to be discovered close to 180 years later by another famous graph theorist: König (1884–1944)(see Fig. 1). We celebrated recently the 123rd anniversary of his birth.

To formulate the concepts needed we will use the graph theoretical definitions of Berge [1]. For more information on the bridges of Königsberg, the reader is referred to [5].

We insist that the purpose of this note is not to give a new algorithm for coloring the edges of a bipartite multigraph but to show the close connection between two famous results in elementary graph theory.

This connection is implicit in some papers which indeed use Eulerian decompositions to derive polynomial time algorithms for edge coloring bipartite multigraphs (see for instance [4,8]).

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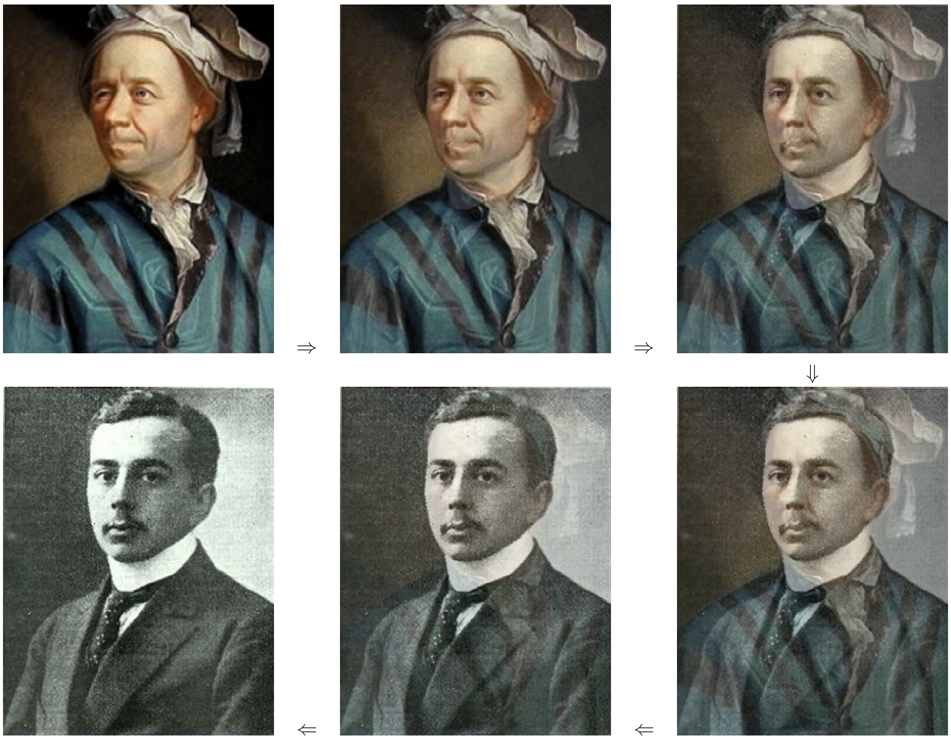

FIGURE 1. From Euler to König.

2. Eulerian partitions

In a multigraph $G = (V, E)$ a chain (or a cycle) is *Eulerian* if it uses every edge of E exactly once.

We give below the so called theorem of Euler; in fact it is well known that Euler proved only the necessity of the condition and the final proof of sufficiency (giving an algorithm for constructing a Eulerian cycle (or chain) was given by Hierholzer [6]. The result is nevertheless often attributed to Euler.

Theorem 2.1 (Euler's theorem [3])**.** *A connected graph has a Eulerian cycle if and only if all degrees are even.*

Such a connected graph containing vertices of odd degree is usually called Eulerian.

While Euler considered partitions of the edge set of a graph into a minimum number of chains and cycles (possibly just one), more than a century later the Hungarian mathematician Dénes König considered partitions of the edge set of a bipartite multigraph $G = (V, E)$ into a minimum number of matchings (sets of mutually non adjacent edges). He showed in fact that this minimum number of matchings (now called the **chromatic index**) is equal to the maximum degree of the vertices in the bipartite multigraph G:

Theorem 2.2 (König's theorem [7]). *The chromatic index of a bipartite multigraph is equal to the maximum degree of its vertices.*

We shall now show how by means of a simple parity argument König's theorem can be derived from Euler's theorem.

We shall need a preliminary property; if $M \subseteq E$ is a set of edges in a multigraph $G = (V, E), m(v)$ will denote the number of edges in M which are incident to vertex v.

Lemma 2.1 [2]. *A connected multigraph $G = (V, E)$ has a partition M_1, M_2 of its edge set satisfying $|m_1(v) - m_2(v)| \leq 1$ for every vertex v if and only if G is not a Eulerian graph with $|E|$ odd and all degrees even.*

Proof. Let $G = (V, E)$ be a connected multigraph. If every vertex of G has even degree and $|E|$ is even, then alternately coloring the edges of a Eulerian cycle with colors 1 and 2 results in a partition (M_1, M_2) satisfying $m_1(v) = m_2(v)$ for every vertex v.

If G has vertices of odd degree, let G' be the Eulerian graph obtained from G by adding a new vertex v_o and joining it to every such vertex. Coloring the edges of G' as above yields a partition (M'_1, M'_2) satisfying $m'_1(v) = m'_2(v)$ for every vertex if $|E'|$ is even. If $|E'|$ is odd we start numbering the edges from any edge incident to the new vertex v_o and we will have $m'_1(v) = m'_2(v)$ for every vertex $v \neq v_o$ and $m'_1(v_o) - m'_2(v_o) = 2$. In any case the partition (M_1, M_2) of E induced by (M'_1, M'_2) satisfies $m_1(v) = m_2(v)$ for every even vertex and $|m_1(v) - m_2(v)| = 1$ for every odd vertex of G.

Conversely if G is a connected graph with all degrees even we must have $m_1(v) = m_2(v)$ for each vertex v, which implies $|M_1| = |M_2|$. This is clearly impossible when $|E|$ is odd. □

So numbering the edges in a Eulerian cycle and separating them according to their parity will always give the required bicoloring in a bipartite graph. This is the main role of Eulerian cycles in the following proof.

3. Using Eulerian partitions to derive the König's theorem

We now present a proof of König's theorem based on lemma 2.3 and hence on the use of Eulerian cycles. Consider a bipartite multigraph $G = (V, E)$ with maximum degree $\Delta(G) \geq 2$ Let $\mathcal{M} = (M_1, M_2, \ldots, M_{\Delta(G)})$ be a partition of E into subsets $M_1, M_2, \ldots, M_{\Delta(G)}$ called color classes such that the following quantity $\widehat{e}(\mathcal{M})$ is minimized.

We define $\widehat{e}(\mathcal{M}) = \max_{v \in V}\ e(v)$ where $e(v) = \max_{p,q}|m_p(v) - m_q(v)| \geq 0$. If $\widehat{e}(\mathcal{M}) \leq 1$ it follows that for each vertex v and for any two colors we have $|m_p(v) - m_q(v)| \leq 1$; since we have $\Delta(G)$ colors in the partition we must necessarily have $m_p(v) \leq 1$ for each vertex v and each color p. So the sets $M_1, \ldots, M_{\Delta(G)}$ are matchings and we are done.

So assume there is a vertex v_o and two colors i, j such that $\widehat{e}(\mathcal{M}) = e(v_o) = |m_i(v_o) - m_j(v_o)| \geq 2$.

Consider the connected component $C_{ij}(v_o)$ of $M_i \bigcup M_j$ containing v_o; since G is bipartite, $C_{ij}(v_o)$ is not a connected graph with all degrees even and $|E(C_{ij}(v_o))|$ odd.

We can apply the above lemma to get (by recoloring of $C_{ij}(v_o)$) a new partition M_i', M_j' of $M_i \bigcup M_j$ (keeping the same colors i, j outside of $C_{ij}(v_o)$); we have $|m_i'(v) - m_j'(v)| \leq 1$ for every vertex v in $C_{ij}(v_o)$ and $|m_i'(v) - m_j'(v)| = |m_i(v) - m_j(v)|$ for all remaining vertices v. For v_o we have $|m_i'(v_o) - m_j'(v_o)| \leq 1 < |m_i(v_o) - m_j(v_o)|$; we set $M_k' = M_k$ for all colors $k \neq i, j$. So for every vertex v we have $e'(v) = \max_{p,q} |m_p'(v) - m_q'(v)| \leq \max_{p,q} |m_p(v) - m_q(v)|$. Furthermore for vertex v_o, the number of pairs p, q of colors for which $|m_p'(v_o) - m_q'(v_o)| \leq e(v_o) - 1$ has increased by at least one (since for the pair i, j, we have $|m_i'(v_o) - m_j'(v_o)| \leq 1 \leq e(v_o) - 1$. We can repeat this until we get a partition $\mathcal{M}''$ with $e''(v_o) < e(v_o)$. Considering every vertex w with $e(w) = \widehat{e}(\mathcal{M})$ we can apply the same procedure to obtain finally a partition $\overline{\mathcal{M}} = \overline{M}_1, \overline{M}_2, \ldots, \overline{M}_\Delta(G)$ with $\widehat{e}(\overline{\mathcal{M}}) < \widehat{e}(\mathcal{M})$ which is a contradiction. So this case is not possible.

4. Final remarks

We remind the reader that the above proof technique is not meant to provide a practical coloring algorithm; the reader will find in [8] an efficient procedure with complexity $O(\Delta(G).|E(G)|)$.

The above proof based on the use of Eulerian cycles combined with a simple parity argument shows in fact that for any number $k \geq 2$ of colors a bipartite multigraph G has an equitable edge $k-$coloring $(M_1, \ldots, M_k)$, *i.e.*, satisfying for each vertex x and any two colors i, j

$|m_i(x) - m_j(x)| \leq 1$ (see [2]).

The same technique (numbering of the edges in the Eulerian cycles constructed in the above proof) shows also that such partitions $(M_1, \ldots, M_k)$ can be found with the additional property that

$-1 \leq |M_i| - |M_j| \leq +1$ for all pairs i, j of colors.

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References

[1] C. Berge, Graphes. *Gauthier-Villars*, Paris (1983).

[2] D. de Werra, Equitable colorations of graphs. *Revue Française d'Informatique et de Recherche Opérationnelle* **R-3** (1971) 3–8.

[3] L. Euler, Solutio problematis ad geometriam situs pertinentis, *Commentarii Academiae Scientiarum Imperialis Petropolitanae* **8** (1736) 128–140. Reprinted in: Leonhardi Euleri – Opera Omnia – Series Prima – Opera Mathematica – Commentationes Algebraicae, L.G. du Pasquier Ed., Teubner, Leipzig (1923) 1–10.

[4] H.N. Gabow, Using Euler partitions to edge color bipartite multigraphs. *Int. J. Parallel Prog.* **5** (1976) 345–355.

[5] I. Gribkovskai, Ø. Halskan and G. Laporte, The bridges of Königsberg – a historical perspective. *Networks* **49** (2007) 199–203.

[6] C. Hierholzer, Über die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren. *Math. Ann.* **6** (1873) 30–32.

[7] D. König, Graphok és alkalmazásuk a determinánsok és a halmazok elméletére (Hungarian). *Mathematikai és Természettudományi Értesitö* **34** (1916) 104–119.

[8] A. Schrijver, Bipartite edge coloring in $0(\delta m)$time. *SIAM J. Comput.* **28** (1998) 841–846.