RAIRO-Oper. Res. 44 (2010) 185–205
DOI: 10.1051/ro/2010014

RAIRO Operations Research
www.rairo-ro.org

KERNEL-FUNCTION BASED PRIMAL-DUAL ALGORITHMS FOR $P_*(\kappa)$ LINEAR COMPLEMENTARITY PROBLEMS

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Abstract. Recently, [Y.Q. Bai, M. El Ghami and C. Roos, SIAM J. Opt. **15** (2004) 101–128] investigated a new class of kernel functions which differs from the class of self-regular kernel functions. The class is defined by some simple conditions on the growth and the barrier behavior of the kernel function. In this paper we generalize the analysis presented in the above paper for $P_*(\kappa)$ Linear Complementarity Problems (LCPs). The analysis for LCPs deviates significantly from the analysis for linear optimization. Several new tools and techniques are derived in this paper.

Keywords. Interior-point, central paths, Kernel functions, primal-dual method, large update, small update, linear complementarity problem.

Mathematics Subject Classification. 65K05, 90C33.

1. Introduction

In this paper we consider the following linear complementarity problem:

$$\begin{aligned} s &= Mx + q, \\ xs &= 0, \\ x, s &\geq 0, \end{aligned} \tag{1.1}$$

where $M \in \mathbf{R}^{n\times n}$ is a $P_*(\kappa)$ matrix and q, x, s are vectors of $\mathbf{R}^n$, and xs denotes the componentwise product (Hadamard product) of vectors x and s. Linear complementarity problems have many applications in mathematical programming and

Received February 23 , 2009. Accepted April 15, 2010.

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equilibrium problems. Indeed, it is known that by exploiting the first-order optimality conditions of the optimization problem, any differentiable convex quadratic program can be formulated into a monotone linear complementarity problem, *i.e.* $P_*(0)$ LCP, and *vice versa* [15]. Variational inequality problems are widely used in the study of equilibrium in economics, transportation planning, and game theory, and have a close connection to the $LCPs$. The reader can refer to Section 5.9 in [5] for the basic theory, algorithms, and applications.

The primal-dual IPM for linear optimization (LO) problems was first introduced in [8,11] and extended to various class of problems, *e.g.*; [3,13]. Kojima *et al.* [8] and Monteiro *et al.* [11] first proved the polynomial computational complexity of the algorithm for LO problem independently, and since then many other algorithms have been developed based on the primal-dual strategy. Kojima *et al.* [9] proved the existence of the central path for any $P_*(\kappa)$ LCP, generalized the primal-dual interior-point algorithm in [8] to $P_*(\kappa)$ LCP and proved the same complexity results. Miao [10] extended the Mizuno-Todd-Ye predictor-corrector method to $P_*(\kappa)$ $LCPs$. His algorithm uses the l_2-neighborhood of the central path and has $O\left((1+\kappa)\sqrt{n}L\right)$ iteration complexity. Recently, Illés and Nagy [7] give a version of the Mizuno-Todd-Ye predictor-corrector interior point algorithm for the $P_*(\kappa)$ LCP and show that the complexity of the algorithm is $O\left((1+\kappa)^{\frac{3}{2}}\sqrt{n}L\right)$. They choose τ and τ' neighborhood parameters in such a way that at each iteration a predictor step is followed by one corrector step. For larger value of κ the values of τ and τ' decrease fast, therefore the constant in the complexity results is increasing.

Most of the polynomial-time interior point algorithms for LO are based on the use of the logarithmic barrier function [8,14]. Peng *et al.* [13] introduced self-regular barrier functions for primal-dual interior-point methods ($IPMs$) for LO, semidefinite optimization (SDO), second order cone optimization (SOCO) and also extended to $P_*(\kappa)$ $LCPs$ and proved that the complexity for large-update primal-dual $IPMs$ for $P_*(\kappa)$ $LCPs$ is the same as the one obtained for LO. Recently in [2] the authors proposed a new primal-dual IPM for LO based on a new class of kernel functions which are not logarithmic and not necessarily self-regular barrier functions.

In this paper we propose a new large-update primal-dual IPM which generalizes the results obtained in [2] to $P_*(\kappa)$ $LCPs$. We use a new search direction based on kernel functions which are neither logarithmic nor self-regular barrier. The new analysis which is derived in this paper is different from the one used in early papers [7,9,10,13]. Furthermore, our analysis provides a simpler way to analyze primal dual $IPMs$.

We use the following notational conventions. Throughout the paper, $\|\cdot\|$ denotes the 2-norm of a vector. The nonnegative orthant and positive orthant are denoted as $\mathbf{R}_+^n$ and $\mathbf{R}_{++}^n$, respectively. If $z \in \mathbf{R}_+^n$ and $f : \mathbf{R}_+ \to \mathbf{R}_+$, then $f(z)$ denotes the vector in $\mathbf{R}_+^n$ whose i-th component is $f(z_i)$, with $1 \leq i \leq n$. Finally, for $x \in \mathbf{R}^n$, $X = \operatorname{diag}(x)$ is the diagonal matrix from vector x, and $J = \{1, 2, ..., n\}$ is the index set.

This paper is organized as follows. In Section 2 we recall basic concepts and the notion of the central path. In Section 3 we review known results relevant for the development of the analysis. Section 4 contains new results to compute the feasible step size and the study of the amount of decrease of the proximity function during an inner iteration. Section 5 combiners the results from Section 3 and the derived results in Section 4 to show the bound for the total number of iterations of the algorithm. Finally, concluding remarks are given in Section 6.

2. Preliminaries

In this section we introduce the definition of $P_*(\kappa)$ matrix and it's properties [9].

Definition 2.1. Let Y be an open convex subset of $\mathbf{R}^n$ and $\kappa \geq 0$. A matrix $M \in \mathbf{R}^{n\times n}$ is called a $P_*(\kappa)$-matrix on Y if and only if

$$(1+4\kappa) \sum_{i\in J_+(x)} x_i\,(Mx)_i + \sum_{i\in J_-(x)} x_i\,(Mx)_i \geq 0,$$

for all $x \in Y$, where

$$J_+(x) = \{i \in J : x_i\,(Mx)_i \geq 0\} \quad \text{and} \quad J_-(x) = \{i \in J : x_i\,(Mx)_i < 0\}.$$

Further, M is called a P_*-matrix if it is a $P_*(\kappa)$-matrix for some $\kappa \geq 0$.

Note that the class of P_*-matrices is the union of all $P_*(\kappa)$-matrices for $\kappa \geq 0$, and contains the class of positive semi-definite matrices, *i.e.* symmetric matrices M satisfying $\sum_{i\in J} x_i(Mx)_i \geq 0$ for all $x \in \mathbf{R}^n$, by choosing $\kappa = 0$. The class of P_* matrices also contains matrices with all positive principal minors. In the following we recall some results which are essential in our analysis.

Proposition 2.1 (Lemma 4.1 in [9])**.** If $M \in \mathbf{R}^{n\times n}$ is a P_* matrix, then

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices X, $S \in \mathbf{R}^{n\times n}$.

We use the following corollary of Proposition 2.1 to prove that the modified Newton system has a unique solution.

Corollary 2.1. *Let $M \in \mathbf{R}^{n\times n}$ be a P_* matrix and $x,\ s \in \mathbf{R}^n_{++}$. Then for all $a \in \mathbf{R}^n$ the system*

$$\begin{aligned} -M\triangle x + \triangle s &= 0, \\ S\triangle x + X\triangle s &= a, \end{aligned}$$

has a unique solution $(\triangle x, \triangle s)$.

The basic idea of primal-dual interior-point methods is to replace the second equation in (1.1) by the nonlinear equation $xs = \mu e$, where e is the all-one vector, and $\mu > 0$. Thus we have the following parameterized system:

$$\begin{aligned} s &= Mx + q, \\ xs &= \mu e, \\ x &\geq 0, \quad s \geq 0, \end{aligned} \tag{2.1}$$

where $\mu > 0$. We assume that there exists strictly positive x and s that satisfy (1.1).

Since M is a $P_*(\kappa)$ matrix and (1.1) is strictly feasible, then the parameterized system (2.1) has a unique solution $(x(\mu), s(\mu))$ for each $\mu > 0$. $(x(\mu), s(\mu))$ is called μ-center of (2.1), the set of μ-centers ($\mu > 0$) defines a homotopy path, which is called the *central path* of (2.1). If $\mu \to 0$ the limit of the central path exists. This limit satisfies the complementarity condition, and belongs to the solution set of (1.1) [9].

Let (x, s) be a strictly feasible point and $\mu > 0$. We define the vector

$$v := \sqrt{\frac{xs}{\mu}}. \tag{2.2}$$

Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$.

Let $\Psi(v)$ be a smooth, strictly convex function defined for all $v > 0$, which is minimal at $v = e$, with $\Psi(e) = 0$. Following [1,2,4,13] we define search directions Δx, Δs by

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ s\Delta x + x\Delta s &= -\mu v \nabla\Psi(v). \end{aligned} \tag{2.3}$$

Since M is a P_* matrix, the system (2.3) uniquely defines $(\Delta x, \Delta s)$ for any $x > 0$ and $s > 0$. Note that $\Delta x = 0$, $\Delta s = 0$, if and only if $v = e$, because the right-hand sides in (2.3) vanish if and only if $\nabla\Psi(v) = 0$, and this occurs if and only if $v = e$.

Let (x, s) be a strictly feasible point. We define the vector p by

$$p := \sqrt{\frac{x}{s}}. \tag{2.4}$$

Generic Primal-Dual Algorithm for LCP

```
Input:
  a proximity function Ψ(v);
  a threshold parameter τ ≥ 1;
  an accuracy parameter ε > 0;
  a barrier update parameter θ, 0 < θ < 1;
begin
  x := x^0; s := s^0; μ := μ^0;
  while nμ ≥ ε do
  begin
    μ := (1 − θ)μ;
    while Ψ(v) > τ do
    begin
      Solve (Δx, Δs) from (2.3)
      x := x + αΔx;
      s := s + αΔs;
      v := sqrt(xs/μ);
    end
  end
end
```

FIGURE 1. The generic primal-dual interior-point algorithm for LCP.

Introducing the following notations

$$\bar{M} := PMP \text{ and } P := \operatorname{diag}(p),\ V := \operatorname{diag}(v) \text{ where } v = \sqrt{\frac{xs}{\mu}}, \tag{2.5}$$

and

$$d_x := \frac{v\Delta x}{x}, \qquad d_s := \frac{v\Delta s}{s}, \tag{2.6}$$

system (2.3) can be reformulated as

$$\begin{aligned} -\bar{M}d_x + d_s &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{2.7}$$

From the solution d_x and d_s, the vectors Δx and Δs can be computed from (2.6).

Note that the vectors d_x and d_s are not orthogonal. So our analysis in this paper will deviate significantly from the analysis used for *LO* in [2].

The algorithm considered in this paper is described in Figure 1. The inner while loop in the algorithm is called *inner iteration* and the outer while loop *outer* *iteration*. So each outer iteration consists of an update of the barrier parameter and a sequence of one or more inner iterations. We assume that (1.1) is strictly feasible, and the starting point $\left(x^0, s^0\right)$ is strictly feasible for (1.1). Choose τ and $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}}$ initial strictly feasible point such that $\Psi\left(v^0\right) \leq \tau$ where τ is threshold value in Figure 1. We then decrease μ to $\mu := (1-\theta)\mu$, for some $\theta \in (0,1)$. In general this will increase the value of $\Psi(v)$ above τ. To get this value smaller again, and getting closer to the current μ-center, we solve the scaled search directions from (2.7), and unscaled these directions by using (2.3). By choosing an appropriate step size α, we move along the search direction, and construct a new pair (x_+, s_+) with

$$x_+ = x + \alpha \triangle x \quad s_+ = s + \alpha \triangle s. \tag{2.8}$$

If necessary, we repeat the procedure until we find iterates such that $\Psi(v)$ no longer exceed the threshold value τ, which means that the iterates are in a small enough neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1-\theta$ and we apply the same procedure targeting at the new μ-centers. This process is repeated until μ is small enough, *i.e.* until $n\mu \leq \epsilon$. At this stage we have found an ϵ-solution of (1.1). Just as in the *LO* case, the parameters τ, θ, and the step size α should be chosen in such a way that the algorithm is 'optimized' in the sense that the number of iterations required by the algorithm is as small as possible. Obviously, the resulting iteration bound will depend on the kernel function underlying the algorithm, and our main task becomes to find a kernel function that minimizes the iteration bound.

3. Properties of Kernel functions

This section is a review of parts of [2] needed in the analysis. We call $\psi : (0,\infty) \to [0,\infty)$ a kernel function if ψ is twice differentiable and the following conditions are satisfied.

(*i*) $\psi'(1) = \psi(1) = 0$;
(*ii*) $\psi''(t) > 0, \quad \text{for all} \quad t > 0.$

In this paper we restrict our selves to functions that are coercive, *i.e.*,

(*iii*) $\lim_{t\downarrow 0} \psi(t) = \lim_{t\to\infty} \psi(t) = \infty$.

Clearly, (*i*) and (*ii*) imply that $\psi(t)$ is a nonnegative strictly convex function such that $\psi(1) = 0$, and $\psi(t)$ is completely determined by its second derivative:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, \mathrm{d}\zeta \mathrm{d}\xi. \tag{3.1}$$

Moreover, by (*iii*), $\psi(t)$ has the so called barrier property. In [2] the additional conditions are imposed on the Kernel function, namely $\psi \in C^3$ and

$$\begin{aligned} t\psi''(t) + \psi'(t) &> 0, \quad t < 1, && \text{(2-a)} \\ \psi'''(t) &< 0, \quad t > 0, && \text{(2-b)} \\ 2\psi''(t)^2 - \psi'(t)\psi'''(t) &> 0, \quad t < 1, && \text{(2-c)} \\ \psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t) &> 0, \quad t > 1, \quad \beta > 1. && \text{(2-d)} \end{aligned}$$

The first property (2-a) is related to Definition 1 and Lemma 2.1.2 in [13]. This property is equivalent to convexity of the composed function $z \mapsto \psi(e^z)$ and this holds if and only if $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for any $t_1, t_2 > 0$. Following [1], we therefore say that ψ is exponentially convex, or shortly, e-convex, whenever $t > 0$.

We denote by $\varrho : [0, \infty) \to [1, \infty)$ and $\rho : [0, \infty) \to (0, 1]$ the inverse functions of $\psi(t)$ for $t \geq 1$, and $-\frac{1}{2}\psi'(t)$ for $t \leq 1$, respectively. In other words

$$s = \psi(t) \quad \Leftrightarrow \quad t = \varrho(s), \quad t \geq 1, \tag{3.3}$$

$$s = -\tfrac{1}{2}\psi'(t) \quad \Leftrightarrow \quad t = \rho(s), \quad t \leq 1. \tag{3.4}$$

We recall from [2] two theorems that are needed later in the analysis of the algorithm presented in this paper.

Theorem 3.1 (Thm. 3.2 in [2]). *For any positive vector v and any $\beta > 1$, we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

It follows from this theorem that if $\Psi(v) \leq \tau$ and $\beta = \frac{1}{\sqrt{1-\theta}}$ then

$$L_\psi(n, \theta, \tau) := n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \tag{3.5}$$

is an upper bound for $\Psi(\frac{v}{\sqrt{1-\theta}})$, the value of $\Psi(v)$ after the update of μ.

The following theorem gives a lower bound of the norm-based proximity measure $\delta(v)$, defined by

$$\delta(v) := \tfrac{1}{2}\|\psi'(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n} \psi'(v_i)^2} = \frac{1}{2}\|d_x + d_s\|, \tag{3.6}$$

in terms of $\Psi(v)$. Since $\Psi(v)$ is strictly convex and attains its minimal value zero at $v = e$, we have

$$\Psi(v) = 0 \quad \Leftrightarrow \quad \delta(v) = 0 \quad \Leftrightarrow \quad v = e.$$

i	Kernel functions ψ_i	ψ_i'	ψ_i''	$\psi_i'''(t)$
1	$\frac{t^2-1}{2} - \log t$	$t - \frac{1}{t}$	$1 + \frac{1}{t^2}$	$-\frac{2}{t^3}$
2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1)$	$t - 1 - \frac{t^{-q}-1}{q}$	$1 + t^{-q-1}$	$-(q+1)\, t^{-q-2}$
3	$\frac{1}{2}\left(t - \frac{1}{t}\right)^2$	$t - \frac{1}{t^3}$	$1 + \frac{3}{t^4}$	$-\frac{12}{t^5}$
4	$\frac{t^2-1}{2} + e^{\frac{1}{t}-1} - 1$	$t - \frac{e^{\frac{1}{t}-1}}{t^2}$	$1 + \frac{1+2t}{t^4}\, e^{\frac{1}{t}-1}$	$-\frac{1+6t+6t^2}{t^6} e^{\frac{1}{t}-1}$
5	$\frac{t^2-1}{2} - \int_1^t e^{\frac{1}{\xi}-1} d\xi$	$t - e^{\frac{1}{t}-1}$	$1 + \frac{e^{\frac{1}{t}-1}}{t^2}$	$-\frac{1+2t}{t^4} e^{\frac{1}{t}-1}$
6	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1},\ \ q > 1$	$t - t^{-q}$	$1 + qt^{-q-1}$	$-q\,(q+1)\, t^{-q-2}$
7	$t - 1 + \frac{t^{1-q}-1}{q-1},\ \ \ q > 1$	$1 - t^{-q}$	qt^{-q-1}	$-q\,(q+1)\, t^{-q-2}$

Table 1. Seven kernel functions and first three derivatives.

Theorem 3.2 (Thm. 4.9 in [2]). *One has*

$$\delta(v) \geq \tfrac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

3.1. Seven kernel function

By way of example we consider in this paper the seven kernel functions studied in [2] , as listed in Table 1. Note that some of these kernel functions depend on a parameter (*e.g.*, $\psi_2(t)$ depends on the parameter $q > 1$), and hence when the parameter is not specified, it represents a whole class of kernel functions.

Note that all kernel functions in Table 1 satisfy the conditions (2-a)...(2-d) [2].

4. Analysis of the algorithm

In this section, we show how to compute a feasible step size α of a Newton step with the decrease of the barrier function. Since d_x and d_s, are not orthogonal the analysis in this paper is different from that of LO case. After a damped step, with step size α, using (2.2) and (2.6) we have

$$x_+ = x + \alpha\Delta x = \frac{x}{v}\left(v + \alpha d_x\right), \quad s_+ = s + \alpha\Delta s = \frac{s}{v}\left(v + \alpha d_s\right).$$

Thus we obtain

$$v_+^2 = \frac{x_+ s_+}{\mu} = \left(v + \alpha d_x\right)\left(v + \alpha d_s\right). \tag{4.1}$$

In the sequel we use the following notation:

$$\nu := \min_{i \in J} v_i, \quad \delta := \delta(v), \quad \sigma_+ := \sum_{i \in J_+} d_{x_i} d_{s_i}, \quad \sigma_- := -\sum_{i \in J_-} d_{x_i} d_{s_i}. \tag{4.2}$$

Since M is a $P_*(\kappa)$ matrix, we have

$$(1+4\kappa)\sum_{i\in J_+}\Delta x_i(M\Delta x)_i+\sum_{i\in J_-}\Delta x_i(M\Delta s)_i\geq 0,$$

where $J_+=\{i\in J:\Delta x_i(M\Delta x)_i\geq 0\}$, $J_-=J-J_+$. Using the first equation in (2.3) we have for $\Delta x\in\mathbf{R}^n$, $M\Delta x=\Delta s$, and

$$(1+4\kappa)\sum_{i\in J_+}\Delta x_i\Delta s_i+\sum_{i\in J_-}\Delta x_i\Delta s_i\geq 0.$$

From (2.6) it follows that $d_xd_s=\frac{v^2\Delta x\Delta s}{xs}=\frac{\Delta x\Delta s}{\mu}$ with $\mu>0$, and

$$(1+4\kappa)\sum_{i\in J_+}d_{x_i}d_{s_i}+\sum_{i\in J_-}d_{x_i}d_{s_i}=(1+4\kappa)\,\sigma_+-\sigma_-\geq 0. \tag{4.3}$$

The next lemma gives an upper bound of σ_+ and σ_-

Lemma 4.1. *One has*

$$\sigma_+\leq\delta^2,\quad \textit{and}\quad \sigma_-\leq(1+4\kappa)\,\delta^2.$$

Proof. By definition of σ_+, σ_- and δ, we have

$$\sigma_+=\sum_{i\in J_+}d_{x_i}d_{s_i}\leq\frac{1}{4}\sum_{i\in J_+}(d_{x_i}+d_{s_i})^2\leq\frac{1}{4}\sum_{i\in J}(d_{x_i}+d_{s_i})^2=\frac{1}{4}\left\|d_{x_i}+d_{s_i}\right\|^2=\delta^2.$$

Since M is a $P_*(\kappa)$ matrix, using (4.3), we get

$$(1+4\kappa)\,\sigma_+-\sigma_-\geq 0.$$

Thus

$$\sigma_-\leq(1+4\kappa)\,\sigma_+\leq(1+4\kappa)\,\delta^2.$$

This proves the lemma. □

The following lemma gives an upper bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 4.2. *One has*

$$\sum_{i=1}^{n}\left(d_{x_i}^2+d_{s_i}^2\right)\leq 4\,(1+2\kappa)\,\delta^2,\quad \|d_x\|\leq 2\sqrt{1+2\kappa}\;\delta,\quad \textit{and}\quad \|d_s\|\leq 2\sqrt{1+2\kappa}\;\delta.$$

Proof. From the definitions (3.6) and (4.2), we have

$$\delta=\frac{1}{2}\left\|d_x+d_s\right\|,\quad\text{and}\quad\sum_{j\in J}d_{x_i}d_{s_i}=\sigma_+-\sigma_-,$$

then

$$2\delta = \|d_x + d_s\| = \sqrt{\sum_{i=1}^{n} (d_{x_i} + d_{s_i})^2} = \sqrt{\sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right) + 2\,(\sigma_+ - \sigma_-)}.$$

Using (4.3), and Lemma 4.1, we get

$$2\delta \geq \sqrt{\sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right) + 2\left(\frac{1}{1+4\kappa}\sigma_- - \sigma_-\right)} = \sqrt{\sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right) - \frac{8\kappa}{1+4\kappa}\sigma_-}.$$

Then, we get

$$4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right).$$

Using again Lemma 4.1, we have

$$4\,(1+2\kappa)\,\delta^2 \geq 4\delta^2 + \frac{8\kappa}{1+4\kappa}\sigma_- \geq \sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right).$$

Thus

$$\|d_x\| \leq \sqrt{\sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right)} \leq 2\sqrt{1+2\kappa}\;\delta.$$

Using the same argument we can prove that

$$\|d_s\| \leq 2\sqrt{1+2\kappa}\;\delta.$$

Thus the lemma follows. □

Our aim is to find an upper bound for

$$f(\alpha) := \Psi\,(v_+) - \Psi\,(v) := \Psi\left(\sqrt{(v+\alpha d_x)\,(v+\alpha d_s)}\right) - \Psi\,(v)\,,$$

where $\Psi : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i). \tag{4.4}$$

To do this, the next four technical lemmas are needed. It is clear that $f(\alpha)$ is not necessarily convex in α. To simplify the analysis we use a convex upper bound for $f(\alpha)$. Such a bound is obtained by using that $\psi(t)$ satisfies the condition (2-a). Hence, $\psi(t)$ is e-convex. This implies

$$\Psi\,(v_+) = \Psi\left(\sqrt{(v+\alpha d_x)\,(v+\alpha d_s)}\right) \leq \tfrac{1}{2}\left[\Psi\,(v+\alpha d_x) + \Psi\,(v+\alpha d_s)\right].$$

Thus we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \tfrac{1}{2}\left[\Psi\left(v + \alpha d_x\right) + \Psi\left(v + \alpha d_s\right)\right] - \Psi\left(v\right)$$

is a convex function of α, since $\Psi(v)$ is convex. Obviously, $f(0) = f_1(0) = 0$. Taking the derivative of $f_1(\alpha)$ to α, we get

$$f_1'(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi'\left(v_i + \alpha d_{xi}\right) d_{xi} + \psi'\left(v_i + \alpha d_{si}\right) d_{si}\right).$$

This gives, using last equation in (2.7) and (3.6),

$$f_1'(0) = \tfrac{1}{2}\nabla\Psi(v)^T\left(d_x + d_s\right) = -\tfrac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{4.5}$$

Differentiating once more, we obtain

$$f_1''(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi''\left(v_i + \alpha d_{xi}\right) d_{xi}^{\,2} + \psi''\left(v_i + \alpha d_{si}\right) d_{si}^{\,2}\right). \tag{4.6}$$

The following lemma gives an upper bound of $f_1(\alpha)$ in terms of δ and $\psi''(t)$.

Lemma 4.3. *One has*

$$f_1''(\alpha) \leq 2\left(1 + 2\kappa\right)\delta^2\,\psi''\left(\nu - 2\alpha\sqrt{1+2\kappa}\;\delta\right).$$

Proof. Using Lemma 4.2 and the definition of ν as given in (4.2),

$$v_i + \alpha d_{x_i} \geq \nu - 2\alpha\sqrt{1+2\kappa}\;\delta, \quad 1 \leq i \leq n, \tag{4.7}$$

$$v_i + \alpha d_{s_i} \geq \nu - 2\alpha\sqrt{1+2\kappa}\;\delta, \quad 1 \leq i \leq n. \tag{4.8}$$

Since ψ'' is monotonically decreasing, using the above inequalities, we get

$$\psi''(v_i + \alpha d_{x_i}) \leq \psi''(\nu - 2\alpha\sqrt{1+2\kappa}\;\delta), \tag{4.9}$$

and

$$\psi''(v_i + \alpha d_{s_i}) \leq \psi''(\nu - 2\alpha\sqrt{1+2\kappa}\;\delta). \tag{4.10}$$

This implies that

$$f_1''(\alpha) \leq \tfrac{1}{2}\,\psi''\left(\nu - 2\alpha\sqrt{1+2\kappa}\;\delta\right)\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right),$$

now using again Lemma 4.2 *i.e.*, $\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right) \leq 4\left(1+2\kappa\right)\delta^2$, then

$$f_1''(\alpha) \leq 2\left(1+2\kappa\right)\delta^2\,\psi''\left(\nu - 2\alpha\sqrt{1+2\kappa}\;\delta\right).$$

This proves the lemma. □

Putting

$$\delta_\kappa := \sqrt{1+2\kappa}\,\delta, \tag{4.11}$$

we have

$$f_1''(\alpha) \le 2\delta_\kappa^2\,\psi''\left(\nu - 2\alpha\delta_\kappa\right), \tag{4.12}$$

Since $f_1(\alpha)$ is convex, we will have $f_1'(\alpha) \le 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal, and *vice versa*. In this respect the next result is important.

Lemma 4.4. *One has $f_1'(\alpha) \le 0$ if α satisfies the inequality*

$$-\psi'\left(\nu - 2\alpha\delta_\kappa\right) + \psi'\left(\nu\right) \le \frac{2\delta_\kappa}{(1+2\kappa)}. \tag{4.13}$$

Proof. We may write, using Lemma 4.3, and also (4.5),

$$\begin{aligned}
f_1'(\alpha) &= f_1'(0) + \int_0^\alpha f_1''(\xi)\,\mathrm{d}\xi \\
&\le -2\delta^2 + 2\delta_\kappa^2 \int_0^\alpha \psi''\left(\nu - 2\xi\delta_\kappa\right)\mathrm{d}\xi \\
&= -2\delta^2 - \delta_\kappa \int_0^\alpha \psi''\left(\nu - 2\xi\delta_\kappa\right)\mathrm{d}\left(\nu - 2\xi\delta_\kappa\right) \\
&= -2\delta^2 - \delta_\kappa\left(\psi'\left(\nu - 2\alpha\delta_\kappa\right) - \psi'\left(\nu\right)\right).
\end{aligned}$$

Hence, $f_1'(\alpha) \le 0$ will certainly hold if α satisfies

$$-\psi'\left(\nu - 2\alpha\delta_\kappa\right) + \psi'\left(\nu\right) \le \frac{2\delta^2}{\delta_\kappa} = \frac{2\delta_\kappa}{(1+2\kappa)},$$

the last equality follows from (4.11), which proves the lemma. □

The next lemma uses the inverse function $\rho : [0,\infty) \to (0,1]$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0,1]$, as defined in (3.4).

Lemma 4.5. *The largest value of the step size α satisfying (4.12) is given by*

$$\bar{\alpha} := \frac{1}{2\delta_\kappa}\left[\rho\left(\delta\right) - \rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa\right)\right]. \tag{4.14}$$

Moreover

$$\bar{\alpha} \ge \frac{1}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa\right)\right)}. \tag{4.15}$$

Proof. We want α such that (4.13) holds, with α as large as possible. Since $\psi''(t)$ is decreasing, the derivative to ν of the expression at the left in (4.13) (*i.e.* $-\psi''(\nu - 2\alpha\delta_\kappa)+\psi''(\nu)$) is negative. Hence, fixing δ_κ, the smaller ν is, the smaller α will be. One has

$$\delta = \tfrac{1}{2}\left\|\nabla\Psi(v)\right\| \geq \tfrac{1}{2}\left|\psi'(\nu)\right| \geq -\tfrac{1}{2}\psi'(\nu).$$

Equality holds if and only if ν is the only coordinate in v that differs from 1, and $\nu \leq 1$ (in which case $\psi'(\nu) \leq 0$). Hence, the worst situation for the step size occurs when ν satisfies

$$-\tfrac{1}{2}\psi'(\nu) = \delta. \tag{4.16}$$

The derivative to α of the expression at the left in (4.13) equals

$$2\delta_\kappa\psi''(\nu - 2\alpha\delta_\kappa) \geq 0,$$

and hence the left-hand side is increasing in α. So the largest possible value of α satisfying (4.13), satisfies

$$-\tfrac{1}{2}\psi'(\nu - 2\alpha\delta_\kappa) = \frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa. \tag{4.17}$$

Due to the definition of ρ, (4.16) and (4.17) can be written as

$$\nu = \rho(\delta), \qquad \nu - 2\alpha\delta_\kappa = \rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa\right).$$

This implies,

$$\alpha = \frac{1}{2\delta_\kappa}\left(\nu - \rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa\right)\right) = \frac{1}{2\delta_\kappa}\left(\rho(\delta) - \rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa\right)\right),$$

proving (4.14).

Using the definition of ρ,

$$-\psi'(\rho(\delta)) = 2\delta.$$

Taking the derivative to δ, we find

$$-\psi''(\rho(\delta))\,\rho'(\delta) = 2,$$

which implies that

$$\rho'(\delta) = -\frac{2}{\psi''(\rho(\delta))} < 0. \tag{4.18}$$

Hence ρ is monotonically decreasing in δ. An immediate consequence of (4.14) and (4.18) is

$$\bar{\alpha} = \frac{1}{2\delta_\kappa}\int_{\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa}^{\delta} \rho'(\sigma)\,\mathrm{d}\sigma = \frac{1}{\delta_\kappa}\int_{\delta}^{\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\delta_\kappa} \frac{\mathrm{d}\sigma}{\psi''(\rho(\sigma))}. \tag{4.19}$$

To obtain a lower bound for $\bar{\alpha}$, we want to replace the argument of the last integral by its minimal value. So we want to know when $\psi''(\rho(\sigma))$ is maximal, for $\sigma \in [\delta, \frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa]$. Due to (2-b), ψ'' is monotonically decreasing. So $\psi''(\rho(\sigma))$ is maximal when $\rho(\sigma)$ is minimal for $\sigma \in [\delta, \frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa]$. Since ρ is monotonically decreasing this occurs when $\sigma = \frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa$. Therefore

$$\bar{\alpha} = \frac{1}{\delta_\kappa}\int_{\delta}^{\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa} \frac{\mathrm{d}\sigma}{\psi''(\rho(\sigma))} \geq \frac{\delta_\kappa}{\delta_\kappa(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa\right)\right)} = \frac{1}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa\right)\right)},$$

which proves the lemma. □

For future use we define

$$\widetilde{\alpha} := \frac{1}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa\right)\right)}, \tag{4.20}$$

as the default step size. By Lemma 4.5 this step $\widetilde{\alpha}$ satisfies (4.13). By (4.15) we have $\bar{\alpha} \geq \tilde{\alpha}$. We recall without proof the following lemma from [12].

Lemma 4.6 (Lem. 3.12 in [12])**.** *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$ then*

$$h(t) \leq \frac{th'(0)}{2}, \quad 0 \leq t \leq t^*.$$

Lemma 4.7. *If the step size α satisfies (4.13) then*

$$f(\alpha) \leq -\alpha\,\delta^2. \tag{4.21}$$

Proof. Let $h(\alpha)$ be defined by

$$h(\alpha) := -2\alpha\delta^2 + \alpha\delta_\kappa\psi'(\nu) - \frac{1}{2}\psi(\nu) + \frac{1}{2}\psi(\nu - 2\alpha\delta_\kappa).$$

Then

$$h(0) = f_1(0) = 0, \quad h'(0) = f_1'(0) = -2\delta^2, \quad h''(\alpha) = 2\delta_\kappa^2\,\psi''(\nu - 2\alpha\delta_\kappa).$$

Due to Lemma 4.3, $f_1''(\alpha) \le h''(\alpha)$. As a consequence, $f_1'(\alpha) \le h'(\alpha)$ and $f_1(\alpha) \le h(\alpha)$. Taking $\alpha \le \bar{\alpha}$, with $\bar{\alpha}$ as defined in Lemma 4.5, we have

$$\begin{aligned} h'(\alpha) &= -2\delta^2 + 2\delta_\kappa^2 \int_0^\alpha \psi''(\nu - 2\xi\delta_\kappa)\, d\xi \\ &= -2\delta^2 - \delta_\kappa \left(\psi'(\nu - 2\alpha\delta_\kappa) - \psi'(\nu)\right) \le 0. \end{aligned}$$

Since $h''(\alpha)$ is increasing in α, using Lemma 4.6, we may write

$$f_1(\alpha) \le h(\alpha) \le \tfrac{1}{2}\alpha h'(0) = -\alpha\delta^2.$$

Since $f(\alpha) \le f_1(\alpha)$, the proof is complete. □

Theorem 4.8. *Let ρ be defined in (3.4) and $\widetilde{\alpha}$ in (4.20). Then*

$$f(\widetilde{\alpha}) \le -\frac{\delta^2}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{1+2\kappa}\,\delta_\kappa\right)\right)} = -\frac{\delta^2}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\,\delta\right)\right)}, \tag{4.22}$$

and the right-hand side expression in (4.22) is monotonically decreasing in δ.

Proof. By combining (4.15) in Lemma 4.5 and results in Lemma 4.7, using also (4.11) we obtain (4.22).

Putting $t = \rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\,\delta\right)$, which implies $t \le 1$, and which is equivalent to $2\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\right)\delta = -\psi'(t)$, t is monotonically decreasing if δ increases. Hence, the right-hand expression in (4.22) is monotonically decreasing in δ if and only if the function

$$g(t) := \frac{(\psi'(t))^2}{4\left(1+\sqrt{1+2\kappa}\right)^2\,\psi''(t)}$$

is monotonically decreasing for $t \le 1$. Note that $g(1) = 0$ and

$$g'(t) = \frac{2\psi'(t)\psi''(t)^2 - \psi'(t)^2\psi'''(t)}{4\left(1+\sqrt{1+2\kappa}\right)^2\,\psi''(t)^2}.$$

Hence, since $\psi'(t) < 0$ for $t < 1$, $g(t)$ is monotonically decreasing for $t \le 1$ if and only if

$$2\psi''(t)^2 - \psi'(t)\psi'''(t) \ge 0, \quad t \le 1.$$

The last inequality is satisfied, due to condition (2-c). Hence the theorem is proved. □

5. Iteration bounds

In this section we derive the complexity bounds for large-update methods and small-update methods. To do this we need to recall the following lemma from [12].

Lemma 5.1 (Prop. 2.2 in [12]). *Let $t_0, t_1, \ldots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \qquad k = 0, 1, \ldots, K-1, \tag{5.1}$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \left\lfloor \frac{t_0^\gamma}{\beta\gamma} \right\rfloor$.

Lemma 5.2. *If K denotes the number of inner iterations, we have*

$$K \leq \frac{\Psi_0^\gamma}{\beta\gamma}.$$

Proof. The definition of K implies $\Psi_{K-1} > \tau$ and $\Psi_K \leq \tau$ and

$$\Psi_{k+1} \leq \Psi_k - \beta \Psi_k^{1-\gamma}, \qquad k = 0, 1, \ldots, K-1.$$

Yet we apply Lemma 5.1, with $t_k = \Psi_k$. This yields the desired inequality. $\square$

Thus the upper bound on the total number of iterations is given by

$$\frac{\Psi_0^\gamma}{\theta\beta\gamma} \log \frac{n}{\epsilon} \leq \frac{1}{\theta\beta\gamma} \left(n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} \right) \right)^\gamma \log \frac{n}{\epsilon}, \tag{5.2}$$

where $\Psi_0 \leq L_\psi(n, \theta, \tau)$ denote the value of $\Psi(v)$ after the $\mu-$update, and $L_\psi(n, \theta, \tau)$ as defined in (3.5).

5.1. Application to the seven kernel functions

It may be clear that we can use the scheme of Figure 2 to analyze the behavior of our algorithm for $P_*(\kappa)$ LCP, as given in Figure 1. We recall from [2] two lemmas without proof that are needed for the analysis.

Lemma 5.3 (Lem. 6.2 in [2]). *When $\psi(t) = \psi_i(t)$ and $1 \leq i \leq 6$, then*

$$\sqrt{1+2s} \leq \varrho(s) \leq 1 + \sqrt{2s},$$

with $\varrho(s)$ is the inverse function of $\psi(t)$ for $t \in [1, \infty)$ obtained by solving t from the equation $s = \psi(t)$, $t \geq 1$, as defined in (3.3).

Lemma 5.4 (Lem. 6.3 in [2]). *Let $1 \leq i \leq 7$. Then one has*

$$\Psi_0 \leq \frac{\psi''(1)}{2} \frac{\left(\sqrt{2\tau} + \theta\sqrt{n}\right)^2}{1-\theta}.$$

Hence, if $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$, then $\Psi_0 = O(\psi''(1))$.

Step 0: Specify a kernel function $\psi(t)$; an update parameter θ, $0 < \theta < 1$; a threshold parameter τ; and an accuracy parameter ϵ.

Step 1: Solve the equation $-\frac{1}{2}\psi'(t) = s$ to get $\rho(s)$, the inverse function of $-\frac{1}{2}\psi'(t), t \in (0,1]$. If the equation is hard to solve, derive a lower bound for $\rho(s)$.

Step 2: Calculate the decrease of $\Psi(v)$ during an inner iteration in terms of δ for the default step size $\tilde{\alpha}$ from

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\;\delta\right)\right)}.$$

Step 3: Solve the equation $\psi(t) = s$ to get $\varrho(s)$, the inverse function of $\psi(t), t \geq 1$. If the equation is hard to solve, derive lower and upper bounds for $\varrho(s)$.

Step 4: Derive a lower bound for δ in terms of $\Psi(v)$ by using

$$\delta(v) \geq \tfrac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

Step 5: By Theorem 4.8, using the results of step 3 and step 4 find a valid inequality of the form

$$f(\tilde{\alpha}) \leq -\beta\Psi(v)^{1-\gamma},$$

for some positive constants β and γ, with $\gamma \in (0,1]$ as small as possible.

Step 6: Calculate the upper bound of Ψ_0 from

$$\Psi_0 \leq L_\psi(n,\theta,\tau) = n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq \frac{n}{2}\psi''(1)\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2.$$

Step 7: Derive an upper bound for the total number of iterations by using that

$$\frac{\Psi_0^\gamma}{\theta\beta\gamma}\log\frac{n}{\epsilon}$$

is an upper bound for this number.

Step 8: Set $\tau = O(n)$ and $\theta = \Theta(1)$ to calculate a complexity bound for large-update methods, and set $\tau = O(1)$ and $\theta = \Theta(\frac{1}{\sqrt{n}})$ to calculate a complexity bound for small-update methods.

FIGURE 2. Scheme for analyzing a kernel-function-based algorithm.

5.2. Example: Analysis of methods based on $\psi_2(t)$

Consider the case where $\psi(t) = \psi_2(t)$

$$\psi(t) = \frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1), \qquad q > 0.$$

Step 1: To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0,1]$ we need to solve t from the equation

$$-t + 1 + \frac{t^{-q}-1}{q} = 2s, \quad t \in (0,1].$$

Using that $\frac{t^{-q}-1}{q} = 2s + t - 1 \leq 2s$, this implies

$$\rho(s) \geq \frac{1}{(1+2qs)^{\frac{1}{q}}}.$$

Step 2: It follows that

$$\begin{aligned} f(\tilde{\alpha}) \leq -\frac{\delta^2}{(1+2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\ \delta\right)\right)} &= -\frac{\delta^2}{(1+2\kappa)\left(1+\frac{1}{\rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\ \delta\right)^{q+1}}\right)} \\ &\leq -\frac{\delta^2}{(1+2\kappa)\left(1+\left(1+2q\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\ \delta\right)\right)^{\frac{q+1}{q}}\right)} \\ &\leq -\frac{\delta^2}{(1+2\kappa)\left(1+(1+4q\delta)^{\frac{q+1}{q}}\right)}. \end{aligned}$$

The last inequality follows, because $\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}} \leq 2$ for all $\kappa \geq 0$.

Step 3: By Lemma 5.3 the inverse function of $\psi(t)$ for $t \in [1,\infty)$ satisfies

$$\sqrt{1+2s} \leq \varrho\,(s) \leq 1 + \sqrt{2s}.$$

Thus we have,

$$\varrho\,(\Psi(v)) \geq \sqrt{1+2\Psi(v)}.$$

Step 4: Now using that $\delta(v) \geq \frac{1}{2}\psi'\left(\varrho\,(\Psi(v))\right)$, since $\tau \geq 1$, we have at the start of each inner iteration that $\Psi(v) \geq \tau \geq 1$, we obtain

$$\delta \geq \frac{1}{2}\left(\sqrt{1+2\Psi} - 1 + \frac{1}{q}\left(1 - \frac{1}{(1+2\Psi)^q}\right)\right) \geq \frac{1}{2}\left(\sqrt{1+2\Psi} - 1\right) = \frac{\Psi}{1+\sqrt{1+2\Psi}}.$$

Step 5: Substituting this, after some elementary reductions we arrive at

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{(1+2\kappa)\left(1+(1+4q\delta)^{\frac{q+1}{q}}\right)} \leq -\frac{\Psi^{\frac{q-1}{2q}}}{54\,q\,(1+2\kappa)}.$$

Thus it follows that

$$\Psi_{k+1} \leq \Psi_k - \beta {\Psi_k}^{1-\gamma}, \qquad k = 0, 1, \ldots, K-1,$$

with $\beta = \frac{1}{54\,q(1+2\kappa)}$ and $\gamma = \frac{q+1}{2q}$, and where K denotes the number of inner iterations. Hence the number K of inner iterations is bounded above by

$$K \leq \frac{\Psi_0^\gamma}{\beta\gamma} = \frac{108\,q^2\,(1+2\kappa)}{q+1}\,\Psi_0^{\frac{q+1}{2q}} \leq 108\,q\,(1+2\kappa)\ \Psi_0^{\frac{q+1}{2q}}.$$

Step 6: To estimate Ψ_0 we use Lemma 5.4, with $\psi''(1) = 2$. Thus we obtain,

$$\Psi_0 \leq \frac{\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2}{1-\theta},$$

Step 7: The total number of iterations is bounded above by

$$\frac{108\,q\,(1+2\kappa)}{\theta}\left(\frac{\left(\theta\sqrt{n}+\sqrt{2\tau}\right)^2}{1-\theta}\right)^{\frac{q+1}{2q}} \log\frac{n}{\epsilon}.$$

Step 8: For large-update methods (with $\tau = O(n)$ and $\theta = \Theta(1)$) the right hand side expression is

$$O\left((1+2\kappa)\,q\,n^{\frac{q+1}{2q}}\ \log\frac{n}{\epsilon}\right).$$

For small-update methods (with $\tau = O(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$) the right hand side expression is $O\left((1+2\kappa)\,q\sqrt{n}\,\log\frac{n}{\epsilon}\right)$.

We have proved that for any given kernel function $\psi(t)$, this will yield a different complexity results from LO case. For the sake of completeness we summarize these results in Table 2, both for small-update and for large-update methods.

6. Concluding remarks

In this paper we extended the results obtained for kernel-function-based IPMs in [2] for LO to $P_*(\kappa)$ linear complementarity problems. The observation that the vectors d_x and d_s are not in general orthogonal implies that the analysis in [2] does not hold. The analysis in this paper is new and different from the one using for *LO* and semidefinite optimization (*SDO*). Several new tools and techniques are derived in this paper. The resulting iteration bounds for $P_*(\kappa)$

i	Kernel functions ψ_i	Large-update	Small-update
1	$\frac{t^2-1}{2} - \log t$	$O\left((1+2\kappa)\, n \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\sqrt{n} \log \frac{n}{\epsilon}\right)$
2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1)$	$O\left((1+2\kappa)\, qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\, q\sqrt{n} \log \frac{n}{\epsilon}\right)$
3	$\frac{1}{2}\left(t - \frac{1}{t}\right)^2$	$O\left((1+2\kappa)\, n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\sqrt{n} \log \frac{n}{\epsilon}\right)$
4	$\frac{t^2-1}{2} + e^{\frac{1}{t}-1} - 1$	$O\left((1+2\kappa)\sqrt{n}\, \log^2 n \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\sqrt{n} \log \frac{n}{\epsilon}\right)$
5	$\frac{t^2-1}{2} - \int_1^t e^{\frac{1}{\xi}-1} d\xi$	$O\left((1+2\kappa)\sqrt{n}\, \log^2 n \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\sqrt{n} \log \frac{n}{\epsilon}\right)$
6	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1},\ q > 1$	$O\left((1+2\kappa)\, qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\, q^2\sqrt{n} \log \frac{n}{\epsilon}\right)$
7	$t - 1 + \frac{t^{1-q}-1}{q-1},\quad q > 1$	$O\left((1+2\kappa)\, qn \log \frac{n}{\epsilon}\right)$	$O\left((1+2\kappa)\, q^2\sqrt{n} \log \frac{n}{\epsilon}\right)$

Table 2. Complexity results for large- and small-update methods for LCPs.

linear complementarity problems depend on the parameter κ. For $\kappa = 0$, the iteration bounds are the same as the bounds that were obtained in [2] for LO and SDO in [6].

Acknowledgements. The authors kindly acknowledge the help of the editor and two anonymous referees in improving the readability of the paper.

References

[1] Y.Q. Bai, M. El Ghami and C. Roos, A new efficient large-update primal-dual interior-point method based on a finite barrier. *SIAM J. Opt.* **13** (2003) 766–782.

[2] Y.Q. Bai, M. El Ghami and C. Roos, A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization. *SIAM J. Opt.* **15** (2004) 101–128.

[3] E.M. Cho, Log-barrier method for two-stagequadratic stochastic programming. *Appl. Math. Comput.* **164** (2005) 45–69.

[4] Gyeong-Mi Cho and Min-Kyung Kim, A new Large-update interior point algorithm for $P_*(\kappa)$ LCPs Based on kernel functions. *Appl. Math. Comput.* **182** (2006) 1169–1183.

[5] R. Cottle, J.S. Pang and R.E. Stone, *The Linear Complementarity Problem.* Academic Press, Boston (1992).

[6] M. El Ghami, Y.Q. Bai and C. Roos, Kernel Function Based Algorithms for Semidefinite Optimization. *Int. J. RAIRO-Oper. Res.* **43** (2009) 189–199.

[7] T. Illés and M. Nagy, The Mizuno-Todd-Ye predictor-corrector algorithm for sufficient matrix linear complementarity problem. *Alkalmaz. Mat. Lapok* **22** (2005) 41–61.

[8] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, A primal-dual interior point algorithm for linear programming, in: *Progress in Mathematical Programming; Interior Point Related Methods,* **10**, edited by N. Megiddo. Springer Verlag, New York (1989) pp. 29–47.

[9] M. Kojima, N. Megiddo, T. Noma and A. Yoshise, A unified approach to interior point algorithms for linear complementarity problems, *Lect. Notes Comput. Sci.* **538** (1991).

[10] J. Miao, A quadratically convergent $o((1+k)\sqrt{n}l$-iteration algorithm for the $P_*(k)$-matrix linear complementarity problem. *Math. Program.* **69** (1995) 355–368.

[11] R.D.C. Monteiro and I. Adler, Interior path following primal-dual algorithms. Part I: Linear programming. *Math. Program.* **44** (1989) 27–41.

[12] J. Peng, C. Roos and T. Terlaky, Self-regular functions and new search directions for linear and semidefinite optimization. *Math. Program.* **93** (2002) 129–171.

[13] J. Peng, C. Roos and T. Terlaky, *Self-Regularity: A New Paradigm for Primal-Dual Interior-Point Algorithms*. Princeton University Press (2002).

[14] C. Roos, T. Terlaky and J.-Ph. Vial, *Theory and Algorithms for Linear Optimization. An Interior-Point Approach*. Springer Science (2005).

[15] S.J. Wright, *Primal-Dual Interior-Point Methods*. SIAM, Philadelphia, USA (1997).