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# ALMOST HIGHER ORDER STOCHASTIC DOMINANCE*

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**Abstract.** In this paper, we develop the concept of almost stochastic dominance for higher order preferences and investigate the related properties of this concept.

**Keywords.** Almost stochastic dominance, expected-utility maximization, higher-order preferences..

**Mathematics Subject Classification.** 90B50, 91B06.

## 1. Introduction

Leshno and Levy [1] develop the theory of almost stochastic dominance (ASD) as a relaxation of the stochastic dominance (SD). This theory plays an important role in several fields particularly in financial research. There are numerous applications based on this concept, see, *e.g.*, [2–5]. Lizyayev and Ruszczynski [6] propose a new almost stochastic dominance concept that is computationally tractable and enjoys many favourable features. Lizyayev and Ruszczynski [6] define the almost first and second order stochastic dominance. However, in the economic and financial literature higher-order preferences are believed to be important, see, *e.g.*, [7,8]. In this paper, we aim to extend Lizyayev and Ruszczynski's work [6] to higher order and study the related properties of the almost higher order stochastic

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dominance. For completeness of the presentation, we next introduce Lizyayev and Ruszczynski's Almost Stochastic Dominance concept [6].

Random variables, denoted by $X$ and $Y$, defined on $\Omega = [a, b]$ are considered together with their corresponding distribution functions $F$ and $G$, and their corresponding probability density functions $f$ and $g$, respectively. The following notations will be used throughout this paper: $\mu_F = \mu_X = E(X) = \int_a^b t\, d\, F(t), \mu_G = \mu_Y = E(Y) = \int_a^b t\, d\, G(t), H^0(x) = h(x)$, where $h = f$ or $g$ and $H = F$ or $G$. In addition, we define

$$H^{(j)}(x) = \int_a^x H^{(j-1)}(y)\, \mathrm{d}y, \quad \text{for} \quad H = F \quad \text{or} \quad G,$$

Lizyayev and Ruszczynski [6] give the following definition for almost second order stochastic dominance.

**Definition 1.1.** For $0 \leq \epsilon < 1/2$,

**$\epsilon$-ASSD:** $X$ is said to dominate $Y$ by $\epsilon$-ASSD, denoted by $X \succeq_2^{almost(\epsilon)} Y$, if and only if $E(X) \geq E(Y)$ and

$$F^{(2)}(t) - G^{(2)}(t) \leq \epsilon, \quad \forall t \in \Omega.$$

Inspired by the above definition and notice the fact that $E(X) - E(Y) = G^{(2)}(b) - F^{(2)}(b)$, we develop the concept of almost stochastic dominance for higher order preferences as follows:

**Definition 1.2.** For $0 \leq \epsilon < 1/2$,

**$\epsilon$-AkSD:** $X$ is said to dominate $Y$ by $\epsilon$-AkSD, denoted by $X \succeq_k^{almost(\epsilon)} Y$, if and only if $G^{(n)}(b) \geq F^{(n)}(b), n = 2, 3, \ldots, k, k \geq 3$ and

$$F^{(k)}(t) - G^{(k)}(t) \leq \epsilon, \quad \forall t \in \Omega. \tag{1.1}$$

Note that if $\epsilon$ is taken to be zero, then we return to the classical $k$th order stochastic dominance(kSD) concept, see [2] for more details.

## 2. Main results

We are now ready to present the main results related to the properties of the almost higher order stochastic dominance defined above.

**Theorem 2.1.** *If $X$ dominates $Y$ by $\epsilon$-AkSD $\left(X \succeq_k^{almost(\epsilon)} Y\right)$, there exists a nonnegative random variables $Z$ such that $E(Z^{k-1}) \leq \epsilon(k-1)!$ and $X + Z$ dominates $Y$ by kSD, here $k \geq 3$.*

*Proof.* From Proposition 1 in Ogryczak and Ruszczynski [9], we know that

$$F^{(k)}(\eta) = \frac{1}{(k-1)!}\int_{-\infty}^{\eta}(\eta - x)^{k-1}\mathrm{d}F(x) = \frac{1}{(k-1)!}E(\eta - X)_+^{k-1}$$

here the function $t \mapsto (t)_+ = \max(0, t)$ and $k \geq 3$.

When $X$ is said to dominate $Y$ by $\epsilon$-AkSD, we can have

$$E(\eta - X)_+^{k-1} \leq E(\eta - Y)_+^{k-1} + \epsilon(k-1)!, \quad \forall \eta \in R.$$

Let $d$ be such that $E(d - X)_+^{k-1} = \epsilon(k-1)!$. Defining $Z = (d - X)_+$, we can see that $X + Z = \max(d, X)$ and $\eta - (X + Z) = \eta - \max(d, X)$.

If $\eta \leq d$, then $(\eta - (X + Z))_+ = (\eta - \max(d, X))_+ = 0$ and thus

$$E(\eta - (X + Z))_+^{k-1} \leq E(\eta - Y)_+^{k-1},$$

and the Theorem holds.

Now we turn to consider the case with $\eta > d$, in this case, we can have

$$(\eta - \max(d, X))_+ = \begin{cases} (\eta - X)_+ & \text{if } X \geq d, \\ (\eta - X)_+ - (d - X) & \text{if } X < d. \end{cases}$$

As a result, we can have

$$\begin{aligned} E(\eta - (X + Z))_+^{k-1} &= \int_d^{\infty}(\eta - X)_+^{k-1}\mathrm{d}F(x) + \int_{-\infty}^{d}[(\eta - X)_+ - (d - X)]^{k-1}\mathrm{d}F(x) \\ &\leq \int_d^{\infty}(\eta - X)_+^{k-1}\mathrm{d}F(x) + \int_{-\infty}^{d}(\eta - X)_+^{k-1}\mathrm{d}F(x) \\ &\quad - \int_{-\infty}^{d}(d - X)^{k-1}\mathrm{d}F(x) \\ &= E(\eta - X)_+^{k-1} - \epsilon(k-1)! \leq E(\eta - Y)_+^{k-1}. \end{aligned}$$

The first inequality follows from the fact that for $k \geq 2$, $(d_1 - d_2)^k \leq {d_1}^k - {d_2}^k$ when $0 < d_2 < d_1$. The proof is finished. □

The above Theorem provides a characteristic of almost higher order stochastic dominance. It links the almost stochastic dominance concept with the traditional stochastic dominance. Besides, it gives an interesting interpretation of the value of $\epsilon$. To be precise, it is the smallest value of the $k - 1$ order moment of a random variable over $(k-1)!$ that needs to be added to a random variable $X$ in order for it to dominate a given benchmark $Y$.

Below, we prove an equivalent $\epsilon$-AkSD formulation in terms of utility functions. We first define the utility function set $U_k$ which represents high order preferences. It's defined as follows:

$$U_k = \{u : (-1)^i u^{(i)} \leq 0\,,\ i = 1, \ldots, k\}$$

where $u^{(i)}$ is the $i$th derivative of the utility function $u$. Since scaling of a utility function does not change the optimal portfolio that maximizes the expected value of that function, we may without loss of generality restrict the set $U_k$ to the following set:

$$\tilde{U}_k = \{u \in U_k : (-1)^k u^{(k-1)}(t) \leq 1\} \tag{2.1}$$

here indeed, any function $u \in U_k$ defined on $\Omega = [a, b]$, can be substituted with $\tilde{u}(t) = u(t)/u^{(k-1)}(a)$ which will preserve the optimal solution.

**Theorem 2.2.** *Under the condition that $G^{(n)}(b) \geq F^{(n)}(b), n = 2, 3, \ldots, k, k \geq 3$, a random variable $X$ $\epsilon$-AkSD dominates a random variable $Y$ if and only if*

$$E[u(X)] + \epsilon(k-1)! \geq E[u(Y)], \quad \forall u \in \tilde{U}_k.$$

*Proof.* We first consider the if part. For a fixed $\eta \in R$, define the following utility function

$$u_\eta(t) = -(\eta - t)_+^{k-1}, \quad t \in R.$$

From Definition 1.2, we can know that the $\epsilon$-AkSD dominance is equivalent to the relation:

$$\begin{aligned} E[u_\eta(X)] + \epsilon(k-1)! &= -E[(\eta - X)_+^{k-1}] + \epsilon(k-1)! \\ &\geq -E[(\eta - X)_+^{k-1}] \\ &= E[u_\eta(Y)]. \end{aligned} \tag{2.2}$$

As $u_\eta \in \tilde{U}_k$, thus the sufficiency is proved.

To prove the necessity, consider an arbitrary $u \in \tilde{U}_k$. For every $\delta > 0$, we can find a finite collection of numbers $\eta_l$ and $\alpha_l \geq 0, l = 1, \ldots, L$, and a constant $c$ such that $\sum_{l=1}^{L} \alpha_l = 1$ and the function

$$\omega(t) = c + \sum_{l=1}^{L} \alpha_l u_{\eta_l}(t)$$

has the following properties:

$$\begin{aligned} E[|u(X) - \omega(X)|] &\leq \delta, \\ E[|u(Y) - \omega(Y)|] &\leq \delta. \end{aligned}$$

This collection can be constructed by a sufficiently accurate piecewise polynomial approximation of the function $u(\cdot)$. Since the $\alpha_l$'s are nonnegative and total 1, $\omega \in \tilde{U}_k$. Adding inequalities (2.2) multiplied by $\alpha_l$ for each $u_{\eta_l}(t)$, we obtain

$$E[\omega(X)] + \epsilon(k-1)! \geq E[\omega(Y)].$$

Then

$$E[u(X)] + \epsilon(k-1)! + 2\delta \geq E[u(Y)].$$

As $\delta > 0$ is arbitrary, the necessity is proved. □

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