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SOME ASPECTS OF BALKING AND RENEGING IN FINITE BUFFER QUEUES

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Abstract. In this paper, a single server finite buffer Markovian queuing system is analyzed with the additional restriction that customers may balk as well as renege. Reneging considered in literature is usually of position independent type where the reneging rate is constant irrespective of the position of the customer in the system. However there are many real world situations where this assumption does not hold. This paper is an attempt to model balking with position dependent reneging. Explicit closed form expressions of a number of performance measures are presented. A typical problem is discussed to demonstrate the usefulness of results derived in the paper.

Keywords. Balking, finite buffer queue, impatience, position dependent reneging, queuing, reneging.

Mathematics Subject Classification. 60K25, 68M20, 90B22.

1. Introduction

These days service oriented organizations pay a lot of importance to positive customer experiences at the point of service delivery. This, it is said, can generate customer loyalty. Being in a fast-paced world, these days customers are very hard pressed for time and hence would usually prefer not to be involved in the act of waiting in any form. However queuing and waiting for service is unavoidable in real life. Consequently customers react to waiting for service in different ways. This reaction is a reflection of their impatience.

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In queuing literature two of the most common reaction modes of reflecting customer's intolerance to waiting are balking and reneging. By balking, we mean the phenomenon of customers arriving for service into a non-empty queue and leaving without joining the queue. There is no balking from an empty queue. Haight [20] has provided a rationale which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined. The other commonly observed customer behavior reflecting impatience is reneging. If a customer does not balk and joins a queuing system, it is possible that the customer gets impatient while waiting and departs without having completed the act of receiving service. Such impatient behavior is known as reneging.

Reneging can be of two types – reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). R_BOS can be observed in queuing systems where customers can renege only as long as they are in the queue. Once they begin receiving service, they do not renege. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service starts *i.e.* hair cut begins, the customer cannot leave till hair cut is over. On the other hand, R_EOS can be observed in queuing systems where customers can renege not only while waiting in queue but also while receiving service. An example is processing or merchandising of perishable goods. The patience time commences from the moment the customer joins the system. In case the reneging discipline is R_BOS, the customer will renege *i.e.* leave the system in case service does not begin before expiry of his patience time. On the other hand in case of R_EOS, the customer would renege if service is not complete before the expiry of his patience time. Thus in case of R_EOS, the customer may depart either from the queue or from the service with partial and incomplete service whereas in case of R_BOS, the customer can renege only from the queue. Both types of reneging R_BOS and R_EOS are treated separately in this paper.

As for modeling reneging phenomenon, one of the approaches in literature is to assume that each customer joining the system has a Markovian patience time. It has also been assumed that customers joining a queuing system are not aware of their position in the same so that the reneging rate is position independent. However, it is one's common day observation that there are systems where the customers are aware of their position in the system. For example, customers queuing at the O.P.D. (out patient department) clinic of a hospital would know of their position in the queue. This invariably causes waiting customers to have higher rates of reneging if their position in the queue is towards the end. It is not unreasonable then to expect that such customers, who are positioned at a distance from the service facility, would have reneging rates which are higher than the reneging rates of customers who are near the service facility. In other words we assume that customers are 'position aware' and in this paper we model the reneging phenomenon in such a manner that the Markovian reneging rate is a function of the position of the customer in the system. Customers will be assumed to have monotonic reneging rates.

The subsequent sections of this paper are structured as follows. Section 2 contains a brief review of the literature. Sections 3 and 4 contain model description, derivation of steady state probabilities and performance measures. We perform sensitivity analysis in Section 5. A numerical example is discussed in Section 6. Concluding remarks are presented in Section 7. The appendix contains some derivation.

2. Literature survey

Reneging considered in literature is of three types-deterministic, Markovian and general. The bulk of the work considers deterministic or Markovian reneging. One of the earliest work on reneging was by Barrer [7] where he considered deterministic reneging with single server Markovian arrival and service rates. Customers were selected randomly for service. In his subsequent work, Barrer [8] also considered deterministic reneging (of both R_BOS and R_EOS type) in a multi server scenario with FCFS discipline. Another early work on reneging was by Haight [21]. Ancker and Gafarian [5] carried out an early work on Markovian reneging with Markovian arrival and service pattern.

Haghighi *et al.* [19] considered a Markovian multiserver queuing model with balking as well as reneging. Each customer had a balking probability which was independent of the state of the system. Reneging discipline considered by them was R_BOS. Liu *et al.* [25] considered an infinite server Markovian queuing system with reneging of type R_BOS. Customers had a choice of individual service or batch service, batch service being preferred. Brandt and Brandt [11] considered a s-server system with two FCFS queues where the arrival rate at the queue and the service may depend on number of customers n in service or in the first queue. The service rate was assumed to be constant for $n > s$. Customers in the first queue were assumed impatient with deterministic reneging.

Ke and Wang [24] considered the machine repair problem in which failed machines balked with probability $(1 - b)$ and renege according to a negative exponential distribution. Zhang *et al.* [35] considered an M/M/1/N framework with Markovian reneging where they derived the steady state probabilities and formulated a cost model. Some performance measures were also discussed. A numerical example was discussed to demonstrate how various parameters of the cost model influence the optimal service rates of the system. Choudhury [13] analyzed a single server Markovian queuing system with reneging. He assumed that the individual patience times were independent and identically distributed exponential random variables. Reneging till beginning of service was considered. A detailed and lucid derivation of the distribution of virtual waiting time in the system was presented. Some performance measures were also presented. El-Paoumy [16] also derived the analytical solution of $\mathrm{M^x}$/M/2/N queue for batch arrival system with Markovian reneging. In this paper, the steady state probabilities and some performance measures of effectiveness were derived in explicit forms. Another paper on Markovian reneging was by Altman and Yechiali [4].

Jouini *et al.* [23] considered two multi-class call center models with and without reneging. They assumed that customers had different priorities and the content of different types of calls was assumed as similar allowing their service times to be identical. Choudhury [14] analyzed a single server finite buffer queuing system (M/M/1/K) where he assumed that customers are of reneging type. Both rules of reneging were considered and various performance measures presented under both rules of reneging.

Other attempts at modeling reneging phenomenon include those by Baccelli *et al.* [6], Martin and Artalejo [26], Shawky [30], Choi *et al.* [12], and Singh *et al.* [33], El-Sherbiny [18] and El-Paoumy and Ismail [17] etc.

An early work on balking was carried out by Haight [20]. Another work using the concepts of balking and reneging in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem [2]. Jouini *et al.* [22] modeled a call center as an M/M/s queue with Markovian reneging and endogenized customer reactions to announcements. They assumed that customers react by balking upon hearing the delay announcement and may subsequently renege if they realized waiting time exceeds the delay that was originally announced to them. They calculated the waiting time distribution *i.e.* announcement coverage and subsequent performance in terms of balking and reneging. Al-Seedy *et al.* [3] presented an analysis for the M/M/c queue with balking and reneging. They assumed that arriving customers balked with a fixed probability and reneged according to a negative exponential distribution. To obtain the transient solution of system, the generating function technique was used. Yue *et al.* [34] analyzed an M/M/2 queuing system with balking and two heterogeneous servers, server 1 and server 2. They assumed that customers arrived according to a Poisson process and form a single waiting line where two parallel servers provided heterogeneous exponential service on a first-come first-served basis. It is also assumed that server 1 is perfectly reliable and server 2 is subject to breakdowns. They obtained the stationary condition where the system reaches a steady state and derived the steady state probabilities in a matrix form by using matrix-geometric solution method. They produced explicit expressions of some performance measures such as the mean system size, the average balking rate and the probabilities that server 2 is in various states. They also provided some numerical illustrations.

There have been some papers in which both balking and reneging were considered. Here mention may be made of the work by Shawky and El-Paoumy [31,32], El-Paoumy [15, 16], El-Sherbiny [18], Shawky and El-Paoumy [32], Pazgal *et al.* [29].

3. The model and system state probabilities

The model we deal with is the traditional M/M/1/k model with the restriction that customers may balk from a non-empty queue as well as may renege after they join the queue. The importance of this queuing model stems from the fact that in the classical M/M/1 model, "it is assumed that the system can accommodate any number of units. In practice, this may seldom be the case. We have thus to consider the situation such that the system has limited waiting space and can hold a maximum number of k units (including the one being served)" Medhi [27].

We shall assume that the inter arrival and service rates are λ and μ respectively. As for balking, we shall assume that each customer arriving at the system has a probability 'p' of balking from a non-empty queue. Customers joining the system will assume to be of Markovian reneging type. We shall also assume that on joining the system, the customer is aware of its position in it. Consequently the reneging rate will be taken as a function of the customer's position. In particular, a customer who is at position 'n' will be assumed to have random patience time following $\exp(\nu_n)$. Under R_BOS, we shall assume that

$$\nu_n = \begin{cases} 0 & \text{for } n = 0, 1. \\ \nu + c^{n-1} & \text{for } n = 2, 3, \ldots, k \end{cases}$$

and under R_EOS,

$$\nu_n = \begin{cases} 0 & \text{for } n = 0. \\ \nu + c^{n-1} & \text{for } n = 1, 2, \ldots, k \end{cases}$$

where c is a constant ($c \geqslant 0$ and $c \neq 1$).

Our aim behind this formulation is to ensure that customers at higher positions have monotonic reneging rate. As a customer progresses in the system from position n to $(n-1)$, the reneging distribution will shift from $\exp(\nu_n)$ to $\exp(\nu_{n-1})$. In view of the memory less property, this shifting of reneging distribution is mathematically tractable as we shall demonstrate in the subsequent sections.

Our work stands out on a number of counts. First, one can observe from Section 2 that existing reneging literature does not analyze the case where the reneging behavior is position dependent. All such Markovian reneging rules assumed that the reneging rate was constant irrespective of the position of the customer. To the best of our knowledge, formulation of position dependent reneging rule has not been attempted in literature. However, as mentioned in Section 1, there are many systems where customers are position aware and hence have variable reneging rates. This formulation is an important focus of this paper. Second, even though one can observe reneging and balking in our day-to-day life, very little work has analyzed these two features together. This has been attempted here. Third, the derivation of explicit closed form expressions of performance measures which can be used 'off the shelf' by practitioners is an important focus of this paper. Reneging and balking literature seldom provide explicit closed form expressions; much less when reneging and balking are both involved as in the case of this paper. The expressions we shall present here are flexible enough to incorporate traditional assumption of position independent reneging rule as a special case.

We derive the steady state probabilities by the Markov process method. Under R_BOS, let p_n denote the probability that there are 'n' customers in the system.

The steady state equations are given below.

$$\lambda p_0 = \mu p_1 \tag{3.1}$$

$$\lambda p_0 + (\mu + \nu + c)p_2 = \lambda(1-p)p_1 + \mu p_1 \tag{3.2}$$

$$\begin{aligned}\lambda(1-p)p_{n-1} + \{\mu + n\nu + c(c^n-1)/(c-1)\}\, p_{n+1} &= \lambda(1-p)p_n \\ + \left\{\mu + (n-1)\nu + c(c^{n-1}-1)/(c-1)\right\} p_n;\quad n &= 2,\ldots,k-1\end{aligned} \tag{3.3}$$

$$\lambda(1-p)p_{k-1} = \{\mu + (k-1)\nu + c(c^{k-1}-1)/(c-1)\}p_k. \tag{3.4}$$

Solving recursively, we get (under R_BOS)

$$p_n = \frac{\lambda^n(1-p)^{n-1}}{\prod_{r=1}^{n}\{\mu + (r-1)\nu + c(c^{r-1}-1)/(c-1)\}}p_0;\quad n = 1,2,\ldots,k \tag{3.5}$$

where p_0 is obtained from the normalizing condition $\sum_{n=0}^{k} p_n = 1$ and is given as

$$p_0 = \left[1 + \sum_{n=1}^{k}\frac{\lambda^n(1-p)^{n-1}}{\prod_{r=1}^{n}[\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}]}\right]^{-1}. \tag{3.6}$$

The steady state probabilities satisfy the recurrence relation

$$p_n = \left[\frac{\lambda(1-p)}{[\mu + (n-1)\nu + \{c(c^{n-1}-1)/(c-1)\}]}\right]p_{n-1};\quad n = 1,2,\ldots,k.$$

Under R_EOS let q_n denote the probability that there are n customers in the system. Again applying the Markov process method, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu + \nu)q_1 \tag{3.7}$$

$$\lambda q_0 + (\mu + 2\nu + c)\, q_2 = \lambda(1-p)q_1 + (\mu + \nu)q_1 \tag{3.8}$$

$$\begin{aligned}\lambda(1-p)q_{n-1} + \{\mu + (n+1)\nu + c(c^n-1)/(c-1)\}\, q_{n+1} &= \lambda(1-p)q_n \\ + \left\{\mu + n\nu + c(c^{n-1}-1)/(c-1)\right\} q_n;\quad n &= 2,3,\ldots,k-1\end{aligned} \tag{3.9}$$

$$\lambda(1-p)q_{k-1} = \{\mu + k\nu + c(c^{k-1}-1)/(c-1)\}q_{k+1}. \tag{3.10}$$

Solving recursively, we get

$$q_n = \left[\frac{\lambda^n (1-p)^{n-1}}{\prod_{r=1}^{n} [\mu + r\nu + \{c(c^{r-1}-1)/(c-1)\}]} \right] q_0; \quad n = 1, 2, \ldots, k \tag{3.11}$$

where q_0 is obtained from the normalizing condition $\sum_{n=0}^{k} q_n = 1$ and is given by

$$q_0 = \left[1 + \sum_{n=1}^{k} \frac{\lambda^n (1-p)^{n-1}}{\prod_{r=1}^{n} [\mu + r\nu + \{c(c^{r-1}-1)/(c-1)\}} \right]^{-1} \cdot \tag{3.12}$$

The steady state probabilities satisfy the recurrence relation

$$q_n = \left[\frac{\lambda(1-p)}{[\mu + n\nu + \{c(c^{n-1}-1)/(c-1)\}]} \right] q_{n-1}; \quad n = 1, 2, \ldots, k.$$

The particular case of $c = 0$ provides steady state probabilities for the traditional assumption of position independent Markovian reneging rule. For large 'k', it can be shown that our results agree with those reported by Haghighi *et al.* [19].

4. Performance measures

In general, "performance measures are the specific representation of a capacity, process or outcome deemed relevant to the assessment of performance, which are quantifiable and can be documented" (www.iphionline.org). The main objective of any queuing study is to assess some well-defined parameters which are designed at striking a good balance between customer satisfaction and economic considerations. In queuing theory, measures through which the nature of the quality of service can be studied are known as performance measures. Performance measures are important as their analysis allows the cause of queuing issues to be identified and the impact of proposed system changes to be assessed. Some of the performance measures of any queuing system that are of general interest for the evaluation of the performance of an existing queuing system and to design a new system in terms of the level of service a customer receives as well as the proper utilization of the service facilities include mean size, server utilization, customer loss and the like.

An important measure is the mean number of customers in the system, which is traditionally denoted by 'L'. We have presented the derivation of this important performance measure separately for the two reneging disciplines in the appendix. These are denoted by $L_{\text{R\_BOS}}$ and $L_{\text{R\_EOS}}$.

Let $P(s)$ be the p.g.f of the steady state probability under R_BOS rule. Then we note that

$$\begin{aligned} L_{\text{R\_BOS}} &= \sum_{n=0}^{k} np_n \\ &= P'(1) \\ &= \frac{\mathrm{d}}{\mathrm{d}s} P(s)|_{s=1} \end{aligned}$$

(See the appendix for more derivations).

From (A.7) and (B.2), the mean system sizes under the two reneging rules are

$$\begin{aligned} L_{\text{R\_BOS}} &= [\lambda - (\mu - \nu + \lambda p)(1 - p_0) - p_0 - \lambda(1-p)p_k \\ &\quad + \{c - (p_0/p_0(c\lambda, \mu, \nu, k))/(c-1)\}]/\nu \qquad (4.1) \\ L_{\text{R\_EOS}} &= [\lambda - (\mu + \lambda p)(1 - q_0) - q_0 - \lambda(1-p)q_k \\ &\quad + \{c - (q_0/q_0(c\lambda, \mu, \nu, k))/(c-1)\}]/\nu. \qquad (4.2) \end{aligned}$$

The mean queue size for the two cases can now be obtained and are given by

$$\begin{aligned} L_{q(\text{R\_BOS})} &= \sum_{n=2}^{k} (n-1)p_n \\ &= L_{\text{R\_BOS}} - (1 - p_0) \\ &= [\lambda - (\mu + \lambda p)(1 - p_0) - p_0 - \lambda(1-p)p_k \\ &\quad + \{c - (p_0/p_0(c\lambda, \mu, \nu, k))/(c-1)\}]/\nu \end{aligned}$$

where p_0 and $p_0(c\lambda, \mu, \nu, k)$ are defined in (3.6) and (A.4) respectively.

Similarly,

$$\begin{aligned} L_{q(\text{R\_EOS})} &= L_{\text{R\_EOS}} - (1 - q_0) \\ &= [\lambda - (\mu + \nu + \lambda p)(1 - q_0) - q_0 - \lambda(1-p)q_k \\ &\quad + \{c - (q_0/q_0(c\lambda, \mu, \nu, k))/(c-1)\}]/\nu \end{aligned}$$

where q_0 and $q_0(c\lambda, \mu, \nu, k)$ are defined in (3.12) and (B.3) respectively.

Customers arrive into the system at rate λ. However, all the customers who arrive do not join the system because of balking and finite buffer restriction. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\begin{aligned} \lambda^e_{(\text{R\_BOS})} &= \lambda p_0 + \lambda(1-p) \sum_{n=1}^{k-1} p_n \\ &= \lambda p_0 + \lambda(1-p)(1 - p_0 - p_k) \\ &= \lambda(1-p)(1 - p_k) + \lambda p p_0. \qquad (4.3) \end{aligned}$$

Similarly in case of R_EOS

$$\lambda^e_{(\text{R\_EOS})} = \lambda(1-p)(1-q_k) + \lambda p q_0. \tag{4.4}$$

We have assumed that each customer has a random patience time following $\exp(\nu_n)$. As such, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate ($\text{avg}\, rr$) is given by

$$\begin{aligned}
\text{Avg}\, rr_{(\text{R_BOS})} &= \sum_{n=2}^{k} \{(n-1)\nu + c(c^{n-1}-1)/(c-1)\} p_n \\
&= \nu\{p'(1) - p_1\} - \nu\{1 - p_0 - p_1\} \\
&\quad + \{1/(c-1)\} \sum_{n=2}^{k} c^n p_n - \{c/(c-1)\} \sum_{n=2}^{k} p_n \\
&= \lambda - (\mu + \lambda p)(1-p_0) - \lambda(1-p)p_k
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\text{Avg}\, rr_{(\text{R_EOS})} &= \sum_{n=1}^{k} \{n\nu + c(c^{n-1}-1)/(c-1)\} q_n \\
&= \nu Q'(1) + \{1/(c-1)\} \sum_{n=1}^{k} c^n q_n - \{c/(c-1)\} \sum_{n=1}^{k} q_n \\
&= \lambda - (\mu + \lambda p)(1-q_0) - \lambda(1-p)q_k.
\end{aligned} \tag{4.6}$$

In real life, lost customers represent the business lost. Customers are lost to the system in three ways *viz* by balking, by finite buffer restriction and by reneging. Management of any queuing system would like to know the proportion of total customers lost in order to have an idea of total business lost. The mean rate at which customers are lost (under R_BOS) is

$$\lambda - \lambda^e_{(\text{R\_BOS})} + \text{avg}\, rr_{(\text{R\_BOS})} = \lambda - \mu(1-p_0) \tag{4.7}$$

and the mean rate at which customers are lost (under R_EOS) is

$$\lambda - \lambda^e_{(\text{R\_EOS})} + \text{avg}\, rr_{(\text{R\_EOS})} = \lambda - \mu(1-q_0). \tag{4.8}$$

These rates helps in determining the proportion of customers lost. These proportions are $\{\lambda - \lambda^e_{(\text{R\_BOS})} + \text{avg}\, rr_{(\text{R\_BOS})}\}/\lambda$ and $\{\lambda - \lambda^e_{(\text{R\_EOS})} + \text{avg}\, rr_{(\text{R\_EOS})}\}/\lambda$ respectively.

The proportion of customers completing service can now be easily determined from the above proportions.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work the server has to do. Let us call the rate at which customers reach the service station as λ^s. Then under R_BOS

$$\begin{aligned}\lambda^s_{(\text{R\_BOS})} &= \lambda^e_{(\text{R\_BOS})} \text{ (1-proportion of customers lost due to reneging out of those joining the system)}\\ &= \lambda^e_{(\text{R\_BOS})}\left\{1-\sum_{n=2}^{k}(n-1)\nu p_n/\lambda^e_{(\text{R\_BOS})}\right\}\\ &= \lambda^e_{(\text{R\_BOS})} - \text{avg}\, rr_{(\text{R\_BOS})}\\ &= \mu(1-p_0).\end{aligned}$$

In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus

$$\begin{aligned}\lambda^s_{(\text{R\_EOS})} &= \lambda^e_{(\text{R\_EOS})} \text{ (1-proportion of customers lost due to reneging from the queue out of those joining the system)}\\ &= \lambda^e_{(\text{R\_EOS})}\left\{1-\sum_{n=2}^{k}(n-1)\nu q_n/\lambda^e_{(\text{R\_EOS})}\right\}\\ &= \lambda^e_{(\text{R\_EOS})} - \text{avg}\, rr_{(\text{R\_EOS})}\\ &= \mu(1-q_0).\end{aligned}$$

5. Sensitivity analysis

We have assumed that there are essentially four parameters *viz*: λ, μ, ν, k. relating to arrival, service, reneging patterns and system size. Various reasons may influence these parameters to undergo change from time to time. From managerial point of view, an idle server is a waste. Similarly also for low server utilization. It is therefore interesting to examine and understand how server utilization varies in response to change in system parameters. We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

$p_n(\lambda, \mu, \nu, k)$ and $q_n(\lambda, \mu, \nu, k)$ will denote the probability that there are 'n' customers in a system with parameters λ, μ, ν, k in steady state under R_BOS and R_EOS respectively.

(i) To understand the change in server utilization consequent to increase in arrival rate, let us consider two arrival rates λ_0 and λ_1 where $\lambda_1 > \lambda_0$. Then

$$\begin{aligned}&\frac{p_0\,(\lambda_1, \mu, \nu, k)}{p_0\,(\lambda_0, \mu, \nu, k)} < 1\\ &\Rightarrow \frac{(\lambda_0-\lambda_1)}{\mu} + \frac{(1-p)(\lambda_0^2-\lambda_1^2)}{\mu(\mu+\nu+c)}\\ &\quad + \ldots + \frac{(1-p)^{k-1}(\lambda_0^k-\lambda_1^k)}{\mu(\mu+\nu+c)\ldots\{\mu+(k-1)\nu+c(c^{k-1}-1)/(c-1)\}} < 0\end{aligned}$$

which is true. Hence $p_0 \downarrow$ as $\lambda \uparrow$.

Similarly, it can be shown that

(ii) $p_0 \uparrow$ as $\mu \uparrow$.
(iii) $p_0 \uparrow$ as $\nu \uparrow$.
(iv) $p_0 \downarrow$ as $k \uparrow$.

Similarly, under R_EOS, we have

(v) $q_0 \downarrow$ as $\lambda \uparrow$.
(vi) $q_0 \uparrow$ as $\mu \uparrow$.
(vii) $q_0 \uparrow$ as $\nu \uparrow$.
(viii) $q_0 \downarrow$ as $k \uparrow$.

Under R_BOS, these results state that an increase in arrival rate would result in lowering of the fraction of time the server is idle. An increase in service rate would mean the server is able to work efficiently so that it can process same amount of work quickly. This translates to higher server idle time. An increase in reneging rate would mean the server has fewer work to do and hence higher fraction of idle time. An increase in system size translates to lowering of the fraction of time the server is idle. Similar conclusions can be drawn under R_EOS.

6. Numerical example

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen [1] (p. 267 and 273).

"Traffic to a message switching centre for Extraterrestrial Communications Corporation arrives in a random pattern (remember that 'random pattern' means exponential interarrival time) at an average rate of 240 messages per minute. The line has a transmission rate of 800 characters per second. The message length distribution (including control characters) is approximately exponential with an average length of 176 characters. Calculate the principal statistical measures of system performance assuming that a very large number message buffers is provided". "Suppose, however, that it is desired to provide only the minimum number of messages buffers required to guarantee that

$$p_k < 0.005.$$

How many buffers should be provided?"

This is a design problem. Here $\lambda = 4/\text{s}$ and $\mu = 4.55/\text{s}$. As required by the switching centre, we examine the minimum number of message buffers with different choices of k. Though not explicitly mentioned, it is necessary to assume reneging and balking. We assume balking because in telecommunication systems it is known that an incoming message that sees a workload may be admitted into the system with a certain probability and rejected otherwise. Rejection implies balking. We shall assume that the balking probability is 'p' (Boxma *et al.* [10] has analyzed with similar motivation).

TABLE 1. Performance measures assuming $\lambda = 4/\text{s}$, $\mu = 4.55/\text{s}$, $\nu = 0.1/\text{s}$, $p = 0.001$ and $c = 1.1$.

Performance measure	Size of minimum number of message buffers		
	$k = 6$	$k = 7$	$k = 8$
p_k	0.00712	0.00208	0.00053
λ^s (*i.e.* arrival rate of customers reaching service station)	3.07524	3.07831	3.07909
Effective mean arrival rate (λ^e)	3.96885	3.98898	3.99518
Fraction of time server is idle (p_0)	0.32412	0.32345	0.32328
Average number of customers in queue	0.69988	0.71091	0.71425
Average number of customers in system	1.37575	1.38746	1.39097
Average reneging rate	0.89360	0.91066	0.91608
Rate of loss due to finite buffer and balking	0.03115	0.01103	0.00483
Average rate at which customers are lost	0.92476	0.92169	0.92091
Proportion of customers lost due to reneging, balking and finite buffer	0.23119	0.23042	0.23023

We also assume reneging because in telecommunication systems it is also known that messages usually have some real time constraints within which the message has to be processed. Message received after the deadline is considered obsolete and discarded. This can be seen as reneging (Movaghar [28] and Boots and Tijms [9] have analyzed with similar motivation).

We shall assume that reneging distribution is position dependent following $\exp(\nu_n)$ where ν_n is as defined in Section 3. Specifically, we shall assume $\nu = 0.1/\text{s}$ and consider the scenario with $c = 1.1$. We further assume that balking rate is independent of system state and is taken as $p = 0.001$ (one in 1000 message).

Various performance measures of interest computed under different scenarios are given in Table 1. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of k were considered. Results relevant with regard to the requirement that the switching centre should provide only the minimum number of message buffers to guarantee $p_k < 0.005$ are presented in the tables (the units of time of all rates in the table are per second).

From Table 1 it appears that an ideal choice of k could be 7 with $p_k = 0.00208$.

A few interesting observations can be made from Table 1. First, the rate of arrival into the system is 4 messages/s whereas the actual load of the server (λ^s) is 3.71686 (case of $k = 7$). Thus, $4 - 3.07831 = 0.92169$ messages/s are being lost. As a proportion of all message arriving at the centre, this amounts to 0.23042 (=0.92169/4) proportion of lost messages. This can be confirmed from the last row of Table 1. Second, since $\lambda^s = 3.07831$, the centre is required to process 3.07831 messages/s when its capacity is 4.55 messages/s. The centre is therefore idle for 0.32345{=(4.55 − 3.07831)/4.55} proportion of time. This corroborates p_0 in under $k = 7$.

7. Conclusion

The analysis of a single server markovian queuing system with balking and position dependent reneging has been presented. Even though balking and reneging have been discussed by others, explicit expressions are not always available. Besides to the best of our knowledge, modeling of position dependent reneging has not been attempted in literature. We believe this paper makes a humble contribution here. Closed form expressions of a number of performance measures have been derived. It is our sincere belief that the expressions we have presented are 'off the shelf' variety and would find favour for ready use among operational research consultants in industry. To study the change in the system corresponding to change in system parameters, sensitivity analysis has also been presented. A numerical example has been discussed to demonstrate results derived. The numerical example is of indicative nature meant to illustrate the benefits of our theoretical results in a design context. The limitations of this work stem from the Markovian assumptions. Extension of our results for general distribution is a pointer to future research.

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Appendix A. Derivation of $P'(1)$ under R_BOS

Let P(s) denote the probability generating function, defined by $P(s) = \sum_{n=0}^{\infty} p_n s^n$.

From equation (3.3) we have

$$\lambda(1-p)p_{n-1} + \{\mu + n\nu + c(c^n - 1)/(c-1)\}\, p_{n+1} = \lambda(1-p)p_n + \left\{\mu + (n-1)\nu + c(c^{n-1} - 1)/(c-1)\right\} p_n; \quad n = 2, \ldots, k-1.$$

Multiplying both sides of the equation by s^n and summing over n

$$\begin{aligned}\lambda s(1-p)\sum_{n=2}^{k-1} p_{n-1}s^{n-1} - \lambda(1-p)\sum_{n=2}^{k-1} p_n s^n &= \sum_{n=2}^{k-1} \{\mu + (n-1)\nu + (c^n - c)/(c-1)\}p_n s^n \\ &\quad - \frac{1}{s}\sum_{n=2}^{k-1} \{\mu + n\nu + (c^{n+1} - c)/(c-1)\}p_{n+1}s^{n+1}\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&\Rightarrow \lambda s(1-p)\{p_1 s^1 + \ldots + p_{k-2}s^{k-2}\} - \lambda(1-p)(p_2 s^2 + \ldots + p_{k-1}s^{k-1}) \\
&\quad = \mu(p_2 s^2 + \ldots + p_{k-1}s^{k-1}) + \nu\{p_2 s^2 + 2p_3 s^3 + \ldots + (k-2)p_{k-1}s^{k-1}\} \\
&\qquad + \{1/(c-1)\}\{c^2 p_2 s^2 + \ldots + c^{k-1}p_{k-1}s^{k-1} - c(p_2 s^2 + \ldots + p_{k-1}s^{k-1}) \\
&\qquad - (1/s)[\mu(p_3 s^3 + \ldots + p_k s^k\} + \nu\{2p_3 s^3 + \ldots + (k-1)p_k s^k\} \\
&\qquad + \{1/(c-1)\}\{c^3 p_2 s^3 + \ldots + c^k p_k s^k - c(p_3 s^3 + \ldots + p_k s^k) \\
&\Rightarrow \lambda s(1-p)\{P(s) - p_0 - p_{k-1}s^{k-1} - p_k s^k\} - \lambda(1-p)\{P(s) - p_0 - p_1 s - p_k s^k\} \\
&\quad = \mu\{P(s) - p_0 - p_1 s - p_k s^k\} + \nu s\{2p_2 s + 3p_3 s^2 + \ldots + (k-1)p_{k-1}s^{k-1}\} \\
&\qquad - \nu(p_2 s^2 + p_3 s^3 + \ldots + p_{k-1}s^{k-1}) + \{1/(c-1)\}\{P(cs) - p_0 - c^k p_k s^k \\
&\qquad - cP(s) + cp_0 + cp_k s^k\} - (1/s)[\mu\{P(s) - p_0 - p_1 s - p_2 s^2\} \\
&\qquad + \nu s\{3p_3 s^2 + \ldots + kp_k s^{k-1}\} - \nu(p_3 s^3 \ldots + p_k s^k) \\
&\qquad + \{1/(c-1)\}\{P(cs) - p_0 - c^2 p_2 s^2 - cP(s) + cp_0 + cp_2 s^2\} \\
&\Rightarrow \lambda s(1-p)\{P(s) - p_0 - p_{k-1}s^{k-1} - p_k s^k\} - \lambda(1-p)\{P(s) - p_0 - p_1 s - p_k s^k\} \\
&\quad = \mu\{P(s) - p_0 - p_1 s - p_k s^k\} + \nu s\{P'(s) - p_1 - kp_k s^{k-1}\} \\
&\qquad - \nu\{P(s) - p_0 - p_1 s - p_k s^k\} + P(cs)/(c-1) + p_0 \\
&\qquad - [\{c(c^{k-1} - 1)\}/(c-1)]p_k s^k - (\mu/s)\{P(s) - p_0 - p_1 s - p_2 s^2\} \\
&\qquad - \nu\{P'(s) - p_1 - 2p_2 s\} + (\nu/s)\{P(s) - p_0 - p_1 s - p_2 s^2\} \\
&\qquad - \{1/(c-1)s\}[P(cs) - p_0(c-1) - cP(s) - \{c/(c-1)\}p_2 s^2] \\
&\Rightarrow \nu P'(s)(1-s) = \mu P(s) - \mu p_0 - \mu p_1 s - \{\mu - \nu + \nu k + c(c^{k-1} - 1)/(c-1)\}p_k s^k \\
&\qquad - \nu P(s) + \nu p_0 + P(cs)/(c-1) + p_0 - cP(s)/(c-1) - \mu P(s)/s \\
&\qquad + \mu p_0/s + \mu p_1 + (\mu + \nu + c)p_2 s + (\nu/s)P(s) - \nu p_0/s - P(cs)/(c-1)s \\
&\qquad - p_0/s + cP(s)/(c-1)s - \lambda s(1-p)P(s) + \lambda s(1-p)p_0 + \lambda(1-p)p_{k-1}s^k \\
&\qquad + \lambda s(1-p)p_k s^k + \lambda(1-p)P(s) - \lambda(1-p)p_0 - \lambda(1-p)p_1 s - \lambda(1-p)p_k s^k \\
&\Rightarrow \nu P'(s)(1-s) = \{\nu P(s)(1-s)\}/s - \{\mu P(s)(1-s)\}/s + \{\mu p_0(1-s)\}/s \\
&\qquad - \{\nu p_0(1-s)\}/s + \lambda p_0(1-s) - \{P(cs)(1-s)\}/\{(c-1)s\} \\
&\qquad - \{p_0(1-s)\}/s + \{cP(s)(1-s)\}/\{(c-1)s\} + \lambda(1-p)P(s)(1-s) \\
&\qquad - \lambda(1-p)p_0(1-s) - \lambda(1-p)p_k s^k(1-s) \\
&\Rightarrow P'(s) = (1/\nu)[\nu P(s)/s - \mu P(s)/s + \mu p_0/s - \nu p_0/s + \lambda p_0 - P(cs)/(c-1)s \\
&\qquad - p_0/s + cP(s)/(c-1)s + \lambda(1-p)P(s) - \lambda(1-p)p_0 - \lambda(1-p)p_k s^k.
\end{aligned}$$

Now

$$\begin{aligned}
&\Rightarrow \lim_{s\to 1^-} P'(s) = \lim_{s\to 1^-} (1/\nu)[\nu P(s)/s - \mu P(s)/s + \mu p_0/s - \nu p_0/s + \lambda p_0 \\
&\qquad - P(cs)/(c-1)s - p_0/s + cP(s)/(c-1)s + \lambda(1-p)P(s) \\
&\qquad - \lambda(1-p)p_0 - \lambda(1-p)p_k s^k \\
&\Rightarrow P'(1) = (1/\nu)[\nu - \mu + \mu p_0 - \nu p_0 - P(c)/(c-1) - p_0 + c/(c-1) + \lambda(1-p) \\
&\qquad - \lambda(1-p)p_0 - \lambda(1-p)p_k \\
&\Rightarrow P'(1) = (1/\nu)[\lambda - (\mu - \nu + \lambda p)(1-p_0) - p_0 - \lambda(1-p)p_k + \{c - P(c)\}/(c-1)].
\end{aligned} \tag{A.2}$$

Here $P(c) = \sum_{n=0}^{k} p_n(\lambda, \mu, \nu, k)c^n$ where the symbol $p_n(\lambda, \mu, \nu, k)$ is as described in Section 5. We use p_n and $p_n(\lambda, \mu, \nu, k)$ interchangeably. However should any of the parameters λ, μ, ν, k change, it is explicitly stated. To obtain a closed form expression for $P(c)$, let us for the time being consider another queuing system with parameter and assumptions similar to the queuing system we are presently considering except that the arrival rate is '$c\lambda$'. For this new system, the steady state equations are same as (3.1), (3.2), (3.3) and (3.4) with 'λ' replaced by '$c\lambda$'. Denoting the steady state probabilities of this new system by $p_n(c\lambda, \mu, \nu, k)$, we can obtain

$$p_n(c\lambda, \mu, \nu, k) = \frac{(c\lambda)^n (1-p)^{n-1}}{\prod_{r=1}^{n} [\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}]} p_0(c\lambda, \mu, \nu, k); \quad n = 1, 2, \ldots, k \tag{A.3}$$

where

$$p_0(c\lambda, \mu, \nu, k) = \left[1 + \sum_{n=1}^{k} \frac{(c\lambda)^n (1-p)^{n-1}}{\prod_{r=1}^{n} [\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}]}\right]^{-1}. \tag{A.4}$$

Let $P(S; c\lambda, \mu, \nu, k)$ denotes the probability generating function of this new queuing system so that

$$P(S; c\lambda, \mu, \nu, k) = \sum_{n=0}^{k} p_n(c\lambda, \mu, \nu, k)s^n.$$

Now

$$\begin{aligned} P(c) &= \sum_{n=0}^{k} p_n(\lambda, \mu, \nu, k)c^n \\ &= p_0 + \sum_{n=1}^{k} \frac{(c\lambda)^n (1-p)^{n-1} p_0}{[\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}]} \\ \Rightarrow [\{P(c) - p_0\}/p_0] &= \sum_{n=1}^{k} \frac{(c\lambda)^n (1-p)^{n-1}}{[\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}]}. \end{aligned} \tag{A.5}$$

Now putting $S = 1$ in $P(S; c\lambda, \mu, \nu, k)$ we get

$$
\begin{aligned}
&P(1; c\lambda, \mu, \nu, k) = p_0(c\lambda, \mu, \nu, k) + \sum_{n=1}^{k} p_n(c\lambda, \mu, \nu, k) \\
&\Rightarrow 1 = p_0(c\lambda, \mu, \nu, k) \\
&\qquad + \sum_{n=1}^{k} \left[\frac{(c\lambda)^n (1-p)^n}{\prod_{r=1}^{n} [\mu + (r-1)\nu + \{c(c^{r-1}-1)/(c-1)\}} \right] p_0(c\lambda, \mu, \nu, k) \\
&\Rightarrow 1 = p_0(c\lambda, \mu, \nu, k) + \{(P(c) - p_0)/p_0\}\, p_0(c\lambda, \mu, \nu, k) \\
&\Rightarrow P(c) = p_0/p_0(c\lambda, \mu, \nu, k)
\end{aligned}
$$

$$\text{using (A.3) and (A.5)} \qquad \text{(A.6)}$$

Using (A.6) in (A.2) we obtain

$$
\begin{aligned}
P'(1) = (1/\nu)[\lambda - (\mu - \nu + \lambda p)(1 - p_0) - p_0 - \lambda(1-p)p_k \\
+ \{c - p_0/p_0(c\lambda, \mu, \nu, k)\}/(c-1)] \quad \text{(A.7)}
\end{aligned}
$$

where $p_0(c\lambda, \mu, \nu, k)$ is given in (A.4).

Appendix B. Derivation of $Q'(1)$ under R_EOS

Let $Q(s)$ denote the probability generating function, defined by $Q(s) = \sum_{n=0}^{\infty} q_n s^n$.

From equation (3.9)

$$
\begin{aligned}
\lambda(1-p)q_{n-1} + \{\mu + (n+1)\nu + c(c^n - 1)/(c-1)\}\, q_{n+1} = \lambda(1-p)q_n \\
+ \left\{\mu + n\nu + c(c^{n-1}-1)/(c-1)\right\} q_n; \qquad n = 2, \ldots, k-1.
\end{aligned}
$$

Multiplying both sides of this equation by s^n and summing over n from we get

$$
\begin{aligned}
\lambda s(1-p) \sum_{n=2}^{k-1} q_{n-1} s^{n-1} - \lambda(1-p) \sum_{n=2}^{k-1} q_n s^n = \sum_{n=2}^{k-1} \{\mu + n\nu + (c^n - c)/(c-1)\} q_n s^n \\
- \frac{1}{s} \sum_{n=2}^{k-1} \{\mu + (n+1)\nu + (c^{n+1} - c)/(c-1)\} q_{n+1} s^{n+1}. \quad \text{(B.1)}
\end{aligned}
$$

Proceeding in a manner similar to previous section, we obtain,

$$
Q'(1) = (1/\nu)[\lambda - (\mu + \lambda p)(1 - q_0) - q_0 - \lambda(1-p)q_k + \{c - q_0/q_0(c\lambda, \mu, \nu, k)\}/(c-1)]
$$

$$\text{using (A.7)} \quad \text{(B.2)}$$

where

$$q_0(c\lambda,\mu,\nu,k) = \left[1 + \sum_{n=1}^{k} \frac{(c\lambda)^n (1-p)^{n-1}}{\prod_{r=1}^{n} [\mu + r\nu + \{c(c^{r-1}-1)/(c-1)\}]}\right]^{-1}. \qquad \text{(B.3)}$$

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