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AN M^X/G/1 UNRELIABLE RETRIAL QUEUE WITH TWO PHASE SERVICE AND PERSISTENCE BEHAVIOUR OF CUSTOMERS IN SERVICE

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Abstract. This paper describes an unreliable server batch arrival retrial queue with two types of repair and second optional service. The server provides preliminary first essential service (FES) to the primary arriving customers or customers from retrial group. On successful completion of FES, the customer may opt for second optional service (SOS) with probability α. The server is subject to active break downs. The customer under FES (or SOS) during the failure decides, with probability q, to join the orbit(*impatient customer*) and, with complementary probability p, to remain in the server for repair in order to conclude his remaining service (*patient customer*). Both service and repair times are assumed to have general distribution. It is considered that the repair time of server during the presence of patient customer and the repair time of the server while the customer (*impatient customer*) joining the orbit due to failure, are different. For this queueing system, the orbit and system size distributions are obtained. Reliability of the proposed model is analysed. Some particular cases are also discussed. Other performance measures are also obtained. The effects of several parameters on the system are analysed numerically.

Keywords. First essential service, reliability measures, second optional service, unreliable server, persistent customers.

Mathematics Subject Classification. 60K25, 90B22.

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1. Introduction

Queueing system with repeated attempts are characterized by the feature is that arriving customers who find all server busy are obliged to abandon the service area and join a retrial group (called orbit) in order to try their luck again after some random time. For a detailed review of the main results and the literature on this topic the reader is referred to Artalejo [4, 5] and Falin [14–16]. In most of the queueing literature it is assumed that the server is available in the service station on a permanent basis and service station never fails.

In practical system we often meet the case where service stations may fail and can be repaired. Such phenomena always occur in the area of computer communication networks and flexible manufacturing systems, etc. Because the performance of such a system may be heavily affected by service station breakdown, such systems with repairable service station are worth investigating from both the queueing theory point of view and reliability point of view. Recently, there has been vast development in the literature on retrial queues, there are only a very few works taking into account both the retrial phenomenon and unreliability of the server.

The retrial queueing models with active breakdowns where the interrupted customer decides to be patient by waiting for repair to complete his service (*non pre-emptive policy*) or impatient by entering the orbit to re-initiate his request later (*pre-emptive policy*) are taking into account only in very few research works. This paper focuses on the analysis of a single server batch arrival retrial queue with *active server breakdowns, two types of repair, second optional service and impatient behaviour of the customers.*

2. Literature survey

Queueing systems with repairable service station have been studied by many authors. as for example Avi-Itzhak and Naor [10],Tang [28], Yue and Cao [32], Krishnakumar *et al.* [19, 20], Li *et al.* [21], Atencia *et al.* [7, 8] and Ke [17].

There are only a few works taking into account both the retrial phenomenon and unreliability of the server. Interested readers can find the main results and methods about unreliable retrial queues in Aissani [1, 2], Aissani and Artalejo [3], Atencia *et al.* [7, 8], Wang *et al.* [29, 30], Choudhury *et al.* [13], Xiaoyong Wu *et al.* [30] and Yang and Li [31]. The breakdowns may be active or passive according to whether the failures occur in a working or idle period of the server. Besides, failures can take place after a random amount of service time or just before starting the service. Recently, Atencia *et al.* [7] have analysed the retrial queues with active breakdowns and exponential server lifetime where the interrupted customer decides to be patient by waiting for repair to complete his service (*non pre-emptive policy*) or impatient by entering the orbit to re-initiate his request later (*pre-emptive policy*). In practice, it is considered that *the repair time of server during the presence of patient customer and the repair time of the server* *while the customer (impatient customer) joining the orbit due to failure, are different*. But two types of repair time are not considered in all of the above models. Thus this paper concerns about studying the unreliability of the server and the impatience problems under two different types of repair.

To be more realistic, we further assume that the service process can be done with optional phase of service, which covers many practical situations: for example, in client-server communication systems, messages which are transmitted through two stages of service. There have been several contributions considering queueing system with two phases of service. Madan [22] considered the classical M/G/1 queueing system in which the server provides the first essential service to all the arriving customers whereas some of them receive second optional service. The first essential service follows general distribution and second optional service follows exponential distribution. Medhi [23] generalizes the model by considering that the second optional service is also governed by a general distribution. Choudhury [11], Choudhury *et al.* [12, 13] investigate the retrial queueing model with second optional service. Ke [18] extended the result for a multi-optional service system where concept of set-up time is also introduced. Artalejo and Choudhury [6] have analyzed the steady state analysis of the M/G/1 queueing system with repeated attempts and two phase service using Embedded Markov chain method. Senthil Kumar and Arumuganathan [24] have analysed the batch arrival single server retrial queue in which the server provides two phases of heterogeneous service and receives general vacation time under Bernoulli schedule. Senthil Kumar and Arumuganathan [25] have analysed the batch arrival with active breakdowns in which there is no breakdowns while doing SOS. Further, Senthil Kumar [26] has analysed discrete time $Geo^{[x]}/G/1$ with M-additional options for service. Moreover, very few authors have studied the comparable work on the impatient behaviour while server breakdowns in retrial queueing model. This paper analyses a single server batch arrival retrial queue with active server breakdowns, two types of repair and second optional service.

A possible application of bulk arrival impatient behaviour retrial queueing system with two phases of heterogeneous service under active server breakdowns and two types of repair is given as follows:

The queueing system under consideration has an intrinsic interest to model some situations in packet switching network. Consider a computer network which consists of a group of processors connected with a central transmission unit (CTU). If a processor wishes to send a message it first sends the message to the CTU. If the transmission medium is available, the CTU sends immediately message; otherwise the message will be stored in a buffer and the messages in CTU must retry for the transmission some time later. It is noted that most papers on retrial queues assume that the server is available on a permanent basis. In practice, however, these assumptions are apparently unrealistic. CTU may well be subject to lengthy and unpredictable breakdowns like scheduled backups and unpredictable failures, while transmitting the message. If CTU is subject to unpredictable breakdowns (not so lengthy) while transmitting the message, CTU gives the priority to transmit that message after being repaired. In the case of lengthy breakdowns, the messages in buffer must retry for the transmission some time later after being repaired. The above situation can be modelled as a batch arrival single server retrial queueing system with active breakdowns, two types of repair time and second optional service.

The primary focus of this paper is to realize an extensive analysis of this system from both the queueing and reliability point of view. Analytical treatment of this model is obtained by supplementary variable technique. The steady state orbit and system size distribution are found. Also, other performance measures are obtained. Reliability measures of this model are discussed. Finally some numerical examples are shown.

3. Model description

We consider a single server retrial queue in which batches of primary calls arrive according to a Poisson stream with rate $\lambda > 0$. In the batch arrival retrial queue it is assumed that at every arrival epoch a batch of k primary calls arrives with probability g_k. $X(z) = \sum_{k=1}^{\infty} g_k z^k$ is denoted as the generating function of the batch size distribution. The mean batch size is denoted as $E[X] = X'(1)$.

If the server is busy at the arrival epoch, then all these calls join the orbit, whereas if the server is free, then one of the arriving calls begins its service and the others form sources of repeated calls in order to seek service again after a random amount of time. The time between two successive repeated attempts of each call in orbit is assumed to be exponentially distributed with rate v. The server provides preliminary FES and may break down while serving customers. On successful completion of FES, the customer may opt for SOS with probability α. When the server fails while doing FES and SOS, it is sent to repair immediately. It is supposed that the server lifetime follows an exponential law with rate $\gamma > 0$, *i.e.*, the server fails after an exponential time with γ^{-1}. The customer who was being served during the server failure chooses, with probability q, to enter the orbit (impatient customer) and, with complementary probability p, to remain in the server for repair in order to conclude his remaining service (patient customer). Both service and repair times are assumed to have a general distribution. It is natural to consider that the repair time of server during the presence of patient customer and the repair time of the server while the customer (impatient customer) joining the orbit due to failure, are different. Therefore, a patient customer will receive the service first and in this case, the customer has a certain priority in the service. However, an impatient customer can carry out other subsidiary tasks during the repair time in order to exploit this period.

Let $S_1(x)$ $(s_1(x))$ $\tilde{S}_1(\theta)$ $[S_1^0(x)]$ be the cumulative distribution (probability density function) Laplace-Stieltjes transform (LST)[remaining service time] of FES. $R_1(x)$ $(r_1(x))$ $\tilde{R}_1(\theta)$ $[R_1^0(x)]$ be the cumulative distribution (probability density function) Laplace-Stieltjes transform (LST)[remaining repair time] of repair time of the server during the presence of patient customer. $R_2(x)$ $(r_2(x))$ $\tilde{R}_2(\theta)$ $[R_2^0(x)]$

be the cumulative distribution (probability density function) Laplace-Stieltjes transform (LST)[remaining repair time] of repair time of the server while the customer (impatient customer) joining the orbit due to failure. $S_2(x)$ $(s_2(x))$ $\tilde{S}_2(\theta)$ $[S_2^0(x)]$ be the cumulative distribution (probability density function) Laplace-Stieltjes transform (LST)[remaining service time] of SOS. $N(t)$ denotes number of customers in the orbit at time t. The server state is denoted as,

$$C(t) = \begin{cases} 0 & if\ the\ server\ is\ idle \\ 1 & if\ the\ server\ is\ busy\ with\ FES \\ 2 & if\ the\ server\ is\ down\ with\ a\ customer\ in\ the\ server\ while \\ & doing\ FES \\ 3 & if\ the\ server\ is\ down\ without\ a\ customer\ in\ the\ server\ while \\ & doing\ FES \\ 4 & if\ the\ server\ is\ busy\ with\ SOS \\ 5 & if\ the\ server\ is\ down\ with\ a\ customer\ in\ the\ server\ while\ doing \\ & SOS \\ 6 & if\ the\ server\ is\ down\ without\ a\ customer\ in\ the\ server\ while \\ & doing\ SOS \end{cases}$$

The state of the system at time t can be described by the Markov process $K(t) = \{(C(t), N(t), S_1^0(t), S_2^0(t), R_1^0(t), R_2^0(t)); t \geq 0\}$. Now we define the following state probabilities:

$$P_{0,n}(t) = Pr\{C(t) = 0; N(t) = n\}\ n \geq 0$$

$$P_{1,n}(x,t)\mathrm{d}x = Pr\{C(t) = 1; N(t) = n; x < S_1^0(t) \leq x + \mathrm{d}x\}\ n \geq 0, x \geq 0$$

$$P_{2,n}(x,y,t)\mathrm{d}y = Pr\{C(t) = 2; N(t) = n; \\ x = S_1^0(t); y < R_1^0(t) \leq y + \mathrm{d}y\}\ n \geq 0, x, y \geq 0$$

$$P_{3,n}(y,t)\mathrm{d}y = Pr\{C(t) = 3; N(t) = n; y < R_2^0(t) \leq y + \mathrm{d}y\}\ n \geq 1, y \geq 0$$

$$P_{4,n}(x,t)\mathrm{d}x = Pr\{C(t) = 4; N(t) = n; x < S_2^0(t) \leq x + \mathrm{d}x\}\ n \geq 0, x \geq 0$$

$$P_{5,n}(x,y,t)\mathrm{d}y = Pr\{C(t) = 5; N(t) = n; \\ x = S_2^0(t); y < R_1^0(t) \leq y + \mathrm{d}y\}\ n \geq 0, x, y \geq 0$$

$$P_{6,n}(y,t)\mathrm{d}y = Pr\{C(t) = 6; N(t) = n; y < R_2^0(t) \leq y + \mathrm{d}y\}\ n \geq 1, y \geq 0$$

4. Steady state distribution of the queueing system

The following equations are obtained for the queueing system, using supplementary variable technique.

$$P_{0,j}(t+\Delta t) = P_{0,j}(t)(1-\lambda\Delta t - jv\Delta t)+P_{4,j}(0,t)\Delta t+(1-\alpha)P_{1,j}(0,t)\Delta t + (1-\delta_{0,j})(P_{3,j}(0,t)\Delta t+P_{6,j}(0,t)\Delta t) \quad j\geq 0$$

$$P_{1,j}(x-\Delta t, t+\Delta t) = P_{1,j}(x,t)(1-\lambda\Delta t-\gamma\Delta t) + \lambda\Delta t\sum_{k=1}^{j+1} g_k P_{0,j-k+1}(t)s_1(x) + (j+1)vP_{0,j+1}(t)s_1(x)\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{1,j-k}(x,t) + P_2(x,0,t)\Delta t \quad j\geq 0$$

$$P_{2,j}(x,y-\Delta t,t+\Delta t) = P_{2,j}(x,y,t)(1-\lambda\Delta t) + p\gamma P_{1,j}(x,t)r_1(y)\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{2,j-k}(x,y,t) \quad j\geq 0$$

$$P_{3,j}(y-\Delta t,t+\Delta t) = P_{3,j}(1-\lambda\Delta t) + q\gamma r_2(y)\left[\int_0^\infty P_{1,j-1}(x,t)\mathrm{d}x\right]\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{3,j-k}(y,t) \quad j\geq 1$$

$$P_{4,j}(x-\Delta t,t+\Delta t) = P_{4,j}(x,t)(1-\lambda\Delta t-\gamma\Delta t) + \alpha P_{1,j}(0,t)s_2(x)\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{4,j-k}(x,t) + P_{5,j}(x,0,t)\Delta t \quad j\geq 0$$

$$P_{5,j}(x,y-\Delta t,t+\Delta t) = P_{5,j}(x,y,t)(1-\lambda\Delta t) + p\gamma P_{4,j}(x,t)r_1(y)\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{5,j-k}(x,y,t) \quad j\geq 0$$

$$P_{6,j}(y-\Delta t,t+\Delta t) = P_{6,j}(1-\lambda\Delta t) + q\gamma r_2(y)\left[\int_0^\infty P_{4,j-1}(x,t)\mathrm{d}x\right]\Delta t + \lambda\Delta t\sum_{k=1}^{j} g_k P_{6,j-k}(y,t) \quad j\geq 1$$

where $\delta_{a,b}$ denotes Kronecker's delta. To obtain the stationary distributions, let the limiting state probabilities

$$P_{0,j} = \lim_{t\to\infty} P_{0,j}(t);\ P_{1,j}(x) = \lim_{t\to\infty} P_{1,j}(x,t); P_{2,j}(x,y) = \lim_{t\to\infty} P_{2,j}(x,y,t);$$
$$P_{3,j} = \lim_{t\to\infty} P_{3,j}(y,t); P_{4,j}(x) = \lim_{t\to\infty} P_{4,j}(x,t); P_{5,j}(x,y) = \lim_{t\to\infty} P_{5,j}(x,y,t);$$
$$P_{6,j}(y) = \lim_{t\to\infty} P_{6,j}(y,t).$$

As $t \to \infty$, the steady state equations are obtained as difference-differential equations. In order to solve and to obtain the probability generating functions and performance measures in steady state, we have used the Laplace transform and then followed by z- transform.

The steady state equations of the above equations are obtained as follows:

$$(\lambda + jv)P_{0,j} = P_{4,j}(0) + (1-\alpha)P_{1,j}(0) + (1-\delta_{0,j})(P_{3,j}(0) + P_{6,j}(0) \tag{4.1}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}P_{1,j}(x) = -(\lambda+\gamma)P_{1,j}(x) + \lambda\sum_{k=1}^{j+1} g_k P_{0,j-k+1}s_1(x)$$
$$+ (j+1)vP_{0,j+1}s_1(x) + \lambda\sum_{k=1}^{j} g_k P_{1,j-k}(x) + P_{2,j}(x,0) \tag{4.2}$$

$$-\frac{\partial}{\partial y}P_{2,j}(x,y) = -\lambda P_{2,j}(x,y) + \gamma p P_{1,j}(x)r_1(y) + \lambda\sum_{k=1}^{j} g_k P_{2,j-k}(x,y) \tag{4.3}$$

$$-\frac{\mathrm{d}}{\mathrm{d}y}P_{3,j}(y) = -\lambda P_{3,j}(y) + q\gamma\left[\int_0^\infty P_{1,j-1}(x)\mathrm{d}x\right] + \lambda\sum_{k=1}^{j} g_k P_{3,j}(y) \tag{4.4}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}P_{4,j}(x) = -(\lambda+\gamma)P_{4,j}(x) + \alpha P_{1,j}(0)s_2(x) + \lambda\sum_{k=1}^{j} g_k P_{4,j-k}(x) + P_{5,j}(x,0) \tag{4.5}$$

$$-\frac{\partial}{\partial y}P_{5,j}(x,y) = -\lambda P_{5,j}(x,y) + \gamma p P_{4,j}(x)r_1(y) + \lambda\sum_{k=1}^{j} g_k P_{5,j-k}(x,y) \tag{4.6}$$

$$-\frac{\mathrm{d}}{\mathrm{d}y}P_{6,j}(y) = -\lambda P_{6,j}(y) + q\gamma\left[\int_0^\infty P_{4,j-1}(x)\mathrm{d}x\right] r_2(y) + \lambda\sum_{k=1}^{j} g_k P_{6,j-k}(y). \tag{4.7}$$

Let $LST[P_{i,j}] = \tilde{P}_{i,j}(\theta); i = 1,4; LST[LST[P_{i,j}(x,y)]] = \tilde{\tilde{P}}_{i,j}(\theta,s); i = 2,5$
$LST[P_{i,j}(y) = \tilde{P}_{i,j}(s) i = 3,6,$

Taking LST on (2) with respect to remaining service time of FES, then the equation (2) becomes,

$$\theta\tilde{P}_{1,j}(\theta) - P_{1,j}(0) = (\lambda+\gamma)\tilde{P}_{i,j}(\theta) - \lambda\sum_{k=1}^{j+1} g_k P_{0,j-k+1}\tilde{S}_1(\theta) - (j+1)vP_{0,j+1}\tilde{S}_1(\theta) - \lambda\sum_{k=1}^{j} g_k\tilde{P}_{1,j-k}(\theta) - \tilde{P}_{2,j}(\theta,0). \quad (4.8)$$

Taking LST on (3), first with respect to remaining repair time of the server and then with respect to remaining service time of FES, the equation (3) becomes,

$$s\tilde{\tilde{P}}_{2,j}(\theta,s) - \tilde{P}_{2,j}(\theta,0) = \lambda\tilde{\tilde{P}}_{2,j}(\theta,s) - \gamma p\tilde{R}_1(s)\tilde{P}_{1,j}(\theta) - \lambda\sum_{k=1}^{j} g_k\tilde{\tilde{P}}_{2,j-k}(\theta,s). \quad (4.9)$$

Taking LST on (4) with respect to remaining repair time of the server, then equation (4) becomes,

$$s\tilde{P}_{3,j}(s) - P_{3,j}(0) = \lambda\tilde{P}_{3,j}(s) - q\gamma\left[\int_0^\infty P_{1,j-1}(x)\mathrm{d}x\right]\tilde{R}_2(s) - \lambda\sum_{k=1}^{j} g_k\tilde{P}_{3,j-k}(s). \quad (4.10)$$

Taking LST on (5) with respect to remaining service time of SOS, then equation (5) becomes,

$$\theta\tilde{P}_{4,j}(\theta) - P_{4,j}(0) = (\lambda+\gamma)\tilde{P}_{4,j}(\theta) - \alpha P_{1,j}(0)\tilde{S}_2(\theta) - \lambda\sum_{k=1}^{j} g_k\tilde{P}_{4,j-k}(\theta) - \tilde{P}_{5,j}(0). \quad (4.11)$$

Taking LST on (6), first with respect to remaining repair time of the server and then with respect to remaining service time of FES, the equation (6) becomes,

$$s\tilde{\tilde{P}}_{5,j}(\theta,s) - \tilde{P}_{5,j}(\theta,0) = \lambda\tilde{\tilde{P}}_{5,j}(\theta,s) - \gamma p\tilde{R}_1(s)\tilde{P}_{4,j}(\theta) - \lambda\sum_{k=1}^{j} g_k\tilde{\tilde{P}}_{5,j-k}(\theta,s). \quad (4.12)$$

Taking LST on (7) with respect to remaining repair time of the server, then equation (7) becomes,

$$s\tilde{P}_{6,j}(s) - P_{6,j}(0) = \lambda\tilde{P}_{6,j}(s) - q\gamma\left[\int_0^\infty P_{4,j-1}(x)\mathrm{d}x\right]\tilde{R}_2(s) - \lambda\sum_{k=1}^{j} g_k\tilde{P}_{6,j-k}(s). \quad (4.13)$$

To resolve the equations (1) and (8)–(13), we introduce the following generating functions,

$$P_0(z) = \sum_{j=0}^{\infty} P_{0,j} z^j.$$

$$\begin{aligned}
\tilde{P}_1(z,\theta) &= \sum_{j=0}^{\infty} \tilde{P}_{1,j}(\theta) z^j; P_1(z,0) = \sum_{j=0}^{\infty} P_{1,j}(0) z^j \\
\tilde{\tilde{P}}_2(z,\theta,s) &= \sum_{j=0}^{\infty} \tilde{\tilde{P}}_{2,j}(\theta,s) z^j; \tilde{P}_2(z,\theta,0) = \sum_{j=0}^{\infty} \tilde{P}_{2,j}(\theta,0) z^j \\
\tilde{P}_3(z,s) &= \sum_{j=1}^{\infty} \tilde{P}_{3,j}(s) z^j; P_3(z,0) = \sum_{j=1}^{\infty} P_{3,j}(0) z^j \qquad (4.14) \\
\tilde{P}_4(z,\theta) &= \sum_{j=0}^{\infty} \tilde{P}_{4,j}(\theta) z^j; P_4(z,0) = \sum_{j=0}^{\infty} P_{4,j}(0) z^j \\
\tilde{\tilde{P}}_5(z,\theta,s) &= \sum_{j=0}^{\infty} \tilde{\tilde{P}}_{5,j}(\theta,s) z^j; \tilde{P}_5(z,\theta,0) = \sum_{j=0}^{\infty} \tilde{P}_{5,j}(\theta,0) z^j \\
\tilde{P}_6(z,s) &= \sum_{j=1}^{\infty} \tilde{P}_{6,j}(s) z^j; P_6(z,0) = \sum_{j=1}^{\infty} P_{6,j}(0) z^j.
\end{aligned}$$

Using PGF, Equations (1) and (8)–(13) can be written follows,

$$\lambda P_0(z) + vzP_0'(z) = (1-\alpha)P_1(z,0) + P_3(z,0) + P_4(z,0) + P_6(z,0) \quad (4.15)$$

$$(\theta - (\lambda+\gamma) + \lambda X(z))\,\tilde{P}_1(z,\theta) = P_1(z,0) - \lambda \frac{X(z)}{z} P_0(z)\tilde{S}_1(\theta) \quad (4.16)$$

$$-vP_0'(z)\tilde{S}_1(\theta) - \tilde{P}_2(z,\theta,0) \quad (4.17)$$

$$(s - \lambda - \lambda X(z))\tilde{\tilde{P}}_2(z,\theta,s) = \tilde{P}_2(z,\theta,0) - p\gamma\tilde{P}_1(z,\theta)\tilde{R}_1(s) \quad (4.18)$$

$$(s - \lambda + \lambda X(z))\tilde{P}_3(z,s) = P_3(z,0) - q\gamma z\tilde{R}_2(s)\tilde{P}_1(z,0) \quad (4.19)$$

$$(\theta - (\lambda+\gamma) + \lambda X(z))\,\tilde{P}_4(z,\theta) = P_4(z,0) - -\alpha P_1(z,0)\tilde{S}_2(\theta) - \tilde{P}_5(z,\theta,0) \quad (4.20)$$

$$(s - \lambda - \lambda X(z))\tilde{\tilde{P}}_5(z,\theta,s) = \tilde{P}_5(z,\theta,0) - p\gamma\tilde{P}_4(z,\theta)\tilde{R}_1(s) \quad (4.21)$$

$$(s - \lambda + \lambda X(z))\tilde{P}_6(z,s) = P_6(z,0) - q\gamma z\tilde{R}_2(s)\tilde{P}_4(z,0). \quad (4.22)$$

Substituting $s = \lambda - \lambda X(z)$ in (17), (18), (20) and (21), the equations become as follows:

$$\tilde{P}_2(z,\theta,0) = p\gamma \tilde{P}_1(z,\theta)\tilde{R}_1(\lambda - \lambda X(z)) \tag{4.23}$$

$$P_3(z,0) = q\gamma z\tilde{P}_1(z,0)\tilde{R}_2(\lambda - \lambda X(z)) \tag{4.24}$$

$$\tilde{P}_5(z,\theta,0) = p\gamma \tilde{P}_4(z,\theta)\tilde{R}_1(\lambda - \lambda X(z)) \tag{4.25}$$

$$P_6(z,0) = q\gamma z\tilde{P}_4(z,0)\tilde{R}_2(\lambda - \lambda X(z)). \tag{4.26}$$

From equation (23), the equation (17) becomes,

$$\left[\theta - (\lambda+\gamma) + \lambda X(z) + \gamma p\tilde{R}_1(\lambda - \lambda X(z))\right]\tilde{P}_1(z,\theta) = P_1(z,0) - vP_0'(z)\tilde{S}_1(\theta) - \lambda\frac{X(z)}{z}P_0(z)\tilde{S}_1(\theta). \tag{4.27}$$

From equation (24), the equation (19) becomes,

$$\left[\theta - (\lambda+\gamma) + \lambda X(z) + \gamma p\tilde{R}_1(\lambda - \lambda X(z))\right]\tilde{P}_4(z,\theta) = P_4(z,0) - \alpha P_1(z,0)\tilde{S}_2(\theta) \tag{4.28}$$

Substituting $h(z) = \theta = \lambda + \gamma - \lambda X(z) - \gamma p\tilde{R}_1(\lambda - \lambda X(z))$ in (26) and (27), we get,

$$P_1(z,0) = vP_0'(z)\tilde{S}_1(h(z)) + \lambda\frac{X(z)}{z}P_0(z)\tilde{S}_1(h(z)) \tag{4.29}$$

$$P_4(z,0) = \alpha P_1(z,0)\tilde{S}_2(h(z)), \tag{4.30}$$

and therefore,

$$\tilde{P}_1(z,0) = \frac{(1-\tilde{S}_1(h(z)))}{h(z)}\left(\lambda\frac{X(z)}{z}P_0(z) + vP_0'(z)\right) \tag{4.31}$$

$$\tilde{P}_4(z,0) = \frac{\alpha(1-\tilde{S}_2(h(z)))}{h(z)}\left(\lambda\frac{X(z)}{z}P_0(z) + vP_0'(z)\right)\tilde{S}_1(h(z)), \tag{4.32}$$

where $h(z)$ satisfies the following properties:

(i) $h(1) = q\gamma$;

(ii) $h'(1) = -\lambda E(X)(1 + p\gamma E[R_1])$.

From the equations (15), (23), (25), (28) and (29), we can obtain,

$$P_0(z) = P_0(1)\exp\left[\frac{\lambda}{v}\int_1^z \left(\frac{f_1(u)}{f_2(u)}\right)\mathrm{d}u\right], \tag{4.33}$$

where

$$f_1(u) = h(u) - \begin{bmatrix} [\alpha\tilde{S}_2(h(u)) + (1-\alpha)]h(u)\tilde{S}_1(h(u)) \\ +q\gamma u\tilde{R}_2(\lambda - \lambda X(u))[(1 - \tilde{S}_1(h(u))[(1-\alpha) + \alpha\tilde{S}_2(h(u))]] \end{bmatrix} \frac{X(u)}{u},$$

$$f_2(u) = q\gamma u\tilde{R}_2(\lambda - \lambda X(u))[(1 - \tilde{S}_1(h(u))[(1-\alpha) + \alpha\tilde{S}_2(h(u))]] \\ -h(u)(u - [\alpha\tilde{S}_2(h(u)) + (1-\alpha)]\tilde{S}_1(h(u))).$$

From the equations (30) and (32), the marginal generating function of the orbit size when the server is busy with FES is given by

$$\tilde{P}_1(z,0) = \frac{\lambda(1 - X(z))(1 - \tilde{S}_1(h(z)))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda - \lambda X(z))[(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha\tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha\tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.34)$$

Substituting the equations (22) and (33) in (17), the marginal generating function of the orbit size when the server is down with a customer waiting in the server while doing FES is given by

$$\tilde{\tilde{P}}_2(z,0,0) = \frac{p\gamma(1 - \tilde{S}_1(h(z)))(1 - \tilde{R}_1(\lambda - \lambda X(z))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda - \lambda X(z))[(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha\tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha\tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}} \cdot (4.35)$$

Substituting the equations (23) and (33) in (18), the marginal generating function of the orbit size when the server is down without a customer waiting in the server while doing FES is given by,

$$\tilde{P}_3(z,0) = \frac{q\gamma z(1 - \tilde{S}_1(h(z)))(1 - \tilde{R}_2(\lambda - \lambda X(z))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda - \lambda X(z))[(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha\tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha\tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.36)$$

Substituting the equations (24) and (30) in (20), the marginal generating function of the orbit size when the server is busy with SOS is given by

$$\tilde{P}_4(z,0) = \frac{\alpha(1 - \tilde{S}_2(h(z)))\tilde{S}_1(h(z))\lambda(1 - X(z))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda - \lambda X(z))[(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha\tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha\tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.37)$$

Substituting the equations (24) and (36) in (20), the marginal generating function of the orbit size when the server is down with a customer waiting in the server while doing SOS is given by

$$\tilde{\tilde{P}}_5(z,0,0) = \frac{p\gamma(1 - \tilde{S}_2(h(z)))(1 - \tilde{R}_1(\lambda - \lambda X(z))\alpha\tilde{S}_1(h(z))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda - \lambda X(z))[(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha\tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha\tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}} \cdot (4.38)$$

Substituting the equations (25) and (36) in (21), the marginal generating function of the orbit size when the server is down without a customer waiting in the server while doing SOS is given by

$$\tilde{P}_6(z,0) = \frac{q\gamma z(1-\tilde{S}_2(h(z)))(1-\tilde{R}_2(\lambda-\lambda X(z))\alpha\tilde{S}_1(h(z))P_0(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda-\lambda X(z))[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]] \\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.39)$$

Theorem 4.1. *In the steady state, the probability generating function $P(z)$ of number of customers in orbit and the probability generating function $R(z)$ of the system size at an arbitrary epoch is expressed as follows,*

$$P(z) = \frac{P_0(z)(1-z)\begin{pmatrix} \lambda(1-X(z))+\gamma p(1-\tilde{R}_1(\lambda-\lambda X(z))) \\ +\gamma q\tilde{S}_1(h(z))[\alpha\tilde{S}_2(h(z))+(1-\alpha)] \end{pmatrix}}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda-\lambda X(z))[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]] \\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.40)$$

$$R(z) = \frac{P_0(z)(1-z)\tilde{S}_1(h(z))[\alpha\tilde{S}_2(h(z))+(1-\alpha)]h(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda-\lambda X(z))[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]] \\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (4.41)$$

Proof. The probability generating function of the orbit size and system size at an arbitrary epoch are given by

$$\begin{aligned} P(z) &= P_0(z) + \tilde{P}_1(z,0) + \tilde{\tilde{P}}_2(z,0,0) + \tilde{P}_3(z,0) + \tilde{P}_4(z,0) \\ &\quad + \tilde{\tilde{P}}_5(z,0,0) + \tilde{P}_6(z,0) \\ R(z) &= P_0(z) + z\tilde{P}_1(z,0) + z\tilde{\tilde{P}}_2(z,0,0) + \tilde{P}_3(z,0) \\ &\quad + z\tilde{P}_4(z,0) + z\tilde{\tilde{P}}_5(z,0,0) + \tilde{P}_6(z,0). \end{aligned}$$

Using the equations (33)–(38), the probability generating functions $P(z)$ and $R(z)$ can be found. This completes the proof. □

Some interesting steady state probabilities are derived as follows:

(1) the probability that the server is idle is

$$P_0(1) = 1 - \frac{\lambda E[X]\left(1+\gamma[pE[R_1]+qE[R_2]](1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]\right)}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}. \quad (4.42)$$

(2) the probability that the server is busy with FES is

$$\tilde{P}_1(1,0) = \frac{\lambda E[X](1-\tilde{S}_1(q\gamma))}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}. \quad (4.43)$$

(3) the probability that the server is down with a customer in the server

$$\tilde{\tilde{P}}_2(1,0,0) = \frac{\lambda p E[X]E[R_1](1-\tilde{S}_1(q\gamma))}{q\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\cdot \tag{4.44}$$

(4) the probability that the server is down without a customer in the server is

$$\tilde{P}_3(1,0) = \frac{\lambda E[X]E[R_2](1-\tilde{S}_1(q\gamma))}{\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\cdot \tag{4.45}$$

(5) the probability that the server is busy with SOS is

$$\tilde{P}_4(1,0) = \frac{\alpha\lambda E[X](1-\tilde{S}_2(q\gamma))\tilde{S}_1(q\gamma)}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\cdot \tag{4.46}$$

(6) the probability that the server is down with a customer in the server while doing SOS is

$$\tilde{\tilde{P}}_5(1,0,0) = \frac{\lambda p E[X]E[R_1]\alpha(1-\tilde{S}_2(q\gamma))\tilde{S}_1(q\gamma)}{q\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\cdot \tag{4.47}$$

(7) the probability that the server is down without a customer in the server while doing SOS is,

$$\tilde{P}_6(1,0) = \frac{\lambda E[X]E[R_2]\alpha(1-\tilde{S}_2(q\gamma))\tilde{S}_1(q\gamma)}{\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\cdot \tag{4.48}$$

Theorem 4.2. *If T_b and T_c be the length of busy period and busy cycle, then under the steady state conditions, we have*

$$E[T_b] = \frac{\exp\left[\frac{\lambda}{v}\int_0^1\left(\frac{f_1(u)}{f_2(u)}\right)\mathrm{d}u\right]}{g_2} - \frac{1}{\lambda} \tag{4.49}$$

$$\text{and}\qquad E[T_c] = \frac{\exp\left[\frac{\lambda}{v}\int_0^1\left(\frac{f_1(u)}{f_2(u)}\right)\mathrm{d}u\right]}{g_2}\cdot \tag{4.50}$$

where

$$g_2 = \lambda\left(1 - \frac{\lambda E[X](1+\gamma[pE[R_1]+qE[R_2]](1-\tilde{S}_1(q\gamma))[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)])}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\right)\cdot$$

Proof. By applying the argument of alternating renewal process, the results are found directly from the well-known result.

$$E[T_b] = \frac{1}{\lambda}\left(\frac{1}{p_0}-1\right)\; E[T_c] = \tfrac{1}{\lambda}\left(\tfrac{1}{p_0}\right)\cdot$$

From the equation (23) substituting z =0,

$$p_0 = \lambda\left(1 - \frac{\lambda E[X](1+\gamma[pE[R_1]+qE[R_2]](1-\tilde{S}_1(q\gamma))[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)])}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}\right) \times \exp\left[\frac{\lambda}{v}\int_0^1\left(\frac{f_1(u)}{f_2(u)}\right)\mathrm{d}u\right]\cdot \quad \square$$

5. Reliability analysis

This section discusses some reliability indexes of the queueing system under study, specifically the analysis of availability of the server, the failure frequency of the server The server is available when it is either idle or working on a customer. The following results concerning the availability of the server.

(1) From the equations (32), (33) and (36), the marginal generating function of the orbit size when the server is available is given by

$$P_0(z)+\tilde{P}_2(z,0)+\tilde{P}_3(z,0) = \frac{\begin{pmatrix}[1-\tilde{S}_1(h(z))((1-\alpha)+\alpha\tilde{S}_2(h(z))][\lambda(1-X(z))+q\gamma zR_2(\lambda-\lambda X(z))] \\ -h(z)[z-\tilde{S}_1(h(z))[\alpha\tilde{S}_2(h(z))+(1-\alpha)]]\end{pmatrix}P_0(z)}{\begin{bmatrix}q\gamma z\tilde{R}_2(\lambda-\lambda X(z))[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]] \\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z)))\end{bmatrix}}. \tag{5.1}$$

(2) Substituting z=1 in the equation (50), the probability that the server is available is

$$P_0(1)+\tilde{P}_1(1,0)+\tilde{P}_4(1,0) = 1 - \frac{\lambda E[X](pE[R_1]+qE[R_2])(1-\tilde{S}_1(q\gamma))[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)])}{q\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]}\cdot \tag{5.2}$$

(3) The failure frequency of the server is

$$W_f = \sum_{j=0}^{\infty}\int_0^{\infty}\gamma(P_{1,j}(x)+P_{4,j}(x))\mathrm{d}x = \lim_{z\to 1}\gamma[\tilde{P}_1(z,0)+\tilde{P}_4(z,0)]$$
$$= \frac{\lambda(1-\tilde{S}_1(q\gamma))[\alpha\tilde{S}_2(q\gamma)+1-\alpha]}{q\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]}. \tag{5.3}$$

6. Performance characteristics

Some useful results of proposed model are listed below:

(a) The mean number of customers in the orbit LQ is derived using the equation (39)

$$L_Q = \lim_{z\to 1}\frac{\mathrm{d}}{\mathrm{d}z}P(z).$$

Using L'Hospital rule, we have,

$$L_Q = \frac{M_4M_1 - M_3M_2}{2M_1^2}.$$

where

$$\begin{aligned}
M_1 =& (1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha])\,(\gamma[pR_{11}+qR_{21}]\\
&+\lambda E[X]) - q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]\\
M_2 =& q\gamma\Big[(R_{12}+2R_{21})(1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha])\\
&-2h'(1)\Big(S_{11}[\alpha\tilde{S}_2(q\gamma)+1-\alpha]+\alpha S_{21}\tilde{S}_1(q\gamma)\Big)(1+R_{21})\Big]\\
&-h'(1)(1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha])\\
&-2h'(1)\Big[1+h(1)S_{11}[\alpha\tilde{S}_2(q\gamma)+1-\alpha]+h'(1)\tilde{S}_1(q\gamma)S_{21}\alpha\Big]\\
M_3 =& -P_0(1)q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha];\\
M_4 =& -2P_0'(1)q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]+2P_0(1)\\
&\Big(\lambda E(X)+\gamma pR_{11}+\gamma qh'(1)S_{11}[\alpha\tilde{S}_2(q\gamma)+1-\alpha]+\gamma qh'(1)S_{21}\alpha\tilde{S}_1(q\gamma)\Big).
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&R_{11} = \lambda E[X]E[R_1]; R_{21} = \lambda E[X]E[R_2];\\
&R_{12} = \lambda E[X^2]E[R_1] + \lambda^2(E[X])^2E[R_1^2];\\
&R_{22} = \lambda E[X^2]E[R_2] + \lambda^2(E[X])^2E[R_2^2];\\
&S_{11} = \textstyle\int_0^\infty t\mathrm{e}^{-q\gamma t}s_1(t)\mathrm{d}t; \tilde{S}_1(q\gamma) = \int_0^\infty \mathrm{e}^{-q\gamma t}s_1(t)\mathrm{d}t;\\
&S_{21} = \textstyle\int_0^\infty t\mathrm{e}^{-q\gamma t}s_2(t)\mathrm{d}t; \tilde{S}_2(q\gamma) = \int_0^\infty \mathrm{e}^{-q\gamma t}s_2(t)\mathrm{d}t;
\end{aligned}$$

$$P_0'(1) = \frac{\lambda P_0(1)\left(h'(1) - \begin{bmatrix} q\gamma[-h'(1)\alpha S_{21}]\tilde{S}_1(q\gamma)+h'(1)\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha] \\ +q\gamma[-h'(1)S_{11}[\alpha\tilde{S}_2(q\gamma)+1-\alpha] \\ +q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]+q\gamma(1-E[X]) \end{bmatrix}\right)}{vM_1}.$$

(b) Mean waiting time in retrial queue:
Using Little's formula, the mean waiting time in the retrial queue W_Q is obtained as,

$$W_Q = E[W] = \frac{L_Q}{\lambda E[X]}. \tag{6.2}$$

(c) The mean number of customers in the systems:

$$L_s = L_Q + \frac{\lambda E[X](1+\gamma[pE[R_1]+qE[R_2]](1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+1-\alpha]}. \tag{6.3}$$

(d) Mean waiting time in the system

$$W_s = \frac{L_s}{\lambda E[X]}. \tag{6.4}$$

7. Stochastic decomposition

This section investigates the stochastic decomposition law. First, we observe the following relationship between generating functions

$$\lim_{v\to\infty} R(z) = R^{\infty}(z) = \frac{P_0(1)(1-z)\tilde{S}_1(h(z))[\alpha\tilde{S}_2(h(z))+(1-\alpha)]h(z)}{\begin{bmatrix} q\gamma z\tilde{R}_2(\lambda-\lambda X(z))[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]] \\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}} \tag{7.1}$$

where $P_0(1) = 1 - \frac{\lambda E[X]\left(1+\gamma[pE[R_1]+qE[R_2]](1-\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]\right)}{q\gamma\tilde{S}_1(q\gamma)[\alpha\tilde{S}_2(q\gamma)+(1-\alpha)]}$.

$R^{\infty}(z)$ is the generating function of the stationary distribution of the number of customers in the $M^{[x]}/G/1/\infty$ queueing system with active server breakdowns and two types of repair time and second optional service. Therefore, when $v\to\infty$ (high rate of retrials), our model behaves as a standard queueing system with batch arrivals and server breakdowns which agrees the intuitive expectations.

This is to be noted that the generating function of the system size distribution can be written as $R(z) = R^{\infty}(z)\frac{P_0(z)}{P_0(1)}$ where the fraction corresponds to the probability generating function to the system size given that the server is idle. Indeed the above equality provides the stochastic decomposition property for our queueing system in an immediate way. *i.e.*, the number of customers in our system is the sum of two independent random variables: one is the number of customers in the corresponding standard system with batch arrivals and server breakdowns and two phase service, and other is the number of repeated customers given that the server is idle.

8. Particular cases

Case I. In case, if there is no optional second phase of service and single type of repair time and single arrival, then

(i) the PGF of distribution of number of customers in the orbit is

$$P(z) = \frac{\left[\lambda(1-z) + \gamma p(1 - \tilde{R}_1(\lambda - \lambda z) + \gamma q \tilde{S}_1(h(z))\right] P_0(z)(1-z)}{q\gamma z \tilde{R}_1(\lambda - \lambda z)(1 - \tilde{S}_1(h(z))) - h(z)(z - \tilde{S}_1(h(z)))}. \quad (8.1)$$

Equation (58) agrees the PGF of the distribution of number of customers in the orbit of $M/G/1$ retrial queue with active breakdowns and Bernoulli schedule in the server obtained by Atencia *et al.*[7].

Case II. Single server batch arrival retrial queue with Exponential FES time subject to breakdowns, two types of repair and second optional service

Assume that the service time is exponential with probability density function $s_1(t) = \mu_1 e^{-\mu_1 t}$, where μ_1 is the parameter. Hence the PGF of the orbit size is as follows when $\tilde{S}_1(h(z)) = \frac{\mu_1}{\mu_1 + h(z)}; \tilde{S}_2(h(z)) = \frac{\mu_2}{\mu_2 + h(z)}$,

$$P(z) = \frac{P_0(z)(1-z)\begin{pmatrix} \lambda(1 - X(z)) + \gamma p(1 - \tilde{R}_1(\lambda - \lambda X(z))) \\ +\gamma q \frac{\mu_1}{\mu_1 + h(z)}[\alpha \frac{\mu_2}{\mu_2 + h(z)} + (1-\alpha)] \end{pmatrix}}{\begin{bmatrix} q\gamma z \tilde{R}_2(\lambda - \lambda X(z))[(1 - \frac{\mu_1}{\mu_1 + h(z)}[(1-\alpha) + \alpha \frac{\mu_2}{\mu_2 + h(z)}]] \\ -h(z)(z - [\alpha \frac{\mu_2}{\mu_2 + h(z)} + (1-\alpha)]\frac{\mu_1}{\mu_1 + h(z)}) \end{bmatrix}}. \quad (8.2)$$

Case III. Single server batch arrival retrial queue with two types of k-Erlangian repair time and second optional service.

It is assumed that the two repair times is an k-Erlang with probability density function, $r_i(x) = \frac{(ku_i)^k x^{k-1} e^{-ku_i x}}{(k-1)!}; i = 1, 2,\ k > 0$; where u_i is the parameter. Hence the PGF of the retrial queue size distribution as follows when $\tilde{R}_i(\lambda - \lambda X(z)) = \left(\frac{u_i k}{u_i k + \lambda(1 - X(z))}\right)^k; i = 1, 2$

$$P(z) = \frac{P_0(z)(1-z)\begin{pmatrix} \lambda(1 - X(z)) + \gamma p(1 - \left(\frac{u_1 k}{u_1 k + \lambda(1 - X(z))}\right)^k) \\ +\gamma q \tilde{S}_1(h(z))[\alpha \tilde{S}_2(h(z)) + (1-\alpha)] \end{pmatrix}}{\begin{bmatrix} q\gamma z \left(\frac{u_2 k}{u_2 k + \lambda(1 - X(z))}\right)^k [(1 - \tilde{S}_1(h(z))[(1-\alpha) + \alpha \tilde{S}_2(h(z))]] \\ -h(z)(z - [\alpha \tilde{S}_2(h(z)) + (1-\alpha)]\tilde{S}_1(h(z))) \end{bmatrix}}. \quad (8.3)$$

Case IV. Single server batch arrival retrial queue with two types of hyper Exponential repair time and second optional service.

Considering the case of Hyper Exponential repair time random variable, the pdf of Hyper Exponential vacation time is given as follows, $r_i(x) = cu_{1,i}\mathrm{e}^{-u_{1,i}x} + (1-c)u_{2,i}\mathrm{e}^{-u_{2,i}x}; i = 1, 2$. Hence the PGF of the orbit size is given by,

$$P(z) = \frac{P_0(z)(1-z)\left(\begin{array}{l}\lambda(1-X(z))+\gamma p\left(1-\left(\frac{cu_{1,1}}{u_{1,1}+\lambda(1-X(z))}+\frac{(1-c)u_{2,1}}{u_{2,1}+\lambda(1-X(z))}\right)\right)\\ +\gamma q\tilde{S}_1(h(z))[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\end{array}\right)}{\left[\begin{array}{l}q\gamma z\left(\frac{cu_{1,2}}{u_{1,2}+\lambda(1-X(z))}+\frac{(1-c)u_{2,2}}{u_{2,2}+\lambda(1-X(z))}\right)[(1-\tilde{S}_1(h(z))[(1-\alpha)+\alpha\tilde{S}_2(h(z))]]\\ -h(z)(z-[\alpha\tilde{S}_2(h(z))+(1-\alpha)]\tilde{S}_1(h(z)))\end{array}\right]}. \tag{8.4}$$

9. Numerical results

Consider a computer network which consists of a group of processors connected with a central transmission unit (CTU). Typically, the messages arrive at CTU following Poisson stream. If the transmission medium is available, the CTU immediately sends a message; otherwise the message will be stored in a buffer (retrial group) and the messages in CTU must retry for the transmission some time later. CTU may well be subjected to lengthy and unpredictable breakdowns like scheduled backups and unpredictable failures, while transmitting the messages. If CTU is subjected to unpredictable breakdowns (not so lengthy) while transmitting the message, CTU gives the priority to transmit that message after being repaired (type-I repair). In the case of lengthy breakdowns, the messages in buffer must retry for the transmission some time later after being repaired (type-II repair). The above situation can be modelled as a batch arrival single server retrial queueing system with active breakdowns, two types of repair time and second optional service. It is important to study the effect of Bernoulli schedule of the server p and the effect of failure rate γ with mean orbit size L_Q and mean waiting time of a message in orbit W_Q.

9.1. Effect of Bernoulli SOS probability α and the repair rate r_1 on mean orbit size

In Tables 1–3 for repair time parameters $r_2 = 10$, the mean orbit size is compared with varying values of the Bernoulli Schedule probability of the server α and with varying repair rate r_1 when second type of repair, Second type of Repair and Second optional Service time distribution follow Exponential, Erlangian-2 and Hyper-Exponential, respectively. It is observed that:

- the mean buffer size is increased if the Bernoulli schedule SOS probability α increases.
- the mean buffer size is decreased if the repair rate r_1 is increased.

TABLE 1. Mean Orbit Size L_Q *versus* Bernoulli SOS probability α and first type of repair rate r_1(both repair time and SOS time follow Exponential with $r_2 = 10$ and SOS rate $s_2 = 15$).

α	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0	0.6767	0.6489	0.6351	0.6268	0.6213	0.6174	0.6145	0.6122	0.6104
0.1	0.6891	0.6613	0.6474	0.6392	0.6337	0.6298	0.6269	0.6246	0.6228
0.2	0.7008	0.6731	0.6593	0.6511	0.6456	0.6417	0.6388	0.6365	0.6347
0.3	0.7118	0.6843	0.6707	0.6625	0.6570	0.6532	0.6502	0.6479	0.6462
0.4	0.7221	0.6949	0.6814	0.6733	0.6679	0.6639	0.6611	0.6589	0.6571
0.5	0.7314	0.7047	0.6913	0.6833	0.6779	0.6742	0.6713	0.6691	0.6673
0.6	0.7398	0.7136	0.7005	0.6926	0.6874	0.6836	0.6808	0.6786	0.6769
0.7	0.7470	0.7216	0.7088	0.7011	0.6959	0.6923	0.6895	0.6874	0.6856
0.8	0.7530	0.7284	0.7160	0.7085	0.7035	0.6999	0.6973	0.6952	0.6935
0.9	0.7576	0.7341	0.7221	0.7149	0.7101	0.7066	0.7040	0.7019	0.7004

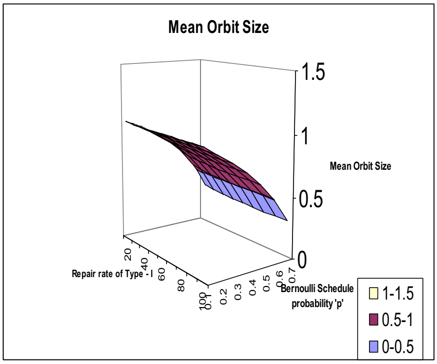
TABLE 2. Mean Orbit Size L_Q *versus* Bernoulli SOS probability α and first type of repair rate r_1(both repair time and SOS time follow Erlangian-2 with $r_2 = 10$ and SOS rate $s_2 = 15$).

α	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0	0.8181	0.7597	0.7319	0.7156	0.7049	0.6975	0.6919	0.6876	0.6841
0.1	0.8277	0.7702	0.7428	0.7268	0.7163	0.7089	0.7034	0.6991	0.6957
0.2	0.8357	0.7795	0.7526	0.7369	0.7266	0.7193	0.7139	0.7098	0.7064
0.3	0.8419	0.7873	0.7612	0.7458	0.7358	0.7287	0.7234	0.7193	0.7160
0.4	0.8463	0.7936	0.7682	0.7534	0.7436	0.7367	0.7316	0.7276	0.7245
0.5	0.8484	0.7980	0.7737	0.7594	0.7501	0.7434	0.7385	0.7346	0.7316
0.6	0.8481	0.8005	0.7774	0.7638	0.7549	0.7485	0.7438	0.7402	0.7373
0.7	0.8452	0.8008	0.7791	0.7664	0.7579	0.7519	0.7475	0.7440	0.7413
0.8	0.8392	0.7986	0.7787	0.7668	0.7589	0.7534	0.7493	0.7460	0.7435
0.9	0.8299	0.7938	0.7757	0.7649	0.7578	0.7527	0.7489	0.7459	0.7436

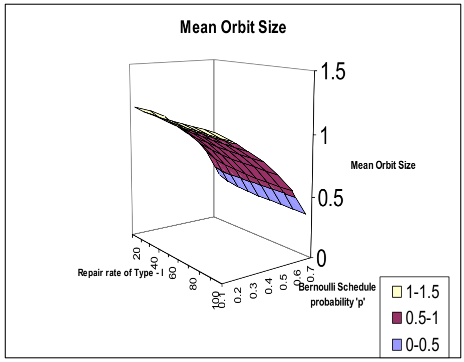
9.2. Effect of Bernoulli schedule probability p and the repair rate r_1 on mean orbit size

In Figures 1–2, for repair time parameters $r_2 = 10$, the mean waiting time of a packet in orbit is compared with varying values of the Bernoulli Schedule probability of the server p and with varying repair rate r_1 when First Type of Repair, Second type of Repair and Second optional Service time distribution follow Exponential, Erlangian-2 and Hyper-Exponential, respectively. It is observed that:

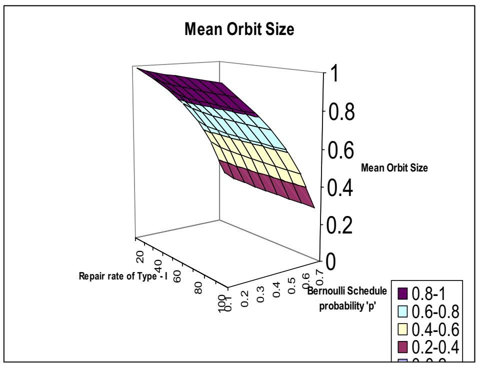
- the mean number of packets in buffer LQ is decreased if the repair rate r_1 increases.
- the mean buffer size LQ is decreased if the Bernoulli schedule probability p is increased.



(a) Exponential



(b) Erlangian-2



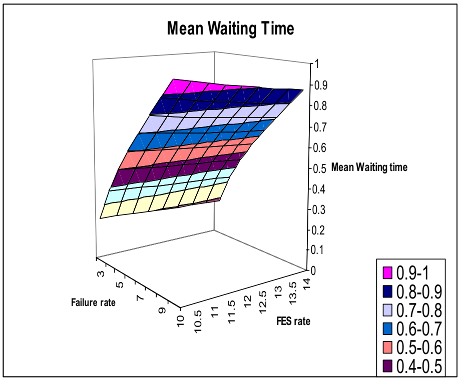
(c) Hyper Exponential

FIGURE 1. Mean Orbit Size *vs* Bernoulli Schedule probability p if first type and second type of repair time and second optional service time follow Exponential $(r_1 = 20; r_2 = 10; s_2 = 15)$.

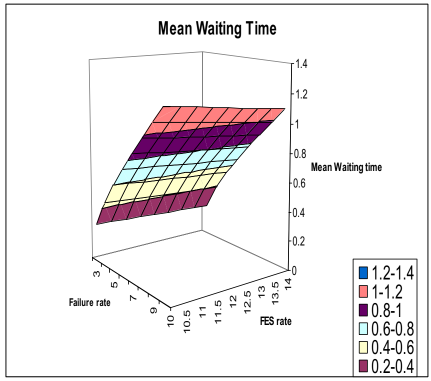
9.3. Effect of Bernoulli schedule probability p and the repair rate r_1 on mean system size L_s

In Tables 4– 6, for second type of repair time with parameter$r_2 = 10$, the mean number of packets in system is compared with varying values of the first type of repair timerr_1 and with varying Bernoulli SOS probability p when First Type of Repair, Second type of Repair and Second optional Service time distribution follow Exponential, Erlangian-2 and Hyper-Exponential, respectively. It is observed that:

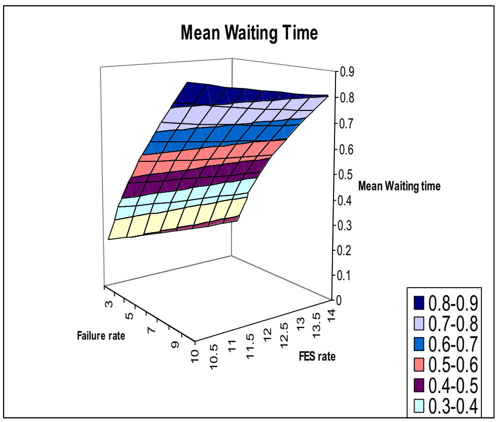
- the mean number of packets in system LS decreases if the first type of repair time r_1 increases.
- the mean number of packets in system LS decreases if Bernoulli SOS probability increases.



(a) Exponential



(b) Erlangian -2



(c) Hyper Exponential

FIGURE 2. Mean Orbit Size *vs* Bernoulli Schedule probability p if first type and second type of repair time and second optional service time follow Erlang-2 ($r_1 = 20; r_2 = 10; s_2 = 15$).

9.4. EFFECT OF BERNOULLI SOS PROBABILITY α AND FAILURE RATE γ ON MEAN WAITING TIME IN ORBIT

In Figures 4–6, for repair time parameters$r_1 = 20$ and$r_2 = 10$, the mean waiting time of a packet in orbit is compared with varying values of FES rate of the server μ_1 and with varying failure rate γ when First Type of Repair, Second type of Repair and Second optional Service time distribution follow Exponential and Hyper-Exponential, respectively. It is observed that:

- the waiting time of a packet in orbit is increased if the failure rate γ increases.
- the waiting time of a packet in orbit decreases if the FES rate is increased.

Table 3. Mean Orbit Size L_Q *versus* Bernoulli SOS probability α and first type of repair rate r_1(both repair time and SOS time follow Hyper-Exponential distribution with $r_2 = 10$ and SOS rate $s_2 = 15$).

α	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0	0.6382	0.6182	0.6084	0.6026	0.5987	0.5959	0.5939	0.5923	0.5911
0.1	0.6508	0.6307	0.6209	0.6150	0.6111	0.6084	0.6063	0.6047	0.6035
0.2	0.6629	0.6428	0.6329	0.6271	0.6232	0.6205	0.6184	0.6168	0.6155
0.3	0.6745	0.6545	0.6447	0.6388	0.6349	0.6322	0.6301	0.6285	0.6272
0.4	0.6855	0.6656	0.6559	0.6500	0.6462	0.6434	0.6414	0.6398	0.6385
0.5	0.6958	0.6762	0.6665	0.6607	0.6569	0.6542	0.6521	0.6506	0.6493
0.6	0.7055	0.6861	0.6766	0.6709	0.6671	0.6644	0.6624	0.6608	0.6595
0.7	0.7143	0.6953	0.6859	0.6803	0.6766	0.6739	0.6719	0.6704	0.6691
0.8	0.7222	0.7037	0.6945	0.6889	0.6853	0.6827	0.6808	0.6793	0.6781
0.9	0.7291	0.7111	0.7022	0.6968	0.6933	0.6907	0.6888	0.6873	0.6862

Table 4. Mean System Size L_S *versus* Bernoulli probability p and first type of repair rate r_1(both repair time and SOS time follow Exponential distribution with $r_2 = 10$ and SOS rate $s_2 = 15$).

p	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0.1	1.7560	1.7488	1.7452	1.7431	1.7417	1.7407	1.7399	1.7393	1.7388
0.2	1.7028	1.6899	1.6836	1.6798	1.6772	1.6754	1.6741	1.6731	1.6722
0.3	1.6383	1.6215	1.6132	1.6082	1.6049	1.6025	1.6008	1.5994	1.5983
0.4	1.5584	1.5396	1.5303	1.5248	1.5211	1.5185	1.5165	1.5149	1.5138
0.5	1.456	1.4376	1.4285	1.4231	1.4195	1.4169	1.4149	1.4135	1.4123
0.6	1.3164	1.3017	1.2944	1.2901	1.2872	1.2852	1.2836	1.2824	1.2815
0.7	1.1059	1.1000	1.0971	1.0954	1.0943	1.0935	1.0929	1.0924	1.0921

Table 5. Mean System Size L_S *versus* Bernoulli probability p and first type of repair rate r_1(both repair time and SOS time follow Erlangian-2 distribution with $r_2 = 10$ and SOS rate $s_2 = 15$).

p	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0.1	1.7645	1.7499	1.7432	1.7393	1.7368	1.7349	1.7337	1.7327	1.7319
0.2	1.7299	1.7039	1.6918	1.6848	1.6802	1.6771	1.6747	1.6729	1.6715
0.3	1.6790	1.6450	1.6292	1.6199	1.6140	1.6099	1.6068	1.6044	1.6025
0.4	1.6074	1.5693	1.5515	1.5412	1.5345	1.5298	1.5263	1.5236	1.5215
0.5	1.5065	1.4690	1.4515	1.4414	1.4349	1.4303	1.4269	1.4243	1.4222
0.6	1.3597	1.3291	1.3149	1.3069	1.3017	1.2979	1.2953	1.2932	1.2916
0.7	1.1277	1.1138	1.1079	1.1048	1.1029	1.1016	1.1007	1.0999	1.0994

TABLE 6. Mean System Size L_S *versus* Bernoulli probability p and first type of repair rate r_1(both repair time and SOS time follow Hyper-Exponential distribution with $r_2 = 10$ and SOS rate $s_2 = 15$).

p	$r_1 = 20$	$r_1 = 30$	$r_1 = 40$	$r_1 = 50$	$r_1 = 60$	$r_1 = 70$	$r_1 = 80$	$r_1 = 90$	$r_1 = 100$
0.1	1.7540	1.7488	1.7462	1.7446	1.7437	1.7429	1.7424	1.7420	1.7417
0.2	1.6952	1.6859	1.6813	1.6786	1.6768	1.6755	1.6746	1.6738	1.6732
0.3	1.6265	1.6143	1.6084	1.6048	1.6025	1.6008	1.5996	1.5986	1.5978
0.4	1.5441	1.5305	1.5239	1.5199	1.5173	1.5155	1.5141	1.5130	1.5121
0.5	1.4411	1.4279	1.4214	1.4176	1.4150	1.4132	1.4119	1.4108	1.4099
0.6	1.3039	1.2933	1.2881	1.2850	1.2830	1.2816	1.2805	1.2797	1.2789
0.7	1.0999	1.0958	1.0938	1.0926	1.0919	1.0913	1.0909	1.0906	1.0904

10. Conclusion

This paper concerns about stability analysis and reliability analysis of the single server batch arrival retrial queue with active breakdowns, second optional service and impatient behaviour. The impatient behaviour of customer during the failure of the server leads to many possible research directions in which the customer allowed to be waited at the service point during breakdowns if the repair time is expected to be short. Such system increases efficiency in such a way that the waiting customer is able to perform useful work during the server repair time. For such systems, numerical illustrations are clearly carried out to illustrate the influence of various system parameters on important performance measures. Some interesting particular cases are also discussed. A natural extension of the foregoing model consists in considering general inter-retrial times and breakdowns independently of the server state. It would be very interesting to examine the discrete time counterpart of our continuous time queueing system, due to the usefulness of the discrete time queueing theory to model many practical problems.

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References

[1] A. Aissani, Unreliable queueing with repeated orders. *Microelectr. Reliab.* **33** (1993) 2093–2106.

[2] A. Aissani, A retrial queue with redundancy and unreliable server. *Queueing Syst.* **17** (1994) 431–449.

[3] A. Aissani and J. Artalejo, On the single server retrial queue subject to breakdowns. *Queueing Syst.* **30** (1998) 309–321.

[4] J.R. Artalejo, Accessible bibliography on retrial queues. *Math. Comput. Model.* **30** (1999a) 1–6.

[5] J.R. Artalejo, A classified bibliography of research on retrial queues: progress in 1990-1999 *Top* **7** (1999b) 187–211.

[6] J.R. Artalejo and G. Choudhury, Steady state analysis of an $M/G/1$ queue with repeated attempts and two phase service. *Quality Technol. Quantitat. Manag.* **1** (2004) 189–199.

[7] I. Atencia, I. Fortes, P. Morena and S. Sanchez, An $M/G/1$ retrial queue with active breakdowns and Bernoulli schedule in the server. *Informat. Manag. Sci.* **17** (2006) 1–17.

[8] I. Atencia, G. Bouza and P. Morena, An retrial queue with server breakdowns and constant rate of repeated attempts. *Annal. Operat. Res.* **157** (2008) 225–243.

[9] I. Atencia and P. Moreno, A single server retrial queue with general retrial times and Bernoulli Schedule. *Appl. Math. Comput.* **162** (2005) 855–880.

[10] B. Avi-Itzhak and P.P. Naor, Some queueing problems with the service station subject to breakdown. *Operat. Res.* **11** (1963) 303–320.

[11] G. Choudhury, Some aspects of an $M/G/1$ queueing system with optional second service. *Top* **11** (2003) 141–150.

[12] G. Choudhury and Kailash Madan, A batch arrival Bernoulli Vacation queue with a random setup time under restricted admissibility policy. *Int. J. Oper. Res.* **2** (2007) 81–97.

[13] G. Choudhury and D. Kandarpa, An $M/G/1$ retrial queue with two phases of service subject to the server breakdowns and repair. *Performance Evaluation* **65** (2008) 714–724.

[14] G.I. Falin, Aggregate arrival of customers in one line queue with repeated calls. *Ukrainian Math J.* **28** (1976) 437–440.

[15] G.I. Falin, A survey of retrial queues. *Queueing Syst.* **7** (1990) 127–168.

[16] G.I. Falin and J.G.C. Templeton, Retrial Queues. Chapman and Hall, London (1997).

[17] J.C. Ke., Batch Arrival queues under vacation policies with server breakdowns and start-up /close-down times. *Appl. Math. Model.* **31** (2007) 1282–1292.

[18] J.C. Ke. An $M^x/G/1$ system with second optional service and J additional options for Service. *Appl. Math. Model.* **32** (2008) 443–458.

[19] B. Krishnakumar, A. Vijayakumar and D. Arivudainambi, An $M/G/1$ retrial queueing system with two-phase service and preemptive resume. *Annal. Operat. Res.* **113** (2002) 61–79.

[20] B. Krishnakumar, S. Pavai Madheswari and A. Vijayakumar, The $M/G/1$ retrial queue with feedback and starting failures. *Appl. Math. Model.* **26** (2002) 1057–1076.

[21] H. Li and Y.Q. Zhao, A retrial queue with constant retrial rate, server downs and impatient customers. *Stoch. Models* **21** (2005) 531–550.

[22] K.C. Madan, An $M/G/1$ queue with second optional service. *Queueing Syst.* **34** (2000) 37–46.

[23] J. Medhi, A single server Poisson input queue with a second optional channel. *Queueing Syst.* **42** (2002) 239–242.

[24] M. Senthil Kumar and R. Arumuganathan, On the single server Batch Arrival Retrial Queue with General vacation Time under Bernoulli schedule and two phases of Heterogeneous service. *Quality Technol. Quantit. Manag.* **5** (2008) 145–160.

[25] M. Senthil Kumar and R. Arumuganathan, An $M^X/G/1$ retrial queue with two-phase service subject to active server breakdowns and two types of repair. *Int. J. Oper. Res.* **8** (2010) 261–291.

[26] M. Senthil Kumar, A discrete-time $Geo^{[X]}/G/1$ retrial queue with general retrial time and M-additional options for service. *RAIRO Operat. Res.* **45** (2011) 131–152.

[27] Y.H. Tang, A single server $M/G/1$ queueing system subject to breakdowns. Some reliability and queueing problems. *Microelectr. Reliab.* **37** (1997) 315–321.

[28] J. Wang, J. Cao and Q. Li, Reliability analysis of the retrial queue with server breakdowns and repairs. *Queueing Syst.* **38** (2001) 363–380.

[29] J. Wang and J. Li, A Repairable $M/G/1$ retrial queue with Bernoulli vacation and Two Phase service. *Quality Technol. Quantit. Manag.* **5** (2008) 179–192.

[30] Wu Xiaoyong, Brill Percy, Hlynka Myron and Wang Jinting, An $M/G/1$ retrial queue with balking and retrials during service. *Int. J. Oper. Res.* **1** (2005) 30–57.

[31] T. Yang and H. Li, The $M/G/1$ retrial queue with the server subject to starting failures. *Queueing Syst.* **16** (1994) 83–96.

[32] D. Yue and J. Cao, Reliability analysis of a $M_1^x, M_2^x/G_1, G_2/1$ queueing system with a repairable service station. *Microelect. Reliab.* **37** (1997) 1225–1231.