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NEW RESULTS ON SEMIDEFINITE BOUNDS FOR ℓ_1-CONSTRAINED NONCONVEX QUADRATIC OPTIMIZATION *

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Abstract. In this paper, we show that the direct semidefinite programming (SDP) bound for the nonconvex quadratic optimization problem over ℓ_1 unit ball (QPL1) is equivalent to the optimal d.c. (difference between convex) bound for the standard quadratic programming reformulation of QPL1. Then we disprove a conjecture about the tightness of the direct SDP bound. Finally, as an extension of QPL1, we study the relaxation problem of the sparse principal component analysis, denoted by QPL2L1. We show that the existing direct SDP bound for QPL2L1 is equivalent to the doubly nonnegative relaxation for variable-splitting reformulation of QPL2L1.

Keywords. Quadratic programming, semidefinite programming, ℓ_1 unit ball, sparse principal component analysis.

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1. Introduction

Semidefinite programming (SDP) relaxation has recently gained much attention due to its tractability in computation to derive both strong bounds and approximation algorithms for combinatorial optimization problems and nonconvex quadratic

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programs, see, for example, [11]. Generally, we have the lifting procedure [6] and the Shor's approach [10] to establish the primal and dual semidefinite relaxation, respectively. But for the non-quadratic representable problems, these approaches can not be automatically applied. In order to obtain SDP relaxations, more efforts such as reformulation should be made firstly.

Pinar and Teboulle [9] obtained a direct SDP relaxation of the following optimization problem, which maximizes a quadratic form over the ℓ_1 unit ball:

$$\begin{aligned} \mathrm{QPL1}(Q) : \max\ & x^T Q x \\ \text{s.t.}\ & \|x\|_1 \le 1. \end{aligned}$$

In the area of nonlinear programming, QPL1(Q) is known as an ℓ_1-norm trust-region subproblem. In compressed sensing, $\|x\|_1$ is used to approximate $\|x\|_0$, the number of nonzero elements of x. In particular, the former is the convex envelope of the latter for $x \in [-1, 1]^n$. If Q is negative semidefinite, QPL1(Q) is a convex program. Besides, if Q is positive semidefinite, it remains trivial, see [9]. Otherwise, it is difficult in general, see [8].

Moreover, it is easy to verify that the set of extreme points of $\{x : \ \|x\|_1 \le 1\}$ is simply $\{e_1, -e_1, \cdots, e_n, -e_n\}$, where e_i is the ith column of the identity matrix I. Throughout this paper, we define

$$\begin{aligned} A &:= [e_1, \cdots, e_n, -e_1, \cdots, -e_n] = [I \ \ -I] \in \Re^{n\times 2n}, \\ \widetilde{Q} &= A^T Q A = \begin{bmatrix} Q & -Q \\ -Q & Q \end{bmatrix}. \end{aligned}$$

Therefore, we can rewrite the ℓ_1 constrained set as

$$\{x : \ \|x\|_1 \le 1\} = \{Ay : \ y \in \Delta_{2n}\}, \tag{1.1}$$

where Δ_{2n} is the simplex in $\Re^{2n}$, *i.e.*,

$$\Delta_{2n} := \left\{ y \in \Re^{2n} : \ e^T y = 1, y \ge 0 \right\}.$$

It follows that QPL1(Q) can be reformulated as

$$\mathrm{QPL1}'\left(\widetilde{\mathrm{Q}}\right) : \quad \max_{y\in\Delta_{2n}} y^T \widetilde{Q} y, \tag{1.2}$$

which is referred to as *standard quadratic program* (QPS) in the literature. QPS admits an exact copositive formulation [3] and has many well-known relaxations, for example, the doubly nonnegative relaxation [3] and the optimal d.c. (difference between convex) relaxation [1]. We refer the reader to the survey [4] for different bounds.

An extension of QPL1(Q) is the following nonconvex quadratic program:

$$\begin{aligned} \mathrm{QPL2L1(Q)} : \max\ & x^T Q x \\ \text{s.t.}\ & \|x\|_2 = 1 \\ & \|x\|_1^2 \le k \end{aligned} \tag{1.3}$$

which arises from a relaxation of the sparse principal component analysis (PCA) problem

$$\begin{aligned}\mathrm{PCA(Q)}: \max\ & x^TQx \\ \text{s.t.}\ & \|x\|_2 = 1 \\ & \mathrm{Card}(x) \le k,\end{aligned}$$

where Card(x) denotes the cardinality of x, and is controlled by a positive parameter k. The equation (1.3) holds true since

$$\|x\|_1 \le \sqrt{\mathrm{Card}(x)}\|x\|_2 \le \sqrt{k},$$

where the first inequality follows from the Cauchy-Schwarz inequality.

d'Aspremont *et al.* [5] proposed an SDP relaxation for the penalized version of QPL2L1(Q). A similar but direct SDP relaxation of QPL2L1(Q) is due to Luss and Teboulle [7]. Applying the above variable-splitting approach for the ℓ_1 unit ball, we can also derive similar SDP relaxations such as the doubly nonnegative relaxation for QPL2L1(Q).

In this paper, we theoretically compare the tightness of the above SDP relaxations. For QPL1(Q), we show the direct SDP bound proposed in [9] is equivalent to the optimal d.c. bound [1] for QPL1$'(\widetilde{\mathrm{Q}})$ (1.2). We then disprove a conjecture [9] stating that the direct SDP bound is tight when $Q_{ij} \ge 0, \forall i \neq j$. For QPL2L1(Q), we show the direct SDP bound presented in [7] is as tight as the doubly nonnegative relaxation for the variable-splitting reformulation of QPL2L1(Q). The remainder of this paper is organized as follows. In Section 1, we first present and then compare the existing SDP relaxations for QPL1(Q). We disprove a conjecture stating that the direct SDP relaxation is tight when all the off-diagonal elements of Q are nonnegative. In Section 2, we compare the tightness of two SDP relaxations for QPL2L1(Q). We give concluding remarks in Section 3.

Throughout the paper, let $v(\cdot)$ denote the optimal value of problem $(\cdot)$. We denote by $\mathcal{S}^n$ the set of all $n \times n$ symmetric matrices. Notation $A \succeq (\preceq) B$ implies that the matrix $A - B$ is positive (negative) semidefinite, whereas $A \ge 0$ indicates that A is componentwise nonnegative. The standard inner product on $\mathcal{S}^n$ is $A \bullet B = \mathrm{trace}\,(AB^T) = \sum_{i,j=1}^n a_{ij}b_{ij}$. We denote a vector of arbitrary dimension with all components equal to one by e and the identity matrix by I. For a matrix A and a vector a, we denote by diag(A) the column vector with its components being the diagonal elements of A, and Diag(a) the diagonal matrix with a being its diagonal vector.

2. SDP relaxations of QPL1(Q)

In this section, we first present the existing SDP relaxations for QPL1(Q). Then we compare their tightness. Finally, we disprove a conjecture in [9] stating that $(\mathrm{SDP_{L1}})$ is an exact bound when $Q \in \mathcal{S}^n$, $Q_{ij} \ge 0$ for all $i \neq j$.

2.1. A direct SDP relaxations of QPL1(Q)

Based on the variational representation of the ℓ_1-norm:

Lemma 2.1. ([9])

$$\|x\|_1^2 = \inf\left\{x^T \mathrm{Diag}\left(v^{-1}\right) x : \ v \in \Gamma\right\}$$

where $v^{-1} = (v_1^{-1}, \cdots, v_n^{-1})^T$ *and*

$$\Gamma = \left\{v : \ e^T v \leq 1, \ v > 0\right\} \tag{2.1}$$

Pinar and Teboulle [9] derived a direct SDP relaxation as follows:

$$\begin{aligned} v(\mathrm{QPL1(Q)}) &= \max\left\{x^T Q x : \ \inf_{v \in \Gamma} x^T \mathrm{Diag}(v^{-1}) x \leq 1\right\} \\ &\leq \sup_{x,v}\left\{x^T Q x : \ x^T \mathrm{Diag}\left(v^{-1}\right) x \leq 1, v \in \Gamma\right\} \\ &= \sup_{x,v}\left\{x^T Q x : \ \mathrm{Diag}(v) - x x^T \succeq 0, v \in \Gamma\right\} &(2.2) \\ &\leq \max_{X,v}\left\{Q \bullet X : \ \mathrm{Diag}(v) - X \succeq 0, X \succeq 0, e^T v \leq 1, v \geq 0\right\} &(2.3) \\ &:= v(\mathrm{SDP_{L1}(Q)}), \end{aligned}$$

where (2.2) follows from Schur's Lemma and the SDP relaxation (2.3) is obtained from the lifting procedure.

Actually, $\mathrm{SDP_{L1}(Q)}$ can be obtained more directly. Notice that for any x in the ℓ_1 unit ball, it holds that

$$|x_i| - x_i^2 = |x_i|(1 - |x_i|) \geq |x_i| \sum_{j \neq i} |x_j| \geq \sum_{j \neq i} x_i x_j.$$

According to the Gerschgorin circle theorem, we have

$$\mathrm{Diag}(|x|) \succeq x x^T, \tag{2.4}$$

which leads to the same relaxation as in (2.3).

Let (X^*, v^*) be an optimal solution to $\mathrm{SDP_{L1}(Q)}$. Suppose $e^T v^* < 1$, then $(X^*, v^* + \frac{1 - e^T v^*}{n} e)$ remains optimal since the objective function of $\mathrm{SDP_{L1}(Q)}$ is independent of v. Notice that $e^T(v^* + \frac{1 - e^T v^*}{n} e) = 1$. Then $\mathrm{SDP_{L1}(Q)}$ (2.3) can be rewritten as

$$v(\mathrm{SDP_{L1}(Q)}) = \max_{X,v}\left\{Q \bullet X : \ \mathrm{Diag}(v) - X \succeq 0, X \succeq 0, e^T v = 1, v \geq 0\right\}. \tag{2.5}$$

It follows that (2.1) in Lemma 2.1 can be strengthened to

$$\Gamma' = \left\{v : \ e^T v = 1, \ v > 0\right\}.$$

2.2. Comparision of the tightness of SDP relaxations

The standard Shor relaxation of QPL1′($\widetilde{\mathrm{Q}}$) (1.2) reads:

$$\mathrm{SQPS}(\widetilde{\mathrm{Q}}): \quad \max\ \widetilde{Q} \bullet Y \\ \text{s.t. } \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \succeq 0, \\ y \in \Delta_{2n},$$

which is equivalent to the dual:

$$\mathrm{DQPS}(\widetilde{\mathrm{Q}}): \quad \min\ \sigma + \mu \\ \text{s.t. } \begin{pmatrix} \sigma & s^T \\ s & -\widetilde{Q} \end{pmatrix} \succeq 0, \\ 2s - \mu e \leq 0.$$

As an improvement of $v(\mathrm{SQPS}(\widetilde{\mathrm{Q}}))$, Anstreicher and Burer [1] derived an optimal d.c. bound, denoted by $\mathrm{DC}(\widetilde{\mathrm{Q}})$. Let $\widetilde{Q} = S - T$, where $S \preceq 0$, $T \preceq 0$. We have

$$\begin{aligned} v(\mathrm{QPL1}(\mathrm{Q})) &\leq \max_{y \in \Delta_{2n}} y^T S y + \max_{y \in \Delta_{2n}} -y^T T y \\ &= \max_{y \in \Delta_{2n}} y^T S y + \max_i \{-T_{ii}\} \\ &= \max_{y \in \Delta_{2n}} y^T S y + \min \{-\theta : \ \theta \leq T_{ii}\} \\ &= \min \quad \sigma + \mu - \theta \qquad (2.6) \\ &\quad\ \text{s.t. } \begin{pmatrix} \sigma & s^T \\ s & -S \end{pmatrix} \succeq 0 \\ &\quad\ 2s - \mu e \leq 0 \\ &\quad\ \widetilde{Q} - S \succeq 0 \\ &\quad\ \theta e \leq \mathrm{diag}\left(S - \widetilde{Q}\right) \\ &= \max \quad \widetilde{Q} \bullet Y \qquad (2.7) \\ &\quad\ \text{s.t. } \begin{pmatrix} 1 & y^T \\ y & Y \end{pmatrix} \succeq 0 \\ &\quad\ Y \preceq \mathrm{Diag}(z) \qquad (2.8) \\ &\quad\ y \in \Delta_{2n}, \ z \in \Delta_{2n}, \\ &:= v\left(\mathrm{DC}\left(\widetilde{\mathrm{Q}}\right)\right), \end{aligned}$$

where (2.6) holds since $v(\mathrm{QPL1}'(\mathrm{S})) = v(\mathrm{DQPS}(\mathrm{S}))$ for convex QPL1′(S) and (2.7) is the conic dual of (2.6). It was further observed in [1] that $\mathrm{DC}(\widetilde{\mathrm{Q}})$ can be equivalently simplified by setting $y = z$ in (2.7).

Furthermore, as shown in [1], $\mathrm{DC}(\widetilde{\mathrm{Q}})$ is dominated by the following double nonnegative relaxation (DNN), which is referred to as 'strengthened Shor relaxation' of QPS:

$$\text{DNN}\left(\widetilde{\text{Q}}\right): \quad \min\ \widetilde{Q} \bullet Y \qquad (2.9)$$
$$\text{s.t.}\ \left(ee^T\right) \bullet Y = 1,$$
$$Y \succeq 0,\ Y \geq 0.$$

Comparing the direct SDP relaxation $\text{SDP}_{\text{L1}}(\text{Q})$ with the above bounds based on QPS reformulation, we have

Theorem 2.2.

$$v(\text{SDP}_{\text{L1}}(\text{Q})) = v\Big(\text{DC}\left(\widetilde{\text{Q}}\right)\Big) \geq v\Big(\text{DNN}\left(\widetilde{\text{Q}}\right)\Big) \geq v(\text{QPL1}(\text{Q})).$$

Proof. It is sufficient to prove the first equality. Let the optimal solution to $\text{DC}(\widetilde{\text{Q}})$ be (Y^*, y^*, z^*). Define $X = AY^*A^T$. Then we have

$$v\left(\text{DC}\left(\widetilde{\text{Q}}\right)\right) = \widetilde{Q} \bullet Y^* = \text{trace}\left(\widetilde{Q}Y^{*T}\right) = \text{trace}\left(QX^T\right) = Q \bullet X.$$

It follows from (2.8) that

$$X = AY^*A^T \preceq A\text{Diag}\left(z^*\right)A^T = \text{Diag}\left(z_1^* + z_{n+1}^*, z_2^* + z_{n+2}^*, \cdots, z_n^* + z_{2n}^*\right).$$

Define $v = (z_1^* + z_{n+1}^*, z_2^* + z_{n+2}^*, \cdots, z_n^* + z_{2n}^*)^T$. Since $z^* \in \Delta_{2n}$, we have $v \in \Delta_n$. Therefore, (X, v) is feasible to $\text{SDP}_{\text{L1}}(Q)$ (2.3) and the corresponding objective value is $v(\text{DC}(\widetilde{\text{Q}}))$. It implies that $v(\text{SDP}_{\text{L1}}(\text{Q})) \geq v(\text{DC}(\widetilde{\text{Q}}))$.

Let (X^*, v^*) be an optimal solution to $\text{SDP}_{\text{L1}}(\text{Q})$ (2.5). Then, $v^* \in \Delta_n$ and

$$0 \preceq X^* \preceq \text{Diag}\left(v^*\right). \qquad (2.10)$$

Define $D = \text{Diag}(\sqrt{v^*})$. Let D^+ be the Moore-Penrose generalized inverse of D, *i.e.*, D^+ is diagonal and

$$D_{ii}^+ = \begin{cases} 1/\sqrt{v_i^*} & \text{if } v_i^* > 0 \\ 0 & \text{otherwise} \end{cases},\ i = 1, \ldots, n.$$

Then, (2.10) implies that

$$0 \preceq D^+X^*D^+ \preceq I.$$

Moreover, suppose there is an index j such that $v_j^* = 0$. Then $D_{jj} = D_{jj}^+ = 0$. It follows from (2.10) that $X_{jj}^* = 0$ and $X_{jk}^* = X_{kj}^* = 0$ for $k = 1, \ldots, j-1, j+1, \ldots, n$. Therefore, we have

$$X^* = D\left(D^+X^*D^+\right)D. \qquad (2.11)$$

Let the eigenvalue decomposition of $D^+X^*D^+$ be $U\Lambda U^T$, where U is orthogonal, $\Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n)$ and the eigenvalues $\lambda_i \in [0, 1]$, $i = 1, \ldots, n$. Then, it is not difficult to verify that

$$\begin{bmatrix} \frac{1}{4}I & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} \frac{1}{4}I + \frac{1}{4}\Lambda & \frac{1}{2}\Lambda \\ \frac{1}{2}\Lambda & \Lambda \end{bmatrix} \preceq \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & I \end{bmatrix}.$$

Moreover, it implies that

$$\begin{bmatrix} \frac{1}{4}D^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} DU & 0 \\ 0 & DU \end{bmatrix} \begin{bmatrix} \frac{1}{4}I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U^TD & 0 \\ 0 & U^TD \end{bmatrix} \tag{2.12}$$

$$\preceq \begin{bmatrix} DU & 0 \\ 0 & DU \end{bmatrix} \begin{bmatrix} \frac{1}{4}I + \frac{1}{4}\Lambda & \frac{1}{2}\Lambda \\ \frac{1}{2}\Lambda & \Lambda \end{bmatrix} \begin{bmatrix} U^TD & 0 \\ 0 & U^TD \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4}D^2 + \frac{1}{4}X^* & \frac{1}{2}X^* \\ \frac{1}{2}X^* & X^* \end{bmatrix}$$

$$\preceq \begin{bmatrix} DU & 0 \\ 0 & DU \end{bmatrix} \begin{bmatrix} \frac{1}{2}I & \frac{1}{2}I \\ \frac{1}{2}I & I \end{bmatrix} \begin{bmatrix} U^TD & 0 \\ 0 & U^TD \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}D^2 & \frac{1}{2}D^2 \\ \frac{1}{2}D^2 & D^2 \end{bmatrix}, \tag{2.13}$$

where (2.11) is used in the first inequality. Define

$$Y = \begin{bmatrix} \frac{1}{4}D^2 + \frac{1}{4}X^* & \frac{1}{4}D^2 - \frac{1}{4}X^* \\ \frac{1}{4}D^2 - \frac{1}{4}X^* & \frac{1}{4}D^2 + \frac{1}{4}X^* \end{bmatrix}$$

and

$$B = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \in \Re^{2n\times 2n}.$$

The above inequalities (2.12)–(2.13) imply that

$$B \begin{bmatrix} \frac{1}{4}v^*v^{*T} & \frac{1}{4}v^*v^{*T} \\ \frac{1}{4}v^*v^{*T} & \frac{1}{4}v^*v^{*T} \end{bmatrix} B^T = \begin{bmatrix} \frac{1}{4}v^*v^{*T} & 0 \\ 0 & 0 \end{bmatrix} \preceq \begin{bmatrix} \frac{1}{4}D^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.14}$$

$$\preceq BYB^T \preceq B \begin{bmatrix} \frac{1}{2}D^2 & 0 \\ 0 & \frac{1}{2}D^2 \end{bmatrix} B^T, \tag{2.15}$$

where the inequality in (2.14) follows from (2.4). Since B is nonsingular, the inequalities (2.14)–(2.15) are equivalent to

$$\begin{bmatrix} \frac{1}{2}v^* \\ \frac{1}{2}v^* \end{bmatrix} \begin{bmatrix} \frac{1}{2}v^* \\ \frac{1}{2}v^* \end{bmatrix}^T \preceq Y \preceq \begin{bmatrix} \frac{1}{2}\mathrm{Diag}(v^*) & 0 \\ 0 & \frac{1}{2}\mathrm{Diag}(v^*) \end{bmatrix}.$$

Define $y^T = z^T = [1/2v^{*T} \ 1/2v^{*T}]$. Then, (Y, y, z) is a feasible solution of $\mathrm{DC}(\widetilde{\mathrm{Q}})$ and the corresponding objective function value is $\widetilde{Q} \bullet Y = Q \bullet (AYA^T) = Q \bullet X^* = v(\mathrm{SDP_{L1}(Q)})$. It follows that $v(\mathrm{DC}(\widetilde{\mathrm{Q}})) \geq v(\mathrm{SDP_{L1}(Q)})$. The proof is complete. □

2.3. The exactness of $v(\mathrm{SDP}_{\mathrm{L1}}(Q))$

The contribution of this subsection is to disprove a conjecture raised in [9] stating that $v(\mathrm{SDP}_{\mathrm{L1}}(\mathrm{Q})) = v(\mathrm{QPL1(Q)})$ under the following assumption:

Assumption 2.3.

$$Q \in \mathcal{S}^n,\ Q_{ij} \geq 0, \forall i \neq j.$$

Proposition 2.4. *Under Assumption 2.3, if* $-Q \succeq 0$*, then* $v(\mathrm{QPL1(Q)}) = 0$*. Otherwise,*

$$0 < v(\mathrm{QPL1(Q)}) = v(\mathrm{QPS(Q)}) := \max_{x \in \Delta_n} x^T Qx.$$

Proof. Assumption 2.3 implies that

$$x^T Qx = \sum_{i=1}^n Q_{ii} x_i^2 + 2\sum_{i<j}^n Q_{ij} x_i x_j \leq \sum_{i=1}^n Q_{ii} |x_i|^2 + 2\sum_{i<j}^n Q_{ij} |x_i||x_j|.$$

Then QPL1(Q) is reduced to

$$\begin{aligned} v(\mathrm{QPL1(Q)}) = \max\ & x^T Qx \\ \text{s.t.}\ & e^T x \leq 1,\ x \geq 0. \end{aligned}$$

Since $x = 0$ is feasible, $v(\mathrm{QPL1(Q)}) \geq 0$. Suppose x^* is the optimal solution but $0 < e^T x^* < 1$, then $y = (\frac{x_i^*}{e^T x^*}) \in \Delta_n$ and $y^T Qy = \frac{x^{*T} Qx^*}{(e^T x^*)^2} > x^{*T} Qx^*$ if $x^{*T} Qx^* > 0$. Therefore,

$$v(\mathrm{QPL1(Q)}) = \max\left\{0, \max_{x \in \Delta_n} x^T Qx\right\}. \tag{2.16}$$

If $-Q \succeq 0$, then

$$x^T(-Q)x \geq 0,\ \forall x \geq 0.$$

It follows that $\max_{x \in \Delta_n} x^T Qx \leq 0$. By (2.16), $v(\mathrm{QPL1(Q)}) = 0$. Now we assume that $-Q \not\succeq$, *i.e.*, there is a vector $y \neq 0$ such that

$$0 > y^T(-Q)y = -y^T Qy \geq -|y|^T Q|y|.$$

Clearly, $e^T|y| > 0$. Let $z = |y|/(e^T|y|)$. We have

$$z \in \Delta_n,\ z^T Qz > 0.$$

It follows from (2.16) that

$$v(\mathrm{QPL1(Q)}) = \max\left\{0, \max_{x \in \Delta_n} x^T Qx\right\} > 0.$$

□

For any $Q \in \mathcal{S}^n$, DNN(Q), defined in (2.9), is the double nonnegative relaxation of QPS(Q), *i.e.*, $v(\mathrm{DNN(Q)}) \geq v(\mathrm{QPS(Q)})$. For $n \leq 4$, it holds that $v(\mathrm{DNN(Q)}) = v(\mathrm{QPS(Q)})$, see [2]. Generally, this is not true. We choose $\overline{Q} \in \mathcal{S}^n$ such that

$$v\left(\mathrm{DNN}\left(\overline{\mathrm{Q}}\right)\right) > v\left(\mathrm{QPS}\left(\overline{\mathrm{Q}}\right)\right). \tag{2.17}$$

Then, for sufficient large $\alpha \in \Re$, we have

$$\left(\overline{Q} + \alpha ee^T\right)_{ij} = \overline{Q}_{ij} + \alpha \geq 0, \ \forall i \neq j$$
$$e^T\left(-\overline{Q} - \alpha ee^T\right) e = e^T\left(-\overline{Q}\right) e - n^2\alpha < 0.$$

Therefore, Assumption 2.3 holds for $\overline{Q} + \alpha ee^T$ and $-(\overline{Q} + \alpha ee^T) \not\succeq 0$. According to Proposition 2.4, we have

$$v\left(\mathrm{QPL1}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right) = v\left(\mathrm{QPS}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right) = v\left(\mathrm{QPS}\left(\overline{\mathrm{Q}}\right)\right) + \alpha. \tag{2.18}$$

It is not difficult to verify that

$$\begin{aligned} v\left(\mathrm{DNN}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right) &= \max_{ee^T \bullet X = 1, X \succeq 0, X \geq 0} \left(\overline{Q} + \alpha ee^T\right) \bullet X \\ &= \max_{ee^T \bullet X = 1, X \succeq 0, X \geq 0} \overline{Q} \bullet X + \alpha \\ &= v\left(\mathrm{DNN}\left(\overline{\mathrm{Q}}\right)\right) + \alpha. \end{aligned} \tag{2.19}$$

It follows from (2.17), (2.18) and (2.19) that

$$v\left(\mathrm{DNN}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right) > v\left(\mathrm{QPL1}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right). \tag{2.20}$$

Let X^* be an optimal solution to $\mathrm{DNN}(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}})$. It follows from the Gerschgorin circle theorem that

$$\mathrm{Diag}(Xe) \succeq X.$$

Therefore, $(X^*, v) = (X^*, Xe)$ is a feasible solution to $\mathrm{SDP_{L1}}(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}))$. The corresponding objective function value is $(\overline{Q} + \alpha ee^T) \bullet X^* = v(\mathrm{DNN}(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}))$. Then it holds that

$$v\left(\mathrm{SDP_{L1}}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right) \geq v\left(\mathrm{DNN}\left(\overline{\mathrm{Q}} + \alpha \mathrm{ee}^{\mathrm{T}}\right)\right). \tag{2.21}$$

Consequently, the conjecture $v(\mathrm{SDP_{L1}(Q)}) = v(\mathrm{QPL1(Q)})$ under Assumption 2.3 is disproved according to (2.20)–(2.21).

3. SDP relaxations of QPL2L1(Q)

In this section, we first present the efficient SDP bound for QPL2L1(Q) due to Luss and Teboulle [7], denoted by $\mathrm{SDP_{L2L1}(Q)}$. Then we show it is equivalent to the double nonnegative relaxation for the variable-splitting reformulation of QPL2L1(Q).

Define

$$\mathcal{U}_s = \left\{U \in \mathcal{S}^n : \ |U|_{ij} \leq s, \forall i, j\right\}.$$

Lemma 3.1. ([7])

$$\|x\|_1^2 = \max\left\{x^T U x:\ U \in \mathcal{U}_1\right\}.$$

Then, it holds that

$$\begin{aligned}
v(\mathrm{QPL2L1(Q)}) &= \max\left\{x^TQx:\ \|x\|_2 = 1, \max_{U\in\mathcal{U}_1} x^TUx \le k\right\}\\
&\le \min_{s\ge 0}\max_{\|x\|_2=1}\left\{x^TQx - s\max_{U\in\mathcal{U}_1} x^TUx\right\} + sk\Big\}\\
&= \min_{s\ge 0}\left\{sk + \max_{\|x\|_2=1}\min\left\{x^T(Q+U)x : U\in\mathcal{U}_s\right\}\right\}\\
&\le \min\left\{sk + \max_{\|x\|_2=1}\left\{x^T(Q+U)x\right\}:\ s\ge 0, U\in\mathcal{U}_s\right\}\\
&= \min_{U\in\mathcal{S}^n}\left\{k\max_{ij}|U_{ij}| + \lambda_{\max}(Q+U)\right\}\\
&:= v(\mathrm{SDP_{L2L1}(Q)}).
\end{aligned}$$

Equivalently, $\mathrm{SDP_{L2L1}(Q)}$ can be reformulated as:

$$\begin{aligned}
\mathrm{SDP_{L2L1}(Q)}:\ \ &\min\ ks + t\\
&\text{s.t. } Q + U \preceq tI\\
&\qquad |U_{ij}| \le s,\ \forall i,j\\
&\qquad U \in \mathcal{S}^n.
\end{aligned}$$

It was observed in [7] that the conic dual of ($\mathrm{SDP_{L2L1}}$) reads

$$\begin{aligned}
&\max\ Q\bullet X\\
&\text{s.t. } \mathrm{trace}(X) = 1\\
&\qquad e^T|X|e \le k\\
&\qquad X \succeq 0,
\end{aligned}$$

which was first proposed by d'Aspremont et al. [5] based on semidefinite relaxation lifting.

Now we present the variable-splitting reformulation of QPL2L1(Q) and the double nonnegative relaxation. Similar to the reformulation (1.1), we have

$$\left\{x:\ \|x\|_1^2 \le k\right\} = \left\{\sqrt{k}Ay:\ y\in\Delta_{2n}\right\}.$$

Notice that the linear constraint $y\in\Delta_{2n}$ can be equivalently rewritten as the quadratic constraints $y^Tee^Ty = 1,\ y_iy_j \ge 0,\ \forall i,j$, if the other constraints and the objective are all even functions. Therefore, we can reformulate (QPL2L1) as the following homogenious quadratic constrained quadratic program:

$$\begin{aligned} \mathrm{QPL2L1}'(\widetilde{\mathrm{Q}}): \quad & \max\ ky^T\widetilde{Q}y && (3.1)\\ & \text{s.t.}\ \ ky^TA^TAy = 1 && (3.2)\\ & \qquad y^Tee^Ty = 1, && (3.3)\\ & \qquad y_iy_j \geq 0,\ \forall i,j. && (3.4) \end{aligned}$$

The Lagrangian dual of $\mathrm{QPL2L1}'(\widetilde{Q})$ is

$$\inf_{\mu,\lambda,S\geq 0}\sup_{y}\left\{L(y,\mu,\lambda,S) := y^T(k\widetilde{Q} - \mu kA^TA - \lambda ee^T + S)y + \mu + \lambda\right\}$$

which has the following new SDP reformulation:

$$\begin{aligned} \mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}}): \quad & \min\ \mu + \lambda \\ & \text{s.t.}\ \ k\widetilde{Q} - \mu kA^TA - \lambda ee^T + S \preceq 0 && (3.5)\\ & \qquad S \in \mathcal{S}^{2n},\ \ S \geq 0. \end{aligned}$$

We remark that if we remove (3.2) and set $k = 1$, QPL2L1'$(\widetilde{\mathrm{Q}})$ reduces to QPL1(Q), and $\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}})$ reduces to the conic dual of DNN(Q) (2.9).

Theorem 3.2.

$$v(\mathrm{SDP_{L2L1}(Q)}) = v\left(\mathrm{DNN_{L2L1}}\left(\widetilde{\mathrm{Q}}\right)\right).$$

Proof. Let (μ,λ,S) be any feasible solution to $\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}})$. Since $AA^T = 2I$ and $Ae = 0$, it follows from (3.5) that

$$0 \succeq A\left(k\widetilde{Q} - \mu kA^TA - \lambda ee^T + S\right)A^T = 4kQ - 4k\mu I + ASA^T$$

Notice that for any i,j, $A(e_i + e_{n+i}) = A(e_j + e_{n+j}) = 0$. Based on (3.5), we have

$$\begin{aligned} 0 &\geq (e_i + e_{n+i})^T\left(k\widetilde{Q} - \mu kA^TA - \lambda ee^T + S)(e_j + e_{n+j}\right)\\ &= -4\lambda + (e_i + e_{n+i})^TS(e_j + e_{n+j})\\ &\geq -4\lambda + |(e_i - e_{n+i})^TS(e_j - e_{n+j})|\\ &= -4\lambda + \left|\left(ASA^T\right)_{ij}\right|. \end{aligned}$$

Therefore, $\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}})$ can be further relaxed to the following lower bound:

$$\begin{aligned} \mathrm{LB(Q)}: \min\ & \mu + \lambda \\ \text{s.t.}\ \ & Q - \mu I + \frac{1}{4k}ASA^T \preceq 0 \\ & \lambda \geq \left|\frac{1}{4}\left(ASA^T\right)_{ij}\right|,\ \forall i,j,\\ & S \in \mathcal{S}^{2n},\ \ S \geq 0. \end{aligned}$$

Let the optimal solution to LB(Q) be (μ^*, λ^*, S^*), define $U = \frac{1}{4k} A S^* A^T$, $t = \mu^*$ and $s = \frac{\lambda^*}{k}$. Then (s, t, U) is feasible to $\mathrm{SDP_{L2L1}}(\widetilde{\mathrm{Q}})$ and $ks + t = \mu^* + \lambda^*$. It follows that $v(\mathrm{LB(Q)}) = \mu^* + \lambda^* \geq v(\mathrm{SDP_{L2L1}}(\widetilde{\mathrm{Q}}))$.

Now it is sufficient to show $v(\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}})) \leq v(\mathrm{SDP_{L2L1}}(\mathrm{Q}))$. Let the optimal solution to $(\mathrm{SDP_{L2L1}}(\mathrm{Q}))$ be (s^*, t^*, U^*), define $\mu = t^*$, $\lambda = ks^*$ and

$$S = \begin{bmatrix} ks^*ee^T + kU^* & ks^*ee^T - kU^* \\ ks^*ee^T - kU^* & ks^*ee^T + kU^* \end{bmatrix} \in \mathcal{S}^{2n}.$$

Since $|U^*|_{ij} \leq s^*$, we have $S \geq 0$. Define

$$B = \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \in \mathcal{S}^{2n}.$$

Then we have

$$BA^T = \begin{bmatrix} 0 \\ 2I \end{bmatrix}, \ Be = \begin{bmatrix} 2e \\ 0 \end{bmatrix}, \ BSB^T = \begin{bmatrix} 4ks^*ee^T & 0 \\ 0 & 4kU^* \end{bmatrix}.$$

Therefore,

$$B\left(k\widetilde{Q} - \mu k A^T A - \lambda ee^T + S\right) B^T = \begin{bmatrix} 0 & 0 \\ 0 & 4k\left(Q - t^* I + U^*\right) \end{bmatrix} \preceq 0.$$

Since B is nonsingular, it holds that

$$k\widetilde{Q} - \mu k A^T A - \lambda ee^T + S \preceq 0.$$

Hence, (μ, λ, S) is feasible to $\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}})$. We have $v(\mathrm{SDP_{L2L1}}(\mathrm{Q})) = t^* + ks^* \geq v(\mathrm{DNN_{L2L1}}(\widetilde{\mathrm{Q}}))$. The proof is complete. □

4. Conclusion

In this paper, we first study semidefinite programming relaxations for the quadratic optimization over ℓ_1 unit ball, denoted by QPL1(Q). We show the direct SDP bound presented in [9] is equivalent to the optimal d.c. bound [1] for the standard quadratic programming reformulation of QPL1(Q). Then we disprove a conjecture in [9] stating that the direct SDP relaxation is an exact bound when $Q_{ij} \geq 0, \forall i \neq j$. Finally, we study semidefinite programming relaxations for QPL2L1(Q), which arises from the relaxation of the sparse principal component analysis and can be regarded as an extension of QPL1(Q). We show that the efficient SDP bound proposed in [7] is equivalent to the double nonnegative relaxation for the variable-splitting reformulation of QPL2L1(Q). Thus, the SDP bound can be further improved by using the approaches strengthening the double nonnegative relaxation, see for example, [4] and [12].

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