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BLOCK DECOMPOSITION APPROACH TO COMPUTE A MINIMUM GEODETIC SET *, **

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Abstract. In this paper, we develop a divide-and-conquer approach, called block decomposition, to solve the minimum geodetic set problem. This provides us with a unified approach for all graphs admitting blocks for which the problem of finding a minimum geodetic set containing a given set of vertices (g-extension problem) can be efficiently solved. Our method allows us to derive linear time algorithms for the minimum geodetic set problem in (a proper superclass of) block-cacti and monopolar chordal graphs. Also, we show that hull sets and geodetic sets of block-cacti are the same, and the minimum geodetic set problem is NP-hard in cobipartite graphs. We conclude by pointing out several interesting research directions.

Keywords. Convexity, geodetic set, hull set, graph classes.

Mathematics Subject Classification. 05C12, 05C85.

1. Introduction

Let $G = (V, E)$ be a connected graph and $D \subseteq V(G)$. The *geodetic closure* of D, denoted by $I[D]$, consists of all vertices which lie on some shortest path between two vertices of D; in this case we say that vertices in $I[D]$ are *covered* (or *generated*)

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by D. If D contains only two vertices, say x and y, then we can write $I[x, y]$ instead of $I[\{x, y\}]$. We say that D is a *geodetic set* if $I[D] = V(G)$. The *geodetic number*, denoted by $g(G)$, is the cardinality of a minimum geodetic set in G, and a *g-set* is a geodetic set of minimum cardinality. The notion of geodetic sets was introduced by Harary *et al.* [17] and it has applications in game theory [4, 18, 25]. Geodetic sets have strong connections to convexity in graphs which is an extensively studied topic in the literature [6, 14, 30]. Convexity and the related concept of g-centroids arises in many practical applications such as telephone switching center, facility location, distributed computing, information retrieval (see [15, 21, 23, 26]), power optimization in mobile ad hoc networks [29], hypercube architecture for parallel processing [24], and communication networks [28]. In this paper, we will restrict our attention to geodetic sets and the geodetic number of graphs.

There have been some work about the computation of the geodetic number in special graph classes. In [10], it was shown that deciding whether the geodetic number is at most k is NP-complete for chordal graphs and chordal bipartite graphs, and the exact value of the geodetic number can be computed for cographs and split graphs in polynomial time. Also, it was proven that the set of simplicial vertices forms the unique g-set in ptolemaic graphs (graphs which are both chordal and distance hereditary) [13, 14] and a g-set of a P_4-sparse graph can be computed in linear time [13]. In [10], the authors give an upper bound on the geodetic number of unit interval graphs (Note that a graph is unit interval if and only if it is proper interval [27] and that they form a proper subclass of chordal graphs). This result has been recently improved in [11] where a polynomial-time algorithm for computing g-sets of proper interval graphs is presented. More recently, a polynomial-time algorithm for computing g-sets of distance hereditary graphs was given in [22]. (Note that a graph is distance hereditary if every induced path is a shortest path.) Geodetic sets and some related concepts are studied in [31] for block-cactus graphs and in [2] for median graphs. There are also some studies about the variation of the geodetic number under some graph operations such as strong product [5], cartesian product [3], and join [7]. Lastly, the geodetic number is also studied from the probabilistic point of view in [8].

Now, let us give the definition of another concept related to convexity in graphs. A subset D of vertices of a graph G is said to be *convex* if every shortest path between two vertices of D lies in D. The *convex hull* of D, denoted by $H[D]$, is the smallest convex set containing D, and D is a *hull set* of G if its convex hull is equal to $V(G)$. The *hull number* of a graph G, denoted by $h(G)$, is the size of a minimum hull set of G.

The difference between hull sets and geodetic sets can be observed in Figure 1 where $I[a, b] \neq V$ since $x \notin I[a, b]$ (and therefore the geodetic number is at least 3) whereas $H[\{a, b\}] = V$ (and therefore the hull number is 2).

The hull number and the geodetic number problems are both defined through convexity, however their computation usually require quite different arguments. It is clear that every geodetic set is a hull set and hence $h(G) \leq g(G)$ for every graph G. However, the converse does not necessarily hold. It was shown that the

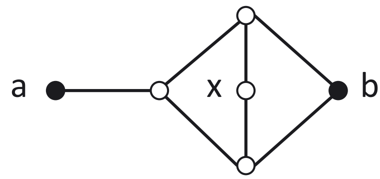

FIGURE 1. A graph whose hull number and geodetic number are different.

difference between these two parameters can be made arbitrarily large [19]. In this paper, we show that in block-cactus graphs hull sets and geodetic sets are the same. Intuitively, it seems that the geodetic number problem is more difficult than the hull number problem. Indeed, it is known that while the geodetic number problem is NP-hard in chordal graphs [10], the hull number problem can be solved in polynomial time in the same graph class [22]. This fact should support that intuition however there is not much study that will allow us to compare the hardness of these two problems in general. The complexity of the hull number problem is still open in many important graph classes like bipartite graphs [9]. In [9], the authors give a polynomial-time algorithm for computing the hull number of a proper interval graph. However, this result does not imply an efficient algorithm for computing minimum geodetic sets of proper interval graphs; the latter is solved in [11] using a different approach.

Our paper is organized as follows. In Section 2, we prove that the geodetic number problem is NP-hard in cobipartite graphs. Section 3 is devoted to the development of the block decomposition approach and the subsequent results. This approach suggests a simple polynomial time algorithm for the minimum geodetic set problem whenever the so-called g-extension problem can be solved in polynomial time in the blocks of a given graph. Using the block decomposition approach, we show that a minimum geodetic set can be computed in (a proper superclass of) block-cacti and in monopolar chordal graphs (which generalize split graphs). As a byproduct, we also show that hull sets and geodetic sets coincide for block-cacti. We conclude with further research directions in Section 4.

2. Definitions and preliminaries

We consider simple finite undirected graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set E. An edge uv consists of two adjacent vertices. The *neighbourhood* of v, denoted as $N_G(v)$, is the set of vertices of G that are adjacent to v, and we may omit the subscript whenever there is no ambiguity. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and it is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) = \{uv \in E(G) | u, v \in V(H)\}$. The *intersection* of two subgraphs H and H' of G, denoted by $H \cap H'$, is the subgraph of G whose vertex set is $V(H) \cap V(H')$ and edge set is $E(H) \cap E(H')$. For a vertex x of G, $G - x$ denotes the graph obtained from G by deleting vertex x and all edges incident to

x. A set X of vertices of G is a *clique* (respectively *independent set*) of G if the vertices in X are pairwise adjacent (respectively non-adjacent) in G. Say that G is a *complete graph* if $V(G)$ is a clique. A vertex v is called *simplicial* if $N(v)$ forms a clique. A sequence $(y_0, \ldots, y_r)$ of pairwise distinct vertices of G is called a y_0, y_r-*path of length* r in G if $y_i y_{i+1} \in E(G)$ for every $0 \leq i \leq r-1$. For a pair of vertices u, v of G, the *distance* between u and v in G, denoted as $d_G(u,v)$, is the smallest integer k such that there is a u, v-path of length k in G; if G has no u, v-path then $d_G(u,v) =_{\text{def}} \infty$. A *shortest* u, v-*path* in G is a u, v-path in G of length $d_G(u,v)$. A *cycle of length* r in G is a path of length $r-1$ where $y_0 y_{r-1} \in E(G)$. A *chord* in a cycle is an edge between two non-consecutive vertices of the cycle; a chord in a path is defined analogously. A graph G is called *chordal* if every cycle of length at least four in G has a cycle. Graph G is *connected* if G has a u, v-path for every pair of vertices u, v; otherwise, G is *disconnected*. A *connected component* of G is a maximal connected induced subgraph of G.

It is not difficult to see that $x \in I_G[u,v]$ if and only if $d_G(u,v) = d_G(u,x) + d_G(x,v)$, and if $C_1, \ldots, C_s$ are connected components of G then a set $D \subseteq V(G)$ is a geodetic set for G if and only if $D \cap V(C_i)$ is a geodetic set for C_i for every $1 \leq i \leq s$. Therefore, it suffices to restrict to connected graphs when studying the computational complexity of finding geodetic sets.

Another important remark is that, every simplicial vertex necessarily belongs to any geodetic set; indeed, a simplicial vertex cannot be covered by two other vertices. It follows directly that if the set of simplicial vertices forms a geodetic set, then it is both minimum and unique.

We begin by showing that computing a minimum geodetic set is NP-hard even for cobipartite graphs. A graph G is called *bipartite* if its vertex set admits a partition into sets A and B such that A and B are independent sets of G; we call (A, B) a *bi-partition* of G. Analogously, if $V(G)$ admits a partition into A and B such that A and B are cliques of G then G is *cobipartite*. Cobipartite graphs are the complements of bipartite graphs. Our hardness result complements the known hardness results for chordal graphs and chordal bipartite graphs by Dourado *et al.* [10]. The construction follows the same ideas in their paper. The *dominating set problem* is defined as follows: given a graph G and an integer k, decide whether G has a dominating set of size at most k, i.e., whether there is $D \subseteq V(G)$ such that $|D| \leq k$ and every vertex of G that is not in D has a neighbour in D. The dominating set problem is NP-complete on connected bipartite graphs [1].

The *geodetic set problem* is defined as follows: given a graph G and an integer k, decide whether G has a geodetic set of size at most k. Then we have the following:

Theorem 2.1. *The geodetic set problem is* NP*-complete for cobipartite graphs.*

Proof. For a graph G and a set $D \subseteq V(G)$, $I_G[D]$ can be computed in polynomial time. It directly follows that the geodetic set problem is in NP. For the hardness, we reduce from the dominating set problem on connected bipartite graphs [1]. Let G be a connected bipartite graph with bi-partition (A, B). Let a, a', b, b' be new vertices, and let $H =_{\text{def}} (V(G) \cup \{a, a', b, b'\}, E(G) \cup \{ab', a'b, a'b'\} \cup E_1 \cup E_2)$, where E_1 and E_2 are the subsets of edges obtained by making $A \cup \{a, a'\}$ and $B \cup \{b, b'\}$ into cliques, respectively. Note that H is a connected cobipartite graph. It is important to observe that every pair of vertices of H is at distance at most 2. We show that G has a dominating set of size at most k if and only if H has a geodetic set of size at most $k + 2$.

Let D be a dominating set for G. We show that $D \cup \{a, b\}$ is a geodetic set for H. Let x be a vertex of H that is not in $D \cup \{a, b\}$. If $x = a'$ or $x = b'$ then (a, x, b) is a shortest a, b-path in H. If $x \in A$ then there is a vertex $u \in D$ with $ux \in E(G)$. Since G is bipartite, $u \in B$, and (u, x, a) is a shortest u, a-path in H; analogously, if $x \in B$ then there is $u \in D \cap A$ such that (u, x, b) is a shortest u, b-path in H. For the converse, let D be a geodetic set for H. Let x be a vertex of G with $x \notin D$. There is a pair of vertices u, v from D with $x \in I_H[u, v]$. Since $d_H(u, v) \leq 2$, it follows that (u, x, v) is a u, v-path in H, which means that $u, v \in N_H(x)$. Assume that $u, v \notin V(G)$. Then, $u, v \in \{a, a', b, b'\}$, and since $uv \notin E(H)$, it holds that $u, v \in \{a, b\}$. Thus, $x \in \{a', b'\}$. Hence, $D \setminus \{a, a', b, b'\}$ is a dominating set for G. It remains to consider the size of $D \setminus \{a, a', b, b'\}$. If $a \notin D$ then $b' \in D$, if $b \notin D$ then $a' \in D$. It follows that $|D \cap \{a, a', b, b'\}| \geq 2$. □

3. Block decomposition approach

A vertex b of a graph G is called *cut-vertex* of G if there is a connected component C of G such that $C - b$ is disconnected. A *block* of G is a maximal connected induced subgraph of G that has no cut vertex. We will see that cut-vertices provide a divide-and-conquer approach to the computation of geodetic sets. In this section, we give linear time algorithms to compute a minimum geodetic set of a proper superclass of block-cactus graphs and a subclass of chordal graphs called monopolar chordal. This latter generalizes the split graphs, for which a polynomial-time algorithm for computing g-sets is already known [10].

Assume that G is connected and that b is a cut-vertex of G. Let $C_1, \ldots, C_t$ be the connected components of $G - b$. Let $K_i =_{\text{def}} G[V(C_i) \cup \{b\}]$ for every $1 \leq i \leq t$. We call $K_1, \ldots, K_t$ the *b-components* of G. Note that each b-component has at least two vertices.

The *block decomposition* of a graph G is the set of blocks of G. A block decomposition can be obtained by repeatedly choosing a cut-vertex b and computing the b-components. We will see that computing a minimum geodetic set for a graph with cut-vertices can be reduced to computing special geodetic sets for its blocks.

Remark 3.1. Let G be a graph and $B_1, \ldots, B_k$ be its blocks, where $k \geq 2$. Then there is a block which has exactly one cut-vertex of G. Moreover, if x is a cut-vertex and D is a geodetic set of G, then every connected component of $G - x$ contains at least one vertex of D.

Then the following holds for general graphs.

Remark 3.2. A cut-vertex can not belong to a minimal geodetic set.

Indeed, if a geodetic set D contains a cut-vertex x, then $D\setminus\{x\}$ is also a geodetic set. We now state the main theorem about the block decomposition approach.

Theorem 3.3. *Let G be a graph and let A be the set of cut-vertices of G. Let $\{B_1, \dots, B_t\}$ be the block decomposition of G. For $1 \le i \le t$, let D_i be a geodetic set for B_i of smallest possible size such that $A \cap V(B_i) \subseteq D_i$. Then, $(D_1 \cup \cdots \cup D_t) \setminus A$ is a minimum geodetic set for G.*

Proof. First let us show that $(D_1 \cup \cdots \cup D_t) \setminus A$ is a geodetic set. Let x be a vertex of G and assume that x belongs to a block B_i. Then $x \in I[u, v]$ for some $u, v \in D_i$, since D_i is a geodetic set of B_i. Now, if both u and v are in $D_i \setminus A$ then we are done. So assume without loss of generality that u is a cut-vertex. Let G_1 be a u-component of G which does not contain B_i. There is a block B_j of G_1 which has exactly one cut-vertex of G. By Remark 3.1, B_j contains a vertex $u' \in D_j \setminus A$. Thus $x \in I[u', v]$. Similarly if v is a cut-vertex, we can find a vertex v' in $D_k \setminus A$ for some $k \neq j$ such that $x \in I[u', v']$.

Assume for a contradiction that G has a geodetic set D such that $|D| < |(D_1 \cup \cdots \cup D_t) \setminus A|$. Then there is a block B_i such that $|D \cap V(B_i)| < |D_i \setminus A|$ since D does not contain a cut-vertex. Now we show that $V(B_i) \cap (D \cup A)$ is a geodetic set of B_i. Let x be a vertex of $V(B_i)$; then x is on a shortest u, v-path P where u and v are in D. If u is not in B_i, then there is a cut-vertex $u' \in B_i$ which separates x and u. So x lies on a shortest u', v-path. Similarly if v is not in B_i, there is a cut-vertex $v' \in B_i$ separating x and v and which is different from u'. Therefore, x is on a shortest u', v'-path where u' and v' are both in A. Moreover, $V(B_i) \cap (D \cup A)$ is a geodetic set of B_i containing all cut-vertices of B_i and such that its size is strictly less than that of D_i; thus we obtain a contradiction. □

Theorem 3.3 provides a simple algorithm for computing a minimum geodetic set for a graph with cut-vertices, if the subproblem of computing a geodetic set of smallest size containing a given set of vertices can be solved efficiently in the blocks of the input graph. Naturally, this problem cannot be solved efficiently for arbitrary graphs. It is therefore necessary to restrict the blocks to specially structured graphs. Here, we compute the minimum geodetic sets of block-cactus graphs and monopolar chordal graphs by solving efficiently the following minimum geodetic extension problem in the blocks of these graphs.

Let A be a subset of vertices of a graph G. We say that D is a *g-extension of $A* if D is a minimum geodetic set that contains A. Finding such a set D is referred as the *g-extension problem*.

A *block-cactus graph* is a graph whose block decomposition contains only chordless cycles and complete graphs. Block-cactus graphs generalize trees, block graphs and cacti.

Corollary 3.4. *A minimum geodetic set for a block-cactus graph can be computed in linear time.*

Proof. We solve the g-extension problem on each block as follows and apply Theorem 3.3 to conclude. Obviously a complete graph has a unique geodetic set which is equal to all of its vertices; so the g-extension necessarily contains all vertices of a complete graph. Now, we consider cycles. Let C be a cycle of length n and $A \subseteq V(C)$. We shall find a g-extension of A in C. If A is empty and n is even, we take two vertices u and v where $d(u,v) = n/2$ so that $\{u,v\}$ is a minimum geodetic set. If A is empty and n is odd, we take three vertices u, v and w where u and v are adjacent and $d(u,w) = d(v,w) = \lfloor n/2 \rfloor$ so that $\{u,v,w\}$ is a minimum geodetic set. Now assume that A has only one vertex x. If n is even, then we take a vertex y on the cycle where $d(x,y) = n/2$ so that $\{x,y\}$ is the desired geodetic set. If n is odd, then we take two adjacent vertices y and z on the cycle where $d(x,y) = d(x,z) = \lfloor n/2 \rfloor$ so that $\{x,y,z\}$ is the desired geodetic set. For the rest we may assume that A has cardinality larger than one. If there is a u,v-path P on the cycle C where $u,v \in A$, $V(P) \cap A = \{u,v\}$ and the length of P is larger than $\lfloor n/2 \rfloor$, then take any vertex x lying on the middle of P (there is one (respectively two) middle vertex (respectively vertices) if the length of the path P is even (respectively odd)). In this case $A \cup \{x\}$ is the desired geodetic set. If there is no path as described above, then A itself is the desired geodetic set. □

Since every simplicial vertex has to be in a geodetic set and that they form a geodetic set of a ptolemaic graph [14], given a ptolemaic graph G and $A \subseteq V(G)$, the set $A \cup S$, where S denotes the set of simplicial vertices of G is the g-extension of A. We can therefore extend the result of Corollary 3.4: if the block decomposition of a graph G contains only chordless cycles and ptolemaic graphs (note that every complete graph is a ptolemaic graph) then a minimum geodetic set for G can be computed in linear time.

In the sequel, we show that the hull sets of block-cacti are precisely its geodetic sets.

Theorem 3.5. *Hull sets and geodetic sets of a block-cactus graph G coincide.*

Proof. Let $S \subseteq V(G)$. It suffices to show that $I[S]$ is convex, since then it follows that $I[S] = H[S]$. Let $P_{u,v}$ denote a shortest u,v-path for $u,v \in S$ and let P be the subgraph which is the union of all such paths, that is, $P = \bigcup_{u,v \in S} P_{u,v}$. Note that $V(P) = I[S]$. It is easy to see that the intersection of P with each clique block is a clique. Now, we claim that P is a block-cactus graph such that for each cycle block C of G, $P \cap C$ is equal to either C itself or the empty set or a shortest path in C between two vertices of S.

First of all, we show that $P \cap C$ is a connected subgraph of C. Assume that it is disconnected, then take two consecutive disjoint maximal paths P_1, P_2 in $P \cap C$. Let x and y be the end-vertices of P_1 and P_2 respectively such that there is a path joining x and y, say $P_{x,y}$, which has no intersection with P other than x and y. Clearly such vertices exist by the choices of P_1 and P_2. Since x and y are the end-vertices of some maximal paths in $P \cap C$, each one of x and y either belongs to S or is a cut-vertex of G which is covered by some shortest path between two vertices of S. It follows that we have three cases for x and y:

- both x and y are in S;
- one of x and y, say without loss of generality x, is in S and y is a cut-vertex such that $G - y$ contains a connected component H with $V(H) \cap V(C) = \emptyset$ and $V(H) \cap S \neq \emptyset$;
- both x and y are cut-vertices having the same properties as in the previous case.

In all cases, we obtain the contradictions that $P \cap P_{x,y} \neq \{x, y\}$ if $P_{x,y}$ is the shortest x, y-path, and P_1 and P_2 are not disjoint otherwise. Hence, $P \cap C$ must be connected.

By using similar arguments, it is easy to see that $P \cap C$ is a shortest path between two vertices of S unless it is an empty set or C itself. It follows from the above that $V(P)$ is convex and we have the result. □

Although the minimum geodetic set problem is NP-hard in the class of chordal graphs [10], it can be solved in polynomial time in some subclasses of it as mentioned earlier in Section 1. As a further application of Theorem 3.3, we deal with a subclass of chordal graphs. A graph G is called *monopolar* if its vertex set can be partitioned into two sets A and B such that A induces an independent set and B a disjoint union of cliques (that is a P_3-free graph) [12]. We consider the minimum geodetic set problem in *monopolar chordal* graphs, that is graphs which are both chordal and monopolar. It can be easily seen that monopolar chordal graphs generalize split graphs; a *split graph* G is a graph whose vertex set can be partitioned into an independent set I and a clique C. Moreover, it is known that 2-connected components of monopolar chordal graphs are precisely split graphs [12]. (Recall that the blocks of a graph are its isolated vertices, its brigdes and its maximal 2-connected subgraphs.) Therefore, solving the g-extension problem in split graphs provides an algorithm to find a g-set of a monopolar chordal graph. To this end, we generalize the result of finding a g-set in split graphs (see [10]) to the g-extension problem as follows.

Theorem 3.6. *Let G be a connected split graph and $I \cup C$ be a partition of its vertices such that I is an independent set and C is a clique. Let S denote the set of simplicial vertices of G and $A \subseteq V(G)$. Then, a g-extension of A can be found in linear time. Moreover, it has size $|A \cup S|$, $|A \cup S| + 1$ or $|A \cup S| + 2$.*

Proof. First we define a set $\bar{U}$ which consists of vertices $u \in C \setminus (S \cup A)$ such that

- u has exactly one neighbor in I, say u';
- u' is adjacent to all vertices of $C \cap (S \cup A)$;
- u' has a common neighbor with every vertex of I.

By definition of $\bar{U}$, it is clear that $\bar{U} = V(G) \setminus I[S \cup A]$. Obviously, if $\bar{U} = \emptyset$ then $S \cup A$ is a g-extension of A. Hence, for the rest we may assume that $\bar{U} \neq \emptyset$. Now we claim that a g-extension of A has size $|S \cup A| + 1$ if and only if there is a vertex

$u \in \bar{U}$ such that $N(u) \cap N(v) \cap I = \emptyset$ for every $v \in \bar{U} \setminus \{u\}$. If a g-extension of A has size $|S \cup A| + 1$ then there must be a vertex u such that every vertex which is not covered by $S \cup A$ is covered by u and another vertex of $S \cup A$, and the vertex u must belong to $\bar{U}$ since we assume that $\bar{U} \neq \emptyset$. For the converse, if there is a vertex $u \in \bar{U}$ such that $N(u) \cap N(v) \cap I = \emptyset$ for every $v \in \bar{U} \setminus \{u\}$ then $S \cup A \cup \{u\}$ is a g-extension of A. Now we assume that a g-extension of A has size larger than $|S \cup A| + 1$. Then $\bar{U}$ contains at least two vertices. If all vertices of $\bar{U}$ have the same neighbor in I, say x, then there is a vertex y in C which is nonadjacent to x since the vertices of $\bar{U}$ are non-simplicial. And in that case $S \cup A \cup \{x\}$ would be a g-extension of A since all vertices of $\bar{U}$ are covered by $I[x, y]$. Hence, there are two vertices u_1 and u_2 of $\bar{U}$ such that u_1 and u_2 have distinct neighbors in I. Now we prove that $S \cup A \cup \{u_1, u_2\}$ is a g-extension of A. It suffices to show that every vertex u of $\bar{U} \setminus \{u_1, u_2\}$ lies on a shortest path between two vertices of $S \cup A \cup \{u_1, u_2\}$, as $S \cup A$ covers all of $V(G) \setminus \bar{U}$. Let $u \in \bar{U} \setminus \{u_1, u_2\}$ and u', u'_1, and u'_2 be the unique neighbors of u, u_1 and u_2 in I respectively. If u' is not equal to any of u'_1 or u'_2, then $u \in I[u', u_1]$. So without loss of generality assume that $u' = u'_1$. In this case we have $u \in I[u'_1, u_2]$ which completes the proof. □

The following corollary is a direct consequence of Theorems 3.3 and 3.6 since the 2-connected components of monopolar chordal graphs are precisely split graphs [12].

Corollary 3.7. *A minimum geodetic set of a monopolar chordal graph can be computed in linear time.*

One can also note that in the special case of *threshold graphs* which are P_4-free split graphs, the g-extension is reduced to the set $S \cup A$ since S forms a g-set. A minimum geodetic set for P_4-free monopolar chordal graphs easily follows from this remark.

4. Final remarks and related problems

As future research, it would be interesting to determine other classes of graphs for which our block decomposition method would provide polynomial time algorithms for solving the minimum geodetic set problem. To this end, we should identify graph classes for which the g-extension problem can be solved in polynomial time in their blocks. One may also study the complexity of the minimum geodetic set problem in subclasses of chordal graphs in order to enlighten the border between NP-hard and polynomially solvable cases.

We mentioned that the set of simplicial vertices forms the (unique) g-set in ptolemaic graphs. One might be interested in the characterization of graphs for which the set of simplicial vertices forms a geodetic set. It should be first noticed that, unless we require this property to hold for every induced subgraph (i.e., to be a hereditary property), we cannot end up with a nice characterization in terms of classes of graphs defined by some hereditary property such as forbidden induced subgraphs. This fact can be easily observed on a graph obtained by adding a pending edge to each one of the vertices of an arbitrary graph G; indeed, the set of simplicial vertices (formed by the set of degree one vertices) of such a graph is its unique minimum geodetic set, however, this property does not necessarily hold for its subgraph G which is an arbitrary graph. So, let's consider the following hereditary graph property: a graph G is called *unigeodetic* if G and all of its connected (induced) subgraphs admit the set of their simplicial vertices as a minimum geodetic set. Then we have:

Proposition 4.1. *G is unigeodetic if and only if it is a ptolemaic graph.*

Proof. We already know that ptolemaic graphs are unigeodetic (recall that induced subgraphs of ptolemaic graphs are also ptolemaic) [14]. For the other direction, it is enough to make two observations. Firstly, any chordless cycle has no simplicial vertex and therefore unigeodetic graphs should be chordal. Secondly, any minimum geodetic set of a gem (a cycle of length 5 with two non-intersecting chords) contains a non-simplicial vertex; consequently, unigeodetic graphs should not contain an induced gem. It remains to notice that gem-free chordal graphs are exactly ptolemaic graphs [20]. □

Indeed, if we relax the property of being hereditary (in the definition of the unigeodetic graphs), the characterization of graphs such that the set of simplicial vertices forms a geodetic set remains an open question.

We have pointed out that for a given graph, the geodetic sets are also hull sets but the converse is not necessarily true. In this paper, we have shown that they coincide for block-cacti. An interesting further study would be to find necessary conditions for this property to hold or find other graph classes where this property holds. Also, further study should be conducted to enlighten the comparison of the computational difficulties of the hull number problem and the geodetic number problem.

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