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DEREGULATED ELECTRICITY MARKETS WITH THERMAL LOSSES AND PRODUCTION BOUNDS: MODELS AND OPTIMALITY CONDITIONS*

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Abstract. A multi-leader-common-follower game formulation has been recently used by many authors to model deregulated electricity markets. In our work, we first propose a model for the case of electricity market with thermal losses on transmission and with production bounds, a situation for which we emphasize several formulations based on different types of revenue functions of producers. Focusing on a problem of one particular producer, we provide and justify an MPCC reformulation of the producer's problem. Applying the generalized differential calculus, the so-called M-stationarity conditions are derived for the reformulated electricity market model. Finally, verification of suitable constraint qualification that can be used to obtain first order necessary optimality conditions for the respective MPCCs are discussed.

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1. Introduction

With the liberalization of the electricity markets in the previous decades, various models for this specific type of market have been proposed. One of the first models is described in [18] and applied to the market of England and Wales in [11]. Motivated by deregulated electricity markets, a new class of models constructed around the concept of generalized Nash game have been introduced. Such models thus incorporate specific features of electricity markets, such as transmission network and biding mechanism of each producer in the network, and correspond to noncooperative games, in which each producer aims to maximize his benefit by means of

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announcing a bid on the energy production. The market is controlled by a central operator, frequently called the Independent System Operator (ISO), who computes the best response to the producers' bids in order to minimize the global cost of energy, thus aiming at effective electricity production, while taking into account technical parameters of the transmission grid and that the demand at each node of the transmission network must be satisfied. This leads to the so-called multi-leader-common-follower game, *cf. e.g.* [19], in which each producer is in the role of a leader and ISO is the single follower, common to all leaders.

There are several reasons to include production bounds in a model of electricity market. Consider, *e.g.*, some special geographic configuration with extreme nodes of networks such as distant islands. In such a case, either the high thermal loss due to transmission to such nodes and/or a relatively low production capacity at these nodes can result to market equilibrium in which the production capacity is reached. Another situation, in which the capacity of production at a given node is reached, arises on the so-called adjustment markets of some countries where the total production capacity of every producer has to be offered to the ISO (see *e.g.* [24, 32]).

Our aim in this paper is to study variational equilibria of the electricity market model in which losses due to transmission and bounds on production and transmission are present. Such variational equilibria correspond to solutions of the so-called EPCCs (Equilibrium Problems with Complementarity Constraints), a coupled system of MPCCs (Mathematical Programs with Complementarity Constraints). We refer to monographs [21, 27] and ([23], Chap. 5.2) for MPCCs and to [3, 28] for EPCCs. In particular, this paper complements and extends results of [1] on electricity markets with transmission losses and results of [13] where M-stationarity conditions (M-stands for Mordukhovich) for the electricity market model were derived.

The paper is organized as follows. The general formulation of problems of producers and ISO are introduced in Section 2.1. In Section 2.2 we discuss an MPCC reformulation of producer's problem and show that the corresponding variational equilibrium, under weak assumptions, it is equivalent to the generalized Nash equilibrium of the electricity market model. In Section 2.3, we discuss conditions ensuring the so-called single-valued case and compare it with another one used in the literature. Finally, Section 3 is devoted to the first order necessary optimality conditions for the reformulated electricity market model along with discussion on verification of required qualification conditions.

Our notation is basically standard. $\mathbb{B}$ denotes the unit ball. We use $\mathbb{R}_+$ to denote nonnegative reals. For a matrix A, A_i denotes the ith row of A. For an index set $I \subset \{1, \ldots, s\}$ and a vector $d \in \mathbb{R}^s$, d_I denotes a subvector composed of the components $d_i, i \in I$ and diag d denotes a diagonal $s \times s$ matrix with $(\text{diag } d)_{ii} = d_i, i = 1, \ldots, s$. Analogously, for a matrix A with s rows, A_I is the submatrix composed of the rows $A_i, i \in I$. For a set Ω, $\overline{\Omega}$ denotes its closure, and for a closed cone D with vertex at the origin, D° denotes its negative polar cone. By $x \xrightarrow{\Omega} \bar{x}$ we mean that $x \to \bar{x}$ with $x \in \Omega$. $T_\Omega(x)$ signifies the contingent (Bouligand−Severi, tangent) cone to Ω at x.

For the readers' convenience we now state the definitions of several basic notions from modern variational analysis. For a closed set Ω and a point $\bar{x} \in \Omega$, the *Fréchet normal cone* to Ω at $\bar{x}$ is defined by

$$\widehat{N}_\Omega(\bar{x}) := \left\{ x^* \in \mathbb{R}^n \;\middle|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\| x - \bar{x} \|} \le 0 \right\} = \left(T_\Omega(\bar{x})\right)^\circ .$$

The *limiting normal cone* to Ω at $\bar{x}$ is given by

$$N_\Omega(\bar{x}) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}_\Omega(x),$$

where the "Lim sup" stands for the Painlevé−Kuratowski upper (or outer) limit. This limit is defined for a set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ at a point $\bar{x}$ by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} M(x) := \{ y \in \mathbb{R}^m \,|\, \exists x_k \to \bar{x}, \exists y_k \to y \text{ with } y_k \in M(x_k) \}.$$

For a convex set Ω, both normal cones N_Ω and $\widehat{N}_\Omega$ amount to the normal cone of convex analysis, for which we use simply the notation N_Ω.

Given a set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ and a point $(\bar{x}, \bar{y})$ from its graph

$$\text{Gph}M := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in M(x)\},$$

the *limiting (Mordukhovich) coderivative* $D^*M(\bar{x}, \bar{y})[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ of M at $(\bar{x}, \bar{y})$ is defined by

$$D^*M(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\text{Gph}M}(\bar{x}, \bar{y})\}.$$

In this paper, we also employ some notions of stability of multifunctions, namely the Aubin property and calmness.

A set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is said to have the *Aubin (pseudo-Lipschitz, Lipschitz-like) property* around $(\bar{x}, \bar{y}) \in \text{Gph } M$ with modulus $\ell \geq 0$ if there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ such that

$$M(x) \cap \mathcal{V} \subset M(u) + \ell ||x - u|| \mathbb{B}$$

for all $x, u \in \mathcal{U}$, where $\mathbb{B}$ is closed unit ball. The Mordukhovich criterion [22] provides a characterization of the Aubin property through knowledge of the respective coderivative: a set-valued mapping M has Aubin property around $(\bar{x}, \bar{y})$ if and only if

$$D^*M(\bar{x}, \bar{y})(0) = \{0\}.$$

A set-valued mapping $M[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is said to be *calm (pseudo upper Lipschitz)* at $(\bar{x}, \bar{y}) \in \text{Gph } M$ with modulus $L \geq 0$ if there are neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ such that

$$M(x) \cap \mathcal{V} \subset M(\bar{x}) + L\|x - \bar{x}\| \mathbb{B} \quad \text{for all } x \in \mathcal{U}.$$

Clearly, the Aubin property implies calmness, whereas the converse is not true. In the sequel, calmness will be utilized as a suitable qualification condition in the used rules of generalized differential calculus, *cf.* [12, 17].

2. Towards an adapted model for electricity market with transmission losses and production bounds

In this section we provide first a general formulation of the problem and introduce notation of the electricity market model. Using the KKT reformulation of the ISO problem, we provide variational equilibrium reformulation of the problem in Section 2.2 and discuss sufficient conditions for equivalence with the original electricity market problem. We comment on several possible choices of producers' revenue functions. In Section 2.3 we comment on the so-called single-valued case which arises whenever the primal and dual solutions of the ISO problem are unique.

2.1. General electricity market model and notation

In this work we assume that the electricity market is represented by a network where at each node $i = 1, \ldots, N$, there is exactly one producer and the local electricity energy demand D_i is known. Therefore, we do not consider consumers as acting agents in our model, *i.e.* the total amount of electricity demanded by consumers at each node is supposed to match the local demand at that node. Thus, the general model from a class of multi-leaders-common-follower games considered in this work takes into account only two types of players, producers and ISO.

Each producer (leader) i, $i = 1, \ldots, N$, aiming at maximizing his or her profit, bids a cost function $\varphi_i(q_i)$ to the ISO (a follower common to all leaders), where q_i denotes the electricity energy production of producer i (*e.g.* in GWhs). The market regulator, ISO, taking into account all bids of producers, aims at maximizing the so-called social welfare, or alternatively, minimizing the social costs. In here, we consider minimization of the total cost of production while taking into account the requirement that the local demand D_i is satisfied at each node. Later on, we will introduce additional constraints which take into account bounds on transmission and production.

Denoting by R_i and C_i, $i = 1, \ldots, N$, the revenue function and the real cost function of producer i, respectively, the multi-leaders-common-follower game can be formulated as the following general equilibrium problem composed of N producer's optimization problems denoted as P_i, $i = 1, \ldots, N$, solved simultaneously

$$P_i \quad \max_{\varphi_i} \; R_i(\varphi_i(q_i), q_i) - C_i(q_i) \quad \text{s.t.} \; \begin{cases} q \text{ solves } ISO(\varphi), \\ \varphi_i \text{ admissible bid function,} \end{cases}$$

where the ISO problem is considered in the form

$$ISO(\varphi) \quad \min_q \sum_i \varphi_i(q_i) \quad \text{s.t.} \quad \text{demand } D_i \text{ is satisfied at each node } i = 1, \ldots, N.$$

In order to distinguish between the components associated with producer i and components linked to the other producers, we employ the following notation for vector of electricity production $q = (q_i, q_{-i})$ and vector-valued bid function $\varphi = (\varphi_i, \varphi_{-i})$. Later on, we employ this notation also to bid coefficients *etc.*

In the following, we introduce additional elements of the market model. Throughout this paper, let

* $\mathcal{N}$ be the set of nodes (with N elements).
* $\mathcal{L}$ be the set of electricity lines (with M elements).
* $e = ij$ be the line from node i to node j.
* t be the vector of energy flows where components $t_e = t_{ij}, e \in \mathcal{L}$ denote the energy flow along the line $e = ij$ with $t_{ij} > 0$ whenever the electricity energy flows in the direction from i to j, and $t_{ij} < 0$ if the energy flow is in the opposite direction.
* $L_e \geq 0$ be the coefficient of a thermal loss on line $e \in \mathcal{L}$. Following the classical technical specifications, thermal loss on the line e is assumed to be a quadratic function of the energy flow along this line, *i.e.* $L_e t_e^2$.
* $\underline{T}_e$ and $\overline{T}_e$ be the lower and upper transmission bounds on the line $e \in \mathcal{L}$, respectively ($\underline{T}_e \leq 0$ and $\overline{T}_e \geq 0$).
* $\overline{Q}_i$ be the upper bounds on production at node $i \in \mathcal{N}$.
* $D = (D_1, \ldots, D_N)$ be the vector of electricity energy demand.

For better representation of the oriented network, δ_{ie} denotes the coefficient of the incidence matrix defined as $\Delta := (\delta_{ie})_{i \in \mathcal{N}, e \in \mathcal{L}} \in \mathbb{R}^{N \times M}$, where

$$\delta_{ie} := \begin{cases} 1 & \text{if line } e \text{ enters node } i \\ -1 & \text{if line } e \text{ leaves node } i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly the chosen orientation of the network does not correspond to physical constraints but simply allows to consider signed flows.

All along the paper we assume that the network $(\mathcal{N}, \mathcal{L})$ is connected in the following sense:

$$\begin{array}{l} \forall i, j \in \mathcal{N}, \text{ if } i \neq j \text{ , } \exists (i_0, \ldots, i_p) \in \mathcal{N}^{p+1} \text{ such that } \; i_0 = i \text{ , } i_p = j \\ \text{and } \forall k \in \{0, \ldots, p-1\} \text{ , } i_k i_{k+1} \in \mathcal{L} \text{ or } i_{k+1} i_k \in \mathcal{L}. \end{array} \tag{2.1}$$

For each line $e = ij \in \mathcal{L}$ and any flow t_e along this line, the cost $L_e t_e^2$ of thermal losses is covered equally by both producer i and producer j. This is summarized in the thermal loss mapping

$$L(t) = \left(\frac{1}{2} \sum_{e \in \mathcal{L}} |\delta_{1e}| L_e t_e^2, \ldots, \frac{1}{2} \sum_{e \in \mathcal{L}} |\delta_{Ne}| L_e t_e^2 \right).$$

This choice of distribution of the thermal losses costs has also been used in, *e.g.*, [8, 13].

In the real markets, bid functions are composed of an increasing and non-continuous sum of box bids and hourly orders. In here, we consider their quadratic approximations which have been suggested by several authors, *cf. e.g.* [16]. The same holds true for the cost function. In particular, we will consider the (real) cost functions $C_i, i = 1, \dots, N$, in the form $C_i(q_i) = A_i q_i + B_i q_i^2$ with $B_i \geq 0$ and the bid functions $\varphi_i, i = 1, \dots, N$, given by $\varphi_i(q_i) = a_i q_i + b_i q_i^2$ with $b_i \geq 0$.

Clearly, both the real cost function and bid function can be characterized just by their linear and quadratic coefficients, respectively. Therefore, instead of assuming that producers are bidding a function, one can consider bidding respective linear and quadratic coefficients. In the sequel, the bid functions $\varphi_i, i = 1, \dots, N$, are substituted by couples (a_i, b_i). The quantities $\underline{A}_i$, $\overline{A}_i$, $\underline{B}_i$, $\overline{B}_i$ stand for the corresponding bounds on the respective bid coefficients. In the sequel the notation $\mathcal{A}$ and $\mathcal{B}$ will be used to describe sets of admissible bid coefficients $\mathcal{A} = \prod_{i=1}^N [\underline{A}_i, \overline{A}_i]$ and $\mathcal{B} = \prod_{i=1}^N [\underline{B}_i, \overline{B}_i]$.

Taking into account the notation introduced above, the producer i's problem P_i can be restated as the following optimization problem

$$P_i(a_{-i}, b_{-i}) \qquad \max_{a_i, b_i, q, t} R_i(a, b, q, t)\text{-}(A_i q_i + B_i q_i^2)$$
$$\text{s.t.} \begin{cases} \underline{A}_i \leq a_i \leq \overline{A}_i, \\ \underline{B}_i \leq b_i \leq \overline{B}_i, \\ (q, t) \text{ solves } ISO(a, b), \end{cases} \tag{2.2}$$

where $ISO(a, b)$ stands for the following ISO's problem

$$ISO(a, b) \qquad \min_{q,t} \sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)$$
$$\text{s.t.} \begin{cases} q_i \geq 0, \ \forall i \in \mathcal{N} \\ q_i \leq \overline{Q}_i, \ \forall\, i \in \mathcal{N} \\ q_i + \sum_{e \in \mathcal{L}} \left(\delta_{ie} t_e - \frac{L_e |\delta_{ie}|}{2} t_e^2 \right) \geq D_i, \forall\, i \in \mathcal{N} \\ t_e \geq \underline{T}_e, \ \forall\, e \in \mathcal{L} \\ t_e \leq \overline{T}_e, \forall\, e \in \mathcal{L} \end{cases} \tag{2.3}$$

This leads us to the following definition of a solution to the electricity market model.

Definition 2.1. A *generalized Nash equilibrium* of the electricity market is a vector $(a^*, b^*, q^*, t^*) \in \mathcal{A} \times \mathcal{B} \times \mathbb{R}^N \times \mathbb{R}^L$ such that

$$(a_i^*, b_i^*, q^*, t^*) \text{ solves } P_i(a_{-i}^*, b_{-i}^*) \ \ \forall i = 1, \dots, N. \tag{2.4}$$

We would like to point out that since the maximum in $P_i(a_{-i}, b_{-i})$ is considered with respect to (a_i, b_i) and also (q, t), the present formulation of the producer's problem is usually called *the optimistic formulation* of the problem and its solutions are referred to as optimistic. Indeed if, for a given bid couple (a, b), there exist several solutions (q, t) of $ISO(a, b)$, considering the maximum of the objective (profit) function with respect to (q, t) is a clearly "favorable" for producer i. However, using the terminology from [3], the electricity market model (2.4) is referred to as multioptimistic. Problems of this type are frequently ill-posed, see [3] for details and [33] for sufficient condition for well-posedness.

Let us end this subsection with the following technical lemma, precising that some natural bounds exists for the network. The proof is left to the reader.

Lemma 2.2. *Suppose that for all nodes $i \in \mathcal{N}$, one has $a_i > 0$ or $b_i > 0$ and let (q, t) be the solution of $ISO(a, b)$. Then for all $e \in \mathcal{L}$ one has $|t_e| \leq \frac{1}{L_e}$ provided $L_e > 0$.*

Taking into account these natural bounds we assume, for the rest of the paper, that the lower and upper transmission bounds satisfy

$$\overline{T}_e \leq \frac{1}{L_e} \quad \text{and} \quad \underline{T}_e \geq \frac{-1}{L_e}\cdot \tag{2.5}$$

2.2. Variational equilibrium reformulation

Even if the revenue function R_i is not yet specified, in this subsection we shall concern an alternative formulation of electricity market problem. We shall start with few remarks about the ISO problem (2.3). It can be easily verified that whenever $a+b>0$, problem (2.3) admits at least one solution. Moreover, invoking ([1], Lem. 3.1), whenever $a+b>0$, for every $i\in\mathcal{N}$, the demand satisfaction constraint

$$q_i + \sum_{e\in\mathcal{L}} \left(\delta_{ie}t_e - \frac{L_e|\delta_{ie}|}{2}t_e^2\right) \geq D_i \tag{2.6}$$

in the ISO problem (2.3) is active at any solution (q,t) of $ISO(a,b)$.

The following KKT conditions will play a central role in the sequel.

$$KKT(a,b)\begin{cases} 0 = a_i + 2b_iq_i - \mu_i + \bar{\mu}_i - \lambda_i & \forall i\in\mathcal{N},\\ 0\leq \mu_i \perp q_i \geq 0 & \forall i\in\mathcal{N},\\ 0\leq \bar{\mu}_i \perp \bar{Q}_i - q_i \geq 0 & \forall i\in\mathcal{N},\\ 0\leq \lambda_i \perp q_i + \sum_{e\in\mathcal{L}}\left(\delta_{ie}t_e - \frac{L_e|\delta_{ie}|}{2}t_e^2\right) - D_i \geq 0 & \forall i\in\mathcal{N},\\ 0 = -\beta_e + \bar{\beta}_e + \sum_{i\in\mathcal{N}}\lambda_i(\delta_{ie} + |\delta_{ie}|L_et_e) & \forall e\in\mathcal{L},\\ 0\leq \beta_e \perp (t_e - \underline{T}_e) \geq 0 & \forall e\in\mathcal{L},\\ 0\leq \bar{\beta}_e \perp \overline{T}_e - t_e \geq 0 & \forall e\in\mathcal{L}, \end{cases} \tag{2.7}$$

where μ, $\bar{\mu}$, λ, β and $\bar{\beta}$ denote the Lagrange multipliers associated to the inequality constraints of $ISO(a,b)$, respectively.

Since for $b_i\geq 0, i=1,\ldots,N$ the objective function of (2.3) is convex, the corresponding optimal solutions of $ISO(a,b)$ coincide with solutions of the KKT system associated to the ISO's problem if classical qualification conditions holds at this solution. It is important to notice that actually this equivalence of solution sets can fail even for very simple electricity markets. Indeed, let us consider for example a two nodes market with only one line connecting node 1 to node 2. Assume that $\bar{Q}_1 = 1$, $\bar{Q}_2 = 6$, $D_1 = D_2 = 2$, $L_{12} = 1/2$, $\overline{T}_{12} = 2$ and $\underline{T}_{12} = -2$. The unique solution of the ISO problem is $(q_1,q_2,t) = (1,5,2)$ but actually the associated KKT system admits no solution, that is no Lagrange multipliers exists for this point. Clearly no Slater qualification condition holds since, due to the structure of the network, the constraint set of the ISO problem has an empty interior. The situation occuring in the above simple example can actually be encountered for networks in which a part of the network is linked to the rest by a single line.

Now, substituting in problem $P_i(a_{-i},b_{-i})$ the constraint "(q,t) solves $ISO(a,b)$" by "(q,t,ξ) solves $KKT(a,b)$", where $\xi := (\mu,\bar{\mu},\lambda,\beta,\bar{\beta})$, we obtain the desired reformulation which belongs to a class of EPCCs.

Let us denote by H the twice continuously differentiable mapping specifying constraints of the ISO problem (2.3):

$$H(q,t) = \begin{pmatrix} -q \\ q-\bar{Q} \\ -q - \Delta t + L(t) + D \\ \underline{T} - t \\ t - \overline{T} \end{pmatrix}.$$

Then, by convexity, the KKT conditions (2.7) can be written down as the following generalized equation

$$0 \in F(a,b,q,t,\xi) + Q(q,t,\xi), \tag{2.8}$$

where

$$F(a,b,q,t,\xi) = \begin{pmatrix} a + 2(\operatorname{diag} b)q + (\nabla_q H(q,t))^\top \xi \\ (\nabla_t H(q,t))^\top \xi \\ -H(q,t) \end{pmatrix}, \quad Q(q,t,\xi) = \begin{pmatrix} \{0\}_N \\ \{0\}_M \\ N_{\mathbb{R}^s_+}(\xi) \end{pmatrix},$$

and $s = 3N + 2M$.

Therefore the $MPCC(a_{-i}, b_{-i})$ associated with $P_i(a_{-i}, b_{-i})$ can be written as

$$MPCC_i(a_{-i}, b_{-i}) \qquad \max_{a_i, b_i, q, t, \xi} \quad R_i(a,b,q,t) - C_i(q_i) \qquad \text{subject to } \begin{cases} 0 \in F(a,b,q,t,\xi) + Q(q,t,\xi) \\ (a_i, b_i) \in \mathcal{A}_i \times \mathcal{B}_i. \end{cases} \tag{2.9}$$

Definition 2.3. A *variational equilibrium* of the electricity market is a vector $(a^*, b^*, q^*, t^*, \xi^*) \in \mathcal{A} \times \mathcal{B} \times \mathbb{R}^N \times \mathbb{R}^L \times \mathbb{R}^s$ such that

$$(a_i^*, b_i^*, q^*, t^*, \xi^*) \text{ solves } MPCC_i(a_{-i}^*, b_{-i}^*) \ \ \forall i = 1, \ldots, N. \tag{2.10}$$

Now, let us inspect the link between generalized Nash equilibrium and variational equilibrium of the electricity market. It is the aim of the following theorem which invokes recent results [7]. Let us first denote by Λ the set of Lagrange multipliers associated to the solutions of the KKT conditions (2.7)

$$\Lambda(a,b,q,t) = \{\xi \in \mathbb{R}^s \ : \ (q,t,\xi) \text{ solves } KKT(a,b)\}.$$

Theorem 2.4. *Assume that the condition*

$$\text{there exists an element } (q,t) \in \mathbb{R}^N \times \mathbb{R}^L \text{ such that}$$
$$\begin{cases} 0 < q_i < \bar{Q}_i \text{ and } q_i + \sum_{e \in \mathcal{L}} \left(\delta_{ie} t_e - \frac{L_e |\delta_{ie}|}{2} t_e^2\right) > D_i \ \forall i \in \mathcal{N}, \\ \underline{T}_e < t_e < \overline{T}_e \ \forall e \in \mathcal{L}, \end{cases} \tag{2.11}$$

is satisfied for the data of the electricity market. Then the following hold:

(i) *If (a^*, b^*, q^*, t^*) is a generalized Nash equilibrium of the electricity market then, for any $\xi \in \Lambda(a^*, b^*, q^*, t^*)$, $(a^*, b^*, q^*, t^*, \xi)$ is a variational equilibrium of the electricity market.*

(ii) *Conversely, if $(a^*, b^*, q^*, t^*, \xi^*)$ is a variational equilibrium of the electricity market, then (a^*, b^*, q^*, t^*) is a generalized Nash equilibrium of the electricity market.*

Proof. The statement is a direct consequence of ([7], Thm. 2.1) and ([7], Thm. 2.2), taking into account that the Slater-type qualification condition (2.11) does not depend on (a, b) which follows from the fact that the constraints of $ISO(a, b)$ are only in terms of q and t. □

A direct way to guarantee that the qualification condition (2.11) holds is to find a solution (ε, q, t) of the nonlinear system of equations

$$q_i + \sum_{e \in \mathcal{L}} \left(\delta_{ie} t_e - \frac{L_e |\delta_{ie}|}{2} t_e^2\right) = D_i + \varepsilon \ \ \forall i \in \mathcal{N}$$

such that $\varepsilon > 0$ and

$$0 \leq q_i \leq \bar{Q}_i \quad \text{and} \quad \underline{T}_e \leq t_e \leq \overline{T}_e \ \ \forall i \in \mathcal{N}, \forall e \in \mathcal{L}.$$

We close this subsection with examples of revenue function $R_i(a,b,q,t)$. Revenue is income that producer i receives from selling the electricity energy produced. In perfect competition, it is the product of the unit price

of electricity and the quantity of electricity energy produced, where the unit price of electricity energy at node i is the marginal price, which is given by the Lagrange multiplier (shadow price) λ_i associated to demand satisfaction constraint (2.6). However, there is *a priori* no reason for the Lagrange multiplier λ_i to be uniquely determined for a given (q, t). Thus, we can define different variants of R_i thus providing different special cases of the producer i problem $P_i(a_{-i}, b_{-i})$ and its corresponding reformulation $MPCC_i(a_{-i}, b_{-i})$. In particular, one can consider the following two cases:
The *optimistic-pessimistic case* defined by the revenue function R_i

$$R_i(a, b, q, t) = \inf\{\lambda_i q_i : (\mu, \bar{\mu}, \lambda, \beta, \bar{\beta}) \in \Lambda(a, b, q, t)\}. \tag{2.12}$$

The *optimistic-optimistic case* defined by the revenue function R_i

$$R_i(a, b, q, t) = \sup\{\lambda_i q_i : (\mu, \bar{\mu}, \lambda, \beta, \bar{\beta}) \in \Lambda(a, b, q, t)\}. \tag{2.13}$$

In the above denominations, the first "optimistic" term concerns the primal variables of the ISO while for the second term (pessimistic or optimistic) it is the dual variable λ_i which is in scope.

Alternatively, one can also consider a selection of the set $\Lambda(a, b, q, t)$ of Lagrange multipliers, see Escobar−Jofre [9]. This approach is close to the one implemented in the Cosmos software used to determine the clearing price in some European markets (EPEX, APX-ENDEX, Belpex), see [5].

Nevertheless, none of revenue functions R_i described above make sense at any (q, t) such that the set of admissible Lagrange multipliers λ_i is unbounded. This unfavourable situation typically occurs whenever the Mangasarian–Fromowitz constraint qualification is not satisfied for the constraints of $ISO(a, b)$. Since the constraints of $ISO(a, b)$ are described by convex differentiable functions, condition (2.11) implies that for any feasible (a, b, q, t) the set $\Lambda(a, b, q, t)$ of Lagrange multipliers is convex and compact (*cf. e.g.* [2, 14]). Additionally, this set is nonempty if and only if (q, t) is a solution of $ISO(a, b)$. Thus, both types of revenue function R_i defined above are well defined on the constraint set of the producer i problem $P_i(a_{-i}, b_{-i})$.

2.3. Single-valued case

Assume that data of the considered network are such that condition (2.11) is satisfied. The single valued case corresponds to the situation when for every $(a, b) \in \mathcal{A} \times \mathcal{B}$ problem $ISO(a, b)$ admits unique primal and dual solutions. In such case the solution of the generalized equation (2.8) shall be denoted by $(q(a, b), t(a, b), \xi(a, b))$ and the revenue function of producer i by $R_i(a, b) := R_i(a, b, q(a, b), t(a, b))$. Sufficient conditions for the uniqueness of the couple (q, t) of production and flow as well as associated Lagrange multipliers are stated in Propositions 2.5 and 2.6 below.

Proposition 2.5. *Assume that for all producers $i \in \mathcal{N}$, one has $a_i \neq 0$ or $b_i \neq 0$, and, for all lines $e \in \mathcal{L}$, $L_e > 0$. Then $ISO(a, b)$ admits a unique solution (q^*, t^*).*

Proof. Let (q^I, t^I) and (q^{II}, t^{II}) be two solutions of the ISO's problem such that $t^I \neq t^{II}$, and let $\lambda \in (0, 1)$. By convexity of K and the objective function of the ISO's problem, $(q^\lambda, t^\lambda) := (\lambda q^I + (1 - \lambda)q^{II}, \lambda t^I + (1 - \lambda)t^{II})$ is also a solution of the ISO's problem.

Since $t^I \neq t^{II}$ there exists $e \in \mathcal{L}$ such that $t^I_e \neq t^{II}_e$. Let $i \in \mathcal{N}$ be such that $\delta_{ie} \neq 0$. Therefore, the convexity of the functions $\varphi_f[x \longrightarrow |\delta_{if}|L_f x^2]$ for all $f \in \mathcal{L}$ and the strict convexity of the function $x \to |\delta_{ie}|L_e x^2$ imply

$$q^\lambda_i - \sum_{f \in \mathcal{L}} \left(\delta_{if} t^\lambda_f + \frac{L_f|\delta_{if}|}{2}(t^\lambda_f)^2\right) > D_i,$$

which contradicts ([1], Lem. 3.2). Thus we have $t^I = t^{II}$ and, again by Lemma ([1], Lem. 3.2), for any $i \in \mathcal{N}$, $q^I_i = q^{II}_i = D_i - \sum_{e \in \mathcal{L}} \left(\delta_{ie} t^\lambda_e - \frac{L_e|\delta_{ie}|}{2}(t^\lambda_e)^2\right)$. $\square$

Proposition 2.6. *Let $(a,b) \in \mathcal{A} \times \mathcal{B}$ be such that for all producers $i \in \mathcal{N}$, one has $a_i \neq 0$ or $b_i \neq 0$ and there exists a unique (q^*, t^*) solving $ISO(a,b)$. Further, suppose that for all $e \in \mathcal{L}$, $\underline{T}_e < t_e^* < \overline{T}_e$ and that there exists a node $i_0 \in \mathcal{N}$ satisfying $q_{i_0}^* \in (0, \overline{Q}_{i_0})$. Then for each $i \in \mathcal{N}$ there exist unique Lagrange multipliers λ_i^*, μ_i^*, $\bar{\mu}_i^*$ and for each $e \in \mathcal{L}$ there exist unique Lagrange multipliers β_e^* and $\bar{\beta}_e^*$.*

Proof. Let μ, β, $\bar{\beta}$ and λ be the vectors of Lagrange multipliers associated with the solution (q^*, t^*) of $ISO(a,b)$. Clearly, from system (2.7), the Lagrange multiplier λ_{i_0} is uniquely given by $\lambda_{i_0}^* = a_{i_0} + 2b_{i_0} q_{i_0}^*$.

Let $e \in \mathcal{L}$. Since $\underline{T}_e < t_e^* < \overline{T}_e$, one has

$$\sum_{i \in \mathcal{N}} \lambda_i (\delta_{ie} - |\delta_{ie} L_e| t_e^*) = 0. \tag{2.14}$$

Consider two nodes i and j such that $\delta_{ie} = -1$ and $\delta_{je} = 1$, *i.e.* $e = ij$. For all $k \in \mathcal{N} \setminus \{i, j\}$, we have $\delta_{ke} = 0$, then the formula (2.14) gives the following relations:

$$\lambda_i = \lambda_j \frac{1 - L_e t_e^*}{1 + L_e t_e^*} \quad \text{and} \quad \lambda_j = \lambda_i \frac{1 + L_e t_e^*}{1 - L_e t_e^*}.$$

Observe that in both equations above the fractions are well defined due to the general assumption (2.5) and the hypothesis of the proposition, indeed $-\frac{1}{L_e} \leq \underline{T}_e < t_e^* < \overline{T}_e \leq \frac{1}{L_e}$, therefore $-1 < L_e t_e^* < 1$.

Thus, the value of λ_j is uniquely determined by the value of λ_i and vice versa. In general way if $e = ij$ or $e = ji$, then the two above equalities can be resumed into the following equality:

$$\lambda_j = \lambda_i \frac{1 + \delta_{je} L_e t_e^*}{1 + \delta_{ie} L_e t_e^*}. \tag{2.15}$$

Now, take any $i \in \mathcal{N} \setminus \{i_0\}$. Due to the fact that the graph $(\mathcal{N}, \mathcal{L})$ is connected, by (2.1) there exist nodes $i_1, \ldots, i_p \in \mathcal{N}$ such that $i_p := i$ and for all $k \in \{0, \ldots, p-1\}$, $i_k i_{k+1} \in \mathcal{L}$ or $i_{k+1} i_k \in \mathcal{L}$. Since λ_{i_0} is unique, λ_{i_1} is unique because it is uniquely determined by λ_{i_0}. By recursion, $\lambda_i := \lambda_{i_p}$ is unique. Thus, for each $i \in \mathcal{N}$, λ_i is unique.

It remains to observe, that for each $i \in \mathcal{N}$ one of the Lagrange multipliers μ_i and $\bar{\mu}_i$ is always vanishing and the other is uniquely determined by the KKT system (2.7). Also, for each $e \in \mathcal{L}$ either β_e or $\bar{\beta}_e$ vanishes and the other nonvanishing multiplier is also determined uniquely by the KKT system (2.7). This concludes the proof. □

Remark 2.7. In Proposition 2.6, one can observe that at the unique solution (q^*, t^*) of $ISO(a,b)$, the optimal marginal prices are given by $\lambda_{i_0} = a_{i_0} + 2b_{i_0} q_{i_0}^*$ and

$$\forall i \in \mathcal{N},\ \lambda_i = \lambda_{i_0} h_i^{i_0}(t^*).$$

The vector $h^{i_0}(t^*)$ is defined by $h_{i_0}^{i_0}(t^*) = 1$ and:

$$\forall i \in \mathcal{N} \setminus i_0, \qquad h_i^{i_0}(t^*) = \prod_{k=1}^{p_i} \frac{1 + \delta_{i_k e_k} L_{e_k} t_{e_k}^*}{1 + \delta_{i_{k-1} e_k} L_{e_k} t_{e_k}^*}$$

with $\{i_0, \ldots, i_{p_i}\} \subset \mathcal{N}$ which satisfies $i_{p_i} = i$ and for all $k \in \{1, \ldots, p_i\}$, $i_{k-1} i_k \in \mathcal{L}$ or $i_k i_{k-1} \in \mathcal{L}$, $e_k = i_{k-1} i_k$ if $i_{k-1} i_k \in \mathcal{L}$, $e_k = i_k i_{k-1}$ otherwise. This expression of $h_i^{i_0}(t^*)$ results by recursion from the equality (2.15) and does not depend on the choice of the path between i_0 and i, because otherwise we obtain a contradiction with the uniqueness of the marginal price at each node and the positiveness of λ_{i_0}.

In the single-valued case, there is no reason to distinguish between optimistic and pessimistic formulations of the problem. Nevertheless, two possible choices of revenue function $R_i(a,b)$ can still be discussed.

(a) Whenever for any $(a,b) \in \mathcal{A} \times \mathcal{B}$ there is no i such that at solution $ISO(a,b)$ one has $q_i(a,b) = 0$ or $q_i(a,b) = \overline{Q}_i$ or $\overline{Q}_i = +\infty$, the Lagrange multiplier λ_i is simply the derivative of the bid function of player i at $q_i(a,b)$, *i.e.* $\lambda_i(a,b) = a_i + 2b_i q_i(a,b)$. This well known fact can be easily seen from the first equation of the KKT system (2.7). Thus, the revenue function is given by

$$\tilde{R}_i(a,b) = (a_i + 2b_i q_i(a,b))q_i(a,b)$$

and each producer i aims at solving

$$\begin{aligned} &\max_{a_i,b_i} \ (a_i + 2b_i q_i(a,b))q_i(a,b) - (A_i q_i(a,b) + B_i q_i(a,b)^2) \\ &\text{s.t.} \begin{cases} \underline{A}_i \le a_i \le \bar{A}_i \\ \underline{B}_i \le b_i \le \bar{B}_i. \end{cases} \end{aligned}$$

We shall denote the associated Nash equilibrium problem as (NEP_1).

(b) In any other case the revenue function R_i is defined as

$$R_i(a,b) := \lambda_i(a,b)q_i(a,b)$$

and the producer i solves

$$\begin{aligned} &\max_{a_i,b_i} \ \lambda_i(a,b)q_i(a,b) - (A_i q_i(a,b) + B_i q_i(a,b)^2) \\ &\text{s.t.} \begin{cases} \underline{A}_i \le a_i \le \bar{A}_i \\ \underline{B}_i \le b_i \le \bar{B}_i \end{cases} \end{aligned}$$

We denote the associated Nash equilibrium problem as (NEP_2).

From the above discussion it follows that even in the single-valued case, considering production bounds in electricity market model one needs to define the revenue function with caution. We would like to point out that some authors (see *e.g.* [13]) use the derivative of the bid function as the unit price of electricity energy even in the case where the production bounds are considered in the model. Nevertheless, in [13] the main result is derived under the assumption that no (lower and upper) production bound is reached.

Observe that in this special single-valued case, electricity market model turns out to belong to a class of classical Nash Equilibrium problems. This interesting fact has been exploited in [1] to provide explicit formulae of the solution vectors q and t.

Clearly, if for all $i \in \mathcal{N}$ and for all $(a,b) \in \mathcal{A} \times \mathcal{B}$ the upper bound of production is not reached, then both problems (NEP_1) and (NEP_2) admit the same solution. By means of a simple academic example of electricity market we show that a solution of the market model with production bounds (NEP_2) need not be a solution of the model without production bounds (NEP_1).

Example 2.8. Consider a network composed of only two nodes (and thus of two producers) connected by a single line, *i.e.* $\mathcal{N} = \{1,2\}$ and $\mathcal{L} = \{12\}$. Suppose the demands and capacity constraints are $D_1 = 5$, $D_2 = 1.9$, $\overline{Q}_1 = 565/98$ and $\overline{Q}_2 = 10$, respectively, whereas the thermal loss coefficient of line $\{12\}$ is $L = 0.2$. Further suppose that there are no transmission bounds on flow along the line, *i.e.* $\overline{T}_{12} = +\infty$ and $\underline{T}_{12} = -\infty$.

For the sake of computational simplicity, we consider the following bidding process: the producers will bid only the linear costs, hence $b_1 = b_2 = 0$ and $\mathcal{A} = [1,3] \times [2,4]$. We assume that the true costs of production are also linear, with $A_1 = 1$, $A_2 = 2$ and $B_1 = B_2 = 0$.

First, observe that for all $(a_1,a_2) \in \mathcal{A}$, $q_2(a_1,a_2) < \overline{Q}_2$. Indeed, one has $t_{12}(a_1,a_2) \ge -1/L = -5$, *cf.* Lemma 2.2, and thus $q_2(a_1,a_2) = D_2 - t_{12}(a_1,a_2) + (L/2)t_{12}^2(a_1,a_2) \le 9.4 < \bar{Q}_2$. According to Proposition 2.6, this observation ensures uniqueness of both Lagrange multipliers λ_1 and λ_2 for all $(a_1,a_2) \in \mathcal{A}$. We denote

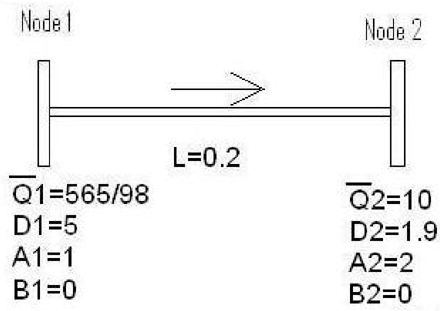

FIGURE 1. The network with data.

these multipliers by $\lambda_1(a_1, a_2)$ and $\lambda_2(a_1, a_2)$, respectively. Moreover, since the production bound $\overline{Q}_2$ is never reached, we have $\lambda_2(a_1, a_2) = a_2$. Using the KKT conditions (2.7), this implies

$$t_{12}(a_1, a_2) = \frac{a_2 - \lambda_1(a_1, a_2)}{L(a_2 + \lambda_1(a_1, a_2))}. \tag{2.16}$$

Thanks to this equality, we can compute $q_1(a_1, a_2)$ and $q_2(a_1, a_2)$ and show that the bid couple $(a_1^*, a_2^*) = (2, 4)$ is not a solution of (NEP_1) while it is a solution of (NEP_2) (see Appendix for details).

3. First order analysis of equilibrium of the electricity market

In this section, using previous results concerning properties of problem $ISO(a, b)$ and its solutions, we will derive the first order necessary optimality conditions for the variational equilibrium problem (2.10). Similarly to [13], we restrict our analysis to the so-called M-stationarity conditions, where M- stands for Mordukhovich. Our goal is to provide explicit necessary optimality conditions formulated in the problem data, similar to ([13], conditions (6.2)−(6.13)) derived for a special case of the problem, however, in our case for a general problem including losses due to transmission and bounds on production and flow. Further, we discuss several possibilities how to ensure the required calmness qualification condition. We will illustrate the application of derived M-stationarity conditions on an academic example with two settlements, *i.e.* example of an network of two nodes connected *via* single transmission line.

3.1. Explicit M-stationarity condition

Recall the reformulation of the KKT conditions (2.7) of problem $ISO(a, b)$ in the form of a generalized equation (2.8) and reformulation of $P_i(a_{-i}, b_{-i})$ to $MPCC(a_{-i}, b_{-i})$ (2.9). The generalized equation (2.8) is sometimes called *enhanced generalized equation*, enhanced by the KKT multipliers. Note that as opposed to [13], we are forced to work with the enhanced generalized equations, due to the fact that the producer's objectives depend on ISO's KKT multipliers λ.

Setting $x := (a, b)$ and $z := (q, t, \xi)$, we can reformulate the *optimistic-optimistic* optimization problem of producer i (2.9) with revenue given by (2.12) into the following parameterized MPCC where the bidding coefficients of other producers play the role of parameters.

$$\begin{aligned} &\underset{x_i, z}{\text{minimize}}\ f_i(x_i, x_{-i}, z) \\ &\text{subject to} \\ &\qquad\qquad 0 \in F(x_i, x_{-i}, z) + Q(z) \\ &\qquad\qquad x_i \in \mathcal{A}_i \times \mathcal{B}_i, \end{aligned} \tag{3.1}$$

where the producer i's objective $f_i(x, z) = -\lambda_i q_i + A_i q_i + B_i q_i^2$.

According to definition 2.3, (x^*, z^*) solves the variational equilibrium of the electricity market whenever (x_i^*, z) solves (3.1) for every $i = 1, \dots, N$.

For the reader's convenience we state a modification of the M-stationarity conditions by Outrata [26] for variational equilibria of electricity market model with the producer i's problem given by (3.1). These conditions are based on M-stationarity conditions for solutions to MPECs by Ye and Ye [34] and Outrata [25].

Proposition 3.1 ([26], Thm. 3.1)**.** *Let for every $i = 1, \dots, N$, f_i and F be continuously differentiable, $\mathcal{A}_i \times \mathcal{B}_i$ be nonempty and closed and for a fixed $\bar{x}_{-i}$ let (x_i^*, z^*) be a local solution of an MPEC (3.1). Further assume that for all $i = 1, \dots, N$, the multifunctions*

$$\Psi_i(p) := \{(x_i, z) | p \in F(x_i, x_{-i}^*, z) + Q(z)\}$$

are calm at $(0, x_i^*, z^*)$. Then for all $i = 1, \dots, N$, there exists vectors v^i such that

$$\begin{aligned} &0 \in \nabla_{x_i} f(x_i^*, x_{-i}^*, z^*) + (\nabla_{x_i} F(x_i^*, x_{-i}^*, z^*))^\top v^i + N_{\mathcal{A}_i \times \mathcal{B}_i}(x_i^*) \\ &0 \in \nabla_z f(x_i^*, x_{-i}^*, z^*) + (\nabla_z F(x_i^*, x_{-i}^*, z^*))^\top v^i + D^* Q(z^*, -F(x_i^*, x_{-i}^*, z^*))(v^i). \end{aligned} \tag{3.2}$$

Recall that D^*Q refers to the coderivative of a multivalued mapping Q.

In the following theorem we provide the explicit version of the M-stationarity conditions for variational equilibria of the electricity market (2.10).

Theorem 3.2. *Let for every $i = 1, \dots, N$, $(a_i^*, b_i^*, q^*, t^*, \mu^*, \bar{\mu}^*, \lambda^*, \beta^*, \bar{\beta}^*)$ be the solution to the problem $MPCC(a_{-i}, b_{-i})$ for a fixed vector (a_{-i}^*, b_{-i}^*) and suppose that the multifunction*

$$\Psi_i(p) := \{(a_i, b_i, q, t, \xi) | p \in F(a_i, a_{-i}^*, b_i, b_{-i}^*, q, t, \xi) + Q(q, t, \xi)\}$$

is calm at $(0, a_i^*, b_i^*, q^*, t^*, \xi^*)$. Then for all $i = 1, \dots, N$, there exist vectors $v_q^i \in \mathbb{R}^N, v_t^i \in \mathbb{R}^M, v_\mu^i \in \mathbb{R}^N, v_{\bar{\mu}}^i \in \mathbb{R}^N, v_\lambda^i \in \mathbb{R}^N, v_\beta^i \in \mathbb{R}^M$, and $v_{\bar{\beta}}^i \in \mathbb{R}^M$ such that the following conditions are satisfied

$$0 \in (v_q^i)_i + N_{\mathcal{A}_i}(a_i^*) \tag{3.3}$$

$$0 \in 2q_i^*(v_q^i)_i + N_{\mathcal{B}_i}(b_i^*) \tag{3.4}$$

$$0 = (-\lambda_i^* + A_i + 2B_i q_i^*)e_i + 2(\mathrm{diag}\ (b_i^*, \bar{b}_{-i}))v_q^i + v_\mu^i - v_{\bar{\mu}}^i + v_\lambda^i \tag{3.5}$$

$$0 = \mathrm{diag}\left\{ L_1 \sum_{i=1}^N \lambda_i^* |\delta_{i1}|, \dots, L_M \sum_{i=1}^N \lambda_i^* |\delta_{iM}| \right\} v_t^i \tag{3.6}$$

$$\quad + (\Delta - \nabla_t L(t^*))^\top v_\lambda^i + v_\beta^i - v_{\bar{\beta}}^i \tag{3.7}$$

$$0 \in -v_q^i + D^* N_{\mathbb{R}_+^N}(\mu^*, -q^*)(v_\mu^i) \tag{3.8}$$

$$0 \in v_q^i + D^* N_{\mathbb{R}_+^N}(\bar{\mu}^*, q^* - \bar{Q})(v_{\bar{\mu}}^i) \tag{3.9}$$

$$0 \in -q_i^* e_i - v_q^i + (\nabla_t L(t^*) - \Delta)v_t^i + D^* N_{\mathbb{R}_+^N}(\bar{\lambda}^*, -q^* - \Delta t^* + L(t^*) + D)(v_\lambda^i) \tag{3.10}$$

$$0 \in -v_t^i + D^* N_{\mathbb{R}_+^M}(\beta^*, \underline{T} - t^*)(v_\beta^i) \tag{3.11}$$

$$0 \in v_t^i + D^* N_{\mathbb{R}_+^M}(\bar{\beta}^*, t^* - \overline{T})(v_{\bar{\beta}}^i). \tag{3.12}$$

Proof. Taking into account the formulas for the Jacobian of F, coderivative of Q, normal cone to $\mathcal{A}_i \times \mathcal{B}_i$, and gradient of f_i

$$\nabla f_i(a, b, q, t, \mu, \bar{\mu}, \lambda, \beta, \bar{\beta}) = \begin{pmatrix} 0 \\ 0 \\ (-\lambda_i + A_i + 2B_i q_i)e_i \\ 0 \\ 0 \\ 0 \\ (-q_i)e_i \\ 0 \\ 0 \end{pmatrix},$$

the statement follows directly from Proposition 3.1. □

Points $(a^*, b^*, q^*, t^*, \mu^*, \bar{\mu}^*, \lambda^*, \beta^*, \bar{\beta}^*)$, such that for all $i = 1, \ldots, N$, the conditions (3.3)−(3.12) are satisfied, are called M(ordukhovich)-stationary.

In the statement of the above theorem, for simplicity of notation, we use coderivatives $D^*N_{\mathbb{R}^N}$ and $D^*N_{\mathbb{R}^M}$, which, in fact, can be easily calculated. In particular, for any $(x, y) \in \text{Gph } N_{\mathbb{R}^n_+}$ and $y^* \in \mathbb{R}^n$ let us introduce the following three index sets

$$\begin{aligned} I_1 &:= \{i \in \{1, \ldots, n\} | x_i > 0, y_i \le 0\} \cup \{i \in \{1, \ldots, n\} | x_i \ge 0, y_i = 0, y_i^* > 0\} \\ I_2 &:= \{i \in \{1, \ldots, n\} | x_i = 0, y_i = 0, y_i^* < 0\} \\ I_3 &:= \{i \in \{1, \ldots, n\} | x_i = 0, y_i < 0, y_i^* = 0\} \cup \{i \in \{1, \ldots, n\} | x_i = 0, y_i = 0, y_i^* = 0\}. \end{aligned}$$

Note that for $(x, y) \in \text{Gph } N_{\mathbb{R}^n_+}$ these three index sets form a complete disjunct decomposition of $\{1, \ldots, n\}$.

Then

$$D^*N_{\mathbb{R}^n_+}(x, y)(y^*) = \begin{cases} \emptyset & \text{if } \exists i : y_i y_i^* \neq 0 \\ \left\{ x^* \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x_i^* = 0 \text{ for } i \in I_1 \\ x_i^* \le 0 \text{ for } i \in I_2 \\ x_i^* \in \mathbb{R} \text{ for } i \in I_3 \end{array} \right\} & \text{else} \end{cases}$$

3.2. Verification of the calmness qualification condition

In order to be able to rely on necessary optimality conditions (3.3)−(3.12), a principle question concerns the constraint qualifications in the form of calmness conditions on multifunctions Ψ_i.

Since the graph of Q is a finite union of polyhedra, in case of linear single-valued mapping F it suffices to invoke a classical result of Robinson [29]. Then, indeed, graph of every multifunction Ψ_i is also a finite union of polyhedra. This implies calmness of Ψ_i at every point of its graph and thus also at $(0, x_i^*, z^*)$. However, mapping F is not linear due to bilinear terms $b_i q_i$ and quadratic $L(t)$. Under assumption of *partial bidding* (in particular the so-called bid-a-only scenario), *cf. e.g.* [16], when b is not considered as the decision variable but rather as parameter known to every producer and ISO, and in the loss-free case ($L_e = 0$ for every $e \in \mathcal{L}$), F becomes linear and calmness of multifunctions Ψ_i follow. Another case for which calmness of Ψ_i can be obtained is when the bids a_i and b_i are positive for all producers and the thermal losses are equal to zero. Indeed, in ([13], Prop. 5.2) calmness of Φ_i has been proved without a direct use neither of Aubin property nor of Robinson's Theorem. Nevertheless, taking into account that bidding in both a and b and positive loss coefficients L_e played important role in our analysis of structural properties of solutions to problem $ISO(a, b)$, both of these cases are restrictive.

We thus provide alternative ways of verifying calmness by checking the stronger Aubin property of Ψ_i. One possibility to check Aubin property of mappings Ψ_i follows from ([25], Prop. 3.2).

Suppose that for $(a^*, b^*, q^*, t^*, \xi^*)$, $I \subset \{1, \ldots, s\}$ be the index set of active components of H. Then calmness of mapping Ψ_i at $(0, a_i^*, b_i^*, q^*, t^*, \xi^*)$ is equivalent to calmness of

$$\tilde{\Psi}_i(p) := \{(a_i, b_i, q, t, \xi) | p \in \tilde{F}(a_i, a_{-i}^*, b_i, b_{-i}^*, q, t, \xi) + \tilde{Q}(q, t, \xi)\}$$

at $(0, a_i^*, b_i^*, q^*, t^*, \xi^*)$, where

$$\tilde{F}(a,b,q,t,\xi) = \begin{pmatrix} a + 2(\text{diag } b)q + (\nabla_q H_I(q,t))^\top \xi_I \\ (\nabla_t H_I(q,t))^\top \xi_I \\ -H_I(q,t) \end{pmatrix}, \quad \tilde{Q}(q,t,\xi) = \begin{pmatrix} \{0\}_N \\ \{0\}_M \\ N_{\mathbb{R}_+^{|I|}}(\xi_I) \end{pmatrix}.$$

Recall that the problem $ISO(a,b)$ with fixed parameters (a^*, b^*) satisfies the strong second-order sufficient conditions (SSOSC) at one of its solutions (q^*, t^*) if

$$\left\langle d, \nabla^2_{(q,t)} \mathcal{L}(q^*, t^*, \xi^*) d \right\rangle > 0 \;\; \forall d \neq 0 : \xi_i^* \nabla H_i(q^*, t^*) d = 0 \;\; i \in I$$

holds for all ξ^* such that $\nabla_{(q,t)} \mathcal{L}(q^*, t^*, \xi^*) = 0$, where

$$\mathcal{L}(q,t,\xi) = \sum_{i=1}^n (a_i^* q_i + b_i^* q_i^2) + (H_I(q,t))^\top \xi_I$$

is the Lagrangian associated with problem $ISO(a^*, b^*)$.

The sufficient criteria for the Aubin property of mappings $\tilde{\Psi}_i, i = 1, \ldots, N$, is given in the following result.

Proposition 3.3. *For every $i = 1, \ldots, N$, let $(a_i^*, b_i^*, q^*, t^*, \xi^*)$ be the solution to the problem of $MPCC(a_{-i}, b_{-i})$ for a fixed vector (a_{-i}^*, b_{-i}^*). Assume that $\nabla H_I(q^*, t^*)$ be surjective and that the problem $ISO(a,b)$ with fixed parameters (a^*, b^*) satisfies SSOSC at (q^*, t^*, ξ^*). Then the multifunctions $\tilde{\Psi}_i$ have the Aubin property.*

Proof. Under assumptions on surjectivity of $\nabla H_I(q^*, t^*)$ and SSOSC of the ISO problem ([30], Thm. 4.1) implies that the generalized equation

$$0 \in \tilde{F}(a,b,q,t,\xi) + \tilde{Q}(q,t,\xi)$$

is strongly regular at $(a^*, b^*, q^*, t^*, \xi^*)$. This means that for every $i = 1, \ldots, N$, the generalized equations

$$0 \in \tilde{F}(a_i, a_{-i}^*, b_i, b_{-i}^*, q, t, \xi) + \tilde{Q}(q,t,\xi)$$

are strongly regular at $(a_i^*, b_i^*, q^*, t^*, \xi^*)$.

Then, by ([25], Prop. 3.2), the mappings

$$(u_1, u_2, u_3, u_4) \to \{(a_i, b_i, q, t, \xi) | u_4 \in \tilde{F}(a_i, a_{-i}^*, b_i, b_{-i}^*, q, t, \xi) + \tilde{Q}(u_1 + q, u_2 + t, u_3 + \xi)\}$$

have the Aubin property around $(0, a_i^*, b_i^*, q^*, t^*, \xi^*)$. Setting $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, it follows that the restricted mappings

$$(0,0,0,u_4) \to \{(a_i, b_i, q, t, \xi) | u_4 \in \tilde{F}(a_i, a_{-i}^*, b_i, b_{-i}^*, q, t, \xi) + \tilde{Q}(q,t,\xi)\},$$

which are in fact mappings $\tilde{\Psi}_i$ have the Aubin property around $(0, a_i^*, b_i^*, q^*, t^*, \xi^*)$. □

Alternatively, instead of relying on the surjectivity of $\nabla H_I(q^*, t^*)$ and SSOSC of the problem $ISO(a,b)$, one can check the Aubin property of mappings Ψ_i *via* Mordukhovich criterion. The constraint qualification ensuring calmness of the multifunction Ψ_i is then replaced by a generalized Mangasarian–Fromowitz constraint qualification condition (GMFCQ), *cf.* [25]: for $w^i \in \mathbb{R}^{N+M+s}$ we have

$$\left. \begin{aligned} 0 \in & (\nabla_{x_i} F(x_i, \bar{x}_{-i}, z))^\top w^i + N_{\mathcal{A}_i \times \mathcal{B}_i}(x_i) \\ 0 \in & (\nabla_z F(x_i, \bar{x}_{-i}, z))^\top w^i + D^* Q(z, -F(x_i, \bar{x}_{-i}, z))(w^i) \end{aligned} \right\} \Rightarrow w^i = 0.$$

The GMFCQ ensures the Aubin property of multifunction Ψ_i around $(0, x_i^*, z^*)$ which, in turn, implies its calmness at that point.

In the following, we derive the GMFCQ in the data of the electricity market model. We thus have to derive the Jacobian of F and coderivative of Q.

It is not difficult to see that

$$\nabla F(a,b,q,t,\xi) = \begin{pmatrix} I & 2(\operatorname{diag} q) & 2(\operatorname{diag} b) & 0 & -I & I & -I & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 & (\nabla_t L(t) - \Delta)^\top & -I & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \Delta - \nabla_t L(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where columns represent partial gradients of F with respect to $a, b, q, t, \mu, \bar{\mu}, \lambda, \beta$ and $\bar{\beta}$, respectively, and $C = \left(\operatorname{diag}\left\{L_1 \sum_{i=1}^N \lambda_i |\delta_{i1}|, \ldots, L_M \sum_{i=1}^N \lambda_i |\delta_{iM}|\right\}\right)$.

Taking $w^i = (w^i_q, w^i_t, w^i_\mu, w^i_{\bar{\mu}}, w^i_\lambda, w^i_\beta, w^i_{\bar{\beta}})$ and invoking ([31], Prop. 6.41) together with the definition of a coderivative,

$$\begin{aligned} &D^*Q((q,t,\mu,\bar{\mu},\lambda,\beta,\bar{\beta}), -F(a,b,q,t,\mu,\bar{\mu},\lambda,\beta,\bar{\beta}))(v^i_q, v^i_t, v^i_\mu, v^i_{\bar{\mu}}, v^i_\lambda, v^i_\beta, v^i_{\bar{\beta}}) \\ &= \begin{pmatrix} \{0\}_N \\ \{0\}_M \\ D^* N_{\mathbb{R}^N_+}(\mu, -q)(v^i_\mu) \\ D^* N_{\mathbb{R}^N_+}(\bar{\mu}, q - \bar{Q})(v^i_{\bar{\mu}}) \\ D^* N_{\mathbb{R}^N_+}(\bar{\lambda}, -q - \Delta t + L(t) + D)(v^i_\lambda) \\ D^* N_{\mathbb{R}^M_+}(\beta, \underline{T} - t)(v^i_\beta) \\ D^* N_{\mathbb{R}^M_+}(\bar{\beta}, t - \overline{T})(v^i_{\bar{\beta}}) \end{pmatrix} \end{aligned}$$

and

$$N_{\mathcal{A}_i \times \mathcal{B}_i}(a_i, b_i) = N_{\mathcal{A}_i}(a_i) \times N_{\mathcal{B}_i}(b_i).$$

Now, using simple linear algebra, we obtain the following explicit version of GMFCQ ensuring Aubin property of Ψ_i around $(0, x^*_i, z^*)$: For $w^i_q \in \mathbb{R}^N$, $w^i_t \in \mathbb{R}^M$, $w^i_\mu \in \mathbb{R}^N$, $w^i_{\bar{\mu}} \in \mathbb{R}^N$, $w^i_\lambda \in \mathbb{R}^N$, $w^i_\beta \in \mathbb{R}^M$ and $w^i_{\bar{\beta}} \in \mathbb{R}^M$, the conditions

$$0 \in (w^i_q)_i + N_{\mathcal{A}_i}(a^*_i) \tag{3.13}$$

$$0 \in 2q^*_i (w^i_q)_i + N_{\mathcal{B}_i}(b^*_i) \tag{3.14}$$

$$0 = 2(\operatorname{diag}\,(b^*_i, b^*_{-i})) w^i_q + w^i_\mu - w^i_{\bar{\mu}} + w^i_\lambda \tag{3.15}$$

$$0 = \operatorname{diag}\left\{L_1 \sum_{i=1}^N \lambda^*_i |\delta_{i1}|, \ldots, L_M \sum_{i=1}^N \lambda^*_i |\delta_{iM}|\right\} w^i_t + (\Delta - \nabla_t L(t^*))^\top w^i_\lambda + w^i_\beta - w^i_{\bar{\beta}} \tag{3.16}$$

$$0 \in -w^i_q + D^* N_{\mathbb{R}^N_+}(\mu^*, -q^*)(w^i_\mu) \tag{3.17}$$

$$0 \in w^i_q + D^* N_{\mathbb{R}^N_+}(\bar{\mu}^*, q^* - \bar{Q})(w^i_{\bar{\mu}}) \tag{3.18}$$

$$0 \in -w^i_q + (\nabla_t L(t^*) - \Delta) w^i_t + D^* N_{\mathbb{R}^N_+}(\bar{\lambda}^*, -q^* - \Delta t^* + L(t^*) + D)(w^i_\lambda) \tag{3.19}$$

$$0 \in -w^i_t + D^* N_{\mathbb{R}^M_+}(\beta^*, \underline{T} - t^*)(w^i_\beta) \tag{3.20}$$

$$0 \in w^i_t + D^* N_{\mathbb{R}^M_+}(\bar{\beta}^*, t^* - \overline{T})(w^i_{\bar{\beta}}) \tag{3.21}$$

imply $w^i_q = 0$, $w^i_t = 0$, $w^i_\mu = 0$, $w^i_{\bar{\mu}} = 0$, $w^i_\lambda = 0$, $w^i_\beta = 0$ and $w^i_{\bar{\beta}} = 0$.

3.3. Application to a simple electricity market

We conclude this section on first order necessary optimality conditions for variational equilibria of the electricity market with the following illustrative academic example.

Example 3.4. Consider a network of two nodes 1 and 2 connected by a single transmission line e, thus $N = 2$ and $M = 1$. Suppose that $\bar{Q}_1 = 5, \bar{Q}_2 = 1, D_1 = 2, D_2 = 1.9, \underline{T}_e = -2, \overline{T}_e = 2$ and $L_e = 0.2$ and set $A_1 = A_2 = B_1 = 1, B_2 = 5$ and $\underline{A}_i = \underline{B}_i = 1, \overline{A}_i = \overline{B}_i = 2, i = 1, 2$.

Clearly, in order to satisfy the demand in both nodes, there needs to be a transmission directed from node 1 to node 2. Taking into account the parameters of both producers, clearly, both of them will be producing in their respectful optimal solutions. It is thus not difficult to see that the variational equilibria of this electricity market model are points $(a^*, b^*, q^*, t^*, \mu^*, \bar{\mu}^*, \lambda^*, \beta^*, \bar{\beta}^*) = (2, a_2^*, 2, b_2^*, 3.1, 1, 1, 0, 0, 0, 15.6, 14.4, 21.6, 0, 0)$, for $a_2^* \in [1, 2]$ and $b_2^* \in [1, 2]$. The non-uniqueness of solutions for producer 2 is due to the fact that producer 2 is forced to produce on maximum capacity, and due to transmission from node 1, the price at node 2 depends on price at node 1, *i.e.* the solution of the problem $ISO(a, b)$ thus depends just on bid of producer 1.

Let us choose, say, $a_2^* = 2, b_2^* = 2$. In the following, we shall verify the calmness constraint qualification and conditions (3.3)–(3.12) at point $(2, 2, 2, 2, 3.1, 1, 1, 0, 0, 0, 15.6, 14.4, 21.6, 0, 0)$.

First, notice that the restrictions on values of coderivatives are common for both producers and remain the same in both GMFCQ conditions and necessary optimality conditions. Denote for $i = 1, 2$

$$\begin{aligned}
\begin{pmatrix} w_1^i \\ w_2^i \end{pmatrix} &\in D^* N_{\mathbb{R}_+^2}(\mu^*, -q^*)(v_\mu^i) \\
\begin{pmatrix} w_3^i \\ w_4^i \end{pmatrix} &\in D^* N_{\mathbb{R}_+^2}(\bar{\mu}^*, q^* - \bar{Q})(v_{\bar{\mu}}^i) \\
\begin{pmatrix} w_5^i \\ w_6^i \end{pmatrix} &\in D^* N_{\mathbb{R}_+^2}(\bar{\lambda}^*, -q^* - \Delta t^* + L(t^*) + D)(v_\lambda^i) \\
w_7^i &\in D^* N_{\mathbb{R}_+}(\beta^*, \underline{T} - t^*)(v_\beta^i) \\
w_8^i &\in D^* N_{\mathbb{R}_+}(\bar{\beta}^*, t^* - \overline{T})(v_{\bar{\beta}}^i).
\end{aligned}$$

We can see that $I_1 = \{4, 5, 6\}, I_2 = \emptyset$ and $I_3 = \{1, 2, 3, 7, 8\}$, thus $(v_\mu^1)_1 = (v_\mu^1)_2 = (v_{\bar{\mu}}^1)_1 = v_\beta^1 = v_{\bar{\beta}}^1 = 0$, $w_4^1 = w_5^1 = w_6^1 = 0$ and remaining variables are arbitrary reals.

To verify the GMFCQ for $i = 1$, conditions (3.13)–(3.21) can be reduced to

$$0 \geq (w_q^1)_1, \tag{3.22}$$
$$0 \geq 6.2(w_q^1)_1, \tag{3.23}$$
$$0 = 4(w_q^1)_1 + (w_\lambda^1)_1, \tag{3.24}$$
$$0 = 4(w_q^1)_2 - (w_{\bar{\mu}})_2 + (w_\lambda^1)_2, \tag{3.25}$$
$$0 = 7.2w_t^1 - 1.2(w_\lambda^1)_1 + 0.8(w_\lambda^1)_2, \tag{3.26}$$
$$0 = -(w_q^1)_1 + w_1^1, \tag{3.27}$$
$$0 = -(w_q^1)_2 + w_2^1, \tag{3.28}$$
$$0 = (w_q^1)_1 + w_3^1, \tag{3.29}$$
$$0 = (w_q^1)_2, \tag{3.30}$$
$$0 = -(w_q^1)_1 + 1.2w_t^1, \tag{3.31}$$
$$0 = -(w_q^1)_2 - 0.8w_t^1, \tag{3.32}$$
$$0 = -w_t^1 + w_7^1, \tag{3.33}$$
$$0 = w_t^1 + w_8^1. \tag{3.34}$$

Starting with equation (3.30), it follows from (3.24–3.26), (3.31) and (3.32) that all variables vanish. Thus GMFCQ is satisfied which implies calmness of Ψ_i at the required point. Analogously for $i = 2$.

The necessary optimality conditions (3.3)–(3.12) for producer 1 reduce to

$$\begin{aligned}
&0 \geq (v_q^1)_1,\\
&0 \geq 6.2(v_q^1)_1,\\
&0 = -7.2 + 4(v_q^1)_1 + (v_\lambda^1)_1,\\
&0 = 4(v_q^1)_2 - (v_{\bar{\mu}})_2 + (v_\lambda^1)_2,\\
&0 = 7.2v_t^1 - 1.2(v_\lambda^1)_1 + 0.8(v_\lambda^1)_2,\\
&0 = -(v_q^1)_1 + w_1^1,\\
&0 = -(v_q^1)_2 + w_2^1,\\
&0 = (v_q^1)_1 + w_3^1,\\
&0 = (v_q^1)_2,\\
&0 = -3.1 - (v_q^1)_1 + 1.2v_t^1,\\
&0 = -(v_q^1)_2 - 0.8v_t^1,\\
&0 = -v_t^1 + w_7^1,\\
&0 = v_t^1 + w_8^1.
\end{aligned}$$

It can be easily checked that this system of equalities and inequalities is satisfied for $(v_q^1, v_t^1, v_\mu^1, v_{\bar{\mu}}^1, v_\lambda^1, v_\beta^1, v_{\bar{\beta}}^1) = (-3.1, 0, 0, 0, 0, 0, 29.4, 19.6, 29.4, 0, 0)$. Analogously, the necessary optimality conditions (3.3)–(3.12) for producer 2 are satisfied for $(v_q^2, v_t^2, v_\mu^2, v_{\bar{\mu}}^2, v_\lambda^2, v_\beta^2, v_{\bar{\beta}}^2) = (-1.5, 0, -1.25, 0, 0, 0, -12.85, 6, -2.25, 0, 0)$. △

Note that the derived necessary optimality conditions can be applied for general variational equilibrium of the electricity market. However, similarly to [13], one can introduce restrictions to a certain classes of solutions, *e.g.* to a class of solutions specified by Proposition 2.6, for which one could simplify the conditions (3.3)–(3.12) accordingly.

Appendix

In this section we will provide detail calculations concerning Example 2.8 of electricity market in Section 2.3.

We show that the vector of bids $(2, 4)$ is a solution of (NEP_2) but it is not a solution of (NEP_1). Recall that we assume uniqueness of the primal and dual solution of the problem $ISO(a, b)$. Thus we can work with the implicit reformulation of electricity market model.

Statement 1: point $(a_1^*, a_2^*) = (2, 4)$ *is not a solution of* (NEP_1).

By Lemma 2.1, one has $q_1(a_1, a_2) = D_1 + t_{12}(a_1, a_2) + \frac{L}{2} t_{12}(a_1, a_2)^2$. Invoking formula (2.16), it follows that

$$q_1(a_1, a_2) = D_1 + \frac{3a_2^2 - \lambda_1(a_1, a_2)^2 - 2\lambda_1(a_1, a_2)a_2}{2L(a_2 + \lambda_1(a_1, a_2))^2}. \tag{A.1}$$

If follows that $q_1(3, 4) = \overline{Q}_1$. Assume for contradiction that $q_1(3, 4) < \overline{Q}_1$. This implies $\lambda_1(3, 4) = 3$ and thus from (A.1) we obtain

$$q_1(3, 4) = \frac{565}{98} = \overline{Q}_1,$$

which concludes the contradiction.

Since $\overline{Q}_1$ is the upper bound of production for producer 1 and the function $q_1(\cdot,4)$ is non-increasing, one has $q_1(a_1,4) = \overline{Q}_1$, for all $a_1 \in \mathcal{A}_1$. Consequently, $(2,4)$ is not a solution of (NEP_1), since $R_1(a_1,a_2) = (a_1 - A_1)q_1(a_1,a_2)$ and therefore $R_1(2,4) < R_1(3,4)$.

Statement 2: point $(a_1^*, a_2^*) = (2,4)$ *is a solution of* (NEP_2).

In order to prove that $(2,4)$ is a solution of (NEP_2) we need to show that $2 \in \arg\max_{\mathcal{A}_1} \tilde{R}_1(\cdot,4)$ and that $4 \in \arg\max_{\mathcal{A}_2} \tilde{R}_2(2,\cdot)$.

First, let us show that the profit function $\tilde{R}_1(\cdot,4)$ of producer 1 is constant over $\mathcal{A}_1$. As proved before, the function $q_1(\cdot,4)$ is clearly constant over $\mathcal{A}_1$. This immediately implies that the functions $q_2(\cdot,4)$ and $t_{12}(\cdot,4)$ are also constant over $\mathcal{A}_1$. Thus, the KKT (2.7) conditions give

$$\lambda_1(a_1,4) = \frac{1 - Lt_{12}(a_1,4)}{1 + Lt_{12}(a_1,4)}\lambda_2(a_1,4),\ \forall a_1 \in [1,3).$$

Now, since the function $t_{12}(\cdot,4)$ is constant and $\lambda_2(a_1,4) = 4$ for all $a_1 \in \mathcal{A}_1$, we deduce that $\lambda_1(\cdot,4)$ is constant over $\mathcal{A}_1$ and thus the profit function $\tilde{R}_1(\cdot,4) := (\lambda_1(\cdot,4) - A_1)q_1(\cdot,4)$ is also constant. The bid coefficient $a_1^* = 2$ is therefore a trivial solution of producer 1's problem $\max_{\mathcal{A}_1} \tilde{R}_1(\cdot,4)$.

Now, let us show that $4 \in \arg\max_{\mathcal{A}_2} \tilde{R}_2(2,\cdot)$. Whenever $q_2(2,\cdot)$ is constant over $\mathcal{A}_2$ then, since $\lambda_2(2,a_2) = a_2$ for all $a_2 \in \mathcal{A}_2$, we immediately deduce that the function $\tilde{R}_2(2,\cdot)$ is an increasing linear function and the conclusion follows. Assume now that the function $q_2(2,\cdot)$ is not constant over $\mathcal{A}_2$. We will show that also in this case the function $\tilde{R}_2(2,\cdot)$ is increasing over $\mathcal{A}_2$. The function $q_2(2,\cdot)$ is continuous and non-increasing on $\mathcal{A}_2$, *cf.* ([8], Lem. 1) and ([1], Prop. 3), respectively. Thus there is $\bar{a}_2 \in [2,4]$ such that for all $a_2 \in [2,\bar{a}_2)$, one has $q_2(2,a_2) > q_2(2,4)$, while for all $a_2 \in [\bar{a}_2,4]$, $q_2(2,a_2) = q_2(2,4)$.

Let us show that for all $a_2 \in [2,\bar{a}_2)$, we have $q_1(2,a_2) < \bar{Q}_1$. This inequality follows immediately for $q_1(2,a_2) < D_1$ since $D_1 < \overline{Q}_1$. Assuming that $q_1(2,a_2) > D_1$, one has $t_{12}(2,a_2) > 0$ because node 1 produces more the demand D_1 at node 1. Lemma 2.1 applied at node 2 along with the fact that the function $g : t \to -t + (L/2)t^2$ is decreasing and bijective between $[0,1/L]$ and $[-1/(2L),0]$ implies that $t_{12}(2,a_2) = g(q_2(2,a_2) - D_2) < g(q_2(2,4) - D_2) = t_{12}(2,4)$. Now, from Lemma 2.1 at node 1 and node 2 we obtain

$$\begin{aligned} q_1(2,a_2) &= Lt_{12}(2,a_2)^2 + D_1 + D_2 - q_2(2,a_2) \\ &< Lt_{12}(2,4)^2 + D_1 + D_2 - q_2(2,4) \\ &= q_1(2,4) = \overline{Q}_1 \end{aligned}$$

and thus $\lambda_1(2,a_2) = 2$.

Invoking formula (2.16) together with the nodal energy balance equation, we have

$$q_2(2,a_2) = \frac{12 - 4a_2 - a_2^2}{2L(2+a_2)^2} + 1.9,\ \forall a_2 \in [2,\bar{a}_2)$$

and therefore, for any $a_2 \in [2,\bar{a}_2)$, one has

$$\partial_2\tilde{R}_2(2,a_2) = \frac{1176 - 3a_2^3 - 18a_2^2 + 236a_2}{5(a_2+2)^3}.$$

Now, we have

$$L(\bar{a}_2 + 2)t_{12}(2,\bar{a}_2) = \bar{a}_2 - 2. \tag{A.2}$$

Indeed, according to Lemma 2.2, for any $a_2 \in \mathcal{A}_2$, $t_{12}(2,a_2) \in [-5,5]$ and $t_{12}(2,a_2)$ is solution of the equation $q_2(2,a_2) = D_2 - t_{12}(2,a_2) + 0.1t_{12}(2,a_2)^2$. But on $[-5,5]$, the map $t \mapsto -t + 0.1t^2$ is continuously invertible and thus the continuity of $q_2(2,a_2)$ also implies the continuity of $t_{12}(2,a_2)$. Now the announced equality is a direct consequence of (2.16).

On the other hand, $q_1(2,4) = \overline{Q}_1$ and therefore $0.1t_{12}(2,4)^2 + t_{12}(2,4) + 5 = \overline{Q}_1$ yields $t_{12}(2,4) = 7/5$. Since $q_2(2,\cdot)$ is constant over $[\bar{a}_2, 4]$, it is the case also for $q_1(2,\cdot)$ and $t_{12}(2,\cdot)$ and thus $t_{12}(2,\bar{a}_2) = 7/5$. Equality (A.2) immediately gives $\bar{a}_2 = 8/3$ and thus for all $a_2 \in [2, \bar{a}_2)$,

$$\partial_2 \tilde{R}_2(2, a_2) \geq \frac{1176 - 3\bar{a}_2^3 - 18\bar{a}_2^2 + 236\bar{a}_2}{5(\bar{a}_2 + 2)^3} > 0,$$

i.e. the function $\tilde{R}_2(2,\cdot)$ is increasing over $[2, \bar{a}_2)$. It is increasing also over $[\bar{a}_2, 4]$ since $\tilde{R}_2(2, a_2) = (a_2 - A_2)q_2(2, \bar{a}_2)$ for any $a_2 \in [\bar{a}_2, 4]$. Finally, by continuity of $q_2(2,\cdot)$, $4 \in \arg\max_{\mathcal{A}_2} \tilde{R}_i(2,\cdot)$. This proves that $(a_1^*, a_2^*) = (2,4)$ is a solution of the equilibrium problem with production bounds (NEP_2).

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