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A NUMERICAL OPTIMAL CONTROL METHOD FOR SOLVING A LARGE THERMIC PROCESS

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Abstract. This work deals with the optimal regulation of a large thermal process when the final state is fixed and the control is subject to some constraints, for which we propose a relaxation method coupled with the shooting one. We study the behavior of this method. The studied example concerns the optimal control law for two ovens with three and twelve heating zones.

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1. Introduction

The goal of the present work is to propose a relaxation method coupled with the shooting one for solving an optimal control problem; the behavior of this iterative method is also studied. The proposed method is applied to the optimal control of a thermal process. The thermal process considered here, is described by a linear state equation with a quadratic criterion to minimize. Furthermore, the terminal state and the size of the horizon are fixed and, the control is subjected to some constraints. In the present study, the considered oven is splitted in n heating zones to control. This system with n controls, presents large internal couplings, due to the natural convection in the chimney and to thermal conduction; the objective is to maintain, in spite of perturbations, a prescribed repartition of temperature on a vertical bar placed in the chimney and to minimize the energy consumed. The observations are considered at n points and the constraints on the control must be verified. The goal is to obtain on one hand good precision and on the other hand the minimized energy consumption. The most simple representation of the linearized process for a given command is a $2n$ dimensional model. The Hamilton−Pontryagin conditions which characterize the optimal conditions, provides $4n$ differential equations and $2n$ inequations. In this paper, we consider the case of vertical oven having three and twelve heating zones. In the case where the control is subject to some constraints, we use the notion of subdifferentiability (see [1,4]) to obtain the optimality conditions. We formalize theoretically the necessary optimal conditions of the Pontryagin principle and then we have to solve a multivalued differential-algebraic problem. The solution of such a problem is obtained by the projection on the convex set of the constraints. The state equation is modeled by a differential system with an initial condition and a final condition. The costate is introduced by applying

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the Hamilton−Pontryagin condition. The costate equations are not equipped with initial or terminal condition which is useful for an automatic computation. Consequently, in order to determine the initial condition of the costate, we will use the shooting method [10]. Under suitable assumptions, we show the convergence of the relaxation iterative algorithm (see [3, 5–8]) coupled with the shooting method. In the case of two oven with three and twelve heating zones, we show that the proposed algorithm is very efficient, since the convergence is fast and the computational time is short.

This paper is organized as follows. In Section 2, the optimal control problem and the Pontryagin principle in the constrained case are presented. Section 3 is devoted to the description of the the relaxation method coupled with the shooting one. In Section 4, a convergence result of the proposed method is given. In Section 5, we present the results of the numerical experiments.

2. The problem to solve

2.1. Formulation of the problem

Let us consider a vertical oven constituted with a chimney having n heating zones. The problem is to bring in a finite time T, the state of a bar located in the chimney, to a desired temperature z_d. In the sequel, $x \in \mathbb{R}^n$ represents the temperature of the chimney in n points. $u \in \mathbb{R}^r$ is the vector of control representing the intensity of currents applied to the n heating zones. For notations consistency, $z \in \mathbb{R}^n$ is the vector of temperature of the n points of the bar. The problem is to obtain a control u that make it possible to maintain the temperature of the bar to z_d in spite of the perturbations. Let $y = (z_1, x_1, \ldots, z_i, x_i, \ldots, z_n, x_n) \in \mathbb{R}^{2n}$ the state vector of the dynamic system at time $t \in [0, T]$. The mathematical model is obtained by linearizing the heat equation; so we obtain:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = Ay(t) + Bu(t), \\ y(0) = y_0, y(T) = y_f, t \in [0, T], \end{cases} \tag{2.1}$$

where $y(0) = y_0$ is the given initial state vector of the system and $y(T) = y_f$ is the final state; $A \in \mathbb{R}^{2n\times 2n}$ is an M-matrix where the entries are given, $B \in \mathbb{R}^{2n\times r}$ is a given constant matrix.
The observation is given by

$$z = Cy, \tag{2.2}$$

where $C \in \mathbb{R}^{n\times 2n}$ is also a given constant matrix. The problem is to minimize the functional $J(u)$:

$$J(u) = \frac{1}{2}\int_0^T \left[\frac{\alpha}{\|z_d\|_2^2}\|z - z_d\|_2^2 + \frac{\beta}{\|u_d\|_2^2}\|u - u_d\|_2^2\right] \mathrm{d}t, \tag{2.3}$$

subject to the constraint (2.1); u_d is the control which leads asymptotically to the prescribed temperature z_d and, it can be shown that:

$$u_d = -\left(CA^{-1}B\right)^{-1} z_d.$$

The two dimensionless coefficients α and β control the comparative weight given in the cost function concerning the accuracy and the energy expenditure. Furthermore, the components of the control vector must verify the following constraints:

$$u_i^m \leq u_i \leq u_i^M, \forall i \in \{1, \ldots, r\}, \forall t \in [0, T], \tag{2.4}$$

and in the sequel $\mathcal{U}_{ad}$ denotes the set of admissible controls; $\mathcal{U}_{ad}$ is obviously convex, closed and non empty.

The problem is, then, to determine $u \in \mathcal{U}_{ad}$ realizing the minimum of the functional J:

$$J(u) = \min_{v \in \mathcal{U}_{ad}} J(v), \tag{2.5}$$

subject to the constraints (2.1) and (2.4). The Hamiltonien of the system is defined by:

$$H(y, p, u, t) = \frac{1}{2}\left[\frac{\alpha}{\|z_d\|_2^2}\|z - z_d\|_2^2 + \frac{\beta}{\|u_d\|_2^2}\|u - u_d\|_2^2\right] + p^t \cdot [Ay + Bu],$$

where p is the costate vector. Then, we have to find $\hat{u}$ such that:

$$H(\hat{y}, \hat{p}, \hat{u}) \leq H(y, p, u), \forall u \in \mathcal{U}_{ad}.$$

A characterization of the solution of the problem in the case where the control is subject to some constraints defined by equations (2.1)$-$(2.5) will be given by an adaption of the Pontryagin principle in which we use the notion of subdifferential which will be recalled in Annex 1.

2.2. The Pontryagin principle in the constrained case

We give now a formulation of the Pontryagin principle, in the case where the control is subject to some constraints. Consider the closed domain $\mathcal{U}_{ad} = \{u \in \mathbb{R}^r \ / \ u_i^m \leq u_i \leq u_i^M\}$ and let $(\Psi_{\mathcal{U}_{ad}})_i$ be the indicator function of $\mathcal{U}_{ad}$ which satisfies:

$$(\Psi_{\mathcal{U}_{ad}})_i = \begin{cases} 0, & \text{if } u_i^m \leq u_i \leq u_i^M, \\ \\ +\infty, \text{otherwise.} \end{cases}$$

The subdifferential is given by:

$$(\partial\Psi_{\mathcal{U}_{ad}})_i = \begin{cases} \emptyset, & \text{if } u_i < u_i^m, \\ ]-\infty, 0], & \text{if } u_i = u_i^m, \\ 0, & \text{if } u_i^m < u_i < u_i^M, \\ [0, +\infty[, & \text{if } u_i = u_i^M, \\ \emptyset, & \text{if } u_i > u_i^M, \end{cases}$$

and the corresponding graph is presented on Figure 1.

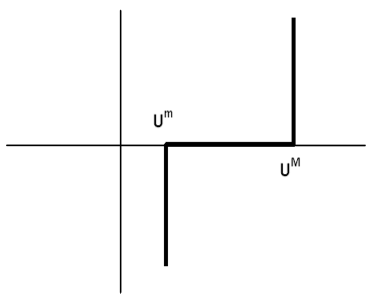

FIGURE 1. Subdifferential of the function $\Psi_{\mathcal{U}_{ad}}$.

Note that the subdifferential $\Psi_{\mathcal{U}_{ad}}$ is monotone. Let us apply the result of Lemma A.4 to our problem. Thus, we have to find $\hat{u} \in \mathcal{U}_{ad}$ which minimizes the Hamiltonien H; so, since H is a continuous operator (see [4]), we have to solve the following problem:

$$0 \in \partial(H + \Psi_{\mathcal{U}_{ad}})(\hat{u}) = \partial H(\hat{u}) + \partial\Psi_{\mathcal{U}_{ad}}(\hat{u}).$$

The new formulation of the necessary conditions of optimality is then:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = Ay + Bu;\ y(0) = y_0,\ y(T) = y_f, \forall t \in [0,T], \\ -\frac{\mathrm{d}p}{\mathrm{d}t} = A^T p + C^T C y - C^T z_d,\ p(0) \text{ to be determined}, \\ \partial(H + \Psi_{\mathcal{U}_{ad}})(\hat{u}) = \partial H(\hat{u}) + \partial \Psi_{\mathcal{U}_{ad}}(\hat{u}) \ni 0, \end{cases} \tag{2.6}$$

which in our case leads to:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = Ay + Bu;\ y(0) = y_0,\ y(T) = y_f, \forall t \in [0,T], \\ -\frac{\mathrm{d}p}{\mathrm{d}t} = A^T p + C^T C y - C^T z_d,\ p(0) \text{ to be determined}, \\ B^T p + k(u - u_d) + \partial \psi_{U_{ad}} \ni 0, \end{cases} \tag{2.7}$$

where:

- $p(t)$ is the costate,
- $k = \frac{\|z_d\|^2}{\|u_d\|^2} \cdot \frac{\beta}{\alpha}$,
- $u_d = -(CA^{-1}B)^{-1} z_d$,
- $\partial \psi_{\mathcal{U}_{ad}}$ is the subdifferential mapping of the indicator function of the convex $\mathcal{U}_{ad}$.

Note that $\partial \psi_{\mathcal{U}_{ad}}$ is identically zero in the case without constraint.

3. Algorithm

3.1. The shooting method

Due to the fact that the final state is fixed, we have also to use the shooting method, which allows to obtain the value of $p(0)$ necessary to the solution of the problem characterized by the Pontryaguin principle. The idea of the shooting method is to introduce an unknown, the initial value of the adjoint state p_0 and to solve a non-linear system of equations:

$$y^{p_0}(T) - y_f = 0,$$

where $y^{p_0}(t)$ is obtained by solving the system of differential equations:

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial H}{\partial p},\ \ y(0) = y_0,\ y(T) = y_f, \forall t \in [0,T], \\ -\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial H}{\partial y},\ \ p(0) = p_0; \end{cases}$$

this system can be numerically solved using an integrator such as the Euler or Runge Kutta methods. By denoting $G(p_0) = y^{p_0}(T) - y_f$, we define a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. G is an implicit nonlinear system of n equations and n unknown variables satisfying:

$$G(p_0) = 0.$$

For solving the previous system, we used the Newton's method. The principle of the Newton's method is described as follows: in the qth step, let p_0^q be a given approximation of the zero p_0 of G; therefore p_0 can be written by $p_0 = p_0^q + \Delta p_0^q$ and then we have:

$$0 = G(p_0) = G(p_0^q + \Delta p_0^q) = G(p_0^q) + J_G(p_0^q) \cdot (p_0 - p_0^q) + o(p_0 - p_0^q),$$

where $J_G(p_0^q)$ is the Jacobian matrix of the application $p_0 \mapsto G(p_0)$ computed when $p_0 = p_0^q$; then the Newton's method leads to the solution of the linear system:

$$J_G(p_0^q) \cdot (p_0 - p_0^q) = -G(p_0^q).$$

Note that the mapping $p_0 \to G(p_0)$ is not explicitly known but it can numerically be computed. Hence, we will use a method of numerical differentiation based on the finite difference method. To avoid the computation of $J_G(p_0^q)$, we will simply find an approximation of $J_G(p_0^q)$. According to [9], we have two typical finite difference approximations which are frequently used:

$$\frac{\partial G_i}{\partial p_{0j}}(p_0) \approx \frac{1}{h_{ij}} \left[G_i \left(p_0 + \sum_{l=1}^{j} h_{il} e^l \right) - G_i \left(p_0 + \sum_{l=1}^{j-1} h_{il} e^l \right) \right],$$

or

$$\frac{\partial G_i}{\partial p_{0j}}(p_0) \approx \frac{1}{h_{ij}} \left[G_i(p_0 + h_{ij} e^j) - G_i(p_0) \right],$$

where h_{ij} are the discretization step of the ith equation with respect to the jth variable and e^l is the lth vector of the canonical basis; note that, classically, we can always choose the values of h_{ij} equal each other to a common value h. Let $\Delta_{ij}(p_0, h)$ be an approximation of $\frac{\partial G_i}{\partial p_{0j}}(p_0)$; if the finite difference approximation is consistent then:

$$\lim_{h \to 0} \Delta_{ij}(p_0, h) = \frac{\partial G_i}{\partial p_{0j}}(p_0), i, j = 1, \ldots, n.$$

Let us denote by

$$J(p_0, h) = (\Delta_{ij}(p_0, h)),$$

the difference matrix. The following process

$$p_0^{q+1} = p_0^q - J(p_0^q, h)^{-1} \cdot G(p_0^q),\ q = 0, 1, \ldots$$

is called a discretized Newton iteration. The problem of convergence of this iterative process is solved thanks to a result of Ortega and Rheinboldt's book [9]. Indeed, if the discretization parameters h_{ij} are small and tend to zero, the convergence of the iterative process is ensured.

3.2. The coupled shooting relaxation methods

To solve the considered problem, we propose the relaxation method (see [3, 5, 8]) coupled with the shooting method [10]. The numerical solution of the problem leads to the following iterative steps:

(1) Choose a first approximation of the costate $p^0(0)$ and the corresponding values of u^0, $t \in [0, T]$.
(2) $r \leftarrow 0$ where r is the number of the current iteration.
(3) **While** convergence $> \varepsilon$ (where ε defines the convergence threshold) **do**:
   - Determine the state variable y^r and the costate variable p^r, by integration of the state equations and of the costate equations:

$$\begin{cases} \dfrac{\mathrm{d}y^r}{\mathrm{d}t} = Ay^r + Bu^r, \\ y^r(0) = y_0. \end{cases} \tag{3.1}$$

$$\begin{cases} -\dfrac{\mathrm{d}p^r}{\mathrm{d}t} = A^T p^r + C^T C y^r - C^T z_d, \\ p^r(0) \end{cases} \tag{3.2}$$

   where $p^r(0)$ is computed by the shooting method.

- Determine the control u^{r+1} :

$$u^{r+1} \leftarrow \mathrm{Proj}\left(u_d - \frac{1}{k} B^T p^r\right), \tag{3.3}$$

where Proj(.) is the projected operator on the closed convex $\mathcal{U}_{ad}$.
- Convergence $\leftarrow |u^{r+1} - u^r|$.
- Determine the shooting function:

$$G(p) = y^r(T) - y_f.$$

- Solve the shooting equation by Newton's method and find the new value of $p(0)$:

$$p^{r+1}(0) \leftarrow p^r(0) + \text{ correction },$$

- $r \leftarrow r + 1$

End while.

Remark 3.1. Steps (3.1)–(3.3) of the loop correspond to the relaxation method while the other steps correspond to the implementation of the shooting method.

4. Convergence of the method

The optimality equation can be written as follows

$$\begin{pmatrix} \frac{\mathrm{d}y}{\mathrm{d}t} \\ -\frac{\mathrm{d}p}{\mathrm{d}t} \\ \partial\Psi_{\mathcal{U}_{ad}} \end{pmatrix} + \begin{pmatrix} \bar{A} & 0 & -B \\ -Q & \bar{A}^T & 0 \\ 0 & B^T & kI \end{pmatrix} \begin{pmatrix} y \\ p \\ u \end{pmatrix} \ni \begin{pmatrix} 0 \\ -C^T z_d \\ k u_d \end{pmatrix},$$

where $\bar{A} = -A$, $k = \frac{\|z_d\|^2}{\|u_d\|^2} \cdot \frac{\beta}{\alpha}$, $Q = C^T C$ and I is the identity matrix. Thus, the problem can be written as the sum of a linear system perturbed by a diagonal mapping. Let

$$\Theta = \begin{pmatrix} \bar{A} & 0 & -B \\ -Q & \bar{A}^T & 0 \\ 0 & B^T & kI \end{pmatrix}.$$

We recall the following definitions:

Definition 4.1. A nonsingular matrix $\bar{A}$ is called an M-matrix if $\bar{A}^{-1} \succeq 0$ and $\bar{a}_{ij} \preceq 0$ for $i \neq j$.

Definition 4.2. A matrix Θ is an H-matrix if the matrix with diagonal elements entries $|\theta_{ii}|$ and with the off-diagonal entries $-|\theta_{ij}|$ is an M-matrix.

Remark 4.3. M-matrices have many important properties; particularly the spectral radius of the Jacobi matrix $J = I - \bar{D}^{-1} \cdot \bar{A}$ is lower than one, where $\bar{D}$ is the diagonal of $\bar{A}$; in the sequel, we will use this property.

Proposition 4.4. *If the following conditions are satisfied:*

- $\overline{A}$ *is a M-matrix,*
- $k \geq k_0 > 0$,
- $p^2(0) - p^2(T) > 0$,

then the relaxation method coupled with the shooting method for the numerical computation of the optimal control defined by equations (2.1), (2.4) *and* (2.5), *converges for all initial value of* u^0.

Proof. We will give briefly the main lines of the proof, since the proof is similar to the one used in [8] in a distinct framework. Indeed, we have seen in Lemma A.5 that the subdifferential is a monotone continuous mapping. Moreover, if $y(0)$ is zero; this is always possible by a change of variable, we have

$$\left\langle \frac{\mathrm{d}y}{\mathrm{d}t}, y \right\rangle = \frac{1}{2}\int_0^T \frac{\mathrm{d}y^2}{\mathrm{d}t}\mathrm{d}t = \frac{1}{2}[y^2(T) - y^2(0)] = \frac{1}{2}y^2(T) > 0,$$

where $\langle , \rangle$ is the standard inner product in the space of continuous functional. Moreover, the final costate value of $p(t)$ is in general different from zero and according to the second assumption, we have:

$$\left\langle -\frac{\mathrm{d}p}{\mathrm{d}t}, p \right\rangle = -\frac{1}{2}\int_0^T \frac{\mathrm{d}p^2}{\mathrm{d}t}\mathrm{d}t = -\frac{1}{2}[p(T)^2 - p(0)^2] > 0.$$

Let $Y = (y, p, u)$ be the exact solution of the following system of equations:

$$\begin{cases} \frac{\mathrm{d}y_i}{\mathrm{d}t} + \bar{a}_{ii}y_i + \sum\limits_{j\neq i} \bar{a}_{ij}y_j - \sum\limits_j b_{ij}u_j = 0, \\ -\frac{\mathrm{d}p_i}{\mathrm{d}t} + \bar{a}_{ii}p_i + \sum\limits_{j\neq i} \bar{a}_{ij}p_j - \sum\limits_j q_{ij}y_j = -\sum\limits_j C_j^T z_{jd}, \\ ku_i - ku_{di} + \sum\limits_{j\neq i} b_{ij}^T p_j + \partial\Psi_i \ni 0. \end{cases} \tag{4.1}$$

Let $W = (\omega, \pi, \nu)$ be the iterate values obtained by an iterative method such that Jacobi or Gauss-Seidel algorithms; then we have:

$$\begin{cases} \frac{\mathrm{d}y_i^r}{\mathrm{d}t} + \bar{a}_{ii}y_i^r + \sum\limits_{j\neq i} \bar{a}_{ij}\omega_j - \sum\limits_j b_{ij}\nu_j = 0, \\ -\frac{\mathrm{d}p_i^r}{\mathrm{d}t} + \bar{a}_{ii}p_i^r + \sum\limits_{j\neq i} \bar{a}_{ij}\pi_j - \sum\limits_j q_{ij}\omega_j = -\sum\limits_j C_j^T z_{jd}, \\ ku_i^r - ku_{di} + \sum\limits_{j\neq i} b_{ij}^T \pi_j + \partial\bar{\Psi}_i \ni 0. \end{cases} \tag{4.2}$$

By subtracting the previous equations (4.1) and (4.2) and multiplying by $(y_i - y_i^r), (p_i - p_i^r)$ and $(u_i - u_i^r)$ respectively, we obtain:

$$\begin{cases} \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(y_i - y_i^r), y_i - y_i^r \right\rangle + \bar{a}_{ii}|y_i - y_i^r|^2 = \sum\limits_{j\neq i} a_{ij}\langle y_j - \omega_j, y_i - y_i^r\rangle \\ \qquad\qquad + \sum\limits_j b_{ij}\langle u_j - \nu_j, y_i - y_i^r\rangle, \\ \langle -\frac{\mathrm{d}}{\mathrm{d}t}(p_i - p_i^r), p_i - p_i^r\rangle + \bar{a}_{ii}|p_i - p_i^r|^2 = \sum\limits_{j\neq i} a_{ij}\langle p_j - \pi_j, p_i - p_i^r\rangle \\ \qquad\qquad + \sum\limits_j q_{ij}\langle x_j - \omega_j, p_i - p_i^r\rangle, \\ k\langle u_i - u_i^r, u_i - u_i^r\rangle + \sum\limits_j b_{ij}^T\langle p_j - \pi_j, u_i - u_i^r\rangle + \langle \partial\Psi_i - \partial\bar{\Psi}_i, u_i - u_i^r\rangle \ni 0, \end{cases}$$

Note that due to the property of monotony of the subdifferential we have $\langle Z_i - \bar{Z}_i, u_i - u_i^r\rangle \geq 0$ for $Z_i \in \partial\Psi_i, \bar{Z}_i \in \partial\bar{\Psi}_i$. By taking into account the monotonicity of the previous three diagonal operators, we obtain the following inequalities:

$$\begin{cases} |y_i - y_i^r| \leq \sum\limits_{j\neq i} \frac{|a_{ij}|}{\bar{a}_{ii}}|y_j - \omega_j| + \sum\limits_j \frac{|b_{ij}|}{\bar{a}_{ii}}|u_j - \nu_j|, \\ |p_i - p_i^r| \leq \sum\limits_{j\neq i} \frac{|a_{ij}|}{\bar{a}_{ii}}|p_j - \pi_j| + \sum\limits_j \frac{|q_{ij}|}{\bar{a}_{ii}}|y_j - \omega_j|, \\ |u_i - u_i^r| \leq \sum\limits_j \frac{|b_{ij}^T|}{k}|p_j - \pi_j|, \end{cases}$$

which can be written as:

$$|Y_i - Y_i^r| \leq \sum_{j \neq i} \frac{|\theta_{ij}|}{\theta_{ii}} |y_j - w_j|.$$

If k is greater than a given number $k_0 > 0$, then the matrix Θ is a H-matrix [5]. Thus, we can define the uniform weighted norm:

$$\|Y - Y^r\|_J = \max_j \frac{|Y_j - Y_j^r|}{\mu_j},$$

where μ_j are the components of the eigenvector associated with the spectral radius $\rho(J)$ of the Jacobi matrix J associated with the matrix $\bar{\Theta}$. From the Perron–Frobenius theorem [9], μ has strictly positive components and we have:

$$J\mu \leq \rho(J)\mu, \qquad \text{with} \quad 0 \leq \rho(J) < 1.$$

Thus, we have the following contraction:

$$\|Y - Y^r\|_J \leq \rho(J) \|Y - Y^{r-1}\|_J,$$

since $\rho(J) < 1$ and then the convergence of the method is ensured. □

Remark 4.5. The proof of the convergence remains valid in both cases with and without constraints on the control. Indeed, in the latter case the subdifferential of the indicator function is identically zero and the proof is still valid in both cases.

5. Numerical experiments

The numerical experimentations were performed on the regulation of two thermal processes of large dimensions. This corresponds to the oven with three and twelve heating zones.

5.1. The oven with three heating zones

The studied example is connected with an optimal control law for a vertical oven with three heating zones which has six state variables and three control variables; in our case $n = 6$ and $r = 3$. The time is provided in minutes and the controls are in calories per minute, $T = 180$ mn, $z_d = 30°$c and $u_d = (372.3915, 193.3312, 419.6856)$. Numerical values of the matrices A, B and C, are given below.

$$A = \begin{pmatrix} -0.030 & 0.013 & 0.0077 & 0.0071 & 0.00017 & 0.00065 \\ 0.0017 & -0.012 & 0.00009 & 0.00033 & 0.00008 & 0.00029 \\ 0.0075 & 0 & -0.040 & 0.016 & 0.0077 & 0.00073 \\ 0 & 0.0030 & 0.0019 & -0.014 & 0.00009 & 0.0033 \\ 0 & 0 & 0.0075 & 0 & -0.029 & 0.012 \\ 0 & 0 & 0 & 0.0030 & 0.0014 & -0.013 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0.00125 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.00125 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.00125 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

5.1.1. Case without constraint

The main experimental results are summarized in Table 1, Figure 2.

TABLE 1. Number of iterations necessary for convergence and CPU time in seconds for the unconstrained case.

β/α	$k = 0.0077 \times (\beta/\alpha)$	CPU time	Number of iterations
1/10	7.6664×10^{-4}	1.0140	10
1/4	0.0019	1.0452	10
1/2	0.0038	1.0764	11
1/1	0.0077	1.0920	11
2/1	0.0153	1.1388	12
4/1	0.0307	1.1856	12
10/1	0.0767	1.2168	13

5.1.2. Case with constraints

The values of the constraints of the control are given in Table 2.

TABLE 2. Values of the control.

i	u_i^m	u_i^M
1	180	300
2	0	450
3	0	500

The main experimental results are summarized in Table 3, Figure 3.

TABLE 3. Number of iterations necessary for convergence and CPU time in seconds for the constrained case.

β/α	$k = 0.0077 \times (\beta/\alpha)$	CPU time	Number of iterations
1/10	7.6664×10^{-4}	0.9984	10
1/4	0.0019	1.0764	10
1/2	0.0038	1.0608	11
1/1	0.0077	1.1700	11
2/1	0.0153	1.1388	12
4/1	0.0307	1.1544	12
10/1	0.0767	1.1388	13

Note that the problem is solved by a satisfactory way since the constraint related to the value of the final state is well satisfied. Moreover, in both cases the convergence is fast and CPU-time is short.

5.2. The oven with twelve heating zones

The studied example is connected with an optimal control law calculation for a vertical oven with twelve heating zones which has twenty four state variables and twelve control variables; in our case $n = 24$ and $r = 12$.

The constants of time are in minutes and the controls are in calories per minute, $T = 40$ mn, $z_d = 30$ °C and $u_d = (1075.7, 826.3, 840.9, 845.7, 842.7, 850.7, 858.8, 872.2, 894.3, 930.7, 959.7, 1484.1)$. The state matrix A is given by:

$$A = \begin{pmatrix} D_1 & S+\varepsilon & \varepsilon S & \varepsilon^2 S & . & \ldots\varepsilon^k S & . & . & . & \varepsilon^{10} S \\ \theta & D & S+\theta & \varepsilon S & . & \ldots & . & . & \varepsilon^8 S & \varepsilon^9 S \\ 0 & \theta & D & S+\theta & . & \ldots & . & . & . & . \\ 0 & 0 & \theta & D & S+\theta & \ldots & . & . & . & . \\ . & . & . & . & . & \ldots & . & . & . & . \\ . & . & . & . & . & \ldots & . & . & . & . \\ . & . & . & . & . & \ldots & . & . & . & . \\ 0 & 0 & 0 & . & . & \ldots & . & S+\varepsilon & \varepsilon S & \varepsilon^2 S \\ . & . & . & . & . & \ldots & . & . & S+\theta & \varepsilon S \\ . & . & . & . & . & \ldots & . & \theta & D & S+\theta \\ 0 & 0 & 0 & . & . & \ldots & 0 & 0 & \theta & D_{12} \end{pmatrix}$$

where each blocks D_1, D, D_{12}, S and θ are defined by the following 2×2 matrices:

$$D_1 = \begin{pmatrix} -0.38 & 0.196 \\ 0.0068 & -0.0629 \end{pmatrix} \quad D = \begin{pmatrix} -0.39 & 0.196 \\ 0.0068 & -0.0682 \end{pmatrix} \quad D_{12} = \begin{pmatrix} -0.44 & 0.196 \\ 0.0068 & -0.0559 \end{pmatrix}$$

$$S = \begin{pmatrix} 0.0054 & 0.012 \\ 0.00033 & 0.00074 \end{pmatrix} \quad \theta = \begin{pmatrix} 0.065 & 0 \\ 0 & 0.0031 \end{pmatrix} \quad S+\theta = \begin{pmatrix} 0.0704 & 0.012 \\ 0.0003 & 0.0038 \end{pmatrix}$$

with $\varepsilon = 0.67$. The elements of the matrices B and C have the following values:

$$b_{ij} = \begin{cases} 0.00195, \text{ if } i \text{ is even and } j = i/2; \\ 0, \qquad \text{else.} \end{cases} \qquad c_{ij} = \begin{cases} 1, \text{ if } i = 1 \text{ to } 24 \text{ and } j = 2i-1; \\ 0, \text{ else.} \end{cases}$$

5.2.1. Case without constraints

The main experimental results are summarized in Table 4 and from Figures 4 to 7.

TABLE 4. Number of iterations necessary for convergence and CPU time in seconds for the unconstrained case.

β/α	$k = 0.00098 \times (\beta/\alpha)$	CPU time	Number of iterations
1/10	9.8325×10^{-5}	6.7704	15
1/4	2.4581×10^{-4}	7.2696	17
1/2	4.9162×10^{-4}	6.8796	15
1/1	9.8325×10^{-4}	6.0372	14
2/1	0.0020	6.6300	15
4/1	0.0039	7.1448	16
10/1	0.0098	6.5832	14

5.2.2. *Case with constraints*

The values of the constraints of the control are given in Table 5.

TABLE 5. Values of the control.

i	u_i^m	u_i^M	i	u_i^m	u_i^M	i	u_i^m	u_i^M
1	100	1200	5	500	850	9	500	900
2	835	870	6	500	850	10	500	1000
3	500	850	7	500	900	11	500	1000
4	500	850	8	500	900	12	100	1000

The experimental results are summarized in Table 6 and from Figures 8 to 11.

TABLE 6. Number of iterations necessary for convergence and CPU time in seconds in the constrained case.

β/α	$k = 0.00098 \times (\beta/\alpha)$	CPU time	Number of iterations
1/10	9.8325×10^{-5}	6.9888	15
1/4	2.4581×10^{-4}	7.5660	17
1/2	4.9162×10^{-4}	6.8328	15
1/1	9.8325×10^{-4}	6.2088	14
2/1	0.0020	6.4428	15
4/1	0.0039	6.6300	16
10/1	0.0098	6.2868	14

5.3. General comments on numerical experiments

The comments concern both cases of oven with three or twelve heating zones and are general for the case without or with constraints. We can remark that the convergence is fast, since from ten to 17 iterations are necessary to reach convergence. Moreover, in both cases the elapsed time for the global computation is about one second for the oven with three heating zones and about seven seconds for the other oven. When we compare the results obtained by the two simulations concerning the two mathematical models which describe the evolution of the thermic process, we note that the results obtained with the oven with twelve heating zones are more realistic. Indeed the splitting in twelve areas is sharper compared with the splitting in three areas. Note also, that in these cases, there exists convection phenomenon in the first and the last areas. Then, the proposed and original method combining the shooting method with the relaxation algorithm is well adapted for an online control of a thermic process, in both cases for constrained and unconstrained control.

6. CONCLUSION

In this paper, we have proposed a relaxation method coupled with the shooting method for solving an optimal control problem. The numerical experimentations were applied on the regulation of two large thermal processes constituted by two ovens with three and twelve heating zones. Note that the convergence is fast and the CPU-time is short. Therefore, the proposed algorithm is well adapted to the online control of a thermic process. Moreover, the proposed method of computation is also efficient when the control is subject to some constraints. Note also that the present study can be extended without any problem to the case where the state and the control are subjected to some constraints.

Annex 1: Basic theoretical results

Subdifferential mapping

The notion of subdifferential mapping will play a major role in this study. Thus we recall hereafter this notion and the main properties associated.

Definition A.1. Given a convex function χ on E and a point $\mu \in E$, we denote by $\partial\chi(\mu)$ the set of all $\mu' \in E'$ such that

$$\chi(v) \geq \chi(\mu) + \langle v - \mu, \mu' \rangle, \; for\ every\ v \in E, \tag{A.1}$$

where $\langle , \rangle$ denotes the pairing between E and E', E' being the topological dual space of E. Such elements μ' are called subgradients of χ at μ and $\partial\chi(\mu)$ is called the subdifferential of χ at μ.

Remark A.2. Recall that the pairing between E and E' is a bilinear form, from $E \times E'$ to $\mathbb{R}$. If E is a Hilbert space, then the pairing is the inner product of E.

Remark A.3. Let χ be a Gateaux differentiable (or Frechet differentiable) mapping at μ, then $\partial\chi(\mu)$ is a single element, namely the Gateaux (or Frechet) differential of χ at μ [1]. From (A.1), it is obvious that $\partial\chi(\mu)$ is a closed convex set (possibly empty, see [1]).

In the sequel, we will use a multivalued formulation of the constrained minimization problem.

Lemma A.4. *Let $\mu \in E$; μ is such that $\chi(\mu) = \min_{v\in E}(\chi(v))$ if and only if $0 \in \partial\chi(\mu)$.*

Proof. Let $\mu \in E$ such that $\chi(\mu) = \min_{v\in E}(\chi(v))$; then we have $\chi(v) \geq \chi(\mu) + \langle v - \mu, 0\rangle$, and then $0 \in \partial\chi(\mu)$. $\square$

Lemma A.5. *The subdifferential $\partial\chi(\mu)$ is a monotone operator (in general multivalued) from E to E'.*

Proof. Let $w' \in \partial\chi(w)$; then $\chi(v) \geq \chi(w) + \langle v - w, w'\rangle, \forall v \in E$. Let also $\mu' \in \partial\chi(\mu)$, then $\chi(v) \geq \chi(\mu) + \langle v - \mu, \mu'\rangle, \forall v \in E$. Consider the first inequality for $v = \mu$ and the second for $v = w$; then by adding the two previous inequalities, we obtain

$$\langle w - \mu, w' - \mu'\rangle \geq 0. \quad \square$$

The indicator function of the convex subset K will also play an important role in the sequel. The indicator function of the convex subset K is defined as follows.

Definition A.6. Let K be a closed convex subset of E and let Ψ_K be the indicator function of the convex subset K; then Ψ_K is defined by

$$\Psi_K(\mu) = \begin{cases} 0, & \text{si } \mu \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly, $\Psi_K(\mu)$ is convex.

Consequent A.7. It follows from Lemma A.4 that the minimum of χ on $K \subset E$ leads to solve a multivalued equation $0 \in A(v)$, where $A = \partial(\chi + \Psi_K)$, Ψ_K is the indicator function of the convex set K. By considering the definition of the subdifferential, we have (see [1]),

$$\partial\Psi_K(v) = \{v' \in E' / \langle v - w, v' \rangle \geq 0, \ for\ every\ w \in K\}.$$

This shows that $D(\partial\Psi_K) = D(\Psi_K) = K$ and $\partial\Psi_K(v) = \{0\}$ for each $v \in int(K)$. Moreover, if v lies on the boundary of K, then $\partial\Psi_K(v)$ coincides with the normal cone K at point v.

Annex 2

The oven with three heating zones

Case without constraint

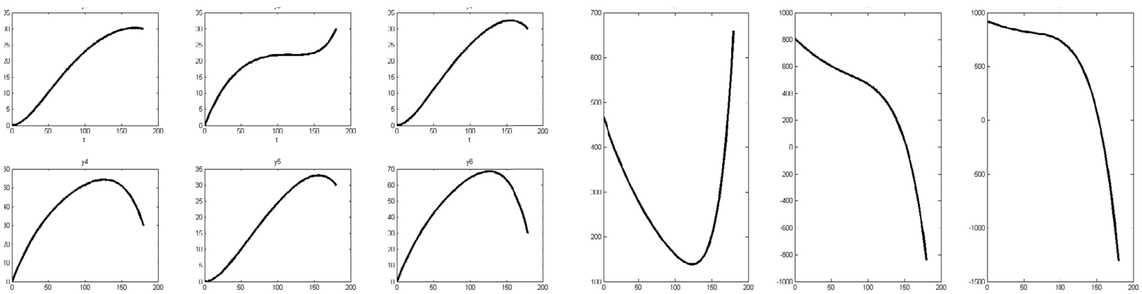

Figure 2. State and control for $\beta = 1$ and $\alpha = 4$.

Case with constraints

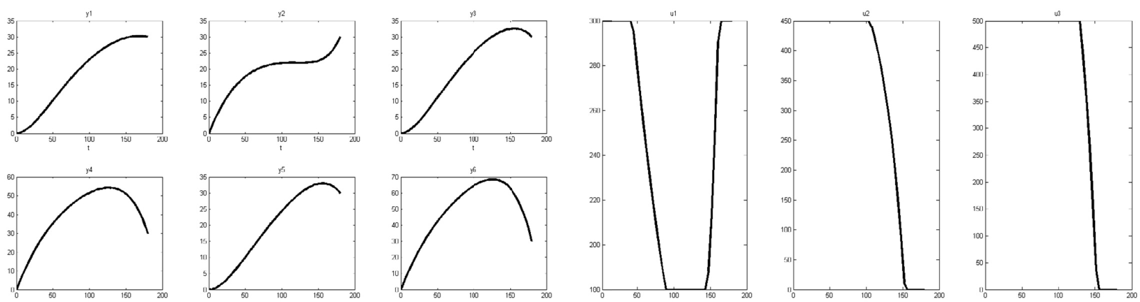

Figure 3. State and control for $\beta = 1$ and $\alpha = 4$.

Annex 3

The oven with twelve heating zones

Case without constraints

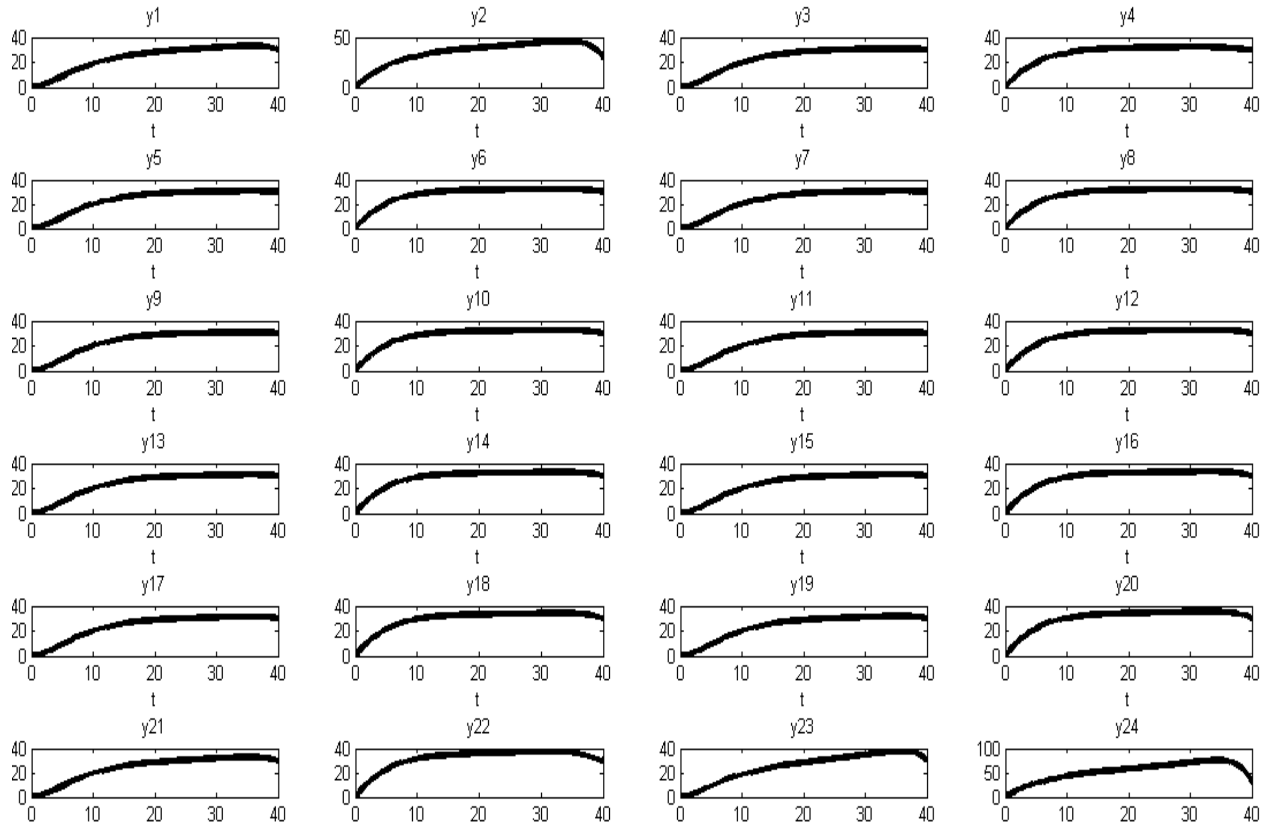

Figure 4. State for $\beta = 1$ and $\alpha = 10$.

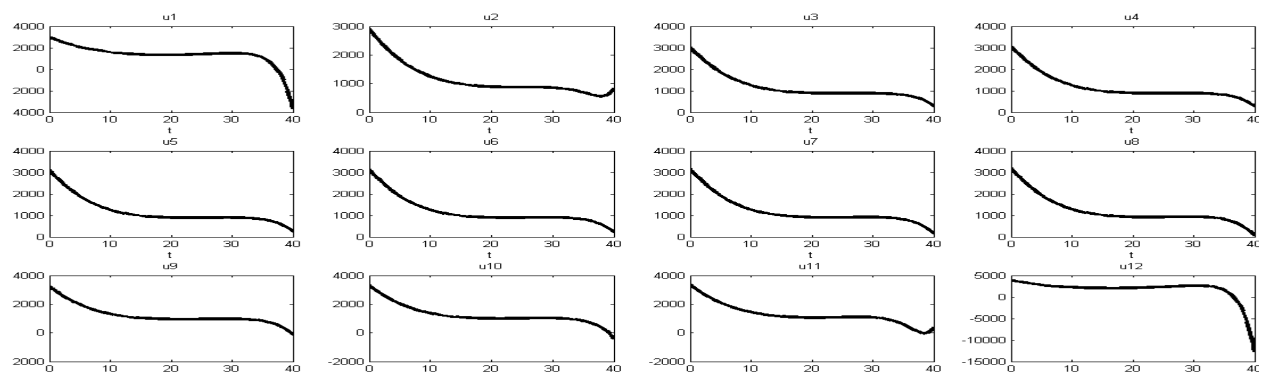

Figure 5. Control for $\beta = 1$ and $\alpha = 10$.

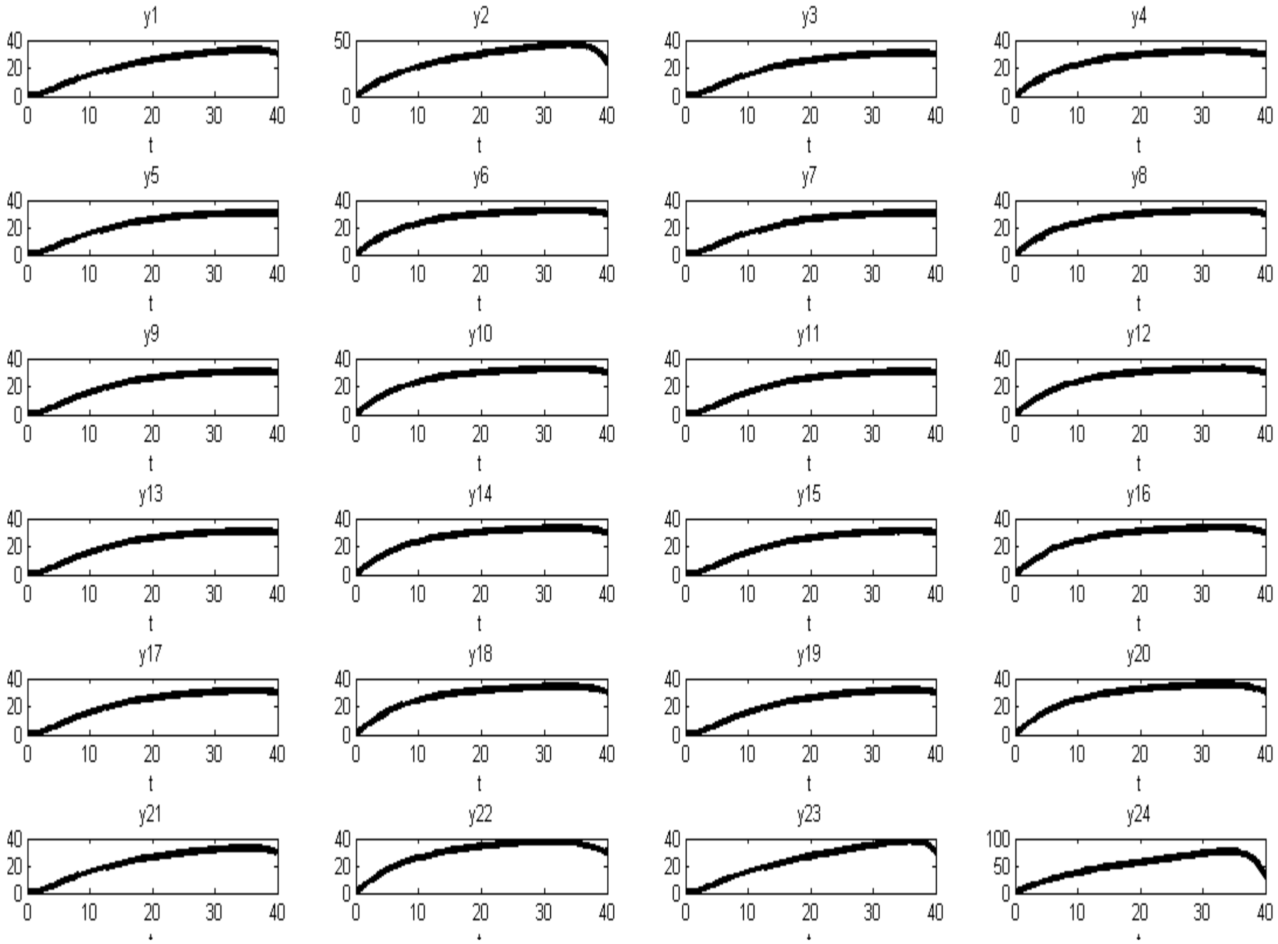

FIGURE 6. State for $\beta = 1$ and $\alpha = 4$.

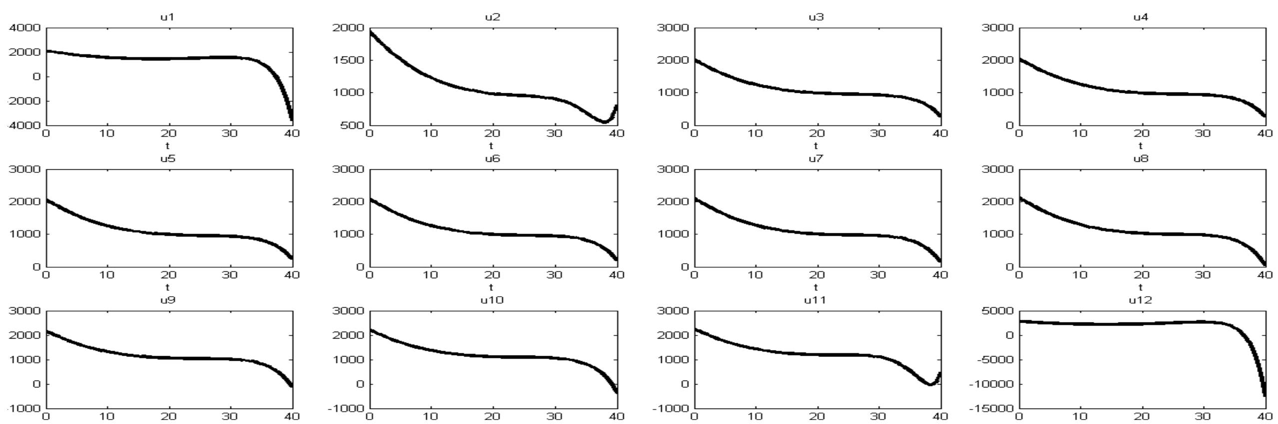

FIGURE 7. Control for $\beta = 1$ and $\alpha = 4$.

Case with constraints

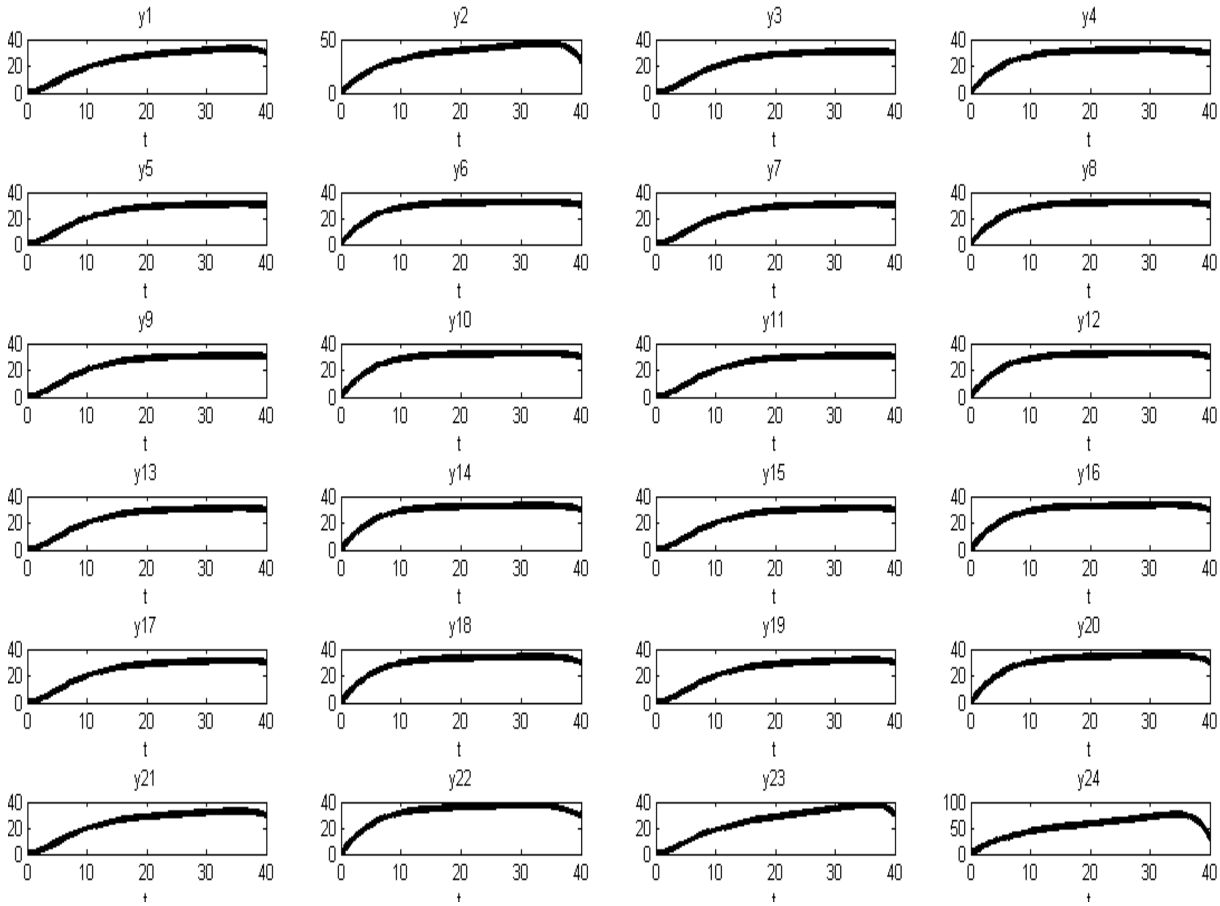

FIGURE 8. State for $\beta = 1$ and $\alpha = 10$.

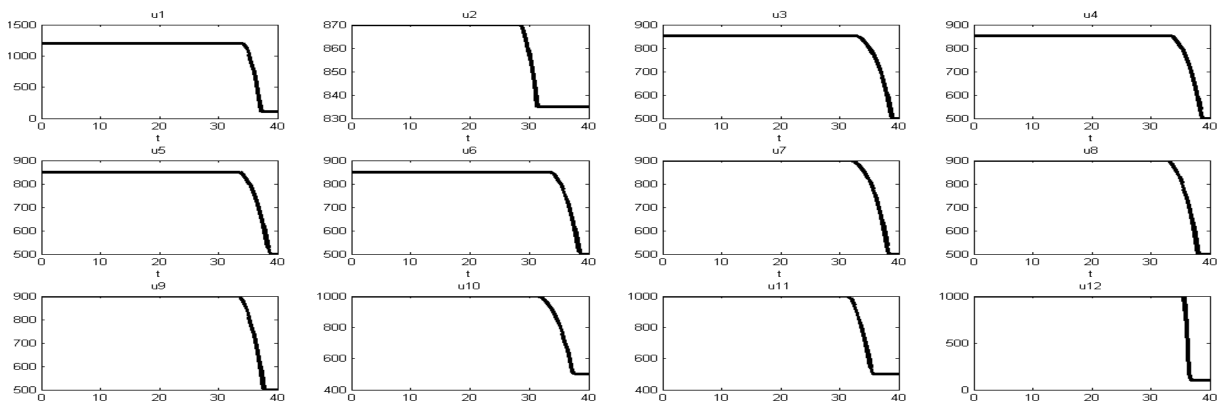

FIGURE 9. Control for $\beta = 1$ and $\alpha = 10$.

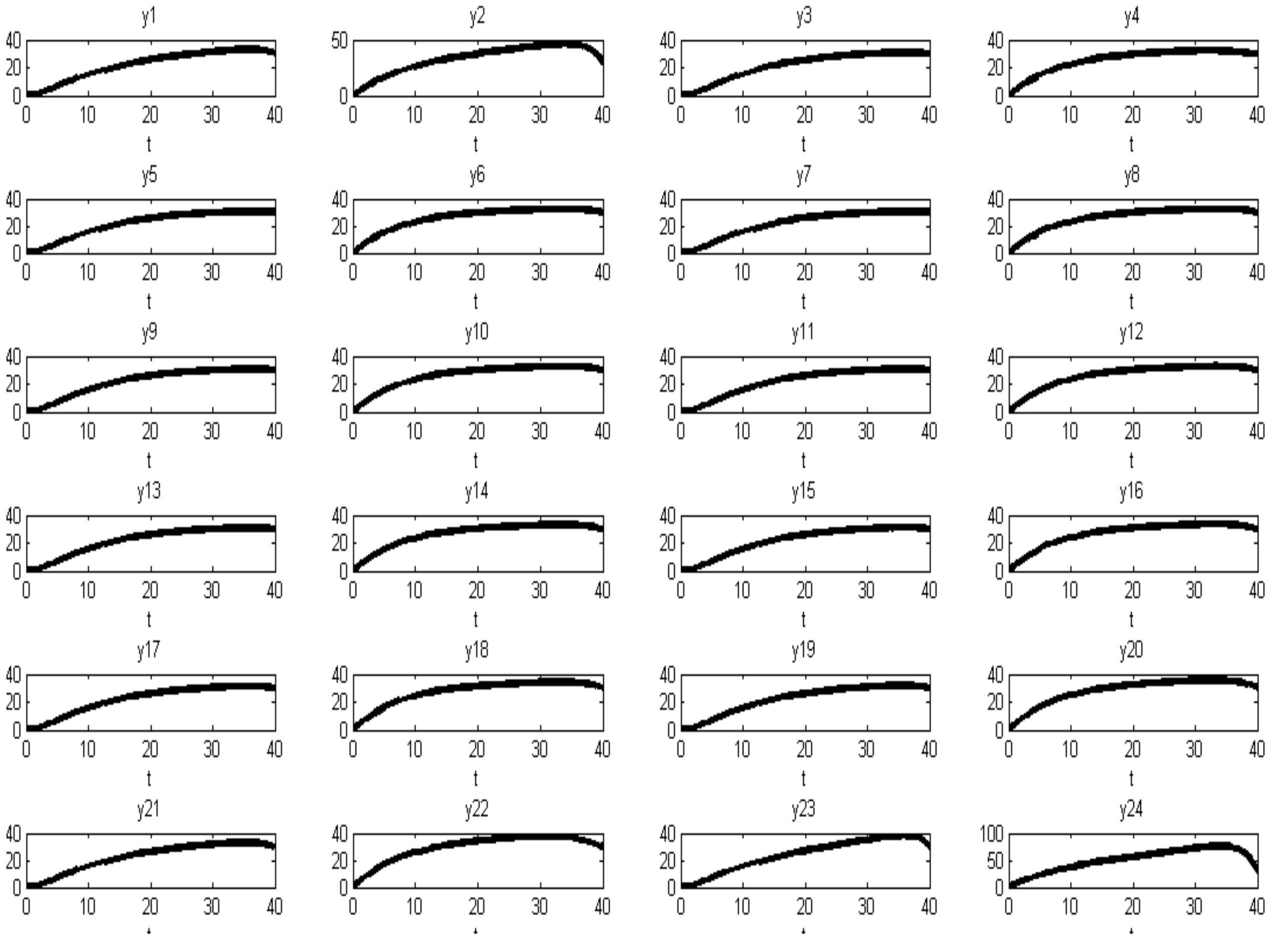

FIGURE 10. State for $\beta = 1$ and $\alpha = 4$.

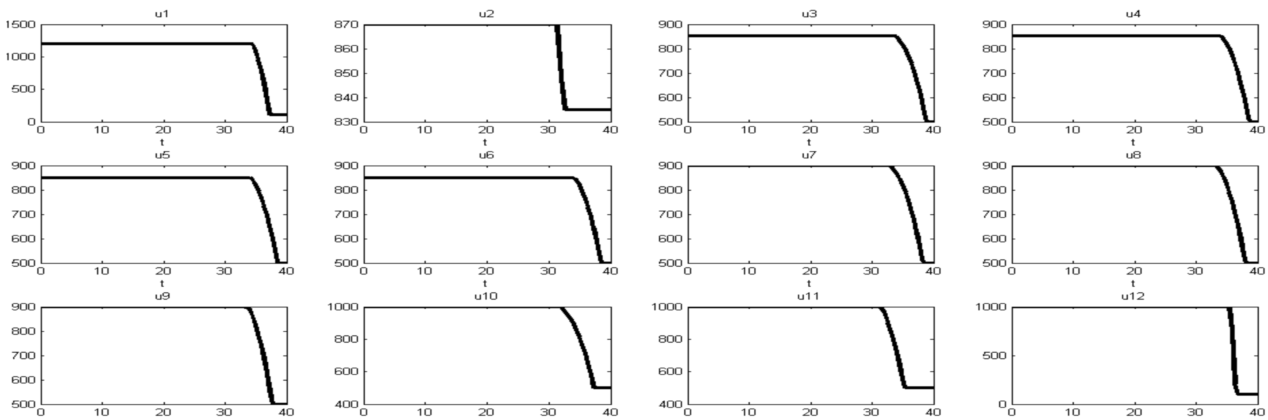

FIGURE 11. Control for $\beta = 1$ and $\alpha = 4$.

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