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OPTIMAL CONTROL POLICY OF AN INVENTORY SYSTEM WITH POSTPONED DEMAND *

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Abstract. This paper deals with the problem of controlling the selection rates of the pooled customer of a single commodity inventory system with postponed demands. The demands arrive according to a Poisson process. The maximum inventory level is fixed at S. The ordering policy is (s, S) policy that is as and when the inventory level drops to s an order for $Q(= S - s)$ items is placed. The ordered items are received after a random time, which is distributed as exponential. We assume that the demands that occur during stock out period either enter a pool of finite size or leave the system according to a Bernoulli distribution. Whenever the on-hand inventory level is positive, customers are selected one-by-one and the selection rate can be chosen from a given set. The problem is to determine a decision rule that specifies the rate of these selections as a function of the on-hand inventory level and the number of customers waiting in the pool at each instant of time to minimise the long-run total expected cost rate. The problem is modelled as a semi-Markov decision problem. The optimal policy is computed using Linear Programming algorithm and the results are illustrated numerically.

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1. Introduction

In most of the inventory models considered in the literature, the demanded items are directly delivered from the stock (if available). The demands that occurred during the stock-out period are either lost (lost sales) or satisfied only after the arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) that occurred during the stock-out period are satisfied. The often quoted review articles Nahmias [5], Raafat [7], Shah and Shah [8] and Goyal and Giri [2] and the recent review article and Bijvank and Vis [1] provide excellent summaries of many of these modelling efforts. In the case of backlogging, the backlogged demands are cleared, partially or fully depending on the availability on replenishment of inventory, immediately when the ordered items are materialized. But in some real life situations the backlogged demand may have to wait even after the replenishment.

Keywords. Inventory control, semi-Markov decision process, Postponed demands.

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Analysis of inventory system with postponed demands has received attention of researchers in the last two decade. Krishnamoorthy and Islam [3] considered an inventory system with Poisson demand, exponential lead time and the pooled customers were selected according to an exponentially distributed time lag. Sivakumar and Arivarignan [9] considered an inventory model in which the demand occurr according to a Markovian arrival process, lead time was distributed as a phase type, exponential life time for the items in the stock and the pooled customers were selected exponentially. Paul Manuel *et al.* [6] dealt an inventory system in which the positive and negative demand occurr according to independent Markovian arrival processes, lead time of the reorder, life time of the items, inter-selection time of customers from the pool and the reneging time points of the customers in the pool were independent exponential distribution. Sivakumar and Arivarignan [10] considered an inventory system with independent Markovian arrival processes for positive and negative demands, exponential lead time for the reorders, exponential life time for each item in the stock and the pool size was infinite.

In all the above models, the objective of the authors were to determine the optimal inventory level and the optimal reorder point that minimize the long-run expected cost function. However, there are practical situation where it is possible to control the selection rates in order to deal with random fluctuations in demands or the lead times. The dynamic control of selection rates gives the system manager great flexibility in dealing with the uncertainty in future demands.

In this article, we analyse a continuous review inventory system with exponential lead time and Poisson demands given maximum inventory level and prefixed reorder level. During the stock-out period, an arriving demand may join the pool of finite size or leave the system. We select the customers from the pool one-by-one with the inter-selection time is assumed to be exponential distribution. We focus our study on a system where speeding up or slowing down the selection rate is possible. This problem is modelled as a semi-Markov decision problem and the optimal solution is obtained using linear programming method.

The rest of the paper is organized as follows. In Section 2, we formulate the problem. In Section 3, we present the steady state analysis of the problem and calculate the total expected cost rate. In Section 4, we derive the linear programming formulation of the problem. Numerical illustration of the results, which provide insights of the behaviour of the system, are provided in the final section.

2. Problem formulation

Consider an inventory system with a maximum stock of S units. The customers arrive according to a Poisson process with rate $\lambda(>0)$ and demand a single item. As and when the inventory level drops to a prefixed level s $(0 \leq s < S)$ an order for $S - s(=Q > s+1)$ units are placed. The lead time for the order is exponentially distributed with parameter $\mu(>0)$. Those customers who arrive during the stock-out period are offered the choice of either postpone their demand until the ordered items are received or leave the system immediately. We assume that the customer accepts the offer of postponement according to an independent Bernoulli trials with probability p, $0 \leq p < 1$. With probability $q(=1-p)$, the customer declines the offer, and he is considered to be lost. The customer who accepts the offer of postponement of their demand enters into a place, called pool, of finite size N. The customer who encounters N customers in the pool has to leave the system, and he is considered to be lost forever.

These pooled customers are selected one-by-one according to a FCFS queue discipline whenever the inventory level is positive. The inter-selection times follow an exponential distribution with parameter depending on the state of the system (on the inventory level and the number of customer in the pool) and can be chosen from a given set of $m+1$ values. More specifically, let $L(t)$ and $X(t)$, denote, respectively, the inventory level and the number of customers in the pool at time t. Let $Y = \{Y(t); t \geq 0\} = \{(L(t), X(t)); t \geq 0\}$ denote the vector process whose state space $E = \{0, 1, \ldots, S\} \times \{0, 1, \ldots, N\}$. Then the inter-selection time distribution parameter is $\alpha(i,j)$ when $L(t) = i$ and $X(t) = j$. Each $\alpha(i,j), (i,j) \in E$ can be chosen from a set $m+1$ values $\{\alpha_0, \alpha_1, \ldots, \alpha_m\}$, where $\alpha_0 = \alpha(i,0) = \alpha(0,j) = 0$ for $(i,j) \in E$ and $\alpha_k > 0, k = 1, 2, \ldots, m$. The parameter α_0 is introduced primarily to take care of the situation when there is no customer in the pool or the on-hand inventory level is zero.

In this paper, our objective is to find a policy that specifies the expected selection rates to be adopted for a given inventory level and the number of customers in the poll, at any instant, so as to minimize the long-run expected cost rate. The costs associated with the system's operation have the following components:

- c_h: inventory carrying cost per unit per unit time.
- c_s: replenishment cost per order.
- c_w: waiting time cost per unit per unit time.
- c_b: cost per customer lost due to stock-out.
- c_l: cost per customer lost due to pool being full.
- β_n: cost associated with using selection rate α_n; $\beta_0 = 0$.

3. Analysis

Let $(L^R, X^R) = \{(L^R(t), X^R(t)); t \geq 0\}$ denote the process $\{(L(t), X(t)), t \geq 0\}$ when policy R is adopted. From our assumptions made on the input and output processes, it can be seen that the controlled process (L^R, X^R) is a finite state semi-Markov decision process. A policy R is called a stationary policy if it is randomized, time invariant and Markovian. Furthermore, a process is said to be completely ergodic, if every stationary policy gives rise to an irreducible Markov chain. From our assumptions, it can be seen that for every stationary policy $f, (L^f, X^f)$ is completely Ergodic. Since the action space is also finite, a stationary optimal policy exists. Hence, we consider the class, $\mathscr{F}$, of all stationary policies.

Denote by (k) the action choosing rate $\alpha_k (k = 0, 1, \ldots, m)$. Whenever $L(t) = 0$ or $X(t) = 0$, we have to necessarily choose the rate α_0, hence the set of actions for all state is not the same. Based on the choice of actions for each state, the state space is partitioned as follows:

$$E_1 = \{(i, j) : i = 1, 2, \ldots, S, j = 0; i = 0, j = 0, 1, \ldots, N\}$$

and

$$E_2 = \{(i, j) : i = 1, 2, \ldots, S, j = 1, 2, \ldots, N\}.$$

We note that $E = E_1 \cup E_2$.

Let $D_k (k = 1, 2)$ represent the set of all possible actions of the system when it belongs to the set E_k. Then, we have

$$D_1 = \{0\}, D_2 = \{k : k = 1, 2, \ldots, m\}$$

and $D = D_1 \cup D_2$.

A decision rule from the class $\mathscr{F}$ is equivalent to function $f : E \to D$ and is given by

$$f(i, j) = \{k : (i, j) \in E_l, k \in D_l, l = 1, 2\}.$$

The long-run expected cost rate, when policy f is adopted, is given by:

$$TC^f(s, S, N) = c_h \eta_I^f + c_s \eta_R^f + c_w \eta_W^f + c_b \eta_B^f + c_l \eta_L^f + \eta_A^f \tag{3.1}$$

where, in the steady state, for a given policy f,

η_I^f : the expected inventory level.
η_R^f : the expected reorder rate.
η_W^f : the expected number of customers waiting in the system.
η_B^f : the expected balking rate.
η_L^f : the expected customer lost due to stock-out.
η_A^f : the expected cost per unit time associated with using the different selection rates of pooled customers.

Our objective is to find an optimal policy f^* for which $TC^{f^*}(s,S,N) \leq TC^f(s,S,N)$ for every f.

For any fixed $f \in \mathscr{F}$ and $(i,j),(k,l) \in E$, define

$$P^f((i,j),(k,l);t) = Pr[L^f(t) = k, X^f(t) = l | L^f(0) = i, X^f(0) = j], \\ (i,j),(k,l) \in E.$$

Then $P^f((i,j),(k,l);t)$ satisfies the Kolmogorov forward differential equations. As each policy f, results in an irreducible Markov chain. We also note that the state space and the action set are finite. Hence the limit

$$\pi^f_{(k,l)} = \lim_{t\to\infty} P^f((i,j),(k,l);t)$$

exists and is independent of the initial conditions.

When $L(t) \geq 1$, and an arrival from outside takes place, obviously an item is served to the customer with probability one. When $L(t) = 0$ or $X(t) = 0$ the question of selection of customer from the pool does not arise. The decision of the selection rate $\alpha(i,j)$ of customer from pool are essentially related to picking customers from pool when both the inventory as well as the customers in the pool are available. The balance equations can be obtained by using the fact that transition out of a state is equal to transition into a state. For example, let us consider a typical state (i,j) that lies in the range $1 \leq i \leq s; 1 \leq j \leq N-1$. This state is represented in (3.6). When (i,j) is in this range, there is an order is pending, and hence transition out of this state can be only due to a demand or a selection of a customer from the pool or replenishment of the item, This fact is reflected in the left-hand side of (3.6). A selection of customer from the pool in state $(i+1,j+1)$ will decrease both the inventory level and the number of customers in the pool by one unit, thus bringing it to state (i,j). The state (i,j) can also be reached from $(i+1,j)$ when a customer arrives. These are the only two possible ways of reaching state (i,j) and are reflected on the right-hand side of (3.6).

$$(p\lambda+\mu)\pi^f_{(0,0)} = \lambda\pi^f_{(1,0)} + \alpha^f(1,1)\pi^f_{(1,1)}, \tag{3.2}$$

$$(p\lambda+\mu)\pi^f_{(0,j)} = \lambda\left(p\pi^f_{(0,j-1)} + \pi^f_{(1,j)}\right) + \alpha^f(1,j+1)\pi^f_{(1,j+1)}, \quad j = 1,2,\ldots,N-1, \tag{3.3}$$

$$\mu\pi^f_{(0,N)} = \lambda\left(p\pi^f_{(0,N-1)} + \pi^f_{(1,N)}\right), \tag{3.4}$$

For $i = 1,2,\ldots,s$,

$$(\lambda+\mu)\pi^f_{(i,0)} = \lambda\pi^f_{(i+1,0)} + \alpha^f(i+1,1)\pi^f_{(i+1,1)}, \tag{3.5}$$

$$\left(\lambda+\mu+\alpha^f(i,j)\right)\pi^f_{(i,j)} = \lambda\pi^f_{(i+1,j)} + \alpha^f(i+1,j+1)\pi^f_{(i+1,j+1)}, \quad j = 1,2,\ldots,N-1, \tag{3.6}$$

$$\left(\lambda+\mu+\alpha^f(i,N)\right)\pi^f_{(i,N)} = \lambda\pi^f_{(i+1,N)}, \tag{3.7}$$

For $i = s+1,s+2,\ldots,Q-1$,

$$\lambda\pi^f_{(i,0)} = \lambda\pi^f_{(i+1,0)} + \alpha^f(i+1,1)\pi^f_{(i+1,1)}, \tag{3.8}$$

$$\left(\lambda+\alpha^f(i,j)\right)\pi^f_{(i,j)} = \lambda\pi^f_{(i+1,j)} + \alpha^f(i+1,j+1)\pi^f_{(i+1,j+1)}, \quad j = 1,2,\ldots,N-1, \tag{3.9}$$

$$\left(\lambda+\alpha^f(i,N)\right)\pi^f_{(i,N)} = \lambda\pi^f_{(i+1,N)}, \tag{3.10}$$

For $i = Q, Q+1, \ldots, S-1$,

$$\lambda\pi^f_{(i,0)} = \lambda\pi^f_{(i+1,0)} + \alpha^f(i+1,1)\pi^f_{(i+1,1)} + \mu\pi^f_{(i-Q,0)}, \tag{3.11}$$

$$\left(\lambda + \alpha^f(i,j)\right)\pi^f_{(i,j)} = \lambda\pi^f_{(i+1,j)} + \alpha^f(i+1,j+1)\pi^f_{(i+1,j+1)} + \mu\pi^f_{(i-Q,j)}, \tag{3.12}$$

$$j = 1, 2, \ldots, N-1,$$

$$\left(\lambda + \alpha^f(i,N)\right)\pi^f_{(i,N)} = \lambda\pi^f_{(i+1,N)} + \mu\pi^f_{(i-Q,N)}, \tag{3.13}$$

For $i = S$,

$$\lambda\pi^f_{(i,0)} = \mu\pi^f_{(i-Q,0)}, \tag{3.14}$$

$$\left(\lambda + \alpha^f(i,j)\right)\pi^f_{(i,j)} = \mu\pi^f_{(i-Q,j)}, \tag{3.15}$$

$$j = 1, 2, \ldots, N-1,$$

$$\left(\lambda + \alpha^f(i,N)\right)\pi^f_{(i,N)} = \mu\pi^f_{(i-Q,N)}. \tag{3.16}$$

The above set of equations together with the condition

$$\sum_{(i,j)\in E} \pi^f_{(i,j)} = 1, \tag{3.17}$$

determine the steady-state probabilities uniquely.

3.1. Expected inventory level

As $\pi^f_{(i,j)}$ equals the (long-run) proportion of time the system contains exactly i-items in stock and the j-customers in the pool under the policy f, the expected inventory level η^f_I is given by:

$$\eta^f_I = \sum_{i=1}^{S}\sum_{j=0}^{N} i\pi^f_{(i,j)}. \tag{3.18}$$

3.2. Expected reorder rate

The expected reorder rate η^f_R is given by:

$$\eta^f_R = \lambda\sum_{j=0}^{N}\pi^f_{(s+1,j)} + \sum_{j=1}^{N}\alpha(s+1,j)\pi^f_{(s+1,j)}. \tag{3.19}$$

3.3. Expected number of customers waiting in the pool

The expected number of customers waiting in the pool η^f_W is given by:

$$\eta^f_W = \sum_{i=0}^{S}\sum_{j=1}^{N} j\pi^f_{(i,j)}. \tag{3.20}$$

3.4. Expected balking rate due to the pool full

The expected balking rate η^f_B due to the pool is full under the policy f is given

$$\eta^f_B = \sum_{i=0}^{S}\lambda\pi^f_{(i,N)}. \tag{3.21}$$

3.5. Expected balking rate due to the stock-out

The expected balking rate η_L^f due stock-out under the policy f is given

$$\eta_B^f = \sum_{j=0}^{N-1} q\lambda\pi_{(0,j)}^f. \tag{3.22}$$

3.6. Expected cost for using different selection rate

The expected cost due to using different selection rate from the customers in the pool, η_A^f, under policy f is given by:

$$\eta_A^f = \sum_{i=0}^{S}\sum_{j=0}^{N} \gamma_{(i,j)}^f \pi_{(i,j)}^f, \tag{3.23}$$

where $\gamma_{(i,j)}^f = \beta_k$ if $f(i,j) = k$.

3.7. Expected cost rate

Substituting the values of η's from (3.18) to (3.23) in (3.1), we get the long run total expected cost rate under the policy f is given by:

$$\begin{aligned} TC^f(s,S,N) = {} & c_h\left(\sum_{i=1}^{S}\sum_{j=0}^{N} i\pi_{(i,j)}^f\right) + c_s\left(\lambda\sum_{j=0}^{N}\pi_{(s+1,j)}^f + \sum_{j=1}^{N}\alpha(s+1,j)\pi_{(s+1,j)}^f\right) \\ & + c_w\left(\sum_{i=0}^{S}\sum_{j=1}^{N} j\pi_{(i,j)}^f\right) + c_b\left(\sum_{i=0}^{S}\lambda\pi_{(i,N)}^f\right) + c_l\left(\sum_{j=0}^{N-1} q\lambda\pi_{(0,j)}^f\right) \\ & + \sum_{i=0}^{S}\sum_{j=0}^{N}\gamma_{(i,j)}^f\pi_{(i,j)}^f. \end{aligned} \tag{3.24}$$

4. Linear programming model

Let us define the variables $D(i,j,k)$ as:

$$D(i,j,k) = Pr\left(\text{decision is } k|\text{state is}(i,j)\right).$$

Then for any stationary policy f, we have $D(i,j,k) = 0$ or 1. Suppose $D(i,j,k)$ were a continuous variable instead of integers, then the semi-Markov decision problem can be reformulated as a linear programming problem. For this purpose we consider the class of all randomized, time homogeneous Markovian policies for which the probability functions $D(i,j,k)$ satisfy

$$0 \le D(i,j,k) \le 1 \quad \text{and} \quad \sum_{k\in A_l} D(i,j,k) = 1, (i,j)\in E, l = 1,2.$$

The linear programming problem is best expressed in terms of the variables $\phi(i,j,k)$ which are defined as:

$$\phi(i,j,k) = D(i,j,k)\pi_{(i,j)}^f. \tag{4.1}$$

Since $\phi(i,j,k) = Pr(\text{State is } (i,j) \text{ and decision is } k)$, for any given f, we have

$$\pi_{(i,j)}^f = \sum_{k\in A}\phi(i,j,k) \quad (i,j)\in E. \tag{4.2}$$

Expressing $\pi^f_{(i,j)}$ in terms of $\phi(i,j,k)$ in the balance equations, we obtain the following linear programming problem.

Minimize

$$
\begin{aligned}
TC(s,S,N) = c_h & \left(\sum_{k=1}^{m} \sum_{i=1}^{S} \sum_{j=1}^{N} i\phi(i,j,k) + \sum_{i=1}^{S} i\phi(i,0,0) \right) \\
& + c_s \left(\lambda \sum_{k=1}^{m} \sum_{j=1}^{N} \phi(s+1,j,k) + \lambda\phi(s+1,0,0) + \sum_{k=1}^{m} \sum_{j=1}^{N} \alpha_k \phi(s+1,j,k) \right) \\
& + c_w \left(\sum_{k=1}^{m} \sum_{i=1}^{S} \sum_{j=1}^{N} j\phi(i,j,k) + \sum_{j=1}^{N} j\phi(0,j,0) \right) + c_b\lambda \left(\sum_{k=1}^{m} \sum_{i=1}^{S} \phi(i,N,k) + \phi(0,N,0) \right) \\
& + c_l q\lambda \left(\sum_{j=0}^{N-1} \phi(0,j,0) \right) + \sum_{(i,j)\in E_2} \sum_{k=1}^{m} \beta_k \phi(i,j,k). \qquad (4.3)
\end{aligned}
$$

The constraints of the linear programming problem are as follows:

- From (4.1), we have
$$\phi(i,j,k) \geq 0, (i,j) \in E_n, k \in A_n, n = 1,2. \qquad (4.4)$$
- Since $\sum_{(i,j)\in E} \pi^f_{(i,j)} = 1$, we have from (4.2)
$$\sum_{n=1}^{2} \sum_{(i,j)\in E_n} \sum_{k\in A_n} \phi(i,j,k) = 1. \qquad (4.5)$$
- The remaining constraints are the balance equations. After discarding the last equation as redundant in the set as equations, we have

$$(p\lambda + \mu)\phi(0,0,0) = \lambda\phi(1,0,0) + \sum_{k=1}^{m} \alpha_k \phi(1,1,k), \qquad (4.6)$$

$$(p\lambda + \mu)\phi(0,j,0) = p\lambda\phi(0,j-1,0) + \sum_{k=1}^{m} \left(\lambda\phi(1,j,k) + \alpha_k\phi(1,j+1,k)\right), \qquad (4.7)$$

$$j = 1,2,\ldots,N-1,$$

$$\mu\phi(0,N,0) = p\lambda\phi(0,N-1,0) + \lambda \sum_{k=1}^{m} \phi(1,N,k), \qquad (4.8)$$

For $i = 1,2,\ldots,s$,

$$(\lambda + \mu)\phi(i,0,0) = \lambda\phi(i+1,0,0) + \sum_{k=1}^{m} \alpha_k \phi(i+1,1,k), \qquad (4.9)$$

$$\sum_{k=1}^{m} (\lambda + \mu + \alpha_k)\, \phi(i,j,k) = \sum_{k=1}^{m} \left(\lambda\phi(i+1,j,k) + \alpha_k\phi(i+1,j+1,k)\right), \qquad (4.10)$$

$$j = 1,2,\ldots,N-1,$$

$$\sum_{k=1}^{m} (\lambda + \mu + \alpha_k)\, \phi(i,N,k) = \sum_{k=1}^{m} \lambda\phi(i+1,N,k), \qquad (4.11)$$

For $i = s+1, s+2, \ldots, Q-1$,

$$\lambda\phi(i,0,0) = \lambda\phi(i+1,0,0) + \sum_{k=1}^{m} \alpha_k \phi(i+1,1,k), \tag{4.12}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,j,k) = \sum_{k=1}^{m} \left(\lambda\phi(i+1,j,k) + \alpha_k \phi(i+1,j+1,k)\right), \quad j = 1,2,\ldots,N-1, \tag{4.13}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,N,k) = \sum_{k=1}^{m} \lambda\phi(i+1,N,k), \tag{4.14}$$

For $i = Q$,

$$\lambda\phi(i,0,0) = \lambda\phi(i+1,0,0) + \sum_{k=1}^{m} \alpha_k \phi(i+1,1,k) + \mu\phi(i-Q,0,0), \tag{4.15}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,j,k) = \mu\phi(i-Q,j,0) + \sum_{k=1}^{m} \left(\lambda\phi(i+1,j,k) + \alpha_k \phi(i+1,j+1,k)\right), \quad j = 1,2,\ldots,N-1, \tag{4.16}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,N,k) = \sum_{k=1}^{m} \lambda\phi(i+1,N,k) + \mu\phi(i-Q,N,0), \tag{4.17}$$

For $i = Q+1, Q+2, \ldots, S-1$,

$$\lambda\phi(i,0,0) = \lambda\phi(i+1,0,0) + \mu\phi(i-Q,0,0) + \sum_{k=1}^{m} \alpha_k \phi(i+1,1,k), \tag{4.18}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,j,k) = \sum_{k=1}^{m} \left(\lambda\phi(i+1,j,k) + \alpha_k \phi(i+1,j+1,k) + \mu\phi(i-Q,j,k)\right), \quad j = 1,2,\ldots,N-1, \tag{4.19}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,N,k) = \sum_{k=1}^{m} \left(\lambda\phi(i+1,N,k) + \mu\phi(i-Q,N,k)\right), \tag{4.20}$$

For $i = S$,

$$\lambda\phi(i,0,0) = \mu\phi(i-Q,0,0), \tag{4.21}$$

$$\sum_{k=1}^{m} (\lambda + \alpha_k)\, \phi(i,j,k) = \sum_{k=1}^{m} \mu\phi(i-Q,j,k), \quad j = 1,2,\ldots,N-1. \tag{4.22}$$

From equations (4.1) and (4.2), we have

$$D(i,j,k) = \frac{\phi(i,j,k)}{\sum\limits_{k \in A_n} \phi(i,j,k)}, (i,j) \in E_n, k \in A_n, n = 1,2. \tag{4.23}$$

Since the decision process is completely ergodic, every basic feasible solution to the above linear programming problem has the property that, for each $(i,j) \in E$, there is only one $k \in A$ such that $\phi(i,j,k) > 0$ and $\phi(i,j,k) = 0$ for k otherwise. Hence, any basic feasible solution of the above linear programming problem yields a deterministic policy (Mine and Osaki [4]).

Table 1. Influence of the arrival rate on the optimal policy.

λ	α_1		α_2		α_3		α_4	
	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 6$	$L=2$	$7\le X\le 10$				
	$L=3$	$1\le X\le 5$	$L=3$	$6\le X\le 9$				
1.5	$L=4$	$1\le X\le 4$	$L=4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$3\le L\le 6$	$X=10$		
	$7\le L\le 20$	$1\le X\le 2$	$7\le L\le 20$	$3\le X\le 6$	$7\le L\le 25$	$7\le X\le 10$		
	$21\le L\le 30$	$1\le X\le 3$	$21\le L\le 25$	$4\le X\le 6$	$26\le L\le 30$	$8\le X\le 10$		
			$26\le L\le 30$	$4\le X\le 7$				
	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 6$	$L=2$	$7\le X\le 10$				
	$L=3$	$1\le X\le 5$	$L=3$	$6\le X\le 9$				
1.8	$L=4$	$1\le X\le 4$	$L=4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$3\le L\le 6$	$X=10$		
	$7\le L\le 24$	$1\le X\le 2$	$7\le L\le 24$	$3\le X\le 6$	$7\le L\le 14$	$7\le X\le 10$		
	$25\le L\le 30$	$1\le X\le 3$	$25\le L\le 30$	$4\le X\le 6$	$15\le L\le 30$	$7\le X\le 9$	$15\le L\le 30$	$X=10$
	$1\le L\le 2$	$1\le X\le 9$	$1\le L\le 2$	$X=10$				
	$L=3$	$1\le X\le 5$	$L=3$	$6\le X\le 9$				
	$L=4$	$1\le X\le 4$	$L=4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$				
2.1	$7\le L\le 10$	$X=1$	$7\le L\le 9$	$2\le X\le 6$	$3\le L\le 6$	$X=10$		
	$11\le L\le 28$	$1\le X\le 2$	$L=10$	$2\le X\le 5$	$7\le L\le 8$	$X=10$		
	$29\le L\le 30$	$1\le X\le 3$	$11\le L\le 18$	$3\le X\le 5$	$L=9$	$7\le X\le 9$	$9\le L\le 30$	$X=10$
			$19\le L\le 28$	$3\le X\le 6$	$10\le L\le 18$	$6\le j\le 9$		
			$29\le L\le 30$	$4\le X\le 6$	$19\le L\le 30$	$7\le j\le 9$		
	$1\le L\le 2$	$1\le X\le 9$	$1\le L\le 2$	$X=10$				
	$L=3$	$1\le X\le 5$	$L=3$	$6\le X\le 9$				
	$L=4$	$1\le X\le 4$	$L=4$	$5\le X\le 9$	$3\le L\le 6$	$X=10$		
2.4	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$7\le L\le 24$	$6\le X\le 9$	$7\le L\le 30$	$X=10$
	$7\le L\le 12$	$X=1$	$7\le L\le 12$	$2\le X\le 5$	$25\le L\le 30$	$7\le j\le 9$		
	$13\le L\le 30$	$1\le X\le 2$	$13\le L\le 24$	$3\le X\le 5$				
			$25\le L\le 30$	$3\le X\le 6$				
	$1\le L\le 2$	$1\le X\le 9$	$1\le L\le 2$	$X=10$				
	$L=3$	$1\le X\le 5$	$L=3$	$6\le X\le 9$				
	$L=4$	$1\le X\le 4$	$L=4$	$5\le X\le 9$	$3\le L\le 6$	$X=10$		
2.7	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$7\le L\le 28$	$6\le X\le 9$	$7\le L\le 30$	$X=10$
	$7\le L\le 13$	$X=1$	$7\le L\le 13$	$2\le X\le 5$	$29\le L\le 30$	$7\le j\le 9$		
	$14\le L\le 30$	$1\le X\le 2$	$14\le L\le 28$	$3\le X\le 5$				
			$29\le L\le 30$	$3\le X\le 6$				

$\mu=0.3, p=0.6, c_h=0.1, c_s=10, c_w=1.5, c_b=6, c_l=5.$

5. Numerical illustration

In this section, we illustrate the method described in the above section through numerical examples. We have studied the effect of system parameters and costs on the optimal policy. Some of our results are presented in Table 1 through 8. We will present only two tables in the presentation and the rest of them are in web resource. In all the tables (Tabs. 1 and 2), we use $\alpha_1=0.6, \alpha_2=0.7, \alpha_3=0.8, \alpha_4=0.9, \beta_1=1, \beta_2=2, \beta_3=4, \beta_4=8$.

Table 2. Optimal policy for various setup cost.

c_s	α_1		α_2		α_3		α_4	
5	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 6$	$L=2$	$7\le X\le 9$				
	$3\le L\le 4$	$1\le X\le 4$	$3\le L\le 4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$2\le L\le 6$	$X=10$	$7\le L\le 30$	$X=10$
	$7\le L\le 27$	$1\le X\le 2$	$7\le L\le 27$	$3\le X\le 6$	$7\le L\le 30$	$7\le X\le 9$		
	$28\le L\le 30$	$1\le X\le 3$	$28\le L\le 30$	$4\le X\le 6$				
10	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 6$	$L=2$	$7\le X\le 9$				
	$3\le L\le 4$	$1\le X\le 4$	$3\le L\le 4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$2\le L\le 6$	$X=10$	$7\le L\le 30$	$X=10$
	$7\le L\le 27$	$1\le X\le 2$	$7\le L\le 27$	$3\le X\le 6$	$7\le L\le 30$	$7\le X\le 9$		
	$28\le L\le 30$	$1\le X\le 3$	$28\le L\le 30$	$4\le X\le 6$				
15	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 5$	$L=2$	$6\le X\le 9$				
	$3\le L\le 4$	$1\le X\le 4$	$3\le L\le 4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$2\le L\le 6$	$X=10$	$7\le L\le 30$	$X=10$
	$7\le L\le 27$	$1\le X\le 2$	$7\le L\le 27$	$3\le X\le 6$	$7\le L\le 30$	$7\le X\le 9$		
	$28\le L\le 30$	$1\le X\le 3$	$28\le L\le 30$	$4\le X\le 6$				
20	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 6$	$L=2$	$6\le X\le 9$				
	$3\le L\le 4$	$1\le X\le 4$	$3\le L\le 4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$2\le L\le 6$	$X=10$	$7\le L\le 30$	$X=10$
	$7\le L\le 27$	$1\le X\le 2$	$7\le L\le 27$	$3\le X\le 6$	$7\le L\le 30$	$7\le X\le 9$		
	$28\le L\le 30$	$1\le X\le 3$	$28\le L\le 30$	$4\le X\le 6$				
25	$L=1$	$1\le X\le 9$	$L=1$	$X=10$				
	$L=2$	$1\le X\le 5$	$L=2$	$6\le X\le 9$				
	$3\le L\le 4$	$1\le X\le 4$	$3\le L\le 4$	$5\le X\le 9$				
	$5\le L\le 6$	$1\le X\le 3$	$5\le L\le 6$	$4\le X\le 9$	$2\le L\le 6$	$X=10$	$7\le L\le 30$	$X=10$
	$7\le L\le 27$	$1\le X\le 2$	$7\le L\le 27$	$3\le X\le 6$	$7\le L\le 30$	$7\le X\le 9$		
	$28\le L\le 30$	$1\le X\le 3$	$28\le L\le 30$	$4\le X\le 6$				

$\lambda = 2.4, \mu = 0.5, p = 0.6, c_h = 0.1, c_w = 1.5, c_b = 6, c_l = 5.$

In all tables, L represents the inventory level and X represents the customer level in the pool. In each table, we present the optimal policy for a specific value for the parameter or the cost. For example, in Table 1, when the arrival rate $\lambda = 1.5$, we use the selection rate $\alpha_3 = 0.8$ provided the inventory level L is 3 to 6 and the customer level in the pool X is 10 or the inventory level is 7 to 25 and customer level in the pool is 7 to 10 or the inventory level is 26 to 30 and the customer level in the pool is 8 to 10.

We observe the following from these tables.

- Highest selection rate is employed only when the number of customers waiting in the pool attains its maximum and the inventory level is greater than or equal to the reorder level.
- The optimal policy is insensitive to changes in the cost of replenishment.

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