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AN EFFICIENT CUTTING PLANE ALGORITHM FOR THE MINIMUM WEIGHTED ELEMENTARY DIRECTED CYCLE PROBLEM IN PLANAR DIGRAPHS

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Abstract. In this paper, we study the efficiency (both theoretically and computationally) of a class of valid inequalities for the minimum weighted elementary directed cycle problem (MWEDCP) in planar digraphs with negative weight elementary directed cycles. These valid inequalities are called *cycle valid inequalities* and are parametrized by an integer called inequality's order. From a theoretical point of view, we prove that separating cycle valid inequalities of order 1 in planar digraph can be done in polynomial time. From a computational point of view, we present a cutting plane algorithm featuring the efficiency of a lifted form of the cycle valid inequalities of order 1. In addition to these lifted valid inequalities, our algorithm is also based on a mixed integer linear formulation of the MWEDCP. The computational results are carried out on randomly generated planar digraph instances of the MWEDCP. For all 29 instances considered, we obtain in average 26.47% gap improvement. Moreover, for some of our instances the strengthening process directly displays the optimal integer elementary directed cycle.

Mathematics Subject Classification. 90-XX.

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1. Introduction

Let $D = (V, A)$ be a connected planar digraph with V as vertex set, A as arc set and an arc weight function $w : A \to \mathbb{R}$. A directed cycle is a sequence $(v_0, a_1, \ldots, a_k, v_k)$, where k is an integer (assumed to be ≥ 1), $v_0, v_1, \ldots v_k$ are vertices such that $v_0 = v_k$, for every index i belonging to $\{1, \ldots k\}$, a_i is an arc connecting the vertices v_{i-1} and v_i and, finally, all arcs a_i have the same orientation. An *elementary directed cycle* is a directed cycle $(v_0, a_1, \ldots, a_k, v_k)$ in which each vertex v_i, for every index belonging to $\{0 \ldots k\}$, appears once. We denote by $\mathcal{P}$ the polytope of all *elementary directed cycles* in D. That is, the convex hull of the set of incidence vectors of elementary directed cycles of the digraph D.

The *minimum weighted elementary directed cycle problem* consists in finding of an elementary directed cycle γ^* such that:

$$l(\gamma^*) = \min \{l(\gamma) : \gamma \in \mathcal{C}\} \tag{1.1}$$

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where $\mathcal{C}$ is the set of all elementary directed cycles in D and $l(\gamma) = \sum_{a\in A(\gamma)} w(a)$ where $A(\gamma)$ denotes the arc set of the elementary directed cycle γ.

This problem is known to be NP-hard, since it includes the asymmetric traveling salesman problem as a special case. As a related work, Balas and Oosten (see [1]) present a linear description of the *elementary directed cycle* polytope. They also introduce some facets of the polytope $\mathcal{P}$. In [2], the authors consider the dominant of the polytope $\mathcal{P}$ and derive others facets of $\mathcal{P}$. The polytope $\mathcal{P}$ is also studied in undirected graphs case (see [5]). The integer linear formulation considered in [1,2] is a non compact formulation, since it contains an exponential number of constraints as the so-called *multiple cycle exclusion constraints* and *linear ordering constraints*. Other cycle polytopes with cardinality restrictions defined on graphs and digraphs are already well studied. For an exhaustive list of those publications, one can refer to [7]. Notice that all these researches investigate, in common, a partial description of the polytope $\mathcal{P}$ in general undirected and directed graphs. But they do not treat the resolution question.

From an algorithmic point of view, although the problem is very fundamental, the state of the art for it dates back to a seminal paper by Itai and Rodeh (see [12]) from the 70s, that deals only with the unweighted variant of the problem. Itai and Rodeh present an $O(n^w)$-time algorithm for an n-node unweighted undirected graph and an $O(n^w \log n)$-time algorithm for an n-node unweighted directed graph. In the same paper, Itai and Rodeh pose the question whether similar results exist for weighted graphs. Yuster and Zwick [18] present an algorithm to find the shortest even cycle in unweighted undirected graphs in $O(n^2)$. After, Lingas and Lundell [14] address efficient approximation algorithms for the shortest cycles in unweighted and weighted undirected graphs. In the same way, Roditty and Williams (see [17]) present a $O(Mn^w)$-time algorithms for directed graphs with integral edge weights in $[-M, M]$ (and no negative weight cycles) and for undirected graphs with integral edge weights in $[1, M]$, where w (less than 2.376) is the exponent of square matrix multiplication (see [4]). On the other hand, Letchford and Pearson [13] give a simple $O(n^{3/2} \log n)$ algorithm for finding a minimum weight odd cycle in positive weighted undirected planar graphs.

In this paper, we propose an efficient cutting plane algorithm based, in part, on the linear relaxation of a compact mixed integer linear formulation of the MWEDCP in planar digraphs. Such an algorithm possibly gives the optimal integer solution. The linear formulation of the MWEDCP considered is due to Maculan *et al.* (see [15]). Even in presence of negative weight cycles, at optimality, such a formulation provides the optimal integer elementary directed cycle with an additional vertex $s \notin V$ and arc (s, i), $i \in V$. Generally, the structure of any optimal fractional solution of the linear relaxation of the considered MWEDCP linear formulation is such that by deleting the vertex s from the supporting sub-digraph, one disconnects it. But, each connected component of this supporting sub-digraph is actually strongly connected. From the dual digraph of every connected component of the supporting sub-digraph, we show that to any u-v elementary directed path corresponds to what we call a cycle valid inequality of order 1 (see [8]). Recall that, in strongly connected digraphs, the separation problem of cycle valid inequalities of order β is NP-hard (even in the case $\beta = 1$ (see [8]). We will see that in the case of planar digraphs, valid inequalities that correspond to u-v elementary directed paths in the dual digraph (of the supporting digraph of the optimal fractional solution) can be separated in polynomial time.

As the cycle valid inequalities of order 1 are generated w.r.t. the connected components of the sub-digraph supporting the optimal fractional solution, we therefore resort to a lifting technique to derive valid inequalities for the polytope $\mathcal{P}$. The lifting technique used is close in spirit to the one presented in [11] for the shortest path problem and applied to derive the so-called *simple-lifted-valid-inequalities*. However, in this work (contrary to the *simple-lifted-valid-inequalities*) we do not consider all the arcs of the complementary sub-digraph of the supporting digraph of the optimal fractional solution. After, we present computational results featuring the efficiency of these lifted valid inequalities. Within a cutting plane algorithm, we show that such lifted valid inequalities are capable to strengthen the linear relaxation of the evoked formulation.

The paper is organized as follows. In Section 2, we present a compact linear formulation of the minimum weighted elementary directed cycle problem and we show that the connected components of the subgraph, that supports the optimal fractional solution of its linear relaxation without the additional vertex $s \notin V$ and all the arcs $(s,i),\ i \in V$, are strongly connected. In Section 3, we present a class of *valid inequalities* for the polytope $\mathcal{P}$ and we give a polynomial time algorithm that separates these valid inequalities in case of strongly connected planar digraphs and for $\beta = 1$. We also address an efficient lifting technique of the cycle valid inequalities of order 1 based on the incident arcs of some given vertices. In Section 4, we discuss the computational results.

2. A compact linear formulation for MWEDCP

The compact linear formulation of the minimum weighted elementary directed cycle problem given below is introduced in [15].

Given a planar connected digraph $D = (V, A)$, we build an auxiliary digraph (denoted by D_s) obtained from D by adding an additional vertex s that does not belong to V and zero weighted arcs (s,k) for every vertex k belonging to V and: (i) for every vertex i belonging to $V \cup \{s\}$, we associate a binary variable x_i such that x_i equals 1 if and only if the minimum weighted elementary directed cycle visits the vertex i; (ii) for every arc (i,j) belonging to $A \cup \{(s,k) : k \in V\}$, we associate a binary variable y_{ij} such that y_{ij} equals 1 if and only if the minimum weighted elementary directed cycle passes through the arc (i,j); (iii) we define variables $f_{ij}^k \geq 0$ as the flow passing through the arc (i,j), from the source vertex s to the terminal vertex k.

The mixed integer linear formulation then we obtain is as follow:

$$\text{Min} \sum_{(i,j)\in A} w_{ij} y_{ij}$$

$s.t.$

$$\sum_{j\in\Gamma^+(s)} f_{sj}^k = x_k, \qquad k \in V \tag{2.1}$$

$$\sum_{j\in\Gamma^+(i)} f_{ij}^k - \sum_{j\in\Gamma^-(i)} f_{ji}^k = 0, \qquad k \in V, i \in V \setminus \{k\} \tag{2.2}$$

$$\sum_{j\in\Gamma^+(k)} f_{kj}^k - \sum_{j\in\Gamma^-(k)} f_{jk}^k = -x_k, \qquad k \in V \tag{2.3}$$

$$f_{ij}^k \leq y_{ij}, \qquad (i,j) \in A, k \in V \tag{2.4}$$

$$\sum_{j\in\Gamma^+(k)} y_{kj} = x_k, \qquad k \in V \tag{2.5}$$

$$\sum_{j\in\Gamma^-(k)} y_{jk} = x_k, \qquad k \in V \tag{2.6}$$

$$\sum_{j\in\Gamma^+(s)} y_{sj} = 1, \tag{2.7}$$

$$\sum_{k\in V} x_k \geq 2, \tag{2.8}$$

$$f_{ij}^k \geq 0, \qquad k \in V, (i,j) \in A \cup \{(s,j) : j \in V\} \tag{2.9}$$

$$x_i, y_{ij} \in \{0,1\}, \qquad i \in V \cup \{s\},\ (i,j) \in A \cup \{(s,j) : j \in V\} \tag{2.10}$$

where, for every vertex i, $\Gamma^+(i)$ and $\Gamma^-(i)$ denote the sets of outgoing arcs from the vertex i and ingoing arcs to the vertex i, respectively. Constraints (2.1)–(2.3) and (2.5)–(2.6) are *flow conservation* and *degree* constraints, respectively. The constraints (2.4) establish the link between the flow passing through an arc and the arc. Constraints (2.8) say that every elementary directed cycle may include at most 2 vertices. At optimality, the above model produces an optimal elementary directed cycle with an additional arc of type (s,j), $j \in V$. So, to find the optimal solution of the MWEDCP, we have to delete the additional arc (s,j).

Consider an optimal fractional solution $(\hat{x}, \hat{y}, \hat{f})$ of the MWEDCP, let $D_s[\hat{y}]$ be its supporting directed sub-digraph. That is, the arc set of $D_s[\hat{y}]$ contains only arcs (i, j) such that $\hat{y}_{ij} > 0$. On the other hand, let $D[\hat{y}]$, with $\hat{V}$ as $\hat{A}$ as arc set, be the directed subgraph obtained from the digraph $D_s[\hat{y}]$ by deleting the vertex s and all the arcs of type (s, k) such that $\hat{y}_{sk} > 0$. Notice that the digraph $D[\hat{y}]$ is not necessarily connected. But, as shown in the next lemma, each of its connected component is strongly connected.

Lemma 2.1. *Each component of the directed sub-digraph $D[\hat{y}]$ is strongly connected.*

Proof. Let C be a connected component of $D[\hat{y}]$. Let u and v be two vertices from C. Since the optimal fractional solution $(\hat{x}, \hat{y}, \hat{f})$ satisfies the flow constraints (2.5) and (2.6) then there exists a directed path from u to v and a directed path from v to u. This completes the proof. □

In the next section, we recall what we call cycle valid inequality of order β and discuss its separation problem in planar digraphs and for $\beta = 1$.

3. Separating cycle valid inequalities of order $\beta = 1$ in planar digraphs

3.1. Concept of digraph duality and cycle valid inequalities of order β

In this part, we define the concept of duality of a given planar digraph and we recall what we call cycle valid inequalities introduced in [8].

3.1.1. Concept of planar digraph duality

Let D be a connected planar digraph. Since the digraph D can be drawn on the plane, a *face* of D is defined as the area surrounded by an elementary cycle. An *Unbounded face* is the area surrounding the entire digraph D. Assume that the digraph D has r faces denoted g_k, where k belongs to $\{0, \ldots, r\}$, where g_0 is its unbounded face. The dual digraph D^* associated with D is defined as follows: (i) for every face $g_k, k = 0, \ldots, r$ in D, we associate a vertex $v_k, k = 0, \ldots, r$ in D^*; (ii) for every arc $a = (u, v)$ adjacent to both faces g_k and g_j, $k \neq j$, we associate an arc (v_k, v_j) in D^*. The orientation of the arc (v_k, v_j) is chosen such that it crosses the original arc $a = (u, v)$ from left to right, here left means the left side when we traverse the arc a from its tail u to its head v.

As an example, in Figure 1: the vertex v_1 in the dual digraph D^* corresponds to the elementary directed cycle $(1, 2, 3, 6, 5, 4, 1)$ in D; the vertex v_2 corresponds to the elementary directed cycle $(6, 5, 7, 8, 6)$ in D; the vertex v_3 corresponds to the elementary directed cycle $(7, 8, 10, 9, 7)$ in D and v_0 represents the unbounded face of D. One can observe that all elementary directed cycles in D are represented by directed stars (or *distars*) in the

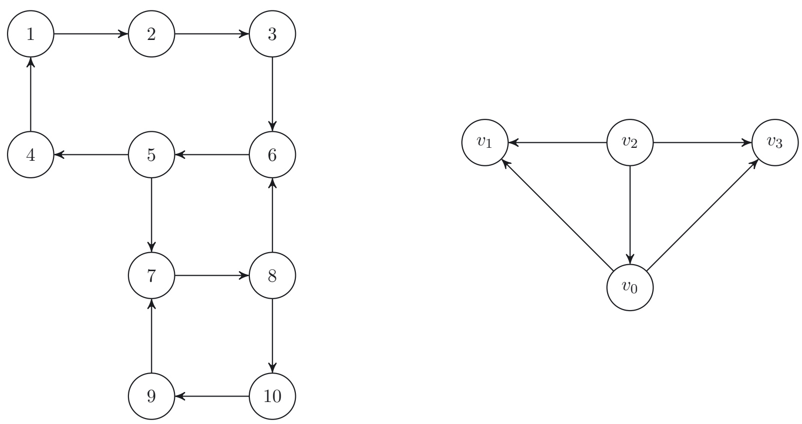

FIGURE 1. A digraph D (*left*) and its dual D^* (*right*).

dual digraph D^*. We mean by a *distar* a set of arcs oriented in the same direction and incident to the same vertex. In Figure 1, with respect to the vertex v_2, the arc set $\{(v_2, v_1), (v_2, v_3), (v_2, v_0)\}$ constitutes a *distar*.

3.1.2. Cycle valid inequalities of order β

Let us consider an arc set F from the planar digraph D such that:

$$|F \cap A(\gamma)| \leq \beta, \ \forall \gamma \in \mathcal{C}, \tag{3.1}$$

where $A(\gamma)$ denotes the set of arcs of the elementary directed cycle γ. Such an arc set F induces *valid inequalities* for the polytope $\mathcal{P}$.

Theorem 3.1 (see [8])**.** *Let β be a nonnegative integer. If F is an arc set satisfying the condition (3.1) then*

$$\sum_{a \in F} y_a \leq \beta \tag{3.2}$$

is a valid inequality for the polytope $\mathcal{P}$.

The valid inequality (3.2), called *cycle valid inequalities* of order β, is introduced and studied in [8]. If the digraph D is strongly connected, then the separation problem of cycle valid inequalities of order β is NP-hard (even in the case $\beta = 1$, see [8]). Indeed, the separation problem of cycle valid inequalities of order 1 can be reduced to the existence of τ-vertex-disjoint simple directed paths in the digraph D (see [6]), with $\tau \geq 2$ and $\tau > \beta$. However, if the digraph D is planar, we show that the separation of the cycle valid inequalities of order 1 can be done in polynomial time.

3.2. Separation algorithm

In the sequel, we prove that separating cycle valid inequalities of order 1 in planar digraphs can be done in polynomial time.

Given an optimal fractional solution $(\hat{x}, \hat{y}, \hat{f})$ of the linear relaxation (2.1)–(2.10) of the MWEDCP and its corresponding sub-digraph $D[\hat{y}]$, the separation problem of the cycle valid inequalities of order β consists in finding in the digraph $D[\hat{y}]$ an arc subset $\hat{F} \subset \hat{A}$ such that:

$$\sum_{a \in \hat{F}} \hat{y}_a > \beta.$$

We recall that $\hat{y}_a$ is the optimal (fractional) value of the arc a. As the sub-digraph $D[\hat{y}]$ is possibly disconnected, let $C_i[\hat{y}], \ i = 1, \ldots, c$ be its different connected components and consider the dual components $C_i^*[\hat{y}], \ i = 1, \ldots, c$ drawn from the different components $C_i[\hat{y}], \ i = 1, \ldots, c$.

Proposition 3.2. *For every vertex pair (u, v) of the dual component $C_i^*[\hat{y}], \ i = 1, \ldots, c$, a u-v elementary directed path in $C_i^*[\hat{y}], \ i = 1, \ldots, c$, defines a cycle valid inequality of order 1 w.r.t the component $C_i[\hat{y}], \ i = 1, \ldots, c$.*

Proof. Let u and v be two vertices belonging to the dual component $C_i^*[\hat{y}]$. Let $P_{u,v}$ be an elementary directed path from u to v in $C_i^*[\hat{y}]$. Let $\hat{F}_i$ be the arc set of the component $C_i[\hat{y}]$ containing all arcs that are crossed by an arc of $A(P_{u,v})$, the arc set of the u-v directed path $P_{u,v}$. Since every arc of $P_{u,v}$ crosses an arc from $C_i[\hat{y}]$ that belongs to two distinct elementary directed cycles of $C_i[\hat{y}]$, then we have

$$|\hat{F}_i \cap A\,(\gamma)\,| \leq 1, \ \forall \ \gamma \tag{3.3}$$

where γ is an elementary directed cycle of the component $C_i[\hat{y}]$. Therefore,

$$\sum_{a \in \hat{F}_i} y_a \leq 1, \ \forall \ i,$$

is a valid inequality w.r.t the component $C_i[\hat{y}]$, this completes the proof. ∎

Remark 3.3. The above inequality is also valid for all elementary directed cycles in $D[\hat{y}] = \bigcup_{i=1}^{c} C_i[\hat{y}]$. Moreover, as the arcs of each $\hat{F}_i$ are in a different component, namely $C_i[\hat{y}]$, we also have the following valid inequality $\sum_{a\in(\bigcup_{i=1}^{c}\hat{F}_i)} y_a \leq 1$.

Theorem 3.4. *The separation of cycle valid inequalities of order* 1 *induced by elementary directed paths of the dual* $D^*[\hat{y}]$ *of a planar sub-digraph* $D[\hat{y}]$ *is solvable in polynomial time.*

Proof. W.l.o.g. we assume that the sub-digraph $D[\hat{y}]$ is connected. By Proposition 3.2, any *u-v elementary directed path* in the dual digraph $D^*[\hat{y}]$ can be associated with a cycle valid inequality of order 1 in $D[\hat{y}]$. Using a depth first search (DFS) algorithm from a vertex $u \in D^*[\hat{y}]$ in view to reach a vertex $v \in D^*[\hat{y}]$, we generate in polynomial time a *u-v* elementary directed path that induces a corresponding cycle valid inequality of order 1, as the complexity time of a DFS algorithm is $O(|V| + |A|)$. To really solve the separation problem, we consider only the *u-v* elementary directed paths of the dual digraph such that the sum of the values of the associated variables of the arcs of the corresponding cycle valid inequality is up to 1. □

In Figure 1, in the dual digraph D^*, the elementary directed paths (v_2, v_0, v_1) and (v_2, v_0, v_3) correspond to the *cycle valid inequalities* of order 1 $y_{3,6} + y_{8,6} \leq 1$ and $y_{8,6} + y_{8,10} \leq 1$ in D, respectively. One can easily verify in D that the arcs $(3,6)$ and $(8,6)$ belong to distincts *elementary directed cycles* in D.

Generally, since the valid inequalities derived on a given sub-digraph, (as $D[\hat{y}]$ the sub-digraph that supports the optimal (fractional) solution $(\hat{x}, \hat{y}, \hat{f})$ of the linear relaxation of the model (2.1)–(2.10)), are not valid considering the entire digraph $D = (V, A)$, we may resort to a lifting technique to extend such valid inequalities for the whole polytope $\mathcal{P}$. This fact is illustrated in what follows

Let the following planar digraph $D = (V, A)$ be an instance of the MWEDCP. Consider $\mathcal{C} = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7\}$ the set of all elementary directed cycles in D. We represent an elementary directed cycle by an ordered sequence of its vertices. Thus, $\gamma_1 = (5, 10, 11, 6, 5)$, $\gamma_2 = (5, 10, 11, 12, 7, 6, 5)$, $\gamma_3 = (5, 10, 11, 12, 13, 8, 7, 6, 5)$, $\gamma_4 = (8, 9, 14, 13, 8)$, $\gamma_5 = (1, 2, 3, 4, 9, 14, 18, 17, 16, 15, 11, 6, 1)$, $\gamma_6 = (1, 2, 3, 4, 9, 14, 18, 17, 16, 15, 11, 12, 7, 6, 1)$ and $\gamma_7 = (1, 2, 3, 4, 9, 14, 18, 17, 16, 15, 11, 12, 13, 8, 7, 6, 1)$.

Consider the sub-digraph $D[\hat{y}]$ of the above instance D of the MWEDCP represented on the left side of Figure 3 and its corresponding dual sub-digraph $D^*[\hat{y}]$ (on the right side). The vertices v_1, v_2, v_3, v_4 of $D^*[\hat{y}]$ are associated to the faces of $D[y]$ delimited by the arc sets $\{(5,10), (10,11), (11,6), (6,5)\}$, $\{(7,6), (11,6), (11,12), (12,7)\}$, $\{(12,13), (13,8), (8,7), (12,7)\}$ and $\{(8,9), (9,14), (14,13), (13,8)\}$, respectively. With v_0 associated to the unbounded face. The elementary directed cycles of $D[\hat{y}]$ are $\gamma_1 = (5, 10, 11, 6, 5)$, $\gamma_2 = (5, 10, 11, 12, 7, 6, 5)$, $\gamma_3 = (5, 10, 11, 12, 13, 8, 7, 6, 5)$ and $\gamma_4 = (8, 9, 14, 13, 8)$. According to the proposition 3.2, from the $v_1 - v_4$ directed path $p_1 = (v_1, v_2, v_3, v_0, v_4)$ of $D^*[\hat{y}]$ represented by an ordered sequence of its vertices, we deduce that the inequality

$$y_{11,6} + y_{12,7} + y_{12,13} + y_{9,14} \leq 1 \tag{3.4}$$

is valid over the sub-digraph $D[\hat{y}]$. Indeed, it is satisfied by the incidence vectors of all elementary directed paths γ_1, γ_2, γ_3 and γ_4 of the sub-digraph $D[\hat{y}]$. However, it is violated by the incidence vectors of the elementary directed cycles γ_5, γ_6 and γ_7 of the entire digraph D. Inequality (3.4) is then not valid w.r.t the digraph D.

In the following lines, we address a lifting technique that can be polynomially implemented and enables to transform the valid inequality (3.4) into the following inequality

$$y_{11,6} - y_{15,11} + y_{12,7} + y_{12,13} + y_{9,14} - y_{4,9} \leq 1 \tag{3.5}$$

that is satisfied by all incidence vectors of the elementary directed cycles of D. Moreover, the lifted valid inequality (3.5) is violated by the fractional solution supported by the sub-digraph represented on the left side of Figure 3. Thus, $\hat{y}_{11,6} - \hat{y}_{15,11} + \hat{y}_{12,7} + \hat{y}_{12,13} + \hat{y}_{9,14} - \hat{y}_{4,9} = 0.5 - 0 + 0.25 + 0.25 + 0.75 - 0 = 1.75 > 1$.

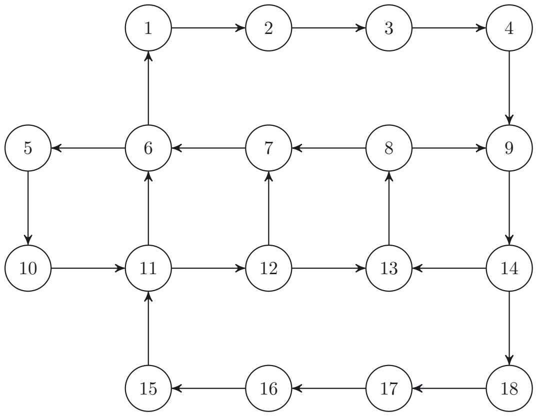

FIGURE 2. An instance D of the MWDECP.

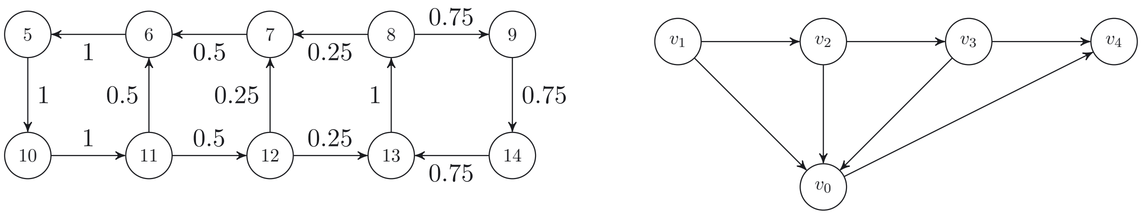

FIGURE 3. A sub-digraph of D (*left*) and its dual (*right*).

Given an instance $D = (V, A)$ of the MWEDCP and its corresponding sub-digraph $D[\hat{y}]$. Consider the arc subset $\hat{F}$ of A such that $\hat{F}$ verifies the condition (3.3). For any arc $a = (u, v) \in \hat{F}$, we denote by $\hat{F}^-$ the set of all arcs in $A \setminus \hat{A}$ such that these arcs have the node u as tail. Similarly, we denote by $\hat{F}^+$ the set of all arcs in $A \setminus \hat{A}$ such that these arcs have v as head. That is $\hat{F}^- = \{(x, u) \in (A \setminus \hat{A}) : (u, v) \in \hat{F}\}$ and $\hat{F}^+ = \{(v, x) \in (A \setminus \hat{A}) : (u, v) \in \hat{F}\}$.

Proposition 3.5. *Let $\hat{F}$ be an arc subset from $\hat{A}$ verifying the condition (3.3) w.r.t. the sub-digraph $D[\hat{y}]$. Then the following lifted inequality*

$$\sum_{a \in \hat{F}} y_a - \sum_{a \in \hat{F}^-} y_a - \sum_{a \in \hat{F}^+} y_a \leq 1 \tag{3.6}$$

is valid for the polytope $\mathcal{P}$.

We call the valid inequality (3.6) the $\hat{F}$-*lifted cycle valid inequality* of order 1. Such a valid inequality is close in spirit to the so-called *simple lifted valid inequalities* introduced in [11] for the *s*-*t* elementary directed path problem. However, here contrary to the *simple lifted valid inequalities*, we do not consider all the arcs of $A^c = A \setminus \hat{A}$, the arcs of the complementary digraph of $D[\hat{y}]$. We recall that $\hat{A}$ is the arc set of the sub-digraph $D[\hat{y}]$.

Proof. According to the condition (3.3), every elementary directed cycle in the sub-digraph $D[\hat{y}]$ intersects with at most one arc of the arc set $\hat{F}$. Consider an elementary directed cycle $\gamma \in \mathcal{C}$, we can distinguish three cases:

1. if γ is an elementary directed cycle of the sub-digraph $D[\hat{y}]$, it is obvious that the inequality (3.6) is valid for the polytope $\mathcal{P}$.
2. if γ is an elementary directed cycle such that $A(\gamma) \cap \hat{A} = \emptyset$, the inequality (15) remains valid. Thus, as $\hat{F} = \emptyset$, the sum of terms on the left hand side of the inequality are non-positive.

3. if γ is an elementary directed cycle such that $A(\gamma) \cap \hat{A} \neq \emptyset$ and $A(\gamma) \cap A^c \neq \emptyset$, it means that the elementary directed cycle γ has some of its arcs in $D[\hat{y}]$ and the others in the complementary subgraph of $D[\hat{y}]$. The inequality (3.6) is satisfied by the incidence vector of such an elementary directed cycle γ as generally every elementary directed cycle γ of $\mathcal{C}$ which passes through an arc $a = (u, v) \in \hat{F}$ has already passed through an arc ingoing to the head of the arc a and/or must pass through an arc outgoing from the tail of a.

This completes the proof. □

Algorithm 1: Strengthening of the linear relaxation $\hat{P}$.

```
Data: P̂ // The linear relaxation of (2.1)–(2.10)
Result: LB // Best lower bound
begin
    Separate ← True; NotIntegerSolution ← True
    Q ← P̂
    while Separate and NotIntegerSolution do
        (x̂, ŷ, f̂) ← solveLP(Q)
        if isInteger((x̂, ŷ)) then
            NotIntegerSolution ← False
        else
            D ← SupportDigraph((x̂, ŷ))
            ℒ ← ConnectedComponents(D)
            nbrNoSeparation = 0
            for C ∈ ℒ do
                C* ← DualDigraph(C)
                if stViolatedPath(C*, π) then
                    addLiftedIneq(Q, π)
                else
                    nbrNoSeparation = nbrNoSeparation + 1
                end
            end
            if nbrNoSeparation = |ℒ| then Separate ← False
        end
    end
    LB ← Weight((x̂, ŷ))
    return LB
end
```

Notice that from a given cycle valid inequality of order 1 generated on the sub-digraph $D[\hat{y}]$, the computational time required to construct its corresponding *$\hat{F}$-lifted cycle valid inequality* of order 1 is $O(|V|)$ in grid digraphs and is $O(|V|^2)$ in complete digraphs.

The steps of our cutting plane procedure are summarized in Algorithm 1 where: the procedure `solveLP` returns an optimal solution of the linear optimization problem over the polyhedron Q given as its argument; the procedure `isInteger` returns `true` if its argument is a binary vector and `false` otherwise; the procedure `SupportDigraph` generates the supporting digraph of the fractional solution given as its argument; the procedure `ConnectedComponents` returns the connected components of the graph given as its argument; the procedure `DualDigraph` builds and returns the dual digraph of the digraph given as its argument; the procedure `stViolatedPath` returns `true` if there is a violated directed path in the dual sub-digraph given as its first argument (the path is returned as its second argument) and `false` otherwise. We say that a directed path is

TABLE 1. Meaning of the entries of Table 2.

Column	Description
`Instances`	Name of the instance. The name follows the format x-y where x indicates the number of vertices and y the instance number.
$(\|V_S\|, \|A_S\|)$	Number of vertices and number of arcs in the supporting digraph of the first optimal fractional solution.
`#NDC`	Number of negative weight elementary directed cycles. The computational time limit is set to 24 hours. The symbol `na` indicates that the value is not available.
`#DC`	Number of elementary directed cycles. The computational time limit is set to 24 hours. The symbol `na` indicates that the value is not available.
$\bar{z}$	Weight of a optimal fractional solution of the linear relaxation (2.1)–(2.10)
z^*	The weight of an optimal integer elementary directed cycle
$\bar{z}_{cvi}$	Value of an optimal solution obtained after strengthening with all $\hat{F}-$lifted cycle valid inequalities of order 1.
`#cvi`	Number of violated $\hat{F}-$lifted cycle valid inequalities of order $\beta = 1$ generated
`gap(%)`	The gap closed (in percent), that is $gap = \frac{\bar{z}_{cvi} - \bar{z}}{z^* - \bar{z}} \times 100$.

violated if its corresponding cycle valid inequalty is violated by the optimal fractional solution $(\hat{x}, \hat{y})$; the procedure `addLiftedIneq` appends to the description of the polyhedron Q the lifted form of the cycle valid inequality of order 1 derived from the violated directed path π given as its second argument. Finally, the procedure `Weight` gives the value of the optimal solution after the cutting plane phase.

4. Computational results

In this section, we present computational results showing the efficiency of the cutting plane algorithm devised for the MWEDCP.

We deal with planar grid digraphs that contain negative weighted elementary directed cycles. We recall that the realistic instances from some libraries like TSPLIB are not planar in general. and does not contain negative weighted elementary cycles, as the weights of its arcs are positive.

Our instances are randomly generated directed grids D of different number of vertices: 50 (with 85 arcs), 100 (with 180 arcs) and 200 (with 370 arcs). As the neighboor set of every vertex in a grid D is known in advance, we have just to choose randomly the orientation of every arc (i, j). For this, we generate uniformly two numbers x and y, if $x < y$ then the arc is oriented from i to j otherwise the arc is oriented from j to i. In regard to the arc weights, with each arc (i, j), we associate a weight $z_1 - 10$ such that $0 \leq z_1 \leq 20$. In this way, the weights have a probability equals to $\frac{1}{2}$ to be whether negative or not. So, all our generated instances contain negative weight elementary directed cycles (see the values of the third column of Table 2 are all different to 0). Therefore, our instances cannot be solved using the algorithms proposed by Roditty and Williams [17]. Notice that among all randomly generated instances, we select only instances having a significant integrality gap (see the fourth and fifth columns of Tab. 2). Our algorithm is implemented in `C` and all computations have been carried out on a computer equipped with a 1.50 GHz Intel (R) core (TM) 2 CPU. All our instances are solved using the open-source software `glpk`. The computational results are presented in Table 2 (the entries of Tab. 2 are given in Tab. 1). We can make the following comments. Consider the first line of Table 2 represented by the

TABLE 2. Efficiency of $\hat{F}$-lifted cycle valid inequalities in strengthening of the linear relaxation of the MWEDCP model (2.1)–(2.10).

Instances	$(\vert V_S\vert, \vert A_S\vert)$	#NDC	#DC	$\bar{z}$	z^*	$\bar{z}_{cvi}$	#cvi	gap(%)
50−1	(29,38)	21	41	−55.5	−38	−46.54	43	51.20
50−2	(8,10)	8	11	−27	−22	−22	8	100
50−3	(32,37)	33	44	−72.5	−41	−70	5	7.94
50−4	(17,21)	9	13	−61	−36	−59	5	8
50−5	(11,14)	10	17	−25.5	−16	−23.5	3	21.05
50−6	(26,33)	26	27	−69.66	−44	−59.5	27	39.59
50−7	(22,24)	70	90	−75.5	−68	−75.5	0	0
50−8	(16,20)	12	17	−44.5	−22	−36.5	5	35.56
50−9	(24,27)	14	27	−51.5	−40	−51.5	3	0
50−10	(32,38)	34	42	−66.75	−51	−60.5	27	39.68
100−1	(33,43)	24	44	−52.25	−26	−50.5	19	6.67
100−2	(30,34)	66	100	−63.66	−57	−61.5	9	32.43
100−3	(17,20)	16	24	−49	−44	−44	11	100
100−4	(26,29)	39	44	−56	−43	−50.25	8	44.23
100−5	(28,31)	212	309	−65	−50	−54.75	8	68.33
100−6	(35,40)	34	53	−56.5	−43	−54.5	6	14.81
100−7	(43,46)	408	902	−77	−61	−74.25	2	17.19
100−8	(25,31)	29	41	−57.25	−51	−56.75	10	8
100−9	(39,43)	57	70	−85.5	−70	−85	5	3.23
100−10	(21,24)	70	87	−65	−57	−65	0	0
200−1	(30,34)	67	84	−104.5	−68	−104.5	0	0
200−2	(15,18)	68	106	−53.5	−46	−51.16	3	31.20
200−3	(44,50)	na	na	−137.5	−106	126.25	14	35.71
200−4	(30,34)	na	na	−48.5	−39	−44	10	47.37
200−5	(39,47)	78	102	−73	−55	−71.62	2	7.67
200−6	(33,40)	na	na	−46.5	−33	−45.66	15	6.22
200−7	(20,22)	na	na	−58	−37	−58	0	0
200−8	(35,45)	na	na	−72.5	−62	−70	46	23.81
200−9	(54,60)	na	na	−113.5	−98	−110.75	9	17.74

instance named 50-1, such an instance has 50 vertices and 85 arcs. After, the resolution of the linear relaxation of model (2.1)–(2.10) applied to 50-1, as shown by the second column of Table 2, the sub-digraph $D_s[\hat{y}]$ that supports the optimal (fractional) solution has 29 vertices, 38 arcs and contains 21 negative weighted directed cycles (see column 3). Assuming that the sub-digraph $D[\hat{y}]$, obtained from the sub-digraph $D_s[\hat{y}]$ by deleting the source vertex s and all arcs $\{(s,j) : \hat{y}_{sj} > 0\}$, is connected and according to Algorithm 1 described above, we build its associated dual sub-digraph $D^*[\hat{y}]$. After, we generated all directed paths in the dual sub-digraph corresponding to cycle valid inequalities of order 1. Among all generated cycle valid inequalities, as shown on the 8th column we only keep 43 that are violated by the optimal (fractional) solution $(\hat{x}, \hat{y}, \hat{f})$. As these valid inequalities are derived w.r.t the sub-digraph $D[\hat{y}]$, we use the lifting technique described in the previous section to transform the generated cycle valid inequalities into valid inequalities for the whole polytope $\mathcal{P}$. The lifted valid inequalities are called the $\hat{F}-$lifted cycle valid inequalities of order 1. Therefore, in a cutting plane process, we resort to the $\hat{F}-$lifted cycle valid inequalities of order 1 to strengthen the linear relaxation of the MWEDCP model (2)–(11). Considering all tested instances, strengthening with corresponding $\hat{F}-$lifted cycle valid inequalities of order 1 produces in average about 26.47% gap improvement showing the efficiency of these lifted inequalities. However, for some of our generated instances, we do not obtain any violated lifted cycle valid inequality. Thus, for such instances (`50-7`, `100-10`, `200-1` and `200-7`), we have $\#cvi = 0$ as shown on the 8th column of Table 2. This explains in part the fact that these instances do not present any gap improvement (see on its corresponding 9th column, $gap = 0$). Nevertheless, we observe that for the instances `50-2` and `100-3` the gap is totally closed (thus, $gap = 100\%$).

5. Conclusion

We address an efficient cutting plane algorithm for the minimum weighted elementary directed cycle problem (MWEDCP) in planar digraphs containing negative weight elementary directed cycles. The algorithm is based on the linear relaxation of a mixed integer linear formulation of the MWEDCP and the use of a lifting technique applied to a class of valid inequalities called cycle valid inequalities that are parametrized by an integer called the order (the inequality right-hand side). We prove that separating cycle inequalities of order 1 in planar digraphs can be done in polynomial time. In addition, as the inequalities are derived over the sub-digraph that supports the optimal (fractional) solution, with respect to the entire digraph, we deal with a lifted form of cycle valid inequalities of order 1 named the $\hat{F}$-lifted cycle valid inequalities of order 1. The lifting technique performed is also efficient, as it is based on some of the incident arcs of every vertex in the considered sub-digraph.

The computational results carried out on randomly generated planar digraphs (grids) of size set between 50 and 200 feature that a strengthening with $\hat{F}$-lifted cycle valid inequalities of order 1 can significantly improve the integrality gap. Indeed, in average, for all 29 instances, we obtain 26.47% gap improvement. Moreover, there exists instances for which the strengthening process directly displays the optimal integer elementary directed cycle.

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