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RENEGING IN A BATCH ARRIVAL TWO PHASES QUEUE WITH RANDOM FEEDBACK AND COXIAN-2 VACATION

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Abstract. In this system we consider a batch arrival Poisson input with two phases of heterogeneous service with random feedback in each servivce. The first phase of service is essential for all customers, but with a probability tagged customer chose second phase, feedback to tail of original queue or leave the system. Also, after completion of the second phase, with a probability the customer leaves the system, or feedback to tail of original queue. At each service completion epoch, the server may apt to take a vacation with a probability or continue to be available in the system for the next service. The service times are assumed to be general. The vacation period of the server has two heterogeneous phases with Coxian-2 distribution. The vacation times are assumed to be general. When the server goes for vacation, service become unavailable and customers may decide to renege at each vacation times. We assume reneging follow exponential distribution. All stochastic processes involved in this system (service and vacation times) are independent of each other. We derive the PGF's of the system and by using them the performance measures are obtained. Some numerical approch are examined the validity of results.

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1. Introduction

In queuing systems when vacation occurs, then it becomes impatience of customers and one the phenomena is reneging from system. In queueing literature, a customer is said to have reneged if it leaves the system without recieving its service entirely. It is a commonly observed phenomenon which customers join to queue and in the process of waiting, either in the queue or while recieving service, get impatient and leave the system. For the first time Barrer [5,6] studied the concept of deterministic reneging in the queue with Markovian arrival and service rates. Haight [12] considered a queue in which a customer having joined may decide to leave and give up service if it appears that the time consumed will exceed some maximum which it available. Untile now there ara many works in this implication that a good litterature survay is in [14]. But the confect of the reneging in $M/G/1$ queueing systems with vacation and heterogeneous services with general distributions is new. Choudhury and Medhi[9] worked on multiserver Markovian system with balking and reneging. Recently two works are inspected with Baruah *et al.* in [5,6]. In this works a two stage batch arrival queueing systems with vacation and reneging are studied.

Keywords. $M^x/G/1$ queue, random feedback, reneging, Coxian-2 vacation, mean queue size, mean response time.

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In many examples such as cars check up systems, production systems, bank services, computer and communication networks, feedback occurs on each service and customers repeat only unsuccessful phase, also he/she feedbacks to the beginning of the this phase. In addition, for overhauling or maintenance of the system, the server may go to vacation. Significant contributions by various authors on queues with feedback and server vacation has been seen in the last few years. Authors like Shahkar and Badamchizadeh [3, 4, 16] have studied queues under different vacation policies.

Choudhury and Paul[10] inspected the $M/G/1$ system with two phases of heterogeneous service and Bernoulli feedback. In this system a tagged customer may have an unsuccessful service , then retried until a successful service is completed. He/She is feedbacked instantaneously to the tail of the queue with probability $p(0 \leqslant p \leqslant 1)$ or departs from the system with probability $q = 1 - p$. Salehirad and Badamchizadeh[15] has extended this model. They worked on a multi-phase $M/G/1$ queue with random feedback.

Also Madan and Choudhury [13] studied a single server queue with two phases of heterogeneous services under Bernoulli schedule and a general vacation time. In this system, without feedback, the server after the completion of the service, may go to vacation with probability θ or remain in system with probability $1 - \theta$.

Our aim is to analyze the steady state condition of a single server queue consisting a batch arrival Poisson input, two phases of heterogeneous service, randomly feedback in services, Bernoulli vacation for server with Coxian-2 policy and reneging in vacation times. In Section 2 we deal with the mathematical model and definitions. Steady-State conditions and generating functions are discussed in Section 3. Mean queue size and mean response time are computed in Section 4, where in Section 5 some special cases are investigated. Finaly using some numerical methods, the validity of model has been examined.

2. Mathematical model and definitions

We consider a queueing system such that:

(i) New customers arrive in batches according to a compound Poisson process with rate λ. Let X_k denote the number of customers belonging to the kth arrival batch, where X_k, $k = 1, 2, 3, \dots$ are with a common distribution

$$Pr[X_k = n] = c_n, \qquad n = 1, 2, 3, \dots$$

and $X(z)$ denotes the probability generating function of X.

(ii) The server provides two phases of heterogeneous service in succession. The service discipline is assumed to be on the basis of first come, first serve(FCFS). The first phase of service is essential for all customers, but as soon as the essential service is completed, a tagged customer moves for second phase with probability γ_1, relapses to tail of original queue with probability η_1 or leaves the system with probability $\zeta_1 = 1 - \gamma_1 - \eta_1$. Similarly after completion of the second phase with probability γ_2 the customer leaves the system or with probability $\zeta_2 = 1 - \gamma_2$ feedback to the tail of original queue. The service times for two phases are independent random variables, denoted by B_1, B_2. Their Laplace–Stieltjes transform (LST)are $B_1^*(s), B_2^*(s)$ where we assume they have finite moments $E(B_i^l)$ for $l \geqq 1$ and $i = 1, 2$.

(iii) After completion of any customer's service, the server may take a vacation with probability θ or may continue to be in the system with probability $1 - \theta$.
We assume that the vacation time has two phases with phase one is compulsory. However, after phase 1 vacation, the server takes phase 2 vacation with probability p or may return back to the system with probability $1-p$. The vacation times are random variables V_i for $i = 1, 2$, follows a general law of probability with distribution function $V_i(x)$, Laplase transform $V_i^*(s)$ and finite moments $E(V_i^k)$ for $k \geq 1$.

(iv) Customers arriving for service may become impatient and renege (leave the system after joining the queue and when the server is on the vacation). At each vacation time the reneging occure and is assumed to follow exponential distribution with parameter α_1 in first phase of vacation and parameter α_2 in second phase. Hence $\alpha_i dt$ is the probability that a customer can renege during a short interval of time $(t, t + \mathrm{d}t]$ in phase i of vacation for $i = 1, 2$.

Definition 2.1. The modified service time or the time required by a customer to complete the service cycle is given by

$$B = \begin{cases} B_s + V_c & \text{with probability } \theta \\ B_s & \text{with probability } (1-\theta), \end{cases} \qquad E(B) = E(B_s) + \theta\ E(V_c) \tag{2.1}$$

where

$$B_s = \begin{cases} B_1 + B_2 & \text{with probability } \gamma_1 \\ B_1 & \text{with probability } (\eta_1 + \zeta_1), \end{cases} \qquad E(B_s) = E(B_1) + \gamma_1 E(B_2) \tag{2.2}$$

and

$$V_c = \begin{cases} V_1 + V_2 & \text{with probability } p \\ V_1 & \text{with probability } (1-p), \end{cases} \qquad E(V_c) = E(V_1) + pE(V_2) \tag{2.3}$$

then the LST $B^*(s)$ of B is given by

$$B^*(s) = \theta B_s^*(s) V_c^*(s) + (1-\theta) B_s^*(s) \tag{2.4}$$

and

$$E(B_s^2) = E(B_1^2) + 2\gamma_1 E(B_1)E(B_2^2) + \gamma_1 E(B_2^2), \qquad E(B^2) = E(B_s^2) + 2\theta E(B_s^2)E(V_c^2) + \theta E(V_c^2) \tag{2.5}$$

B_s is the required time without feedback and the random variable B_f with

$$E(B_f) = \zeta_1 E(B_1) + \gamma_1\gamma_2 [E(B_1) + E(B_2)] \tag{2.6}$$

represents the required time with feedback. Also because of reneging in vacation times we have a factor

$$E(V_R) = \alpha_1 E(V_1) + \alpha_2 E(V_2). \tag{2.7}$$

Further, for $i = 1, 2$ we assume that; $B_i(0) = 0, \quad B_i(\infty) = 1$ and $B_i(x)$ are continuous at $x = 0$, so that

$$\mu_i(x)\mathrm{d}x = \frac{\mathrm{d}B_i(x)}{1 - B_i(x)} \tag{2.8}$$

is the first order differential equation(hazard rate functions) of B_i. Also for $i = 1, 2$ we assume $V_i(0) = 1$, $V_i(\infty) = 1$ and $V_i(x)$ are continuous at $x = 0$. The hazard rate functions of V_i's are

$$\nu_i(x) = \frac{\mathrm{d}V_i(x)}{1 - V_i(x)} \tag{2.9}$$

Definition 2.2. Let $N_q(t)$ be the queue size at time $'t'$ and the supplementary variables are defined as:

$B_1^0(t)[B_2^0(t)] \equiv$ the elapsed first [second] phase of service at time $'t'$
$V_1^0(t)[V_2^0(t)] \equiv$ the elapsed first second] phase of vacation at time $'t'$

Now let us introduce the following random variables:

$$Y(t) = \begin{cases} 0 & \text{if the server is idle at time } 't', \\ 1[2] & \text{if the server is busy with first[second] phase of service at time } 't', \\ 3[4] & \text{if the server is on first [second] phase of vacation at time } 't'. \end{cases} \tag{2.10}$$

Then we have a bivariate Markov process $\{N_q(t), L(t)\}$ where $L(t) = 0$ if $Y(t) = 0$; $L(t) = B_1^0(t)$ if $Y(t) = 1$, $L(t) = B_2^0(t)$ if $Y(t) = 2$, $L(t) = V_1^0(t)$ if $Y(t) = 3$, $L(t) = V_2^0(t)$ if $Y(t) = 4$. Now for $i = 1, 2$ the following probabilities are defined as

$$V_{i,n}(x,t) = \text{Prob}[N_q(t) = n, L(t) = V_i^0(t); x < V_i^0(t) \leqslant x + \mathrm{d}x] \quad x > 0, \quad n \geqslant 0 \tag{2.11}$$

$$P_{i,n}(x,t) = \text{Prob}[N_q(t) = n, L(t) = B_i^0(t); x < B_i^0(t) \leqslant x + \mathrm{d}x] \quad x > 0, \quad n \geqslant 0 \tag{2.12}$$

and

$$R_0(t) = \text{Prob}[N_q(t) = 0, L(t) = 0] \tag{2.13}$$

With the assumption that steady state exists, we let

$$R_0 = \lim_{t\to\infty} R_0(t) \tag{2.14}$$

$$P_{i,n}(x)\mathrm{d}x = \lim_{t\to\infty} P_{i,n}(x,t)\mathrm{d}x \quad i = 1,2 \quad x > 0,\ n \geqslant 0 \tag{2.15}$$

$$V_{i,n}(x)\mathrm{d}x = \lim_{t\to\infty} q_n(x,t)\mathrm{d}x \quad i = 1,2 \quad x > 0,\ n \geqslant 0 \tag{2.16}$$

Now for $i = 1, 2$ the PGF of this probabilities is defined as follows:

$$P_i(x,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(x) \quad |z| \leqslant 1, \quad x > 0 \tag{2.17}$$

$$P_i(0,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(0) \quad |z| \leqslant 1 \tag{2.18}$$

Also

$$V_i(x,z) = \sum_{n=0}^{\infty} z^n V_{i,n}(x) \quad |z| \leqslant 1, \quad x > 0 \tag{2.19}$$

$$V_i(0,z) = \sum_{n=0}^{\infty} z^n V_{i,n}(0) \tag{2.20}$$

3. Steady-state probability generating function

From Kolmogorov forward equations, for $i = 1, 2$ the steady-state conditions can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{i,n}(x) + [\lambda + \mu_i(x)] P_{i,n}(x) = \lambda \sum_{k=1}^{n} c_k P_{i,n-k}(x) \quad n \geq 1, \quad x > 0 \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} P_{i,0}(x) + [\lambda + \mu_i(x)] P_{i,0}(x) = \lambda P_{i,0}(x) \tag{3.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} V_{i,n}(x) + [\lambda + \nu_i(x) + \alpha_i] V_{i,n}(x) = \lambda \sum_{k=1}^{n} c_k V_{i,n-k}(x) + \alpha_i V_{i,n+1}(x) \quad n \geq 1, \quad x > 0 \tag{3.3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} V_{i,0}(x) + [\lambda + \nu_i(x)] V_{i,0}(x) = \alpha_i V_{i,1}(x) \tag{3.4}$$

also

$$\lambda R_0 = (1-\theta)\zeta_1 \int_0^{+\infty} \mu_1(x) P_{1,1}(x)\mathrm{d}x + (1-\theta)\gamma_2 \int_0^{+\infty} \mu_2(x) P_{2,1}(x)\mathrm{d}x + (1-p) \times \int_0^{+\infty} \nu_1(x) V_{1,0}(x)\mathrm{d}x + \int_0^{\infty} \nu_2(x) V_{2,0}(x)\mathrm{d}x \tag{3.5}$$

At $x = 0$, the boundary conditions for $n \geq 0$ are

$$P_{1,n}(0) = \lambda a_{n+1} R_0 + (1-\theta)(1-\gamma_1-\eta_1) \times \int_0^{+\infty} \mu_1(x) P_{1,n+1}(x)\mathrm{d}x + \eta_1 \int_0^{+\infty} \mu_1(x) P_{1,n}(x)\mathrm{d}x + \gamma_2(1-\theta) \int_0^{+\infty} \mu_2(x) P_{2,n+1}(x)\mathrm{d}x + (1-\gamma_2) \times \int_0^{+\infty} \mu_2(x) P_{2,n}(x)\mathrm{d}x + (1-p) \int_0^{+\infty} \nu_1(x) V_{1,n}(x)\mathrm{d}x + \int_0^{\infty} \nu_2(x) V_{2,n}(x)\mathrm{d}x \tag{3.6}$$

$$P_{2,n}(0) = \gamma_1 \int_0^{+\infty} \mu_1(x) P_{1,n}(x)\mathrm{d}x \tag{3.7}$$

also

$$V_{1,n}(0) = \theta(1-\gamma_1-\eta_1) \int_0^{+\infty} \mu_1(x) P_{1,n+1}(x)\mathrm{d}x + \theta\gamma_2 \int_0^{+\infty} \mu_2(x) P_{2,n+1}(x)\mathrm{d}x, \quad n \geqslant 0 \tag{3.8}$$

$$V_{2,n}(0) = p \int_0^{+\infty} \nu_1(x) V_{1,n}(x)\mathrm{d}x \tag{3.9}$$

Finally the normalizing condition is

$$R_0 + \sum_{i=1}^{2}\sum_{n=0}^{\infty} \int_o^{+\infty} P_{i,n}(x)\mathrm{d}x + \sum_{i=1}^{2}\sum_{n=0}^{\infty} \int_0^{+\infty} V_{i,n}(x)\mathrm{d}x = 1 \tag{3.10}$$

Lemma 3.1. *For $i = 1, 2$ from* (3.1) *and* (3.2) *we have*

$$P_i(x,z) = P_i(0,z)[1 - B_i(x)]\mathrm{e}^{-\lambda[1-X(z)]x} \quad x > 0 \tag{3.11}$$

and from (3.3) *and* (3.4)

$$V_i(x,z) = V_i(0,z)[1 - V_i(x)]\mathrm{e}^{-(\lambda[1-X(z)]+\alpha_i-\frac{\alpha_i}{z})x} \quad x > 0 \tag{3.12}$$

For $i = 1, 2$ we define $\Lambda_i(z) = \lambda[1 - X(z)] + \alpha_i - \frac{\alpha_i}{z}$ and

$$B_i^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}B_i(x) \tag{3.13}$$

$$V_i^*(s) = \int_0^{+\infty} \mathrm{e}^{-sx}\mathrm{d}V(x) \tag{3.14}$$

as z-transform of B_i and V_i respectively, then

Theorem 3.2. *If* $b(z) = (\zeta_1 + \gamma_1\gamma_2 B_2^*[(\lambda - \lambda X(z)])B_1^*[(\lambda - \lambda X(z)]$ *and* $a(z) = (1-\theta) + \theta[(1-p) + pV_2^*(\Lambda_2(z))]V_1^*(\Lambda_1(z))$, *then we have*

$$P_2(0,z) = \gamma_1 P_1(0,z) B_1^*[\lambda - \lambda X(z)] \tag{3.15}$$

$$zV_1(0,z) = \theta\{\zeta_1 P_1(0,z)B_1^*[(\lambda - \lambda X(z)] + \gamma_2 P_2(0,z)B_2^*[(\lambda - \lambda X(z)]\} \tag{3.16}$$

$$V_2(0,z) = pV_1(0,z)V_1^*(\Lambda_1(z)) \tag{3.17}$$

$$P_1(0,z) = \frac{R_0(\lambda - \lambda X(z))}{z[1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)] - b(z)a(z)}. \tag{3.18}$$

Proof. By multiplying (3.7) in z^n and summation from $n = 0$ to ∞, using (2.17) and (2.18) the formula (3.15) is obtained.
By multiplying (3.8) in z^n and summation from $n = 0$ to ∞, using (2.19) and (2.20) the formula (3.16) is obtained.
By multiplying (3.9) in z^n and summation from $n = 0$ to ∞, using (2.19) and (2.20) the formula (3.17) is obtained.
By multiplying (3.6) in z^n and summation from $n = 0$ to ∞, using (2.17) and (2.18) and also (3.5), (3.15), (3.16), (3.17) the formula (3.18) is obtained. □

Corollary 3.3. *By using* (3.18) *in* (3.15), (3.16) *and* (3.17) *we have*

$$P_2(0,z) = \frac{R_0\gamma_1(\lambda - \lambda X(z))}{z[1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)] - b(z)a(z)} \tag{3.19}$$

$$V_1(0,z) = \frac{\theta R_0 b(z)(\lambda - \lambda X(z))}{z[1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)] - b(z)a(z)} \tag{3.20}$$

$$V_2(0,z) = \frac{\theta p R_0 b(z)(\lambda - \lambda X(z))}{z[1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)] - b(z)a(z)}. \tag{3.21}$$

3.1. Generating functions

Since for $i = 1, 2$

$$P_i(z) = \int_0^\infty P_i(x,z)\mathrm{d}x$$

and

$$V_i(z) = \int_0^\infty V_i(x,z)\mathrm{d}x$$

we have

Corollary 3.4. *From* (3.11) *using* (3.18)

$$P_1(z) = \frac{R_0[1 - B_1^*(\lambda - \lambda X(z))]}{b(z)a(z) - z\{1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)]\}}. \tag{3.22}$$

From (3.11) *using* (3.19)

$$P_2(z) = \frac{R_0\gamma_1 B_1^*[(\lambda - \lambda X(z)][1 - B_2^*(\lambda - \lambda X(z))]}{b(z)a(z) - z\{1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)]\}}. \tag{3.23}$$

From (3.12) *using* (3.20)

$$V_1(z) = \frac{b(z)\theta R_0(\lambda - \lambda X(z))}{b(z)a(z) - z\left\{1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)]\right\}} \frac{[1 - V_1^*(\Lambda_1(z))]}{\Lambda_1(z)}. \tag{3.24}$$

From (3.12) *using* (3.21)

$$V_2(z) = \frac{b(z)\theta p R_0 V_1^*(\Lambda_1(z))(\lambda - \lambda X(z))}{b(z)a(z) - z\left\{1 - \eta_1 B_1^*[(\lambda - \lambda X(z)] - (1-\gamma_2)\gamma_1 B_1^*[(\lambda - \lambda X(z)]B_2^*[(\lambda - \lambda X(z)]\right\}} \frac{[1 - V_2^*(\Lambda_2(z))]}{\Lambda_2(z)}. \tag{3.25}$$

Remark 3.5. The unknown constant R_0 can be determined by using normalizing condition (3.10) which is

$$R_0 + P_1(1) + P_2(1) + V_1(1) + V_2(1) = 1 \tag{3.26}$$

from (3.22), (3.23), (3.24) and (3.25) by using L'Hopital rule and relations (2.2) and (2.3) we have

$$\begin{aligned}
P_1(1) =& R_0 \frac{\frac{\lambda E(X)E(B_1)}{\zeta_1 + \gamma_1\gamma_2}}{1 - \left\{\frac{\lambda E(X)E(B_s)}{\zeta_1 + \gamma_1\gamma_2} + \theta[\lambda E(X)E(V_c) - (\alpha_1 E(V_1) + \alpha_2 E(V_2))]\right\}} \\
P_2(1) =& R_0 \frac{\frac{\gamma_1 \lambda E(X)E(B_2)}{\zeta_1 + \gamma_1\gamma_2}}{1 - \left\{\frac{\lambda E(X)E(B_s)}{\zeta_1 + \gamma_1\gamma_2} + \theta[\lambda E(X)E(V_c) - (\alpha_1 E(V_1) + \alpha_2 E(V_2))]\right\}} \\
V_1(1) =& R_0 \frac{\theta \lambda E(X)E(V_1)}{1 - \left\{\frac{\lambda E(X)E(B_s)}{\zeta_1 + \gamma_1\gamma_2} + \theta[\lambda E(X)E(V_c) - (\alpha_1 E(V_1) + \alpha_2 E(V_2))]\right\}} \\
V_2(1) =& R_0 \frac{\theta p \lambda E(X)E(V_1)}{1 - \left\{\frac{\lambda E(X)E(B_s)}{\zeta_1 + \gamma_1\gamma_2} + \theta[\lambda E(X)E(V_c) - (\alpha_1 E(V_1) + \alpha_2 E(V_2))]\right\}}
\end{aligned}$$

hence by substituting the above values in (3.26) and simplifying we have $R_0 = 1 - \rho$ where

$$\rho = \frac{\lambda E(X)\left[\frac{E(B_s)}{\zeta_1 + \gamma_1\gamma_2} + \theta E(V_c)\right]}{1 + \theta[\alpha_1 E(V_1) + \alpha_2 E(V_2)]} \tag{3.27}$$

R_0 is the steady-state probability that the server is idle but available in the system, hence $\rho < 1$ can be the stability condition under which the steady state solution exists.

Now the PGF of the queue size distribution at a random epoch is

$$P_q(z) = P_1(z) + P_2(z) + V_1(z) + V_2(z)$$

also the PGF of the system size at random epoch is

$$P(z) = R_0 + P_q(z)$$

Because of the nature of model and the huge form of P_i and V_i the PGF is very huge, hence the calculate of L_q is hard and the result is long. Yet the results as follow.

4. Mean queue size and other measures of system

Let L_q be the mean number of customers in the queue (*i.e.* mean queue size), then we have

$$L_q = \frac{\mathrm{d}P_q(z)}{\mathrm{d}z} \,|_{z=1} \tag{4.1}$$

Proposition 4.1. *From* (3.28) *and using* (3.22), (3.23), (3.24), (3.25) *we have*

$$\begin{aligned} L_q = {} & \frac{\lambda^2 E(X)^2 \left[E(B_s^2) + \theta E(B_f)\left(E(V_c) + E(V_1)\right)\right]}{2(\varsigma_1 + \gamma_1\gamma_2)\left(1 + \theta E(V_R)\right)} + \frac{\lambda E(X(X-1))\left[E(B_s) + \theta(\varsigma_1 + \gamma_1\gamma_2)E(V_c)\right]}{2(\varsigma_1 + \gamma_1\gamma_2)\left(1 + \theta E(V_R)\right)} \\ & + \frac{\theta\lambda E(X)\left[\left(\lambda E(X) - \alpha_1\right)E(V_1^2) + p\left(\lambda E(X) - \alpha_2\right)E(V_2^2)\right]}{1 + \theta E(V_R)} \\ & + \frac{\lambda E(X)\left[E(B_s) + \theta(\varsigma_1 + \gamma_1\gamma_2)E(V_c)\right]}{2(\varsigma_1 + \gamma_1\gamma_2)^2(1-\rho)\left(1 + \theta E(V_R)\right)^2}\Psi \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \Psi = {} & \lambda^2 E(X)^2 E(B_s^2) + 2\lambda E(B_s)\left[E(X^2) + E(X)\right] \\ & + 2\theta\lambda E(X)E(B_f)\left[\lambda E(X)E(V_c) - \left(\alpha_1 E(V_1) + p\alpha_2 E(V_2)\right)\right] \\ & + \theta(\varsigma_1 + \gamma_1\gamma_2)\left\{\left[\left(\lambda E(X) - \alpha_1\right)E(V_1^2) + p\left(\lambda E(X) - \alpha_2\right)E(V_2^2)\right]\right. \\ & \left. + \left[\lambda E(X(X-1))E(V_c) - 2\left(\alpha_1 E(V_1) + p\alpha_2 E(V_2)\right)\right]\right\} \\ & + 2p\theta\left(\lambda E(X) - \alpha_1\right)\left(\lambda E(X) - \alpha_2\right)E(V_1)E(V_2). \end{aligned}$$

Now for computing the mean response time of a test customer in this model, let $W_q^*(s)$ be the LST of DF of waiting time of a tagged customer in this model. Then we have

$$W_q^*(\lambda - \lambda z)B^*(\lambda - \lambda z) = P_q(z) \tag{4.3}$$

where B^* is defined in (2.4).

If W_R denotes the time interval from arrival time to the time when a tagged customer leaves the system after the completion of service, *i.e.* waiting time plus service time, then

$$W_R^*(s) = W_q^*(s)B^*(s) \tag{4.4}$$

and mean response time of a tagged customer is

$$E(W_R) = -\frac{\mathrm{d}W_R^*(s)}{\mathrm{d}s} \,|_{s=0} \tag{4.5}$$

By substituting from (4.3) in (4.4) we have

$$W_R^*(s) = P_q\left(1 - \frac{s}{\lambda}\right) \tag{4.6}$$

By using (3.28) and from (4.5) we have

$$E(W_R) = \frac{1}{\lambda}L_q \tag{4.7}$$

Also the average system size is $L = L_q + \rho$ where ρ is in (3.27).

5. Special cases and numerical results

System without vacation: If $\theta = 0$, then the system is without vacation. Hence $E(B) = E(B_s)$, $E(B^2) = E(B_s^2)$ and

$$\rho = \frac{\lambda E(X)E(B_s)}{\zeta_1 + \gamma_1\gamma_1}$$

also

$$L_q = \frac{\left(\lambda E(X)\right)^2 E(B_s^2) + \lambda E(X(X-1))E(B_s)}{2(\varsigma_1 + \gamma_1\gamma_2)} + \frac{\lambda E(X)E(B_s)}{2(\varsigma_1 + \gamma_1\gamma_2)^2(1-\rho)}\Psi$$

where

$$\Psi = \lambda^2 E(X)^2 E(B_s^2) + 2\lambda E(B_s)[E(X^2) + E(X)]$$

System without reneging: If $\alpha_1 = \alpha_2 = 0$, then system is without reneging. Hence $E(V_R) = 0$ and

$$\rho = \lambda E(X)\left[\frac{E(B_s)}{\varsigma_1 + \gamma_1\gamma_2} + \theta E(V_c)\right]$$

System without feedback: If $\gamma_1 = 1$, $\gamma_2 = 1$, $\eta_1 = 0$ and $\zeta_1 = 0$, then the system is without feedback. Then $E(B_f) = E(B_s)$ and other formulas change naturally.

5.1. Numerical analysis

Analyzing a queueing system via actual cases is very important and a useful method to confirm validity of the model. In this section we selected known distributions for service time and vacation time, so with this, and by some numerical approches the validity of the system is examained. Also this approch explains that our model can function reasonably well for certain practical problems.

Case 1. L_Q vis-à-vis λ. For each $i = 1, 2$, let the distribution of service time be τ_i-Erlang as follows:

$$\mathrm{dB_i}(x) = \frac{(\tau_i\mu_i)^{\mu_i} x^{\tau_i - 1}\mathrm{e}^{-\tau_i\mu_i x}}{(\tau_i - 1)!}\mathrm{d}x \quad x > 0, \tau_i \geq 1$$

hence

$$B_i^*(\lambda - \lambda X(z)) = \frac{(\tau_i\mu_i)^{\tau_i}}{[\lambda(X(z) - 1) + \tau_i\mu_i)]^{\tau_i}}$$

so $E(B_i) = \dfrac{1}{\mu_i}$ and $E(B_i^2) = \dfrac{\tau_i + 1}{\tau_i\mu_i^2}$.

Also for $j = 1, 2$ we assume the distribution of vacation times be ϵ_j-Erlang

$$\mathrm{d}V(x) = \frac{(\epsilon_j\nu_j)^{\nu_j} x^{\epsilon_j - 1}\mathrm{e}^{-\epsilon_j\nu_j x}}{(\epsilon_j - 1)!}\mathrm{d}x \quad x > 0, \epsilon_j \geq 1$$

hence

$$V^*(\lambda - \lambda X(z)) = \frac{(\epsilon_j\nu_j)^{\epsilon_j}}{[\lambda(X(z) - 1) + \epsilon_j\nu_j]^{\epsilon_j}}$$

so $E(V_j) = \dfrac{1}{\nu_j}$ and $E(V_j^2) = \dfrac{\epsilon_j + 1}{\epsilon_j\nu_j^2}$. If we choose geometric distribution for a batch size, *i.e.* $c_n = d(1 - d)^{n-1}, 0 < d < 1$, then $E(X) = \dfrac{1}{d}$, $E(X^2) = \dfrac{2-d}{d^2}$ and

$$E(X(X-1)) = \frac{2(1-d)}{d^2}\cdot$$

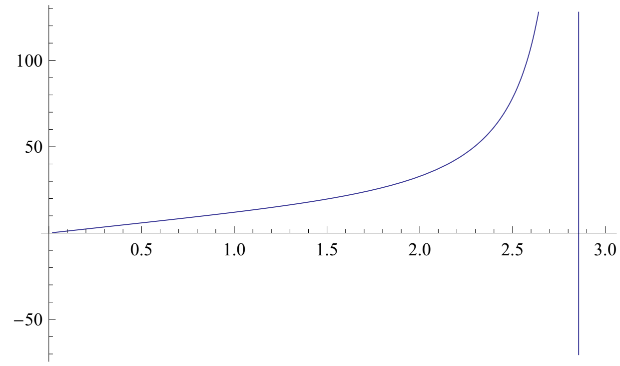

FIGURE 1. L_Q vis-à-vis λ.

TABLE 1. Values of parameters.

μ_1	μ_2	τ_1	τ_2	γ_1	γ_2	ζ_1	θ	p	ν_1	ν_2	ϵ_1	ϵ_2	α_1	α_2	d
5	2	1	2	.3	.5	.3	.8	.7	1	2	1	2	5	7	.8

TABLE 2. Values of parameters.

λ	μ_1	μ_2	τ_1	τ_2	γ_1	γ_2	ζ_1	p	ν_1	ν_2	ϵ_1	ϵ_2	α_1	α_2	d
2	5	3	1	2	.2	.3	.5	.6	2	3	1	2	3	5	.7

Now for numerical results we assume the following values for parametrs such that the steady state condition ($\rho < 1$) can be obtained. These are shown in Table 1.

Now by using above values and (3.27), the steady state condition is $\rho = .35\lambda < 1$, so $\lambda < 2.8$. By using (3.29)

$$L_q = .062\lambda^2 - 1.45\lambda + \frac{.07\lambda}{1 - .35\lambda}\Psi$$

where $\Psi = 11.52\lambda^2 - 83.14\lambda + 191.5$. The graph of model shown in Figure 1.

Case 2: L_Q vis-à-vis θ. In this case we use the values of Table 2.

From (3.27) the steady state condition is $\rho = \frac{1.31 + 1.98\theta}{1 + 3.16\theta} < 1$ or $\theta > .26$. From (3.29) we have

$$L_q = \frac{2.5 + .29\theta}{1 + 3.16\theta} + \frac{1.17 + 1.75\theta}{(1.18\theta - .31)(1 + 3.16)}\Psi$$

where $\Psi = 1.29\theta + 8.98$. The graph of model shown in Figure 2. Unlike the usual queue systems, in this model when θ increase then L_q decrease, this is because of reneging that customers renege the system in vacation times. In Table 3 some values of L_q against θ are computed. L_q decrease until $\theta = .9$, then increase slowly until $\theta = 1$.

L_q vis-à-vis α_1 and α_2: In this case by using the same distributions and values of Table 4 the steady state condition is

$$\rho = \frac{4 + .24\alpha_1 + .12\alpha_2}{1 + .6\alpha_1 + .3\alpha_2} < 1$$

or $.12\alpha_1 + .6\alpha_2 > 1$. Now from(3.29) we have

$$L_q = \frac{59.5 - 4.8\alpha_1 - .39\alpha_2}{1 + .6\alpha_1 + .3\alpha_2} + \frac{585.2 - 52.3\alpha_1 - 31.26\alpha_2 + 6.27\alpha_1\alpha_2}{(.36\alpha_1 + .18\alpha_2 - 3)(1 + .6\alpha_1 + .3\alpha_2)}$$

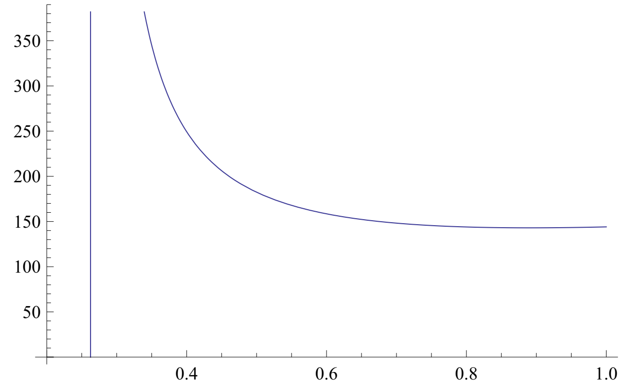

FIGURE 2. L_Q vis-à-vis θ.

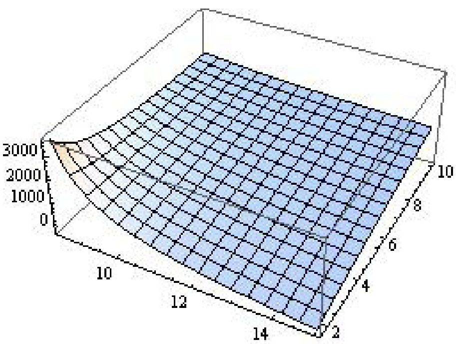

FIGURE 3. L_Q vis-à-vis α_1 and α_2.

TABLE 3. Values of L_Q with respect θ.

θ	.3	.5	.7	.9	.95	1
L_Q	704	182	148	143.013	143.34	144.063

TABLE 4. Values of parameters.

λ	μ_1	μ_2	τ_1	τ_2	γ_1	γ_2	ζ_1	θ	p	ν_1	ν_2	ϵ_1	ϵ_2	d
2	3	1	1	2	.1	.3	.4	.6	.5	1	2	1	3	.5

TABLE 5. Values of L_Q with respect α_i.

α_1	9	10	12	13	14	15
α_2	2	4	5	7	8	9
L_q	1925	1168	775	952	1033	1154

Figure 3 shows the surface of L_q wth respect α_1 and α_2. Because of steady state condition, the domain of L_q is up the line $.12\alpha_1 + .6\alpha_2 = 1$, or $\alpha_1 > 8.3$ and $\alpha_2 > 1.63$. Table 5 shows some values of L_q with respect α_1 and α_2. It seems near $\alpha_1 = 12, \alpha_2 = 5$, there is a relative minimum for L_q. Yet untile α_2 is near 1.63, L_q takes minimum values when α_1 is near 14.5.

6. Concluding remarks

In this paper we have studied a batch arrival two phases queueing system with randomly feedback, server's vacation with Coxian-2 policy and reneging in each vacation times which generalized classical $M/G/1$ queue. Reneging is a usual phenomenon in queueing system with vacation that makes basic variations in effective measures of system. In systems such as this when the probability of vacation incresing, because of reneging, the mean size of system become decrease. An application of this model can be found in mobile network where the messages are in batch form, the service may have many phases such that services may be unaccepted and customer may repeat the services. Our investigations are concerned with not only queue size distribution but also waiting time distribution. This model extends the systems in references. A practical generalization for this system is to consider optional services and k phases of services.

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