RAIRO-Oper. Res. 51 (2017) 157–171
DOI: 10.1051/ro/2016012

RAIRO Operations Research
www.rairo-ro.org

IMPROVEMENTS TO SMOOTH DATA ENVELOPMENT ANALYSIS

Luana C. Brandão[1] and João Carlos C.B. Soares de Mello[2]

Abstract. Classic data envelopment analysis (DEA) models do not provide unique solutions for multipliers of extreme efficient units. To overcome this problem, previous works have proposed the smooth DEA technique. However, multidimensional models with variable returns to scale (BCC) present deficiencies, as they do not fully ensure the frontier's convexity. Therefore, the main contribution of this paper is to correct smooth BCC models, with regard to such property. Moreover, we propose improvements to smooth DEA models, so that all projections for the evaluated units, and their efficiency values, are non-negative. Furthermore, based on the corrected and improved smooth model, we propose a solution to avoid a classic BCC distortion, which may be called efficiency by default. Finally, we evaluate the operational performance of Brazilian airlines in 2010, to show the applicability of our model and to illustrate the practical effect of our contributions.

Mathematics Subject Classification. 90B50.

Received March 3rd, 2015. Accepted February 1st, 2016.

1. Introduction

Data envelopment analysis (DEA) is a non-parametric method based on mathematical programming that measures efficiencies of production units, which are referred to as decision making units (DMUs). There are two classic DEA models: CCR [10] and BCC [7]. The first assumes constant returns to scale and proportionality between inputs and outputs, while the BCC model assumes variable returns to scale. For each classic model, there are two equivalent and dual formulations [13]. One of them, called Multipliers model, provides multipliers of inputs and outputs for each DMU, which may be interpreted as trade-offs [13] or shadow prices [12].

However, in classic models, there are multiple optimal solutions for multipliers of extreme efficient DMUs, *i.e.*, DMUs that are "corners" to the frontier formed by efficient DMUs, called efficient frontier. This problem is of practical importance, since analysts are often interested in multipliers values of the best-evaluated units, to obtain added insight on the overall efficiency score [14]. It is also critical for certain DEA methods, such as cross evaluation [17], used to improve discrimination. In fact, as noted in [17], the problem of multiple optimal solutions for multipliers possibly reduce the usefulness of cross efficiency.

To address such problem, several papers have proposed smooth DEA models [32, 34, 39, 40]. This technique replaces the original efficient frontier, which is piecewise linear, with a frontier that has derivatives at all points.

Keywords. Smooth DEA, DEA properties, BCC distortion, efficiency by default, airlines efficiency.

[1] Post Graduation in Production Engineering, Universidade Federal Fluminense, Rua Passo da Pátria 156, 24.210-240 Niterói/RJ, Brazil; luanabrandao@id.uff.br

[2] Production Engineering Department, Universidade Federal Fluminense, Rua Passo da Pátria 156, 24.210-240 Niterói/RJ, Brazil; jcsmello@pesquisador.cnpq.br

Article published by EDP Sciences

The smooth frontier should be as close as possible to the original and also maintain its basic properties [39]. Besides avoiding multiple optimal solutions, this technique also eliminates Pareto inefficient regions, where Pareto inefficient units are considered efficient in classic DEA. Such units are adequately considered inefficient in smooth DEA.

Nevertheless, this paper proves that smooth BCC models in the found literature do not ensure the model's essential property of convexity, despite affirming otherwise. Therefore, our main contribution is to correct smooth BCC models, ensuring the frontier's convexity. Moreover, we also present improvements to smooth DEA models, so that all DMUs' projections, and consequently, their efficiency values, are non-negative. Despite being present in classic DEA, this characteristic was not ensured in previous smooth models.

Furthermore, we also propose the use of smooth DEA to avoid the following distortion in classic BCC, hereinafter called efficiency by default, as in [22]: if a DMU has the smallest value for any input or the greatest value for any output, it is necessarily efficient [1]. A preliminary version of this solution was proposed in [9] to evaluate team efficiency in a football championship. However, in the present paper, we use the corrected and improved smooth model proposed herein, to provide a more robust and coherent solution for the BCC distortion.

Finally, we evaluate the operational performance of Brazilian airlines in 2010, to show the applicability of our approach. As in [23], this paper considers the airlines' fleet capacity as input, and passenger and cargo transport as outputs, using a variable returns to scale approach. However, in this paper, our purpose is to eliminate the efficiency by default distortion from such approach, thus we apply the smooth BCC model presented herein. We also apply previous and intermediate smooth models, *i.e.*, models with some of this paper's contributions, to illustrate the practical effect of each correction and improvement.

2. Literature review

The problem of multiple optimal solutions for multipliers in classic DEA models could be understood in light of the Theorem of Complementary Slacks. This theorem shows that multipliers correspond to coefficients of the hyperplane that is tangent at each point of the efficient frontier [13]. Since the original frontier is piecewise linear, it has multiple hyperplanes tangent to each of its "corners", where extreme-efficient DMUs are located. Hence, there are multiple optimal sets of multipliers for each of these DMUs.

There have been different techniques proposed in the literature to deal with such problem, most of which provide unsatisfactory solutions [32, 34]. In 1985, the authors of [11] proposed the use of average multipliers based on the barycentres of the concurring hyper-surfaces. For this, however, one must calculate the equations for all facets, requiring an intense load of computer work [18]. Moreover, it is not applicable to DMUs at the start of Pareto inefficient regions or to DMUs that are adjacent to facets of incomplete dimension [33].

The super efficiency model [2] provides a unique set of multipliers for all DMUs. However, there are multiple efficient frontiers, depending on the DMU being studied, and their efficiency values are not limited to the interval [0,1].

Restricted to extreme efficient points, Cooper *et al.* [14] propose a two-step procedure to select a unique solution and also avoid null multipliers in CCR models. First, they select the multipliers set associated with the hyperplanes supported by the maximum number of extreme efficient units, *i.e.*, associated with the facets of highest dimension for each unit, using mixed integer linear programming (MILP). Then, from the multipliers set selected in the first step, they choose the one that maximizes the minimum relative multiplier, using another MILP problem.

Other papers [17, 26, 42] proposed partial solutions to this problem of multiple optimal sets of multipliers, to enable the application of cross evaluation models. Moreover, inspired by Cross Evaluation and Game Theory, Liang *et al.* [27] proposed the DEA game cross efficiency model, which may be regarded as a generalized benevolent approach.

Rosen *et al.* [36] affirm that it is impossible to avoid the multiplicity of optimal solutions, because it is intrinsic to the frontier's piecewise linear nature. This is why smooth DEA [32, 34, 39, 40] replaces the original frontier with a continuously differentiable frontier. The new frontier should contain all efficient DMUs from classic DEA, be as close as possible to the original frontier, and maintain essential properties of classic DEA. These properties are monotonicity of outputs with respect to inputs, limited efficiency values (in the interval [0,1]), allocation of different weights by each DMU, and, for smooth BCC, convexity [32].

Smooth DEA was found to eliminate other classic problems, such as Pareto inefficient regions and non-complete dimension facets, studied in [24]. In [21], the authors employed this technique, not only for its aforementioned benefits, but also to eliminate the need of calculating all facets' equations in Zero Sum Gains DEA [20,28]. According to the authors, there is no efficient algorithm for this problem, of high complexity.

Besides identifying corrections and improvements to Smooth BCC models, this paper proposes the use of such models to correct the BCC distortion detected in [1], called efficiency by default, as in [22]. Other papers have proposed advanced DEA techniques to eliminate different BCC problems. The authors of [43], for example, studied negative efficiencies in cross evaluation with BCC input oriented models and also propose a solution to this problem.

In a different context, certain authors (see, *e.g.*, [4,5,25,29–31,37]) proposed continuously differentiable DEA frontiers to redistribute resources among the DMUs. Their results may be considered a smoothed variant of Zero Sum Gains DEA.

3. Theoretical overview

Smooth DEA models are a Quadratic Problem (QP) that approximates smooth and classic frontiers, considering their arc lengths, with the adequate restrictions. Since the original frontier is piecewise linear and that a straight line has the minimum arc length between two fixed points, the objective function minimizes the smooth frontier's arc length (or its n-dimensional generalization). Actually, we minimize the square of the arc length (or its n-dimensional generalization) because it is simpler to calculate and leads to the same result.

Model (3.1) shows a smooth BCC problem with 2 inputs (x, y) and 1 output (Z). In (3.1), $x_{\text{eff}}, y_{\text{eff}}$ are the input values for extreme efficient DMUs, Z_{eff} are their output values, and $x_{\max}, y_{\max}$ are the greatest inputs of all DMUs.

The frontier is described by a polynomial equation, such as $Z = F(x, y) = a + bx + cy + \mathrm{d}x^2 + exy + fy^2 + \ldots$

$$
\begin{aligned}
&\text{Min } \left\{ \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \left[1 + \left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right] \mathrm{d}y \mathrm{d}x \right\} \\
&\text{subject to} \\
&F\left(x_{\text{eff}}, y_{\text{eff}}\right) = Z_{\text{eff}} \ \forall \text{ extreme efficient DMU} \\
&\frac{\partial F}{\partial x}\left(x_{\max}, y_{\max}\right) \geqslant 0 \\
&\frac{\partial F}{\partial y}\left(x_{\max}, y_{\max}\right) \geqslant 0 \\
&\frac{\partial^2 F}{\partial x^2} \leqslant 0, \quad \forall x, y \\
&\frac{\partial^2 F}{\partial y^2} \leqslant 0, \quad \forall x, y.
\end{aligned}
\tag{3.1}
$$

The objective function minimizes the square of the three-dimensional generalization of the arc length. The first constraint ensures that the smooth frontier contains the same efficient DMUs from classic DEA. The following couple of constraints ensure that the output is an increasing function of the inputs. Finally, the last two constraints supposedly ensure the frontier's convexity (this will be explained in Sect. 4.1). These last two restrictions may be very difficult to calculate, and should be replaced with $d, f \ldots \leqslant 0$ [40]. These are stronger constraints, yet considerably easier to calculate.

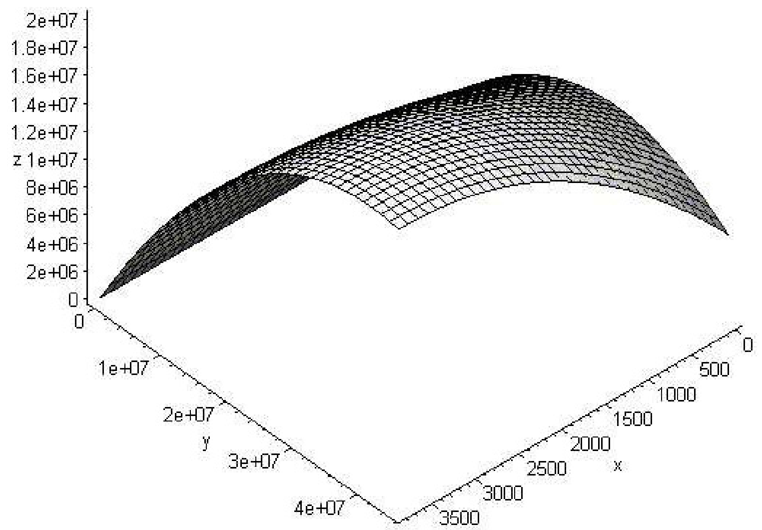

Figure 1. Smooth DEA Frontier [40].

The number of decision variables in the frontier-smoothing problem, *i.e.*, the number of polynomial coefficients, should be greater than the number of equality constraints, *i.e.*, the number of extreme-efficient DMUs [39, 40].

Figure 1 illustrates the smooth frontier for the DEA model in [40], which evaluates Brazilian Airline Companies using two inputs and one output.

Analogously, Model (3.2) shows a smooth BCC problem with 1 input (X) and 2 outputs (z, w). $z_{\text{eff}}, w_{\text{eff}}$ are the output values for extreme efficient DMUs, X_{eff} are their input values, and $z_{\min}, w_{\min}$ are the smallest outputs of all DMUs.

The frontier is described by a polynomial equation, such as $X = H(z, w) = a + bz + cw + dw^2 + ezw + fz^2 + \ldots$

$$\begin{aligned}
&\text{Min } \left\{ \int_{z_{\min}}^{z_{\max}} \int_{w_{\min}}^{w_{\max}} \left[1 + \left(\frac{\partial H}{\partial z} \right)^2 + \left(\frac{\partial H}{\partial w} \right)^2 \right] \mathrm{d}w \mathrm{d}z \right\} \\
&\text{subject to} \\
&H\left(x_{\text{eff}}, y_{\text{eff}}\right) = X_{\text{eff}} \;\forall \text{ extreme efficient DMU} \\
&\frac{\partial H}{\partial z}\left(z_{\min}, w_{\min}\right) \geqslant 0 \\
&\frac{\partial H}{\partial w}\left(z_{\min}, w_{\min}\right) \geqslant 0 \\
&\frac{\partial^2 H}{\partial z^2} \geqslant 0, \quad \forall z, w \\
&\frac{\partial^2 H}{\partial w^2} \geqslant 0, \quad \forall z, w.
\end{aligned} \tag{3.2}$$

In (3.2), the objective function and restrictions have the same purpose as in (3.1). The second derivatives are now positive because the input is a function of the outputs. We may use the theorem of the inverse function to prove that this constraint ensures a concave up frontier [32]. These last two constraints, which supposedly ensure the frontier's convexity (see Sect. 4.1), may also be substituted with $d, f \ldots \geqslant 0$, to enable calculations [40].

4. Proposed corrections and improvements

In this section, we propose corrections and improvements to smooth DEA BCC models. First, we ensure the property of convexity, then positive projections for any data set. Thence, we propose a solution to avoid the efficiency by default BCC distortion. Finally, we present the smooth DEA formulation with the innovations described herein.

4.1. Convexity correction

By definition, smooth DEA models should maintain basic properties from classic DEA. Particularly, in the variable returns to scale model, the frontier's convexity is essential. However, for models with three or more variables, this property may not be entirely present.

First, we must highlight that we are using the term convexity to indicate that the frontier's concave is totally up or down, as in the other works that study smooth DEA, *e.g.*, [32,39].

We may use the following theorems from [19] to ensure the smooth BCC frontier's convexity, in the case of multiple inputs and a single output.

Let $f{:}U \subset R^n \longrightarrow R$ be a function in the class C^2 defined in a convex subgroup that is open in U. f is a concave down function in U if and only if the hessian matrix $D^2 f(\boldsymbol{p})$ is negative-semidefinite for every $\boldsymbol{p} \in U$.

Let A be a symmetrical matrix $n \times n$. A is negative-semidefinite if and only if all of its principal minors of odd order are less or equal to zero and all of its principal minors of even order are greater or equal to zero.

A minor matrix of order k is the determinant of a square submatrix $k \times k$ formed by deleting $n-k$ rows and $n-k$ columns from A. The principal minor of order k is a minor of order k formed by deleting the same $n-k$ rows and columns.

In the case with one input x and one output $z = F(x)$, the Hessian matrix is $[\frac{\mathrm{d}^2 F}{\mathrm{d}x^2}]$. To guarantee convexity in this case, we only need to guarantee $\frac{\mathrm{d}^2 F}{\mathrm{d}x^2} \leqslant 0$. This restriction is already present in the smooth model, thus the two dimensional model has already fully ensured convexity.

In the case with two inputs (x, y) and one output $z = F(x, y)$, the Hessian matrix is:

$$\begin{bmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial y^2} \end{bmatrix}.$$

According to the aforementioned theorems, to ensure convexity in this second case, we must first guarantee that the principal minors of order 1 are less or equal to zero: $\frac{\partial^2 F}{\partial x^2} \leqslant 0$ and $\frac{\partial^2 F}{\partial y^2} \leqslant 0$. This is also present in the traditional smooth models. However, we must also guarantee that the principal minors of order 2 are greater or equal to zero, as in (4.1).

$$\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2 \geqslant 0. \tag{4.1}$$

Smooth models in the found literature, with one output and more than one input, do not include equation (4.1) – or its generalization – among their restrictions. This explains why these frontiers may not be entirely convex.

Nevertheless, equation (4.1) may not be linear. Therefore, to guarantee convexity, we could eliminate all crossed terms from the polynomial equation that describes the frontier, *i.e.*, all crossed terms will be null. With this solution, we have $\frac{\partial^2 F}{\partial x \partial y} = 0$, thus equation (4.1) will always hold, because $\frac{\partial^2 F}{\partial x^2} \leqslant 0$ and $\frac{\partial^2 F}{\partial y^2} \leqslant 0$, in view of restrictions already present in the smooth model, as shown in (3.1).

This condition is simpler, yet stricter than (4.1). This simplification is similar to the solution proposed in [15], to ensure partial convexities, which was previously mentioned. Hence, for the case with 1 output and 2 inputs, the smooth frontier will be described as $Z = F(x, y) = a + bx + cy + \mathrm{d}x^2 + ey^2 + fx^3 + gy^3 + \ldots$

Eliminating all crossed terms also guarantees convexity with a single output and multiple inputs. This is true because the Hessian matrix will be diagonal, as shown below, for the case with n outputs $(x_1, x_2, \ldots, x_n)$ and

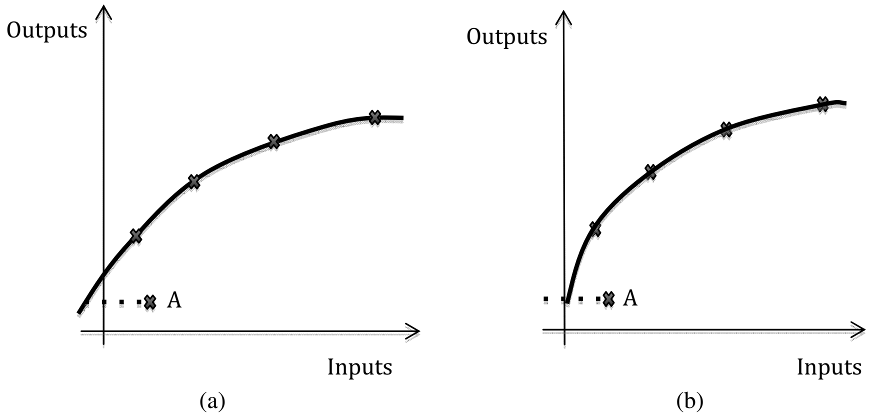

FIGURE 2. Illustrations of negative and non-negative projections.

one output $z = F(x_1, x_2, \ldots, x_n)$.

$$\begin{bmatrix} \frac{\partial^2 F}{\partial {x_1}^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 F}{\partial x_1 \partial x_n} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial {x_2}^2} & \cdots & \frac{\partial^2 F}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_1} & \frac{\partial^2 F}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 F}{\partial {x_n}^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 F}{\partial {x_1}^2} & 0 & \cdots & 0 \\ 0 & \frac{\partial^2 F}{\partial {x_2}^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial^2 F}{\partial {x_n}^2} \end{bmatrix}.$$

According to [19], a diagonal matrix is negative-semidefinite if, and only if all of its elements from the main diagonal are less or equal to zero. This means that the constraint $\frac{\partial^2 F}{\partial x_1^2}, \frac{\partial^2 F}{\partial x_2^2}, \ldots, \frac{\partial^2 F}{\partial x_n^2} \leqslant 0$, already present in smooth models, is necessary and now sufficient to guarantee convexity.

For cases where a single input X is a function of two outputs z and w, similar demonstrations prove that restriction (4.2) is also necessary to guarantee convexity of the smooth frontier.

$$\frac{\partial^2 H}{\partial z^2}\frac{\partial^2 H}{\partial w^2} - \left(\frac{\partial^2 H}{\partial z \partial w}\right)^2 \geqslant 0. \tag{4.2}$$

Since this equation may not be linear either, we may also eliminate all crossed terms from the polynomial equation that describes the frontier, and prove that this ensures its convexity, regardless of the number of outputs z_j.

4.2. Improvement for positive projections

Another relevant characteristic in classic DEA is that DMU targets do not present negative inputs. Consequently, this should be a concern for smooth models, even though it was not ensured in previous papers. Therefore, in this section, we develop a solution that improves smooth DEA models, by securely avoiding this situation.

Figure 2 illustrates a problematic situation in Figure 2a and an adequate situation in Figure 2b. In Figure 2a, the target for DMU A has positive outputs, but negative inputs. This means that the model requires inefficient DMU A to aim at an impractical target, to become efficient. In Figure 2b, the problem was solved and DMU A has now a valid target.

This concern is only present with input orientation, when we reduce inputs from inefficient DMUs until they reach the efficient frontier. With output orientation, we increase outputs from inefficient DMUs until they reach the frontier. Since all outputs are originally positive, they will always be positive.

To avoid projections with negative inputs, we must impose restrictions to ensure that all inputs in the frontier are equal or greater than zero for every output value, as follows.

For the case with n inputs $x_i (i = 1, \ldots, n)$ and a single output z, the equation that describes the frontier is $z_g = F(x_1, x_2, \ldots, x_n) = a + \sum_k a_k x_1^{ci_{1k}} x_2^{ci_{2k}} \ldots x_n^{ci_{nk}}\ \forall ci_{ik}$, where $\sum_{i=1}^{n} ci_{ik} \leqslant g$ and g is the polynomial degree. This may be rewritten as in (4.3).

$$z = F(x_1, x_2, \ldots, x_n) = a + f(x_1, x_2, \ldots, x_n) \tag{4.3}$$

where $f(x_1, x_2, \ldots, x_n)$ has no term independent of x_i, meaning that $f(x_1, x_2, \ldots, x_n) = 0$ if $x_i = 0$, $\forall i = 1, \ldots, n$.

Since $f(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) - a$, therefore $\frac{\partial f}{\partial xi} = \frac{\partial F}{\partial xi}$. Since $\frac{\partial F}{\partial xi} \geqslant 0$ which is a restriction present in the model, $\frac{\partial f}{\partial xi} \geqslant 0$. Therefore, (4.4) always holds

$$f(x_{1A}, x_{2A}, \ldots, x_{nA}) \geqslant f(x_{1B}, x_{2B}, \ldots, x_{nB}) \Leftrightarrow x_{iA} \geqslant x_{iB}\ \forall i = 1, \ldots, n. \tag{4.4}$$

Using (4.3) and (4.4), we may conclude that $f(x_1, x_2, \ldots, x_n) \geqslant 0 \Leftrightarrow x_i \geqslant 0\ \forall i = 1, \ldots, n$. Hence we may impose (4.5) to guarantee that every projection has non negative inputs ($x_i \geqslant 0\ \forall i = 1, \ldots, n$).

$$z = F(x_1, x_2, \ldots, x_n) = a + f(x_1, x_2, \ldots, x_n) \geqslant a. \tag{4.5}$$

To ensure (4.5), we may simply impose the strongest restriction, when $z = z_{\min}$, *i.e.*, $z_{\min} \geqslant a$.

For the case where a single input X is a function of multiple outputs z_j, *i.e.* $X = H(z_1, z_2, \ldots, z_m)$, similar demonstrations show that restriction (4.6) ensures positive projections for every DMU

$$H(z_{1\min}, z_{2\min}, \ldots, z_{m\ \min}) \geqslant 0. \tag{4.6}$$

4.3. Solution to avoid efficiency by default

According to Ali [1], DMU_0 is necessarily efficient if it is the unique DMU with $x_{i0} = \min_{k=1\ldots n} x_{ik}$ or if it is the unique DMU with $y_{j0} = \max_{k=1\ldots n} y_{jk}$, where n is the number of DMUs, x_{ik} is DMU k's value for input i, and y_{jk} is DMU k's value for output j. According to [22], herein we refer to this distortion as efficiency by default, based on the Free Disposal Hull (FDH) approach [16]. This problem affects not only evaluation for falsely efficient DMUs, but also for all DMUs that are on or projected onto the facets that include efficient by default units.

To avoid this distortion, we propose a solution based on [39], which dealt with unfeasibility of the smoothing QP. In this paper, we relax the equality constraints associated with efficient by default DMUs. Instead of having $F(x_{\text{eff}}, y_{\text{eff}}) = Z_{\text{eff}}$ for every BCC efficient DMU, as shown in (3.1), we replace those associated with efficient by default units, by $F(x_{\text{eff}}, y_{\text{eff}}) \geqslant Z_{\text{eff}}\ \forall$ efficient by default DMU. This is shown in the smooth model formulation (4.8).

In other words, efficient by default DMUs will certainly be in the smooth production possibility region, but the new smooth frontier may or may not contain them. If the smooth frontier does contain any of them, they are considered truly efficient, whereas units that are not in the smooth frontier are considered inefficient, or falsely efficient.

It is important to point out that the efficiency by default distortion is only present in the BCC model, and not in the CCR model. In other words, a CCR efficient DMU may never be considered efficient by default. Therefore, if a certain DMU has the smallest of one of the inputs or the greatest of one of the outputs, yet it is CCR efficient, we should not relax its equality constraint in the smooth model.

Moreover, in equation (4.7) we propose a comparative index, similar to the index proposed by Banker and Thrall [6], which compares CCR and BCC efficiencies. Equation (4.7) compares smooth and traditional efficiencies for each DMU. $\%Efficiency_{\text{BCC}}$ is the DMU's efficiency in classic BCC and $\%Efficiency_{\text{Smooth}}$ is the DMU's efficiency in the smooth model proposed in (4.7).

$$Comparative\ Index = \frac{\%Efficiency_{\text{BCC}} - \%Efficiency_{\text{Smooth}}}{\%Efficiency_{\text{BCC}}}. \tag{4.7}$$

Equation (4.7) calculates how much of the efficiency in classic BCC is due to efficiency by default. Therefore, the greater the index, the less efficient the DMU really is. Since the BCC distortion affects other DMUs, including inefficient units, it is possible to use equation (4.7) for all DMUs.

We should mention that the index calculated in (4.7) is not entirely consequence of the efficiency by default distortion. Efficiency values for inefficient DMUs in previous smooth models differ from those in classic DEA, even though such models make no attempt in correcting efficiencies by default. However, because these differences are relatively small, we may consider that the index in (4.7) is reasonably sufficient to measure the degree of efficiency by default in classic BCC.

4.4. Smooth model formulation

In (4.8), we present the smooth model formulation, for cases with multiple inputs x_i, $i = 1, \ldots, n$, and a single output $Z = F(x_1, \ldots, x_n)$, taking into account the contributions from the present work. In (4.8), $x_{i\,\text{eff}}$ $(i = 1, \ldots, n)$ are the values for input i of extreme efficient DMUs, Z_{eff} are their output values, and $x_{i\ \max}$ $(i = 1, \ldots, n)$ are the greatest values for input i of all DMUs.

$$\begin{aligned}
&\text{Min}\left\{\int\limits_{x_{1\ \min}}^{x_{1\ \max}} \ldots \int\limits_{x_{n\ \min}}^{x_{n\ \max}} \sum_{i=1}^{n}\left(\frac{\partial F}{\partial x_i}\right)^2 \mathrm{d}x_n \ldots \mathrm{d}x_1\right\}\\
&\text{subject to}\\
&F\left(x_{1\,\text{eff}}, \ldots, x_{n\,\text{eff}}\right) \geqslant Z_{\text{eff}}\ \forall \text{ efficient by default DMU}\\
&F\left(x_{1\,\text{eff}}, \ldots, x_{n\,\text{eff}}\right) = Z_{\text{eff}}\ \forall \text{ other BCC efficient DMU}\\
&\frac{\partial F}{\partial x_i}\left(x_{1\ \max}, \ldots, x_{n\ \max}\right) \geqslant 0\quad \forall i = 1, \ldots, n\\
&c_1, \ldots, c_n d_1, \ldots, d_n, \ldots \leqslant 0;\quad a \leqslant z_{\min}.
\end{aligned} \tag{4.8}$$

The frontier is described by a polynomial equation with no crossed terms, such as $Z = F(x_1, \ldots, x_n) = a + b_1 x_1 + \ldots + b_n x_n + c_1 x_1^2 + \ldots + c_n x_n^2 + \ldots$, so that the constraint $c_1, c_2, \ldots, d_1, d_2, \ldots \leqslant 0$ in model (4.8) may completely ensure convexity.

The first constraint ensures that efficient by default DMUs belong to the production possibility region, but allows the smooth frontier to contain them or not. The second constraint ensures that the smooth frontier contains the other BCC efficient DMUs. The following couple of constraints ensure that output is an increasing function of inputs, as well as the frontier's convexity. Finally, the last constraint ensures positive targets for all DMUs.

In (4.9), we present the smooth model formulation, for cases with multiple outputs z_j, $j = 1, \ldots, n$, and a single input $X = H(z_1, \ldots, z_m)$, taking into account the contributions from the present work. In (4.9), $z_{j\,\text{eff}}$ $(j = 1, \ldots, m)$ are the values for output j of extreme efficient DMUs, X_{eff} are their input values, and $z_{j\ \min}$

$(j = 1, \ldots, m)$ are the smallest values for output j of all DMUs.

$$\begin{aligned}
&\text{Min } \left\{ \int_{z_{1\ \min}}^{z_{1\ \max}} \cdots \int_{z_{m\ \min}}^{z_{m\ \max}} \sum_{j=1}^{m} \left(\frac{\partial H}{\partial z_j} \right)^2 \mathrm{d}z_m \ldots \mathrm{d}z_1 \right\} \\
&\text{subject to} \\
&H\left(z_{1\,\text{eff}}, \ldots, z_{m\,\text{eff}}\right) \leqslant X_{\text{eff}}\ \forall \text{ DMU efficient by default} \\
&H\left(z_{1\,\text{eff}}, \ldots, z_{m\,\text{eff}}\right) = X_{\text{eff}}\ \forall \text{ other BCC efficient DMU} \\
&\frac{\partial H}{\partial z_j}\left(z_{1\ \min}, \ldots, z_{m\ \min}\right) \geqslant 0 \quad \forall j = 1, \ldots, m \\
&c_1, \ldots, c_m, d_1, \ldots, d_m, \ldots \geqslant 0 \\
&H\left(z_{1\ \min}, \ldots, z_{m\ \min}\right) \geqslant 0.
\end{aligned} \tag{4.9}$$

Model (4.9) is very similar to model (4.8), though the second derivatives are positive in (4.9) because the input is a function of the outputs, as in (3.2). The frontier is now described by a polynomial equation with no crossed terms, such as $X = H\left(z_1, \ldots, z_m\right) = a + b_1 z_1 + \ldots + b_m z_m + c_1 z_1^2 + \ldots + c_m z_m^2 + \ldots$, so that the constraint $c_1, \ldots, c_n, d_1, \ldots, d_n, \ldots \geqslant$ in model (4.9) may completely guarantee convexity.

5. Case study

In this section, we present an application of the model proposed herein, using the 2010 data studied in [23], which evaluates the operational performance of Brazilian airlines. Based on [44], Gomes *et al.* [23] evaluated how airlines use their resources to provide their services. For that, the authors consider the airlines' fleet capacities as input, as done in [15, 35, 38]. As outputs, the authors consider both passenger and cargo transport, *i.e.*, the number of passengers carried, multiplied by the total distance travelled, as well as the total cargo tonnage transported, multiplied by the total distance travelled, for each airline.

Table 1 shows the 2010 input and output data from [23]. To avoid misinterpretation, we should highlight that *Total LA* refers to a single airline called *Total Linhas Aéreas* (Total Airlines, in Portuguese).

As in [23], this paper uses DEA BCC, because there is no presumption of proportionality between inputs and outputs. However, herein we consider output orientation, so that inefficient airlines improve passenger and cargo transport, instead of reducing fleet capacity. Moreover, we normalize data to avoid possible errors.

In this paper, our target is to correct the efficiency by default BCC distortion, thus we use the Smooth BCC model proposed in Section 4. However, before this, we present results for previous smooth BCC models, to illustrate the importance of corrections and improvements proposed herein.

First, Table 2 presents the normalized data, classic BCC efficiency with output orientation, derived from SIAD [3], and results for the smooth BCC model shown in (3.2), calculated with Excel's Solver. Here, x represents input Fleet Capacity, z represents the passenger output and w represents the cargo output. αz and αw are the targets for each output, where $x = H(\alpha z, \alpha w)$. Finally, the smooth output oriented efficiency is $\mathit{Eff}_{\text{out}} = \frac{1}{\alpha}$, in accordance to classical DEA efficiencies [41].

This model does not correct the efficiency by default distortion. In fact, we could observe from Table 2 that all four traditional BCC efficient DMUs remained efficient in the smooth model. The primary objective for such smooth model would therefore be the calculation of unique sets of multipliers, *e.g.*, for a subsequent application of cross efficiency.

The polynomial that describes the frontier was found to be approximately $X = H\left(z, w\right) = 0.0004 + 0.5763z + 0.2765w - 0.0256wz + 0.1723w^2$. Despite not adding the restriction for positive projection shown in (4.6), there were no negative targets in the input oriented model (as noted in Sect. 4.2, only input oriented models may present negative projections).

TABLE 1. Fleet Capacity and passenger and cargo transport data for Brazilian airlines in 2010.

Airlines	Fleet capacity	Pass km	Ton km
Abaeté	11	1 933 487	160 613
Avianca	789	1 856 243 438	171 287 586
Azul	1041	4 306 850 206	357 499 774
Gol	9190	31 402 871 742	2 917 360 340
Meta	27	20 088 021	1 787 191
NHT	40	10 424 543	873 769
Noar	10	5 004 498	1 425 654
Pantanal	205	242 945 923	20 822 599
Passaredo	215	428 592 915	37 400 674
Puma	59	86 803 465	45 216 390
Rico	182	633 899	51 766 818
Sete	29	16 794 473	1 618 528
Sol	7	2 244 622	185 609
TAM	15 114	51 712 453 009	5 010 977 416
Team	17	3 084 331	237 686
Total LA	542	61 991 710	52 739 563
Trip	824	1 547 564 238	136 878 579
Webjet	1322	4 130 647 241	360 627 713

TABLE 2. Normalized data, classic BCC efficiencies, and results from smooth model (3.2).

DMUs	Original Normalized Data			Eff BCC	Results for Smooth BCC (3.2)		
	x: Fleet	z: Pass km	w: Ton km	Output	αz: target z	αw: target w	Eff Output
Abaeté	0.00073	0.00004	0.00003	10%	0.0004	0.0003	10%
Avianca	0.05220	0.03590	0.03418	61%	0.0611	0.0582	59%
Azul	0.06888	0.08328	0.07134	100%	0.0833	0.0713	100%
Gol	0.60805	0.60726	0.58219	99%	0.6541	0.6271	93%
Meta	0.00179	0.00039	0.00036	24%	0.0016	0.0015	24%
NHT	0.00265	0.00020	0.00017	7%	0.0027	0.0024	7%
Noar	0.00066	0.00010	0.00028	58%	0.0002	0.0005	58%
Pantanal	0.01356	0.00470	0.00416	30%	0.0160	0.0141	29%
Passaredo	0.01423	0.00829	0.00746	50%	0.0167	0.0150	50%
Puma	0.00390	0.00168	0.00902	100%	0.0017	0.0090	100%
Rico	0.01204	0.00001	0.01033	60%	0.0000	0.0409	25%
Sete	0.00192	0.00032	0.00032	19%	0.0018	0.0017	19%
Sol	0.00046	0.00004	0.00004	100%	0.0000	0.0000	100%
TAM	1.00000	1.00000	1.00000	100%	1.0000	1.0000	100%
Team	0.00112	0.00006	0.00005	7%	0.0009	0.0007	7%
Total LA	0.03586	0.00120	0.01052	26%	0.0112	0.0987	11%
Trip	0.05452	0.02993	0.02732	47%	0.0647	0.0590	46%
Webjet	0.08747	0.07988	0.07197	80%	0.1039	0.0936	77%

However, this frontier is not entirely convex (in this case, concave), because the condition from equation (4.2) is not followed. In other words, $\frac{\partial^2 H}{\partial z^2}\frac{\partial^2 H}{\partial w^2} \leqslant \left(\frac{\partial^2 H}{\partial z \partial w}\right)^2$ for the frontier derived from model (3.2). Consequently, although outputs increase with inputs, these increments are not always increasing (or constant), as they should be. This problem only happens with regard to output z, as shown in Table 3. This table presents output z in increasing order and $\partial H/\partial z$ for each DMU. DMUs with greater output z, though with a smaller increment $\partial H/\partial z$, compared to the previous DMU, are in bold.

TABLE 3. Increments with regard to output z, evidencing lack of convexity in model (3.2).

DMU	z: Pass.km	$\partial H/\partial z$
Rico	0.00001	0.5761
Abaeté	0.00004	0.5763
Sol	0.00004	0.5763
Team	0.00006	0.5763
Noar	0.00010	0.5763
NHT	0.00020	0.5763
Sete	0.00032	0.5763
Meta	0.00039	0.5763
Total LA	**0.00120**	**0.5761**
Puma	0.00168	0.5761
Pantanal	0.00470	0.5762
Passaredo	**0.00829**	**0.5761**
Trip	**0.02993**	**0.5756**
Avianca	**0.03590**	**0.5755**
Webjet	**0.07988**	**0.5745**
Azul	0.08328	0.5745
Gol	**0.60726**	**0.5614**
TAM	**1.00000**	**0.5508**

TABLE 4. Efficiency values for smooth models that eliminate efficiencies by default.

DMU	Results for Smooth Model A		Results for Smooth Model B	
	Eff Output	Eff Input	Eff Output	Eff Input
Abaeté	2%	–143%	3%	–51%
Avianca	60%	61%	59%	59%
Azul	100%	100%	100%	100%
Gol	Impossible	53%	87%	87%
Meta	12%	–40%	15%	–4%
NHT	5%	–34%	6%	–9%
Noar	10%	–136%	15%	–37%
Pantanal	29%	23%	28%	26%
Passaredo	49%	45%	48%	47%
Puma	100%	100%	100%	100%
Rico	36%	31%	32%	30%
Sete	10%	–40%	12%	–6%
Sol	3%	–224%	4%	–80%
TAM	Impossible	25%	89%	89%
Team	2%	–91%	3%	–32%
Total LA	15%	12%	13%	12%
Trip	47%	47%	47%	46%
Webjet	76%	77%	77%	77%

Such problem becomes much more evident when we use smooth models to avoid efficiencies by default, without introducing corrections and improvements proposed herein. Table 4 shows results for two smooth models that eliminate efficiencies by default, *i.e.*, that relax the equality constraint for DMUs *Sol*, with the smallest input value, and *TAM*, with the greatest value for both outputs. For illustrative purposes, we also present input oriented efficiency results, where $Eff_{in} = \frac{H(z,w)}{x}$, in accordance to classical DEA efficiencies [41].

Despite eliminating efficiencies by default, as proposed in Section 4.3, model A does not present the convexity correction proposed in Section 4.1 or the positive projections improvement proposed in Section 4.2. In addition

TABLE 5. Classic BCC efficiencies and results from smooth model (4.9).

DMU	Eff BCC	Results for Smooth BCC (4.9)				
	Output	αz: target z	αw: target w	Eff Output	X: target x	Eff Input
Abaeté	10%	0.0009	0.0008	4%	0.00001	2%
Avianca	61%	0.0608	0.0579	59%	0.03082	59%
Azul	100%	0.0833	0.0713	100%	0.06888	100%
Gol	99%	0.7060	0.6768	86%	0.52301	86%
Meta	24%	0.0021	0.0020	18%	0.00031	17%
NHT	7%	0.0032	0.0028	6%	0.00015	6%
Noar	58%	0.0004	0.0013	22%	0.00013	20%
Pantanal	30%	0.0162	0.0144	29%	0.00391	29%
Passaredo	50%	0.0169	0.0152	49%	0.00696	49%
Puma	100%	0.0017	0.0090	100%	0.00390	100%
Rico	60%	0.0000	0.0361	29%	0.00344	29%
Sete	19%	0.0022	0.0022	15%	0.00027	14%
Sol	100%	0.0006	0.0005	7%	0.00002	4%
TAM	100%	1.1428	1.1428	88%	0.87506	88%
Team	7%	0.0014	0.0011	4%	0.00003	3%
Total LA	26%	0.0103	0.0907	12%	0.00415	12%
Trip	47%	0.0645	0.0588	46%	0.02530	46%
Webjet	80%	0.1039	0.0936	77%	0.06724	77%

to the negative efficiencies with input orientation, it is impossible to calculate two output oriented efficiencies for this model, because there is no α value, where $x = H(\alpha z, \alpha w)$ in such cases. This is consequence of the frontier's lack of convexity.

Model B presents the convexity correction proposed in Section 4.1, *i.e.*, eliminates all crossed terms from the frontier's polynomial, but does not include restriction (4.6). Thus, it is possible to calculate all output oriented efficiency values, and they are all in the interval [0,1]. Furthermore, the problem shown in Table 3, which is consequence of the frontier's lack of convexity, does not occur. However when considering input orientation, certain efficiency values are negative, as shown in Table 4, *i.e.*, those DMUs have negative input oriented targets. Although our case study is based on output orientation, the DEA frontier should not present negative input values, in the dataset interval.

Therefore, Table 5 presents results for the smooth model proposed in this paper, in (4.9). We present both output and input orientation, to show that all efficiency values are positive, even with input orientation.

The polynomial that describes the frontier was found to be approximately $X = H(z, w) = -0.00002 + 0.54122z + 0.33385w$. This frontier is entirely convex (in this case, concave), so it does not present the problem shown in Table 3.

Moreover, the efficiency by default distortion, highlighted in [23], is corrected in model (4.9), since *Sol* (smallest airline) is only 7% efficient and *TAM* (largest airline) is 88% efficient. Hence, the comparative index (4.7) is 93% for *Sol* and 12% for *TAM*, which shows how much of classic BCC efficiency is consequence of the classic model's distortion. In other words, despite having 4 efficient DMUs in classic BCC, only *Azul* (medium-sized) and *Puma* (small) are truly efficient.

Other DMUs also benefit from the efficiency by default distortion of classic BCC, despite not being efficient, because entire frontier segments are closer to the DMUs, due to such distortion. Thus, we may calculate the comparative index (4.7) for other DMUs, such as *Noar* and *Abaeté*, whose input and output values are very similar to *Sol*'s. In fact, *Noar*'s comparative index is 62%, the second greatest, followed by *Abaeté*, with 59%. Similarly, input and output values for *Gol* are close to *TAM*'s, and *Gol*'s comparative index is 13%.

Comparing Tables 2 and 5, we may observe that the efficiency results from model (3.2) are much closer to classic BCC than those from model (4.9). This is expected, since the model's target is no longer plain proximity with the original frontier, but also the correction of one of its distortions, therefore diverging from the classic model.

As a result, the smooth model proposed in this paper allows a more exact interpretation of the airlines' evaluation, because it eliminates the efficiency by default distortion, still maintaining essential properties from classic DEA.

6. Conclusions

The Smooth DEA technique solves classic DEA problems, particularly multiple optimal solutions for multipliers and Pareto inefficient regions, maintaining basic DEA properties. Although convexity is essential in the BCC approach, this property was not ensured in previous smooth BCC models. Therefore, the main contribution of this paper was to correct smooth BCC models, and ensure the frontier's convexity.

This paper also proposed improvements to smooth DEA models, so that all DMUs' projections are non-negative, as in classic DEA. Consequently, our improved model also ensures non-negative efficiency values, for any orientation. With the corrections and improvements proposed in this paper, smooth models become more robust and coherent.

Moreover, this paper addresses the BCC distortion called efficiency by default [22], in which DMUs that have the smallest of any input or the greatest of any output are necessarily efficient [1]. Based on the corrected and improved smooth BCC model proposed herein, we present a solution for the efficiency by default distortion, thus allowing for more precise evaluations.

Finally, we use the smooth model proposed in this paper to evaluate operational performance of Brazilian airline companies in 2010. We also apply smooth models from the literature, as well as intermediate smooth models, *i.e.*, models with some of the proposed improvements and corrections, to illustrate the practical effect of each contribution.

We verified that the frontier from our smooth model was entirely convex, whereas the frontiers from previous smooth models were not. We also observed that previous smooth models should not be used to eliminate the efficiency by default distortion, because they provide unrealistic results. Thus, compared to classic BCC and to previous smooth models, our model was the only one that avoided the BCC distortion.

Future works may develop smooth models for the multiple input and output case, with the improvements proposed herein. Other studies may also simplify smooth models by transforming the Quadratic Problem shown herein into a Linear Problem. Preliminary results for this simplification are presented in [8].

Acknowledgements. We acknowledge the financial support of CNPq (Brazilian Ministry of Science and Technology).

References

[1] A.I. Ali, Streamlined computation for Data Envelopment Analysis. *Eur. J. Oper. Res.* **64** (1993) 61–67.

[2] P. Andersen and N.C. Petersen, A procedure for ranking efficient units in Data Envelopment Analysis. *Manage. Sci.* **39** (1993) 1261–1264.

[3] L. Angulo Meza, L. Biondi Neto, J.C.C.B. Soares de Mello and E.G. Gomes, ISYDS – Integrated System for Decision Support (SIAD): a Software Package for Data Envelopment Analysis Model. *Pesq. Operac.* **25** (2005) 493–503.

[4] J.V.G. Avellar, A.Z. Milioni and T.N. Rabello, Spherical frontier DEA model based on a constant sum of inputs. *J. Oper. Res. Soc.* **58** (2007) 1246–1251.

[5] J.V.G. Avellar, A.Z. Milioni, T.N. Rabello, and H.P. Simão, On the redistribution of existing inputs using the spherical frontier DEA model. *Pesquisa Oper.* **30** (2010) 1–16.

[6] R.D. Banker and R.M. Thrall, Estimation of returns-to-scale using Data Envelopment Analysis. *Eur. J. Oper. Res.* **62** (1992) 74–84.

[7] R.D. Banker, A. Charnes and W.W. Cooper, Some models for estimating technical and scale inefficiencies in Data Envelopment Analysis. *Manage. Sci.* **30** (1984) 1078–1092.

[8] L.C. Brandao and J.C.C.B. Soares de Mello, Smooth Data Envelopment Analysis – a new model. *Annals of XLVI SBPO*, Salvador/BA, Brazil (2014) 921–931.

[9] L.C. Brandao, F.V.S. Andrade and J.C.C.B. Soares de Mello, UEFA Euro efficiency evaluation based on market expectations. *Proc. of 4th International Conference on Mathematics in Sport.* KULeuven, Leuven (2013) 32–37.

[10] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units. *Eur. J. Oper. Res.* **2** (1978) 429–444.

[11] A. Charnes, W.W. Cooper, B.E. Golany, L. Seiford and J. Stutz, Foundations of Data Envelopment Analysis for Pareto-Koopmans efficient empirical production functions. *J. Econom.* **30** (1985) 91–107.

[12] T.J. Coelli, D.S. Prasada Rao, G.E. Battese, *An Introduction to Efficiency and Productivity Analysis,* 1st edition. Kluwer Academic Publishers, London (1998).

[13] W.W. Cooper, K.S. Park, J.T.P. Ciurana, Marginal rates and elasticities of substitution with additive models in DEA. *J. Prod. Anal.* 13 (2000) 105–123.

[14] W.W. Cooper, J.L. Ruiz, I. Sirvent, Choosing weights from alternative optimal solutions of dual multiplier models in DEA. *Eur. J. Oper. Res.* **180** (2007) 443–458.

[15] T.C.V.D. Correia, J.C.C.B. Soares de Mello and L. Angulo Meza, Assessment of Brazilian airlines technical efficiency: a study using data envelopment analysis and fuzzy sets. *Producao* **21** (2011) 676–683.

[16] D. Deprins, L. Simar and H. Tulkens, Measuring labour efficiency in post offices, in: *The Performance of Public Enterprises: Concepts and Measurements*, edited by M. Marchand, P. Petiseau and H. Tulkens. North-Holland, Amsterdam (1984) 243–267.

[17] J.R. Doyle and R.H. Green, Efficiency and Cross-efficiency in DEA: Derivations, Meanings and Uses. *Eur. J. Oper. Res.* **45** (1994) 567–578.

[18] J.H. Dulá, Computations in DEA. *Pesquisa Oper.* **22** (2002) 165–182.

[19] C.H. Edwards and D.E. Penney, *Multivariable Calculus*, 6th edition. Prentice Hall PTR (2002).

[20] E.G. Gomes and M.P.E. Lins, Modelling undesirable outputs with zero sum gains data envelopment analysis models. *J. Oper. Res. Soc.* **59** (2008) 616–623.

[21] E.G. Gomes, J.C.C.B. Soares de Mello and M.P.E. Lins, Redistribuição de inputs e outputs em modelos de análise de envoltória de dados com ganhos de soma zero. *Pesquisa Oper.* **24** (2004) 269–284.

[22] E.G. Gomes, U.G.P. Abreu, J.C.C.B. Soares de Mello, T.B. de Carvalho and S. de ZEN, DEA performance evaluation of livestock systems in Brazil. In: *Data Envelopment Analysis and Its Applications to Management*, edited by V. Charles and M. Kumar, 1st edition. Cambridge Scholars Publishing, Newcastle upon Tyne (2012) 224–238.

[23] S.F. Gomes Júnior, A.P.S. Rubem, J.C.C.B. Soares de Mello and L. Angulo Meza, Evaluation of Brazilian airlines nonradial efficiencies and targets using an alternative DEA approach. *Intl. Trans. Oper. Res.* **00** (2015) 1–21.

[24] M.C. Gonzalez-Araya, *Projeções Não Radiais em regiões fortemente eficientes da fronteira DEA– Algoritmos e aplicações*, Tese de Doutorado, COPPE/UFRJ, Rio de Janeiro (2003).

[25] E.C. Guedes, A.Z. Milioni, J.V.G. Avellar and R.C. Silva, Adjusted spherical frontier model: allocating input via parametric DEA. *J. Oper. Res. Soc.* **63** (2012) 406–417.

[26] L. Liang, J. Wu, W.D. Cook and J. Zhu, Alternative secondary goals in DEA cross-efficiency evaluation. *Int. J. Prod. Econ.* **113** (2008) 1025–1030.

[27] L. Liang, J. Wu, W.D. Cook and J. Zhu, The DEA Game Cross-Efficiency Model and Its Nash Equilibrium. *Oper. Res.* **56** (2008) 1278–1288.

[28] M.P.E. Lins, E.G. Gomes, J.C.C.B. Soares de Mello and A.J.R. Soares de Mello, Olympic ranking based on a Zero Sum Gains DEA model. *Eur. J. Oper. Res.* **148** (2003) 85–95.

[29] A.Z. Milioni and L.B. Alves, Ten years of research on parametric data envelopment analysis. *Pesquisa Oper.* **33** (2013) 89–104.

[30] A.Z. Milioni, J.V.G. Avellar, E.G. Gomes and J.C.C.B Soares de Mello, An ellipsoidal frontier model: allocating input via parametric DEA. *Eur. J. Oper. Res.* **209** (2011) 113–121.

[31] A.Z. Milioni, J.V.G. Avellar, T.N. Rabello and G.M. Freitas, Hyperbolic frontier model: a parametric DEA approach for the distribution of a total fixed input. *J. Oper. Res. Soc.* **62** (2011) 1029–1037.

[32] F.B. Nacif, J.C.C.B. Soares de Mello and L. Angulo Meza, Choosing weights in optimal solutions for DEA-BCC models by means of a n-dimensional smooth frontier. *Pesquisa Oper.* **29** (2009) 623–642.

[33] O.P. Olesen and N.C. Petersen, Indicators of ill-conditioned data sets and model misspecifications in Data Envelopment Analysis: an extended facet approach. *Manage. Sci.* **42** (1996) 205–219.

[34] E.R. Pereira and J.C.C.B. Soares de Mello, Smoothed frontier to determine a single set of weights in CCR models. *Production* **25** (2015) 585–597.

[35] E.R. Pereira, M.C.C. Chaves and J.C.C.B. Soares de Mello, Evaluation of efficiency of Brazilian airlines using the MCDEA-TRIMAP model. *Proc. of the 2nd Intl. Conf. on Oper. Res. and Enterprise Syst.* Barcelona, Spain (2013) 213–221.

[36] D. Rosen, C. Schaffnit and J. Paradi, Marginal Rates and Two-dimensional Level Curves in DEA. *J. Prod. Anal.* **9** (1998) 202–232.

[37] R.C. Silva and A.Z. Milioni, The adjusted spherical frontier model with weight restrictions. *Eur. J. Oper. Res.* **220** (2012) 729–735.

[38] J.Q. Silveira, J.C.C.B. Soares de Mello and L. Angulo-Meza, Brazilian airlines efficiency evaluation using a data envelopment analysis (DEA) and multiobjective linear programming hybrid model. *Ingeniare* **20** (2012) 331–342.

[39] J.C.C.B. Soares de Mello, M.P.E. Lins and E.G. Gomes, Construction of a smoothed DEA Frontier. *Pesquisa Oper.* **22** (2002) 183–201.

[40] J.C.C.B. Soares de Mello, E.G. Gomes, L. Biondi Neto and M.P.E. Lins, Suavização da fronteira DEA: o caso BCC tridimensional. *Investig. Oper.* **24** (2004) 89–107.

[41] J.C.C.B. Soares de Mello, L. Angulo Meza, E.G. Gomes and L. Biondi Neto, Curso de Análise de Envoltória de Dados. *Annals of XXXVII SBPO*, Gramado/RS, Brazil (2005) 2520–2547.

[42] J.C.C.B. Soares de Mello, L. Angulo Meza and B.P. Branco da Silva, A ranking for the Olympic Games with unitary input DEA models. *IMA J. Manag. Math.* **20** (2009) 201–211.

[43] J.C.C.B. Soares de Mello, L. Angulo Meza, J.Q. Silveira and E.G. Gomes, About negative efficiencies in cross evaluation BCC input oriented models. *Eur. J. Oper. Res.* **229** (2013) 732–737.

[44] M. Schefzyck, Operational performance of airlines: an extension of traditional measurement paradigms. *Strat. Manage. J.* **14** (1993) 301–317.