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COMPUTATIONAL ANALYSIS OF MULTI-SERVER DISCRETE-TIME QUEUEING SYSTEM WITH BALKING, RENEGING AND SYNCHRONOUS VACATIONS

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Abstract. This paper proposes a discrete-time multi-server queue with multiple synchronous vacations under balking and reneging. Arriving customers decide whether to join the system or balk on the basis of some state-dependent joining/balking probabilities, and renege according to a geometric distribution when servers are busy. The servers take a vacation together if there are no customers in the system at a service completion instant. When the servers are on vacation, an arriving customer activates an impatience timer which is geometrically distributed. The inter-arrival times, service times and vacation times are assumed to be independent and geometrically distributed. We obtain closed-form expressions and develop a computational algorithm for calculating the steady-state probabilities. Specifically, we establish the application of the proposed framework in analyzing a multi-server queueing system with synchronous vacation under balking and reneging. Applications of such models can be found in a wide variety of real-time systems including call centers, computer and communication systems, cloud computing, quality control and maintenance in industrial establishments. We develop a cost model to determine the optimal service rate. Various performance measures and numerical examples are sketched out to demonstrate the impact of the proposed method. Some special cases of the model have also been discussed. Finally, we show that in the limiting case the results converge to the corresponding continuous-time counterparts.

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1. Introduction

There is an emerging trend to analyze queueing systems from an economic point of view to deal the customers' dissatisfaction for waiting and their desire for service. When this dissatisfaction gets adequately firm and customers depart before being served, the affected service providers must take measure to minimize the congestion to an extent that customers can tolerate. From business point of view, customer impatience has a very negative impact as the firms lose their potential customers, which affects the business as a whole. The customers may

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have options to balk or join the queue. Service providers may tolerate a certain amount of balking if it provides to a higher level of customer satisfaction.

Customers' impatience in queueing systems can be observed in many real-time systems, such as call centers, communication networks and web services, telecommunication applications, hospital emergency situations, waiting lines at airports, production-inventory systems, public offices, and many other associated areas. Firstly, it is the customer that takes the decision to abandon hastily, most obvious example is a call centre. Secondly, the system may decide to take out customers from the queue. The first scenario may be evident in audio or video streaming applications, when packets affinity to such applications would not reach at their next destination, they are removed from the buffer. The second scenario may be of great significance in inventory management [6]. Multi-server queueing models with impatient customers have been investigated by a number of researchers [1, 2, 11, 17, 20, 24, 27].

Queueing models with impatient customers and server vacations have been studied due to their suitability and applicability. Dequan *et al.* [30] investigated the influence of balking, reneging and server vacations in a single server queueing system. Yue *et al.* [31] and Selvaraju and Goswami [23] studied customers' impatience with working vacations in an Markovian queue. Goswami [16] investigated $GI/M/1/N$ queue with balking, reneging and working vacations. Altman and Yechaili [3, 4] analyzed queueing systems with impatient customers when the servers are on vacation and unavailable for service, respectively. Economou *et al.* [12] considered an optimal balking strategies with general service and vacation times in a single-server queues. Wang *et al.* [26] studies the single server machine repair problem with working vacation. Analysis of a multi-server queue with markovian arrivals in which a group of servers take a simultaneous phase type vacation has been reported in [7]. Yue and Sun [28] considered multi-server queuing system with balking, reneging and synchronous vacations of partial servers. They derived the distributions of the conditional waiting time of the customers who join the system when all the available servers are busy and finally get service. Yue and Yue [29] analyzed an $M/M/c/N$ queueing system with balking, reneging and synchronous vacations of servers together. Ke and Wu [18] presented a multi-server machine repair model with standbys under a synchronous multiple vacations. Several authors have investigated discrete-time multi-server queues under early arrival system (EAS) and late arrival system with delayed access (LAS-DA) [5, 8–10, 13]. Lozano and Moreno [21] studied a discrete-time single-server queue with balking. Research work on balking and reneging in discrete-time queuing systems with vacations comes out to be a recent endeavor. Readers may refer to [15, 25] for the variations and extensions of vacation models. Recently, the analysis of $Geo/Geo/m$ queue with balking has been studied in Goswami [14].

Most of the literature on customers' impatience focus on the continuous-time models. In comparison to the continuous-time case, the discrete-time queues with balking, reneging and vacations received much less attention in literature. In reality, the discrete-time multi-server queueing system with balking and reneging are more suitable for the design and analysis of slotted time communication systems. One can find the continuous-time result from a discrete-time queue in the limiting case but converse is not true. Analysis and modelling of multi-server discrete-time queue with balking and renenging is more engrossed and completely varied than the equivalent continuous-time counterpart. In various business, losses in income due to balking and renenging are huge and hence need to be examined in suitable context. To the best of our knowledge, the discrete-time $Geo/Geo/m$ queue with balking, reneging and synchronous vacations has never been studied. In this paper, we study the discrete-time multi-server queue with or without multiple synchronous vacations under balking and reneging.

The rest of the paper is organized as follows. In Section 2, we present descriptions of the discrete-time multi-server queueing model with balking, renenging and multiple synchronous vacations for LAS-DA. Section 3 gives the steady-state analysis of the model. Section 4 presents algorithm for computation of steady-state probabilities. Section 5 provides various performance measures. Section 6 discusses some special cases of our proposed model. Section 7 presents numerical results to establish the impact of different parameters on system performances. Section 8 Concludes the paper. Finally, a relationship between our model and its continuous counterpart is given in the appendix.

2. Description of the model

We study a multi-server discrete-time queue with balking, reneging and synchronous vacations under late arrival system with delayed access (LAS-DA). In LAS-DA, potential arrivals take place in the interval $(t-, t)$ and potential departures take place in the interval $(t, t+)$. We denote $\bar{x} = 1 - x$ for any real number $x \in [0, 1]$. The assumptions of our proposed queueing system are as follows:

- The inter-arrival times of successive arrivals are independent and geometrically distributed with probability mass function (p.m.f.)

$$a_i = \bar{\lambda}^{i-1}\lambda,\ i \geq 1,\ 0 < \lambda < 1.$$

- There are m servers and service times are assumed to be independent and geometrically distributed with p.m.f.

$$s_i = \bar{\mu}^{i-1}\mu,\ i \geq 1,\ 0 < \mu < 1.$$

  The probability that j services are completed given that there are i busy servers is given by

$$c(j|i) = \binom{i}{j}\mu^j \bar{\mu}^{i-j},\ \text{for } i = 0, 1, 2, \ldots, m,\ j = 0, 1, \ldots, i. \tag{2.1}$$

- All the m servers take a vacation synchronously when they are idle. At a vacation completion epoch, all the m servers take another synchronous vacation if there are no customers; otherwise, the servers start serving the customers. The vacation times V are independent and geometrically distributed with common p.m.f.

$$P(V = i) = \bar{\phi}^{i-1}\phi,\ i \geq 1,\ 0 < \phi < 1.$$

- If all the servers are busy, an arriving customer may either decide to join the system or balk. Let b_i be the probability that customer joins the system (balk with probability $1 - b_i$), when there are $i \geq m$ customers ahead in the system. We assume that $0 \leq b_i \leq 1,\ m \leq i \leq N - 1$. The arrival rate is given by

$$\lambda_i = \begin{cases} \lambda, & 1 \leq i \leq m-1, \\ \lambda b_i, & m \leq i \leq N-1, \\ 0, & i = N. \end{cases}$$

  If the servers are on synchronous vacations, an arriving customer in the system may join {or balk} the system with probability b_i $\{1 - b_i\}$, when there are i $(i \geq 0)$ customers in the system. In this case, the arrival rate is given by

$$\lambda_i = \begin{cases} \lambda b_i, & 0 \leq i \leq N-1, \text{ with } b_0{=}1, \\ 0, & i = N. \end{cases}.$$

- After joining the system, each reneging customer remains a certain length of time before turning impatient and departing from the system. The reneging time is assumed to follow geometrical distribution with common p.m.f. $P(r = i) = \bar{\alpha}^{i-1}\alpha,\ i \geq 1,\ 0 < \alpha < 1$. Let i and n represent the number of busy servers and the number of customers in the system, respectively. The average reneging rate is $\alpha_{n-i} = (n-i)\alpha$, $i{+}1 \leq n \leq N$, $i = 0, m$; as the arrival and departure of impatient customers without service are independent.
- The inter-arrival times, vacation times and service times are mutually independent. The service discipline is presumed to be first-come, first-served (FCFS).

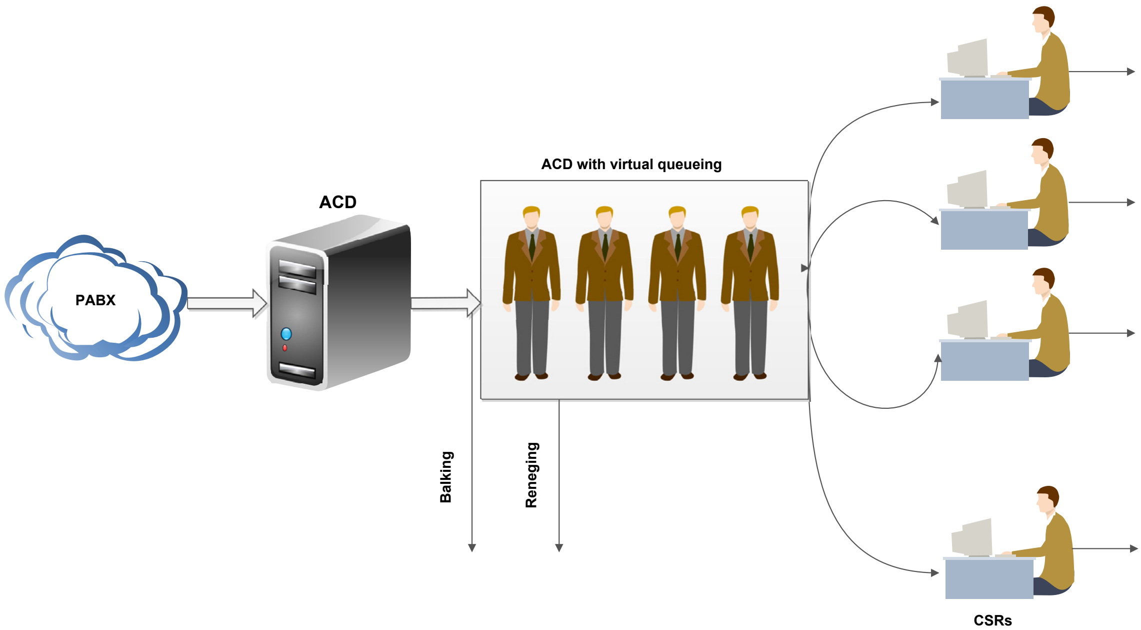

FIGURE 1. Schematic representation of a Call Center.

Practical application of the model

Figure 1 depicts a schematic representation of a call center. If an incoming call finds all trunk lines filled, it receives a busy signal and gets blocked from entering the system [19, 32]. Otherwise, it is either connected or it balks. Let the number of waiting slots at the Automatic Call Distributor (ACD) be N and the number of Customer Service Representatives (CSR) be m. Note that there are $N+m$ trunk lines at the Private Automatic Branch Exchange (PABX). Upon entering the system, if an incoming call finds at least one CSR, it gets service immediately. Otherwise it waits in the queue. A call in the queue may get impatient and abandon (renege) the system by releasing its trunk line. On completion of service, a call releases both the trunk line and the CSR. The system follows FCFS discipline. We consider balking in call centers to be state-dependent, where a calling customer gets information about the state of the system. Based on this information it decides to join the queue or to leave the system. A waiting customer may renege after a random time, if the service has not begun. The customer willingness to wait may depend on various factors. The CSRs take synchronous vacations whenever the system has no calls.

3. Analysis of the model

Let us denote the number of customers present in the system at time $t-$ as $N_s(t-)$ and

$$\zeta(t-) \;=\begin{cases} 0, & \text{if servers are on vacation,}\\ 1, & \text{if servers are in busy period.}\end{cases}$$

Let us define the joint probability as $P_{i,j}(t-) = P\Big\{N_s(t-) = i, \zeta(t-) = j\Big\}, \;\; j \le i \le N, \; j = 0, 1.$

Relating the state of the system at two consecutive time epochs $t-$ and $(t+1)-$, we get the following equations. We use the symbol t instead of $t-$ for the sake of simplicity.

$$P_{0,0}(t+1) = \sum_{j=0}^{m} v_{j,m} c(j|j) P_{j,1}(t) + w_{m+1,m} c(m|m) P_{m+1,1}(t) + w_{1,0} P_{1,0}(t), \tag{3.1}$$

$$P_{k,0}(t+1) = \bar{\phi}\Big(u_{k-1,0}P_{k-1,0}(t) + v_{k,0}P_{k,0}(t) + w_{k+1,0}P_{k+1,0}(t)\Big),\ 1 \le k \le N-1, \tag{3.2}$$

$$P_{N,0}(t+1) = \bar{\phi}\Big(v_{N,0}P_{N,0}(t) + u_{N-1,0}P_{N-1,0}(t)\Big), \tag{3.3}$$

$$\begin{aligned} P_{1,1}(t+1) = {} & \sum_{j=1}^{m+1} v_{j,m} c(j-1|j) P_{j,1}(t) + \sum_{j=1}^{m} u_{j,m} c(j|j) P_{j,1}(t) + \sum_{j=m+1}^{m+2} w_{j,m} c(j-2|m) P_{j,1}(t) \\ & + \phi\Big(u_{0,0}P_{0,0}(t) + v_{1,0}P_{1,0}(t) + w_{2,0}P_{2,0}(t)\Big), \end{aligned} \tag{3.4}$$

$$\begin{aligned} P_{k,1}(t+1) = {} & \sum_{j=k}^{m+k} v_{j,m} c(j-k|j) P_{j,1}(t) + \sum_{\max j=\{m+1,k+1\}}^{m+k+1} w_{j,m} c(j-k-1|m) P_{j,1}(t) \\ & + \sum_{j=k-1}^{m+k-1} u_{j,m} c(j-k+1|j) P_{j,1}(t) + \phi\Big(u_{k-1,0}P_{k-1,0}(t) + v_{k,0}P_{k,0}(t) + w_{k+1,0}P_{k+1,0}(t)\Big), \\ & 2 \le k \le N-m-1, \end{aligned} \tag{3.5}$$

$$\begin{aligned} P_{k,1}(t+1) = {} & \sum_{j=k}^{N-1} v_{j,m} c(j-k|m) P_{j,1}(t) + \sum_{j=k+1}^{N-1} w_{j,m} c(j-k-1|m) P_{j,1}(t) + v_{N,m} c(N-k|m) P_{N,1}(t) \\ & + \sum_{j=k-1}^{N-1} u_{j,m} c(j-k+1|m) P_{j,1}(t) + w_{N,m} c(N-k-1|m) P_{N,1}(t) \\ & + \phi\Big(u_{k-1,0}P_{k-1,0}(t) + v_{k,0}P_{k,0}(t) + w_{k+1,0}P_{k+1,0}(t)\Big),\ N-m \le k \le N-1, \end{aligned} \tag{3.6}$$

$$P_{N,1}(t+1) = v_{N,m} c(0|m) P_{N,1}(t) + u_{N-1,m} c(0|m) P_{N-1,1}(t) + \phi\Big(v_{N,0}P_{N,0}(t) + u_{N-1,0}P_{N-1,0}(t)\Big), \tag{3.7}$$

where

$$u_{i,j} = \begin{cases} \lambda : i = j = 0;\ 1 \le i \le m-1,\ j = m, \\ \lambda_m : i = j = m, \\ \lambda_i \bar{\alpha}_{i-j} : j+1 \le i \le N-1,\ j = 0, m, \end{cases}$$

$$v_{i,j} = \begin{cases} \bar{\lambda} : i = j = 0;\ 1 \le i \le m-1,\ j = m, \\ \bar{\lambda}_m : i = j = m, \\ \bar{\lambda}_i \bar{\alpha}_{i-j} + \lambda_i \alpha_{i-j} : j+1 \le i \le N-1,\ j = 0, m, \\ \bar{\alpha}_{N-j} : i = N,\ j = 0, m, \end{cases}$$

$$w_{i,j} = \begin{cases} \bar{\lambda}_i \alpha_{i-j} : j+1 \le i \le N-1,\ j = 0, m, \\ \alpha_{N-j} : i = N,\ j = 0, m. \end{cases}$$

Let us define in steady-state $P_{i,j} = \lim_{t\to\infty} P_{i,j}(t), \quad j \le i \le N,\ j = 0, 1$. The steady-state equations (3.1)–(3.7) reduce to

$$0 = -\lambda P_{0,0} + \sum_{j=1}^{m} v_{j,m} c(j|j) P_{j,1} + w_{m+1,m} c(m|m) P_{m+1,1} + w_{1,0} P_{1,0}, \tag{3.8}$$

$$0 = -P_{k,0} + \bar{\phi}\Big(u_{k-1,0} P_{k-1,0} + v_{k,0} P_{k,0} + w_{k+1,0} P_{k+1,0}\Big),\ 1 \le k \le N-1, \tag{3.9}$$

$$0 = -P_{N,0} + \bar{\phi}\Big(v_{N,0} P_{N,0} + u_{N-1,0} P_{N-1,0}\Big), \tag{3.10}$$

$$\begin{aligned}0 = &-P_{1,1} + \sum_{j=1}^{m+1} v_{j,m} c(j-1|j) P_{j,1} + \sum_{j=1}^{m} u_{j,m} c(j|j) P_{j,1} + \sum_{j=m+1}^{m+2} w_{j,m} c(j-2|m) P_{j,1}\\ &+\phi\Big(u_{0,0} P_{0,0} + v_{1,0} P_{1,0} + w_{2,0} P_{2,0}\Big),\end{aligned} \tag{3.11}$$

$$\begin{aligned}0 = &-P_{k,1} + \sum_{j=k}^{m+k} v_{j,m} c(j-k|j) P_{j,1} + \sum_{\max j=\{m+1,k+1\}}^{m+k+1} w_{j,m} c(j-k-1|m) P_{j,1} + \phi u_{k-1,0} P_{k-1,0}\\ &+ \sum_{j=k-1}^{m+k-1} u_{j,m} c(j-k+1|j) P_{j,1} + \phi\Big(v_{k,0} P_{k,0} + w_{k+1,0} P_{k+1,0}\Big),\ 2 \le k \le N-m-1,\end{aligned} \tag{3.12}$$

$$\begin{aligned}0 = &-P_{k,1} + \sum_{j=k}^{N-1} v_{j,m} c(j-k|m) P_{j,1} + \sum_{j=k+1}^{N-1} w_{j,m} c(j-k-1|m) P_{j,1} + \sum_{j=k-1}^{N-1} u_{j,m} c(j-k+1|m) P_{j,1}\\ &+\Big(w_{N,m} c(N-k-1|m) + v_{N,m} c(N-k|m)\Big) P_{N,1} + \phi u_{k-1,0} P_{k-1,0} + \phi v_{k,0} P_{k,0}\\ &+\phi w_{k+1,0} P_{k+1,0}, \quad N-m \le k \le N-1,\end{aligned} \tag{3.13}$$

$$0 = -P_{N,1} + v_{N,m} c(0|m) P_{N,1} + u_{N-1,m} c(0|m) P_{N-1,1} + \phi\Big(v_{N,0} P_{N,0} + u_{N-1,0} P_{N-1,0}\Big). \tag{3.14}$$

The solution of equations (3.8)–(3.14) will give the system-length distribution $P_{i,0}$, $(0 \le i \le N)$ and $P_{i,1}$, $(1 \le i \le N)$. To get them, first we need to solve the difference equations (3.10) and (3.9). Then using (3.14), (3.13) and (3.12), we get

$$P_{N-1,0} = \frac{1-\bar{\phi} v_{N,0}}{\bar{\phi} u_{N-1,0}} P_{N,0}, \tag{3.15}$$

$$P_{k-1,0} = \left\{ \prod_{i=k}^{N} \frac{1-\bar{\phi} v_{i,0}}{\bar{\phi} u_{i-1,0}} - \sum_{i=1}^{N-k-1} \frac{w_{N-i+1,0}}{u_{N-i-1,0}} \prod_{j=k}^{N-i-1} \frac{1-\bar{\phi} v_{j,0}}{\bar{\phi} u_{j-1,0}} - \frac{w_{k+1}}{u_{k-1}} \right\} P_{N,0}, \quad 1 \le k \le N-1, \tag{3.16}$$

$$P_{N-1,1} = \frac{1}{u_{N-1,m} c(0|m)} \Big[\Big(1 - v_{N,m} c(0|m)\Big) P_{N,1} - \phi\Big(v_{N,0} P_{N,0} + u_{N-1,0} P_{N-1,0}\Big)\Big], \tag{3.17}$$

$$\begin{aligned}P_{k-1,1} = \frac{1}{u_{k-1,m} c(0|m)} \Big[& P_{k,1} - \sum_{j=k}^{N-1} \Big(v_{j,m} c(j-k|m) + u_{j,m} c(j-k+1|m)\Big) P_{j,1}\\ &-\Big(v_{N,m} c(N-k|m) + w_{N,m} c(N-k-1|m)\Big) P_{N,1} - \sum_{j=k+1}^{N-1} w_{j,m} c(j-k-1|m) P_{j,1}\\ &-\phi\Big(v_{k,0} P_{k,0} + u_{k-1,0} P_{k-1,0} + w_{k+1,0} P_{k+1,0}\Big)\Big], \quad k = N-1, \ldots, N-m,\end{aligned} \tag{3.18}$$

$$P_{k-1,1} = \frac{1}{u_{k-1,m}c(0|k-1)}\Bigg[P_{k,1} - \sum_{j=k}^{m+k} v_{j,m}c(j-k|j)P_{j,1} - \sum_{j=k}^{m+k-1} u_{j,m}c(j-k+1|j)P_{j,1}$$
$$- \sum_{\max j=\{m+1,k+1\}}^{m+k+1} w_{j,m}c(j-k-1|m)P_{j,1} - \phi\Big(v_{k,0}P_{k,0} + u_{k-1,0}P_{k-1,0} + w_{k+1,0}P_{k+1,0}\Big)\Bigg],$$
$$k = N-m-1,\dots,2. \tag{3.19}$$

4. Computational algorithm

In this section, we establish a computational algorithm to compute steady-state probabilities. Based on the analysis of Section 3, we evaluate the probabilities $P_{i,j},\ 0 \le i \le N, j=0;\ 1 \le i \le N, j=1$ in terms of $P_{N,0}$.

Step 1: For $i = 0,1,\dots,N$, calculate $P_{i,0}$ in terms of $P_{N,0}$ as follows

$$P_{i,0} = \psi_i P_{N,0}, \quad 0 \le i \le N,$$

where ψ_i are computed as follows.

- Calculate ψ_i as follows

$$\psi_N = 1,\ \psi_{N-1} = \frac{1-\bar{\phi}v_{N,0}}{\bar{\phi}u_{N-1,0}},$$
$$\psi_{k-1} = \prod_{i=k+1}^{N} \frac{1-\bar{\phi}v_{i,0}}{\bar{\phi}u_{i-1,0}} - \sum_{i=1}^{N-k-1} \frac{w_{N-i+1,0}}{u_{N-i-1,0}} \prod_{j=k}^{N-i-1} \frac{1-\bar{\phi}v_{j,0}}{\bar{\phi}u_{j-1,0}} - \frac{w_{k+1}}{u_{k-1}},\ k = N-1,\dots,1.$$

Step 2: For $i = N, N-1,\dots,1$, calculate $P_{i,1}$ in terms of $P_{N,1}$ and $P_{N,0}$ as follows

$$P_{i,1} = \zeta_i P_{N,1} + \xi_i P_{N,0},\ 1 \le i \le N,$$

where ζ_i and ξ_i are computed as follows.

- Calculate ζ_i and ξ_i as follows.

$$\zeta_N = 1, \quad \xi_N = 0, \quad \zeta_{N-1} = \frac{1-v_{N,m}c(0|m)}{u_{N-1,m}c(0|m)}, \quad \xi_{N-1} = -\phi\left(\frac{v_{N,0}\psi_N + u_{N-1,0}\psi_{N-1}}{u_{N-1,m}c(0|m)}\right),$$

for $k = N-1,\dots,N-m$,

$$\zeta_{k-1} = \frac{1}{u_{k-1,m}c(0|m)}\left[\zeta_i - \sum_{j=k}^{N-1}\left(v_{j,m}c(j-k|m) + u_{j,m}c(j-k+1|m)\right)\zeta_j\right.$$
$$\left. - v_{N,m}c(N-k|m)\zeta_N - \sum_{j=k+1}^{N-1} w_{j,m}c(j-k-1|m)\zeta_j - w_{N,m}c(N-k-1|m)\zeta_N\right],$$

$$\xi_{k-1} = \frac{1}{u_{k-1,m}c(0|m)}\left[\xi_i - \sum_{j=k}^{N-1}\Big(v_{j,m}c(j-k|m) + u_{j,m}c(j-k+1|m)\Big)\xi_j\right.$$
$$- \sum_{j=k+1}^{N-1} w_{j,m}c(j-k-1|m)\xi_j - w_{N,m}c(N-k-1|m)\xi_N - v_{N,m}c(N-k|m)\xi_N$$
$$\left. - \phi\Big(v_{k,0}\psi_k + u_{k-1,0}\psi_{k-1} + w_{k+1,0}\psi_{k+1}\Big)\right],$$

for $k = N-m-1, \ldots, 2$,

$$\zeta_{k-1} = \frac{1}{u_{k-1,m}c(0|k-1)}\left[\zeta_i - \sum_{j=k}^{m+k} v_{j,m}c(j-k|j)\zeta_j - \sum_{j=k}^{m+k-1} u_{j,m}c(j-k+1|j)\zeta_j \right.$$
$$\left. - \sum_{\max j=\{m+1,k+1\}}^{m+k+1} w_{j,m}c(j-k-1|m)\zeta_j\right],$$

$$\xi_{k-1} = \frac{1}{u_{k-1,m}c(0|k-1)}\left[\xi_i - \sum_{j=k}^{m+k} v_{j,m}c(j-k|j)\xi_j - \sum_{j=k}^{m+k-1} u_{j,m}c(j-k+1|j)\xi_j \right.$$
$$\left. - \sum_{\max j=\{m+1,k+1\}}^{m+k+1} w_{j,m}c(j-k-1|m)\xi_j - \phi\Big(v_{k,0}\psi_k + u_{k-1,0}\psi_{k-1} + w_{k+1,0}\psi_{k+1}\Big)\right].$$

Step 3: Compute $P_{N,1}$ in terms of $P_{N,0}$ as follows

$$\pi_{N,1} = \Omega\pi_{N,0}, \quad 1 \le i \le N,$$

where

$$\Omega = \frac{\lambda\psi_0 - \sum_{j=1}^{m} \mu^j \xi_j v_{j,m} - \psi_1 w_{1,0} - \mu^m \xi_{m+1} w_{m+1,m}}{\sum_{j=1}^{m} \mu^j \zeta_j v_{j,m} + \mu^m \zeta_{m+1} w_{m+1,m}}.$$

Step 4: For $i = 1, 2, \ldots, N$, calculate $P_{i,1}$ in terms of $P_{N,0}$ as follows

$$P_{i,1} = (\psi_i + \Omega\zeta_i)P_{N,0}, \quad 1 \le i \le N.$$

Step 5: Using normalization equation $\sum_{i=0}^{N} P_{i,0} + \sum_{i=1}^{N} P_{i,1} = 1$, determine $P_{N,0}$ as

$$P_{N,0} = \left[\sum_{i=0}^{N} \psi_i + \sum_{i=1}^{N}(\xi_i + \Omega\zeta_i)\right]^{-1}.$$

5. Performance measure

Performance measures are significant characteristics of queueing systems as they speculate the efficiency of the model under consideration. We can evaluate the various performance measures, once the state probabilities at various epochs are known. The average number of customers in the queue (L_q) is given by

$$L_q = \sum_{k=1}^{N} kP_{k,0} + \sum_{k=m+1}^{N} (k-m)P_{k,1}.$$

The average number of customers in the system (L_s) is

$$L_s = \sum_{k=1}^{N} k(P_{k,0} + P_{k,1}).$$

The average number of busy servers ($E(B)$) are

$$E(B) = \sum_{k=0}^{m-1} kP_{k,1} + m\sum_{k=m}^{N} P_{k,1} = m - \sum_{k=0}^{m-1}(m-k)P_{k,1} = L_s - L_q.$$

The average number of servers on vacation ($E(V)$) and the average number of idle servers in the system ($E(I)$) are given by

$$E(V) = \sum_{k=0}^{N} P_{k,0}, \quad E(I) = m - E(B) - E(V).$$

In the system, customers arrive at the rate λ. Some of arriving customers may not join the system because of balking. The effective arrival rate λ^e into the system is thus different from the overall arrival rate and is given by

$$\begin{aligned}\lambda^e &= \sum_{i=0}^{N-1} \lambda b_i P_{i,0} + \sum_{i=1}^{m-1} \lambda P_{i,1} + \sum_{i=m}^{N-1} \lambda b_i P_{i,1} \\ &= \mu \sum_{k=1}^{m} kP_{k,1} + \sum_{k=m+1}^{N} (m\mu + (k-m)\alpha)\, P_{k,1} + \sum_{k=1}^{N} k\alpha P_{k,0} = \alpha L_q + \mu E(B).\end{aligned}$$

Using Little's rule, the average waiting time of a customer in the system (W_s) and the average waiting time of a customer in the queue (W_q) is given by

$$W_s = L_s/\lambda^e, \qquad W_q = L_q/\lambda^e.$$

The fraction of customers that enter the system is λ^e/λ and the utilization of the system is $U = \lambda^e/(m\mu)$. The potential (offered) load is $\rho = \lambda/(m\mu)$. The proportion of loss of customers is given by $\lambda - \lambda^e/\lambda$. In a real life situation, customers who are lost to the system due to balking represent the business loss. The probability that a customer balks is $1 - b_i$, the immediate balking rate is $\lambda(1 - b_i)$. The average balking rate (BR) is specified by

$$BR = \sum_{i=1}^{N} \lambda(1-b_i)P_{i,0} + \sum_{i=m}^{N} \lambda(1-b_i)P_{i,1}.$$

If all the servers are busy and there are i customers in the system, then there are $(i - m)$ waiting customers in the queue. The reneging rate is $(i - m)\alpha$ as any of the $(i - m)$ customers in the queue may renege. The average reneging rate (RR) is given as

$$RR = \sum_{i=1}^{N} i\alpha P_{i,0} + \sum_{i=m+1}^{N} (i-m)\alpha P_{i,1} = \alpha L_q.$$

The average rate of customer loss (LR) is the sum of the average balking rate and the average reneging rate. Thus, we get

$$LR = BR + RR.$$

5.1. Cost model

We formulate the total expected cost function per unit time where the service rate μ is a decision variable. Our objective is to minimize the total expected cost function by optimizing the service rate μ. We consider the following cost parameters:

$C_1 \equiv$ service cost per unit time during synchronous vacations,
$C_2 \equiv$ service cost per unit time during busy period,
$C_3 \equiv$ service cost per unit time during idle period,
$C_4 \equiv$ service cost per unit time when customer is waiting,
$C_5 \equiv$ cost per unit time when customer joins the system and is served,
$C_6 \equiv$ customer balking or reneging cost per unit time.

Let $F(\mu)$ be the total expected cost function per unit time. Using the definitions of each cost element and its associated system performance measures, the total expected cost function per unit time is expressed as

$$\text{Minimize } F(\mu) = C_1E(V) + C_2E(B) + C_3E(I) + C_4L_q + C_5(L_s - L_q) + C_6LR,$$

where $E(V), E(B), E(I), L_q$ and LR are as defined above. The objective is to determine the optimal service rate μ^* to minimize the cost function F. As expected cost function is highly complex, it is a difficult task to develop analytic results for the optimum value of μ. We use the quadratic fit search method to solve the above optimization problem [22]. Given a 3-point pattern, the unique optimum x of the quadratic function agreeing with $f(x)$ at (x_0, x_1, x_2) occurs at

$$x = \frac{1}{2}\frac{f(x_0)(x_1^2 - x_2^2) + f(x_1)(x_2^2 - x_0^2) + f(x_2)(x_0^2 - x_1^2)}{f(x_0)(x_1 - x_2) + f(x_1)(x_2 - x_0) + f(x_2)(x_0 - x_1)}\cdot \tag{5.1}$$

6. Special cases

In this section, we derive some known results available in the literature as special cases of our model by taking specific values for the parameters.

Case 1: $\boldsymbol{\alpha \to 0}$. That is, the customers never renege. The model reduces to $Geo/Geo/m/N$ queue with balking and synchronous vacations. Taking $\alpha \to 0$, it is found that $u_{i,j}$, $v_{i,j}$ and $w_{i,j}$ reduce to

$$u_{i,j} = \begin{cases} \lambda, & i = j = 0; 1 \le i \le m-1,\ j = m, \\ \lambda_i & 1 \le i \le N-1,\ j = 0;\ m \le i \le N-1, j = m, \end{cases}$$

$$v_{i,j} = \begin{cases} \bar{\lambda}, & i = j = 0;\ 1 \le i \le m-1,\ j = m, \\ \bar{\lambda}_i, & 1 \le i \le N-1,\ j = 0; m \le i \le N-1,\ j = m, \\ 1, & i = N,\ j = 0, m, \end{cases}$$

$w_{i,j} = 0, \quad 1 \le i \le N-1,\ j = 0;\ m+1 \le i \le N-1,\ j = m.$

Then, using (3.16)–(3.19), we obtain the steady-state probabilities as

$$P_{N-1,0} = \frac{\phi}{\bar{\phi}\lambda_{N-1}} P_{N,0},$$

$$P_{k-1,0} = \prod_{i=k}^{N} \left(\frac{1-\bar{\phi}\bar{\lambda}_i}{\bar{\phi}\bar{\lambda}_{i-1}} \right) P_{N,0}, \quad 1 \le k \le N-1,$$

$$P_{N-1,1} = \frac{1}{\lambda_{N-1} c(0|m)} \Big[\Big(1 - c(0|m)\Big) P_{N,1} - \phi\Big(P_{N,0} + \lambda_{N-1} P_{N-1,0}\Big) \Big],$$

$$P_{k-1,1} = \frac{1}{\lambda_{k-1} c(0|m)} \Big[P_{k,1} - \sum_{j=k}^{N-1} \Big(\bar{\lambda}_j c(j-k|m) + \lambda_j c(j-k+1|m)\Big) P_{j,1} - c(N-k|m) P_{N,1}$$
$$-\phi\Big(\bar{\lambda}_k P_{k,0} + \lambda_{k-1} P_{k-1,0}\Big)\Big], \quad k = N-1, \ldots, N-m,$$

$$P_{k-1,1} = \frac{1}{\lambda_{k-1} c(0|m)} \Big[P_{k,1} - \sum_{j=k}^{m+k-1} \Big(\bar{\lambda}_j c(j-k|m) + \lambda_j c(j-k+1|m)\Big) P_{j,1} - \bar{\lambda}_{m+k} c(m|m) P_{m+k,1}$$
$$-\phi\Big(\bar{\lambda}_k P_{k,0} + \lambda_{k-1} P_{k-1,0}\Big)\Big], \quad k = N-m+1, \ldots, m+1,$$

$$P_{k-1,1} = \frac{1}{\lambda_{k-1} c(0|k-1)} \Big[P_{k,1} - \sum_{j=k}^{m-1} \Big(\bar{\lambda} c(j-k|j) + \lambda c(j-k+1|j)\Big) P_{j,1} - \sum_{j=m}^{m+k} \bar{\lambda}_j c(j-k|m)$$
$$- \sum_{j=m}^{m+k} \lambda_j c(j-k+1|m)\Big) P_{j,1} - \phi\Big(\bar{\lambda}_k P_{k,0} + \lambda_{k-1} P_{k-1,0}\Big)\Big], \quad k = m, \ldots, 2.$$

Case 2: $\phi \to 1$. The model reduces to *Geo*/*Geo*/*m*/*N* queue with balking and reneging. In this case, the vacations probabilities $P_{i,0}$ $(1 \le i \le N)$ does not exist. If we take $P_{0,0}$ as P_0 and $P_{i,1}$ as P_i for $1 \le i \le N$, we get the results of *Geo*/*Geo*/*m*/*N* queue with balking and reneging.

Case 3: $\alpha \to 0$ and $b_i = 1$ $(0 \le i \le N-1)$. The model reduces to *Geo*/*Geo*/*m*/*N* queue with synchronous vacations. Taking $\alpha \to 0$ and $b_i = 1$ $(0 \le i \le N-1)$, it is found that $u_{i,j}$, $v_{i,j}$ and $w_{i,j}$ reduce to $u_{i,j} = \lambda, 0 \le i \le N-1, j = 0, m,$ $v_{i,j} = \bar{\lambda}, 0 \le i \le N-1, j = 0, m,$ $v_{N,j} = 1, j = 0, m$ and $w_{i,j} = 0,\ 1 \le i \le N-1,\ j = 0;\ m+1 \le i \le N-1,\ j = m.$
Using (3.16)–(3.19), the steady-state probabilities, after simplification, are given as

$$P_{k,0} = \left(\frac{\phi}{\bar{\phi}\lambda}\right)\left(\frac{\phi + \bar{\phi}\lambda}{\bar{\phi}\lambda}\right)^{N-k-1} P_{N,0},\ 0 \le k \le N-1,$$

$$P_{N-1,1} = \frac{1}{\lambda c(0|m)} \Big[\Big(1 - c(0|m)\Big) P_{N,1} - \phi\Big(P_{N,0} + \lambda P_{N-1,0}\Big) \Big],$$

$$P_{k-1,1} = \frac{1}{\lambda c(0|m)} \Big[P_{k,1} - \sum_{j=k}^{N-1} \Big(\bar{\lambda} c(j-k|m) + \lambda c(j-k+1|m)\Big) P_{j,1} - c(N-k|m) P_{N,1}$$
$$-\phi\Big(\bar{\lambda} P_{k,0} + \lambda P_{k-1,0}\Big)\Big], \quad k = N-1, \ldots, N-m,$$

$$P_{k-1,1} = \frac{1}{\lambda c(0|k-1)} \Big[P_{k,1} - \sum_{j=k}^{m+k} \bar{\lambda} c(j-k|j) P_{j,1} - \sum_{j=k}^{m+k-1} \lambda c(j-k+1|m) P_{j,1} - \phi\Big(\bar{\lambda} P_{k,0} + \lambda P_{k-1,0}\Big)\Big],$$
$$k = N-m-1, \ldots, 2.$$

Case 4: $\phi \to 1$, $\alpha \to 0$ and $b_i = 1$ $(0 \le i \le N-1)$. The model reduces to $Geo/Geo/m/N$ queue without balking, reneging and synchronous vacations. In this case, the vacations probabilities $P_{i,0}$ $(1 \le i \le N)$ does not exist. If we take $P_{0,0}$ as P_0 and $P_{i,1}$ as P_i for $1 \le i \le N$, we get the results of $Geo/Geo/m/N$ queue without balking, reneging and synchronous vacations.

Case 5: $\phi \to 1$, $\alpha \to 0$ and $b_i = 1$, $0 \le i \le m-1$ and $b_i = b$, $i \ge m$. Assuming $\rho = \lambda/m\mu < 1$ and $N \to \infty$, the model reduces to infinite buffer $Geo/Geo/m$ queue without reneging and synchronous vacations. Note that the synchronous vacation probabilities $P_{i,0}$ $(1 \le i \le N)$ does not exist. Let us define $P_{0,0} = P_0$ and $P_{i,1} = P_i$. Substituting $\phi \to 1$, $\alpha \to 0$ and $b_i = 1$, $0 \le i \le m-1$ and $b_i = b$, $i \ge m$ in (3.16)–(3.19), we obtain

$$\lambda P_0 = \bar{\lambda} \sum_{j=1}^{m-1} c(j|j)P_j + (1-\lambda b)c(m|m)P_m, \tag{6.1}$$

$$P_k = \bar{\lambda} \sum_{j=k}^{m-1} c(j-k|j)P_j + \lambda \sum_{j=k-1}^{m-1} c(j+1-k|j)P_j + (1-\lambda b) \sum_{j=m}^{m+k} c(j-k|m)P_j$$
$$+\lambda b \sum_{j=m}^{m+k-1} c(j+1-k|m)P_j, \quad 1 \le k \le m, \tag{6.2}$$

$$P_k = (1-\lambda b) \sum_{j=k}^{m+k} c(j-k|m)P_j + \lambda b \sum_{j=k-1}^{m+k-1} c(j+1-k|m)P_j, \quad k \ge m+1. \tag{6.3}$$

The system-length distributions at steady-state are given as

$$P_k = \begin{cases} P_k^* \; P_m, & 0 \le k \le m, \\ r^{k-m} P_m, & k \ge m+1, \end{cases}$$

where r, $0 < r < 1$, is the unique real root of the equation $\Big((1-\lambda b)r + \lambda b\Big)(\bar{\mu} + \mu r)^m = r$. Using the normalization condition, we get $P_m = \left(\frac{1}{1-r} + \sum_{k=0}^{m-1} P_i^*\right)$. The $\{P_k^*\}_0^m$ can be obtained from (6.1) and (6.2) using backward recursion. These results match with the results reported in Goswami [14].

7. Numerical results

In this section, some numerical results have been presented in the form of table and graphs. It gives managerial insights on optimal decisions to add the qualitative views of the queueing system under examination through exemplifying numerical results. Certainly, the change of parameters, such as the balking rate, reneging rate and vacation rate in the system, may influence various performance measures of the model.

Table 1 gives the optimum value of μ^*, the minimum cost $F(\mu^*)$ and various performance measures for different values of reneging rate (α). The parameters are taken as $\lambda = 0.5$, $\phi = 0.1$, $m = 3, N = 10$ and $b_i = 1 - i/N$. We observe that as α increases: (i) The optimum μ^* increases. (ii) The LR, $E(V)$ and $E(I)$ increase. (iii) The optimum cost and the other performance indices decrease.

Figure 2 provides the average rate of customer loss (LR) with a change of vacation time ϕ for different balking functions (i) $b_i = 1 - i/N^2$, (ii) $b_i = 1 - i/N$, (iii) $b_i = 1/(i+1)$. The parameters are taken as $\alpha = 0.1, \lambda = 0.5, \mu = 0.0625, m = 10$ and $N = 25$. It is seen that as ϕ increases the average rate of customer loss (LR) decreases for different balking functions. But the average rate of customer loss in the system is lower for the balking function given in (i). Figure 3 presents the impact of offered load (ρ) on the average number of busy servers $E(B)$ and the average number of servers on vacation $E(V)$ for various b_i, where $\mu = 0.2, \alpha = 0.05, \phi = 0.1, N = 10$ and $m = 3$. It is evident from the figure that average number of busy servers increases as offered load increase, where as $E(V)$ decreases. Further, with fixed offered load, the average number of busy servers and the average

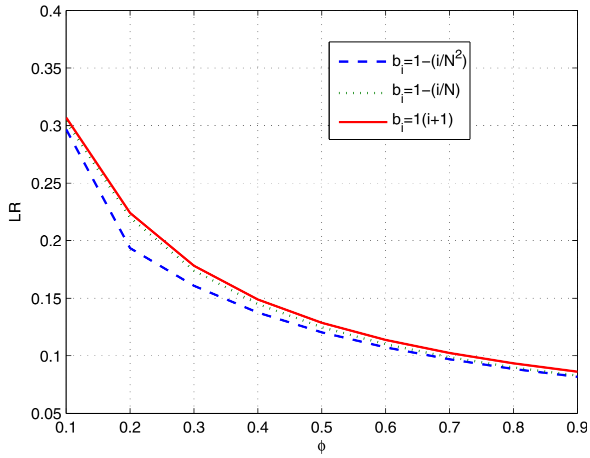

FIGURE 2. Effect of ϕ on LR.

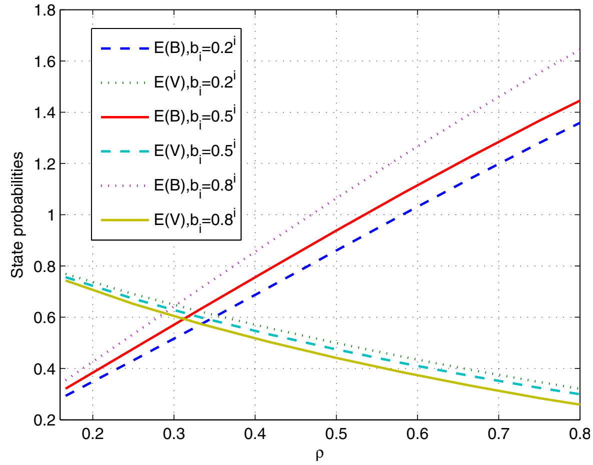

FIGURE 3. Effect of ρ on state probabilities.

TABLE 1. Effect of α on performance characteristics.

	$\alpha = 0.03$	$\alpha = 0.05$	$\alpha = 0.07$	$\alpha = 0.1$	$\alpha = 0.15$
μ^*	0.37200	0.41470	0.46050	0.53430	0.67210
L_q	1.43567	1.40519	1.37492	1.32538	1.23428
L_s	2.40805	2.23016	2.07398	1.87292	1.60438
W_q	3.54665	3.40756	3.28800	3.11790	2.84474
W_s	5.94880	5.40809	4.95975	4.40596	3.69772
LR	0.13743	0.15775	0.17806	0.20745	0.25126
$E(B)$	0.97238	0.82497	0.69906	0.54754	0.37009
$E(V)$	0.50708	0.56368	0.61466	0.67985	0.76292
$E(I)$	1.52054	1.61136	1.68628	1.77261	1.86699
$F(\mu^*)$	340.373	329.019	319.213	306.811	290.577

number of servers on vacation decrease when the balking probability increases. We may setup an admissible offered load and the balking probability to employ servers efficiently.

The parameters for Figures 4 to 7 are taken as $\mu = 0.2, \alpha = 0.1, \phi = 0.1, N = 10, m = 3$ and $b[i] = 1 - i/N$. Figures 4 and 5 illustrate dependence of W_q and L_q on the ϕ and λ, respectively. A decreasing trend is observed in W_q with the increase of ϕ and λ. But, an increasing trend is observed in L_q with the increase of ϕ and λ. We can carefully setup ϕ and λ in the system to ensure the minimum W_q and L_q. The variation in the cost for different values of μ and α is shown in Figure 6. It is observed that the average cost decreases with the increase of μ and α. We may infer from this figure that for lower values of μ the maximal performance gain is rather limited, for higher values of μ, the maximal performance gain is larger, especially when α is moderate or high. When α is low, we observe that their is no significant difference in cost.

Figure 7 depicts dependence of average rate of customer loss (LR) on ρ and ϕ. It is observed that for fixed ϕ, the LR increases when the offered load ρ increases. This is because as ρ increases the average rate of customer loss increases. Further, with fixed offered load ρ, the average rate of customer loss decreases when the vacation parameter ϕ increases. We can carefully setup the offered load and vacation parameter ϕ in the system in order to ensure the minimum customer loss. From the numerical results, we can determine the impact of parameters on the performance measures in the system.

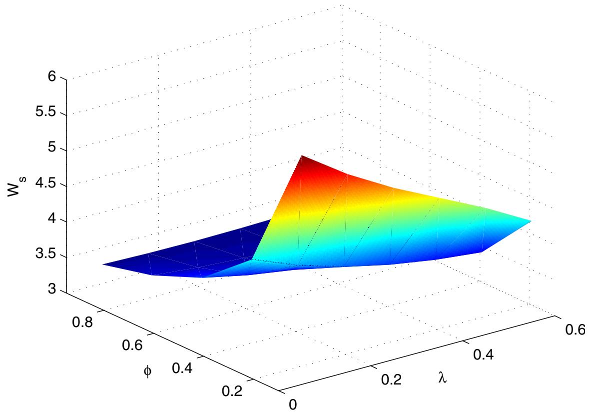

Figure 4. W_q *versus* λ and ϕ.

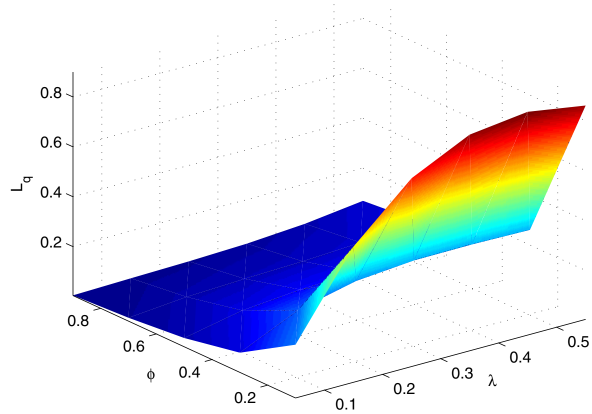

Figure 5. L_q *versus* λ and ϕ.

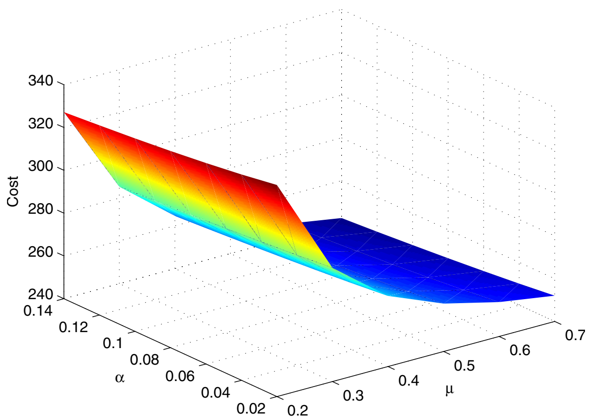

Figure 6. Cost t *versus* μ and α.

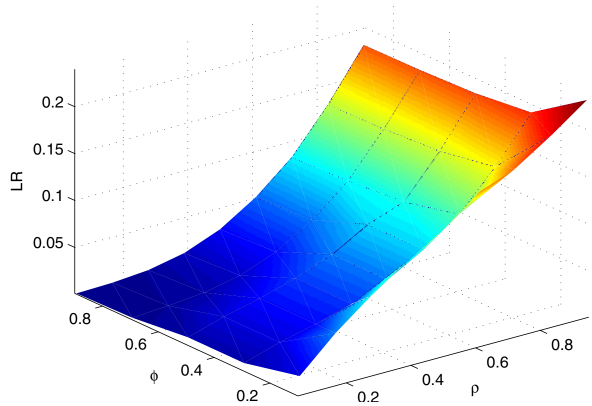

Figure 7. LR *versus* ρ and ϕ.

8. Conclusions

In this paper, we analyzed a discrete-time multi-server queue with multiple synchronous vacations under balking and reneging for late arrival system with delayed access. The presented model may be potentially used in a wide variety of real-time systems including call centers, communication system, cloud computing, quality control and maintenance in industrial establishments. We have obtained a closed-form analytical expressions and developed a computational algorithm for calculating the steady-state probabilities. A method to obtain an optimal service rate that minimizes the total expected cost under a certain cost function was presented. Various performance measures and numerical results in the form of table and graphs are sketched out to display the impact of the system parameters. Some special cases of the model have also been presented. The relationship between our model and the corresponding continuous-time model was also investigated. Finally, it is shown that in the limiting case the results converge to the corresponding continuous-time counterparts. The analytical approach used in this paper may be applied to analyze discrete-time renewal input multi-server queue with multiple working vacations under balking and reneging which is left for future investigation.

Appendix

Here we study the relationship between the discrete-time $Geo/Geo/m/N$ queueing system with balking, reneging and synchronous vacations and its continuous-time counterpart. For the continuous-time $M/M/m/N$ queueing system with balking, reneging and synchronous vacations, we assume that the inter-arrival time, vacation times and impatient timer are exponentially distributed with parameters $\tilde{\lambda}$, $\tilde{\phi}$ and $\tilde{\alpha}$. There are m server and service times are assumed to be independent and exponentially distributed with mean service time $1/\tilde{\mu}$. Let the time axis be slotted into intervals of equal length, so that $\lambda = \tilde{\lambda}\Delta$, $\mu = \tilde{\mu}\Delta$, $\phi = \tilde{\phi}\Delta$ and $\alpha = \tilde{\alpha}\Delta$, where $\Delta > 0$ is sufficiently small. Now, using $\mu = \tilde{\mu}\Delta$ in (2.1), we obtain

$$c(j|i) = \begin{cases} 1, \quad i = j = 0; \\ 1 - \tilde{\mu}\Delta, \quad i = 1,\ j = 0; \\ \tilde{\mu}\Delta, \quad i = j = 1; \\ 1 - \min(i,m)\tilde{\mu}\Delta + o(\Delta), \quad i = 2,3,\dots,m,\ j = 0; \\ \min(i,m)\tilde{\mu}\Delta + o(\Delta), \quad i = 2,3,\dots,m,\ j = 1; \\ o(\Delta), \quad i = 2,3,\dots,m,\ j = 2,3,\dots,\min(i,m); \\ 0, \quad \text{otherwise}, \end{cases} \tag{A.1}$$

where $o(\Delta)$ denotes any function of Δ such that $\lim_{\Delta\to 0} \frac{o(\Delta)}{\Delta} = 0$.
Using $\lambda = \tilde{\lambda}\Delta$, $\mu = \tilde{\mu}\Delta$, $\phi = \tilde{\phi}\Delta$, $\alpha = \tilde{\alpha}\Delta$, $u_{i,j}, v_{i,j}, w_{i,j}$ and (A.1) in (3.15), we obtain

$$P_{N-1,0} = \frac{1 - (1 - \tilde{\phi}\Delta)(1 - N\tilde{\alpha}\Delta)}{(1 - \tilde{\phi}\Delta)\tilde{\lambda}\Delta b_{N-1}(1 - (N-1)\tilde{\alpha}\Delta)} P_{N,0}.$$

Dividing both sides by Δ and taking limit as $\Delta \to 0$ in the above equation, we obtain

$$P_{N-1,0} = \frac{\tilde{\phi} + N\tilde{\alpha}}{\tilde{\lambda} b_{N-1}} P_{N,0}.$$

Using (A.1), putting $\lambda = \tilde{\lambda}\Delta$, $\mu = \tilde{\mu}\Delta$, $\phi = \tilde{\phi}\Delta$, $\alpha = \tilde{\alpha}\Delta$ and taking limit as $\Delta \to 0$ in equations (3.16)–(3.19), we get,

$$\begin{aligned} P_{k-1,0} &= \left[\prod_{i=k}^{N} \frac{\tilde{\phi} + \tilde{\lambda} b_i + i\tilde{\alpha}}{\tilde{\lambda} b_{i-1}} - \sum_{i=1}^{N-k-1} \frac{(N-i+1)\tilde{\alpha}}{\tilde{\lambda} b_{N-i-1}} \prod_{j=k}^{N-i-1} \frac{\tilde{\phi} + \tilde{\lambda} b_j + j\tilde{\alpha}}{\tilde{\lambda} b_{j-1}} - \frac{(k+1)\tilde{\alpha}}{\tilde{\lambda} b_{k-1}} \right] P_{N,0}, \\ &\quad k = N-1, \dots, 1, \\ P_{N-1,1} &= \frac{m\tilde{\mu} + (N-m)\tilde{\alpha}}{\tilde{\lambda} b_{N-1}} P_{N,1} - \frac{\tilde{\phi}}{\tilde{\lambda} b_{N-1}} P_{N,0}, \\ P_{k-1,1} &= \frac{m\tilde{\mu} + \tilde{\lambda} b_k + (k-m)\tilde{\alpha}}{\tilde{\lambda} b_{k-1}} P_{k,1} - \frac{m\tilde{\mu} + (k+1-m)\tilde{\alpha}}{\tilde{\lambda} b_{k-1}} P_{k+1,1} - \frac{\tilde{\phi}}{\tilde{\lambda} b_{k-1}} P_{k,0}, \\ &\quad k = N-1, N-2, \dots, 2, \end{aligned}$$

which match with the relations for the continuous-time $M/M/m/N$ queue with balking, reneging and synchronous vacations reported in [29].

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