RAIRO-Oper. Res. 50 (2016) 575–586
DOI: 10.1051/ro/2016026

RAIRO Operations Research
www.rairo-ro.org

MULTIVARIATE STOCHASTIC DOMINANCE FOR RISK AVERTERS AND RISK SEEKERS

Xu Guo[1,2] and Wing-Keung Wong[3,4,5]

Abstract. This paper first extends some well-known univariate stochastic dominance results to multivariate stochastic dominances (MSD) for both risk averters and risk seekers, respectively, to n order for any $n \geq 1$ when the attributes are assumed to be independent and the utility is assumed to be additively and separable. Under these assumptions, we develop some properties for MSD for both risk averters and risk seekers. For example, we prove that MSD are equivalent to the expected-utility maximization for both risk averters and risk seekers, respectively. We show that the hierarchical relationship exists for MSD. We establish some dual relationships between the MSD for risk averters and risk seekers. We develop some properties for non-negative combinations and convex combinations random variables of MSD and develop the theory of MSD for the preferences of both risk averters and risk seekers on diversification. At last, we discuss some MSD relationships when attributes are dependent and discuss the importance and the use of the results developed in this paper.

Mathematics Subject Classification. D81, G11.

Received February 25, 2015. Accepted March 29, 2016.

1. Introduction

There are two major types of persons: risk averters and risk seekers. Their corresponding utility functions are concave and convex; both are increasing. Many authors have studied their selection rules. For example, Markowitz [25] and others propose the mean-variance selection rules for risk averters and risk seekers. Quirk and Saposnik [29] and many others develop some univariate stochastic dominance (SD) rules for risk averters. On the other hand, Hammond [14] and many others develop the univariate SD rules for risk seekers.

Decisions are often drawn from multidimensional attributes and, in this situation, decision makers will rely on multidimensional utility. Some, if not all, of the outcomes in the decisions with multidimensional consequences could be risky (see, for example, [11]). Consider a problem of decision-making with risky outcomes described by n ($n > 1$) attributes and under the expected-utility framework, if the distribution of the outcomes is known, the main practical difficulty is to define an appropriate n-variate utility function u for the assessment of the expected utility of the outcomes. A simple way for the assessment is to assume that the utility function u

Keywords. Multivariate stochastic dominance, risk averters, risk seekers, ascending stochastic dominance, descending stochastic dominance, utility function.

[1] School of Statistics, Beijing Normal University, Beijing, P.R. China.

[2] College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P.R. China.

[3] Department of Finance, Asia University, Taiwan. alanwkwong@gmail.com

[4] Faculty of Economics, Hong Kong Baptist University, Hong Kong, P.R. China.

[5] Department of Economics, Lingnan University, Hong Kong, P.R. China.

Article published by EDP Sciences

possesses separability property for its attributes and the attributes of the outcomes are independent [21]. Our paper follows this approach.

Some work has been done in the literature to extend the SD concept from a single variable to the multivariate case. For example, Levy [22] assumes stochastically independent attributes whereas Levy and Paroush (1974a) adopt additively separable utilities. In addition, Hazen [16] investigates multivariate SD when simple forms of utility independence can be assumed. Readers may refer to Denuit and Eeckhoudt [9] and the references therein for other studies in the multivariate SD. In this paper we call SD for risk averters ascending SD (ASD) and SD for risk seekers descending SD (DSD).

In this paper, we extend some well-known univariate ASD and DSD results to multivariate ASD and DSD for risk averters and risk seekers, respectively, to n order for any $n \geq 1$ when attribute is assumed to be independent and the utility is assumed to be additively and separable. Under these assumptions, we develop some properties for ASD and DSD, respectively. For example, we prove that ASD and DSD are equivalent to the expected-utility maximization for risk averters and risk seekers, respectively. We show that the hierarchical relationship exists and establish the relationships between multivariate ASD and DSD. We establish some dual relationships between the MSD for risk averters and risk seekers. We develop some properties of non-negative combinations and convex combinations random variables for multivariate stochastic dominance and develop the theory of multivariate SD for the preferences of risk averters and risk seekers on diversification. At last, we discuss some multivariate SD relationships when attributes are dependent.

We begin by introducing notations and definitions in Section 2. In Section 3, we develop the theory of multivariate stochastic dominance for risk averters and risk seekers. We discuss in Section 4 the importance and the use of the results developed in this paper. Section 5 concludes.

2. Definitions and notations

We let X and Y be random variables defined on $\Omega = [a, b]$ with distribution functions F and G, and probability density functions f and g, respectively, satisfying

$$H_j^A(x) = \int_a^x H_{j-1}^A(y)\,\mathrm{d}y \quad , \quad H_j^D(x) = \int_x^b H_{j-1}^D(y)\,\mathrm{d}y \quad \text{for} \quad H = F \quad \textit{or} \quad G.$$

in which $H_0^A(x) = H_0^D(x) = h(x)$ with $h = f$ or g and $H = F$ or G. In addition, we let $\mu_F = \mu_X = E(X) = \int_a^b t\,\mathrm{d}\,F(t)$ and $\mu_G = \mu_Y = E(Y) = \int_a^b t\,\mathrm{d}\,G(t)$

2.1. Univariate stochastic dominance

We next define the N-order ascending (descending) stochastic dominance which is applied to risk averters (seekers)[6]. We first modify the definition used in Jean [19] to obtain the following definition for the N-order ascending stochastic dominance (ASD).

Definition 2.1. Given two random variables X and Y with F and G as their respective distribution functions, X is at least as large as Y in the sense of:

1. FASD(SASD), denoted by $X \succeq_1 Y$ ($X \succeq_2 Y$) if and only if $F_1^A(x) \leq G_1^A(x)$ ($F_2^A(x) \leq G_2^A(x)$) for each x in $[a, b]$,
2. NASD, denoted by $X \succeq_N Y$ if and only if $F_N^A(x) \leq G_N^A(x)$ for each x in $[a, b]$ and $F_k^A(b) \leq G_k^A(b)$ for $k = 2, \ldots, N-1$ for $N \geq 3$,

where FASD, SASD and NASD stand for first-, second-, and N-order ascending SD, respectively.

[6]We call stochastic dominance for risk averters ascending stochastic dominance because its integrals count from the leftmost point in the domain ascending to the rightmost point in the domain. Similarly, we call stochastic dominance for risk seekers descending stochastic dominance because its integrals count from the rightmost point in the domain descending to the leftmost point in the domain. Readers may refer to [31] and the references therein for more information.

We then define the first, second and N-order descending stochastic dominance (DSD) as follows:

Definition 2.2. Given two random variables X and Y with F and G as their respective distribution functions, X is at least as large as Y in the sense of:

1. FDSD(SDSD), denoted by $X \succeq^1 Y$ $(X \succeq^2 Y)$ if and only if $F_1^D(x) \geq G_1^D(x)$ $(F_2^D(x) \geq G_2^D(x))$ for each x in $[a, b]$,
2. NDSD, denoted by $X \succeq^N Y$ if and only if $F_N^D(x) \geq G_N^D(x)$ for each x in $[a, b]$, and $F_k^D(a) \geq G_k^D(a)$ for $k = 2, \ldots, N-1$ for $N \geq 3$,

where FDSD, SDSD, and NDSD stand for first-, second-, and N-order descending stochastic dominance, respectively.

In Definition 2.1, if, in addition, there exists x in $[a, b]$ such that the inequality is strictly, then we say that X is large than Y and F is large than G in the sense of SFASD, SSASD, and SNASD, denoted by $X \succ_1 Y$ or $F \succ_1 G$, $X \succ_2 Y$ or $F \succ_2 G$, and $X \succ_N Y$ or $F \succ_N G$, respectively, where SFASD, SSASD, and SNASD stand for strictly first-, second-, and N-order ASD, respectively. The strictly first, second, and N-order DSD, denoted by SFDSD, SSDSD, and SNDSD can be defined similarly.

2.2. Majorization and Dalton transfer

In this paper we also study the single period portfolio selection for investors to allocate their wealth to the n $(n > 1)$ risks without short selling in order to maximize their expected utilities from the resulting final wealth [1–26]. Let random variable $X \in \mathcal{X}$ be an (excess) return of an asset or prospect. If there are n assets $\vec{X}_n = (X_1, \ldots, X_n)'$, a portfolio of $\vec{X}_n$ without short selling is defined by a convex combination, $\overrightarrow{\lambda_n}'\overrightarrow{X}_n$, of the n assets $\vec{X}_n$ for any $\overrightarrow{\lambda_n} \in S_n^0$ (S_n) where

$$S_n^0 = \left\{(s_1, s_2, \ldots, s_n)' \in \mathbb{R}^n : \, 0 \leq s_i \leq 1 \text{ for any } i, \; \sum_{i=1}^n s_i = 1\right\},$$
$$S_n = \left\{(s_1, s_2, \ldots, s_n)' \in \mathbb{R}^n : 1 \geq s_1 \geq s_2 \geq \ldots \geq s_n \geq 0, \; \sum_{i=1}^n s_i = 1\right\} \, , \tag{2.1}$$

in which $\mathbb{R}$ is the set of real numbers and the i^{th} element of $\overrightarrow{\lambda_n}$ is the weight of the portfolio allocation on the i^{th} asset of return X_i. A portfolio will be equivalent to return on asset i, which we call a *specialized portfolio*, or simply a *specialized asset*, if $s_i = 1$ and $s_j = 0$ for all $j \neq i$. It is a *partially diversified portfolio* if there exists i such that $0 < s_i < 1$ and is the *completely diversified portfolio* if $s_i = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$. We note that we include the condition of $\sum_{i=1}^n s_i = 1$ is only for convenience. All the results developed in this paper work well without this condition. We then follow Hardy, *et al.* [15] to state the following definition to order the elements in S_n.

Definition 2.3. Let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$ in which S_n is defined in (2.1). $\vec{\beta}_n$ is said to majorize $\vec{\alpha}_n$, denoted by $\vec{\beta}_n \succeq_M \vec{\alpha}_n$, if $\sum\limits_{i=1}^k \beta_i \geq \sum\limits_{i=1}^k \alpha_i$, for all $k = 1, 2, \ldots, n$.

Possessing the Dalton Pigou transfer is an important feature for the theory of majorization. We state the definition as follows:

Definition 2.4. [7] For any $\vec{\alpha}_n$, $\vec{\beta}_n \in S_n$, $\vec{\alpha}_n$ is said to be obtained from $\vec{\beta}_n$ by applying a single Dalton (Pigou) transfer, denoted by $\vec{\beta}_n \xrightarrow{D} \vec{\alpha}_n$, if there exist h and k ($1 \le h < k \le n$) such that $\alpha_i = \beta_i$ for any $i \ne h, k$; $\alpha_h = \beta_h - \epsilon$; and $\alpha_k = \beta_k + \epsilon$ with $\epsilon > 0$.

2.3. Multivariate Stochastic Dominance

Now, we are ready to define multivariate stochastic dominance for risk averters and risk seekers for independent assets. If there are two vectors of n assets, $\vec{X}_n = (X_1, \ldots, X_n)'$ and $\vec{Y}_n = (Y_1, \ldots, Y_n)'$, in which $\{X_i\}$ are independent and $\{Y_i\}$ are independent. We have the following definitions:

Definition 2.5. For any integer N, we say $\vec{X}_n \succeq_N \vec{Y}_n$ if and only if $X_i \succeq_N Y_i$ for any $i = 1, 2, \ldots, n$.

Definition 2.6. For any integer N, we say $\vec{X}_n \succeq^N \vec{Y}_n$ if and only if $X_i \succeq^N Y_i$ for any $i = 1, 2, \ldots, n$.

We note that readers may modify the work by Levy [22] and others to set some conditions and establish the statements in Definitions 2.5 and 2.6 as theorems instead of definitions. However, our paper prefer to keep them as definitions.

Before we discuss the theory further, we state the definitions of the sets of utility functions for risk averters and risk seekers, U_j^A and U_j^D, for the MSD as follows:

Definition 2.7. For $J = 1,\ 2, \ldots, N$, $u \in U_J^A$ or U_J^D is an utility function for $\vec{X}_n = (X_1, \ldots, X_n)'$ satisfying

$$u(\vec{X}) = u_1(X_1) + \ldots + u_n(X_n) \tag{2.2}$$

such that:

$$U_J^A = \{u : (-1)^k \partial^k u / \partial X_{i_1} \ldots \partial X_{i_n} \le 0\,,\ \sum i_j = k,\ i_j \le J \ \ \forall j = 1, \ldots, n\}\,, \ \text{ and}$$
$$U_J^D = \{u : \partial^k u / \partial X_{i_1}, \ldots \partial X_{i_n} \ge 0\,,\ \sum i_j = k,\ i_j \le J \ \ \forall j = 1, \ldots, n\},$$

where u_i is the utility on X_i satisfying $(-1)^j u^{(j)} \le 0$ for any integer j if $u \in U_J^A$ and satisfying $u^{(j)} \ge 0$ for any integer j if $u \in U_J^D$.

Readers may refer to Keeney and Raiffa [21] and others for the definition of u in Definition 2.7. We note that u in (2.2) is quasiconcave (quasiconvex) if each argument is concave (convex) (see for example[3]). We also note that the theory developed in this paper fit well for the following additive and separable utility:

$$u(\vec{X}) = \alpha + \sum_{i=1}^{n} \beta_i u_i(X_i), \quad \beta_i \ge 0 \ \ \forall i.$$

However, for simplicity, in this paper we only use u in (2.2).

At last, we assume investors will choose between F and G (which could be univariate or multivariate) in accordance with a consistent set of preferences satisfying the von Neumann–Morgenstern [32] consistency properties such that F is (strictly) preferred to G, or equivalently, X is (strictly) preferred to Y if

$$\Delta Eu \equiv E\big[u(X)\big] - E\big[u(Y)\big] \ge 0 (> 0), \tag{2.3}$$

where $E\big[u(X)\big] \equiv \int_a^b u(x) \mathrm{d}F(x)$ and $E\big[u(Y)\big] \equiv \int_a^b u(x) \mathrm{d}G(x)$.

[7] Some scholars suggest the reverse direction for the definition of a Dalton Pigou transfer. In this paper, we follow Ok and Kranich [27] for the definition.

3. The theory

We first develop some new results for n-dimensional multivariate stochastic dominance (MSD) under the assumptions that the attributes are independent and the utility is additive and separable and thereafter release the assumption of independence.

3.1. Stochastic dominance for independent random variables

Quirk and Saposnik [29] and others have developed some properties of univariate SD for the preferences of risk averters while Hammond [29] and others have extended the work to include properties of univariate SD for the preferences of risk seekers. We extend their work to establish the following important theorem in SD theory.

Theorem 3.1. *Let $\vec{X}_n$ and $\vec{Y}_n$ be two vectors of n independent random variables. Suppose u is a utility function defined in Definition 2.7. For any integer N, we have*

1. $\vec{X}_n \succeq_N \vec{Y}_n$ *if and only if* $E\big[u(\vec{X}_n)\big] \geq E\big[u(\vec{Y}_n)\big]$ *for any u in U_N^A, and*
2. $\vec{X}_n \succeq^N \vec{Y}_n$ *if and only if* $E\big[u(\vec{X}_n)\big] \geq E\big[u(\vec{Y}_n)\big]$ *for any u in U_N^D.*

Proof. We only prove the sufficient part of part 1 of the theorem. The proof of part 2 can be obtained similarly and one could also easily prove the necessary part of the theorem. Recall the definition of u in Definition 2.7, we know that

$$E\big[u(\vec{X}_n)\big] = \sum_{i=1}^{n} E\big[u_i(X_i)\big].$$

Now we assume that $\vec{X}_n \succeq_N \vec{Y}_n$ holds first. According to Definition 2.5, we can obtain that $X_i \succeq_N Y_i$ for any $i = 1, 2, \dots, n$. Following Levy (2006), we then can get $E\big[u_i(X_i)\big] \geq E\big[u_i(Y_i)\big]$. This can yard that

$$E\big[u(\vec{X}_n)\big] = \sum_{i=1}^{n} E\big[u_i(X_i)\big] \geq \sum_{i=1}^{n} E\big[u_i(Y_i)\big] = E\big[u(\vec{Y}_n)\big],$$

and thus, the assertion holds. □

It is well-known that the hierarchical relationship exists in SD (see, for example, [23]). We extend their results to obtain the hierarchical relationship for MSD as stated in the following theorem:

Theorem 3.2. *For any integer N, we have*

1. *if* $\vec{X}_n \succeq_N \vec{Y}_n$, *then* $\vec{X}_n \succeq_{N+1} \vec{Y}_n$, *and*
2. *if* $\vec{X}_n \succeq^N \vec{Y}_n$, *then* $\vec{X}_n \succeq^{N+1} \vec{Y}_n$.

Theorem 3.2 can be obtained by applying Theorem 3.1, the fact that the hierarchical relationship holds for univariate SD, and Definition 2.5.

It is well-known that if $\mu_F = \mu_G$, $F \succeq_2 G$ ($F \succ_2 G$) and if their variances exist, then $\sigma_F^2 \leq \sigma_G^2$ ($\sigma_F^2 < \sigma_G^2$). If $\mu_F = \mu_G$, $F \succeq^2 G$ ($F \succ^2 G$) and if their variances exist, then $\sigma_F^2 \geq \sigma_G^2$ ($\sigma_F^2 > \sigma_G^2$). These reflect the fact that risk averters prefer to invest in prospects or portfolios with smaller variances while risk seekers prefer larger variances. Li and Wong [24], Wong and Li [35], and others establish a similar relation between the first three orders of ASD and DSD for univariate SD. We extend their results by establishing some relationships between multivariate ASD and DSD for any order as shown in the following theorem.

Theorem 3.3.

1. $\vec{X}_n \succeq_1 \vec{Y}_n$ *if and only if* $\vec{X}_n \succeq^1 \vec{Y}_n$.
2. $\vec{X}_n \succeq_i \vec{Y}_n$ *if and only if* $-\vec{Y}_n \succeq^i -\vec{X}_n$.
3. *If* $\vec{X}_n$ *and* $\vec{Y}_n$ *have the same mean which is finite, then* $\vec{X}_n \succeq_2 \vec{Y}_n$ *if and only if* $\vec{Y}_n \succeq^2 \vec{X}_n$.
4. *If* $\mu_X = \mu_Y$ *and* $F_3^A(b) = G_3^A(b)$, *then* $\vec{X}_n \succeq_3 \vec{Y}_n$ *if and only if* $\vec{X}_n \succeq^3 \vec{Y}_n$.
5. *If* $F_k^A(b) = G_k^A(b), k = 2, \ldots, N$, *then*
   (a) *for any even* N, $\vec{X}_n \succeq_N \vec{Y}_n$ *if and only if* $\vec{Y}_n \succeq^N \vec{X}_n$;
   (b) *for any odd* N, $\vec{X}_n \succeq_N \vec{Y}_n$ *if and only if* $\vec{X}_n \succeq^N \vec{Y}_n$.

Proof. One could easily obtain the proof of the first three parts of Theorem 3.3 by using the results from Lemma 3 in [24] and Definitions 2.5 and 2.6. To prove the fourth and the fifth part of Theorem 3.3, we only need to prove the results hold for univariate SD. We first prove the fourth part. Note that

$$\begin{aligned} H_2^D(x) &= \int_x^b H_1^D(y)\mathrm{d}y = \int_x^b (1 - H(y))\mathrm{d}y \\ &= (b-a) - \int_a^b H(y)\mathrm{d}y - (x-a) + \int_a^x H(y)\mathrm{d}y \\ &= b - x - H_2^A(b) + H_2^A(x). \end{aligned}$$

Furthermore, we get

$$\begin{aligned} H_3^D(x) &= \int_x^b H_2^D(y)\mathrm{d}y = \int_x^b (b - y - H_2^A(b) + H_2^A(y))\mathrm{d}y \\ &= \int_x^b (b-y)\mathrm{d}y - H_2^A(b)(b-x) + H_3^A(b) - H_3^A(x). \end{aligned}$$

We then obtain

$$F_3^D(x) - G_3^D(x) = F_3^A(b) - G_3^A(b) - F_3^A(x) + G_3^A(x) + (b-x)(G_2^A(b) - F_2^A(b)). \tag{3.1}$$

In addition, if $F_n^A(b) = G_n^A(b)$ for $n = 2, 3$, we have

$$F_3^D(x) - G_3^D(x) = G_3^A(x) - F_3^A(x).$$

As a result, we can conclude that $X \succeq_3 Y$ if and only if $X \succeq^3 Y$. Now we turn to the general case as stated in the fifth part. We prove by induction on N. Assume that if $F_k^A(b) = G_k^A(b), k = 2, \ldots, N$, then $F_N^D(x) - G_N^D(x) = G_N^A(x) - F_A^N(x)$. The result is true if $N = 3$. Then, we obtain

$$\begin{aligned} F_{N+1}^D(x) &= \int_x^b F_N^D(y)\mathrm{d}y = \int_x^b (G_N^D(y) + G_N^A(y) - F_A^N(y))\mathrm{d}y \\ &= G_{N+1}^D(x) + G_{N+1}^A(b) - F_{N+1}^A(b) + F_{N+1}^A(x) - G_{N+1}^A(x). \end{aligned}$$

For $N+1$, if $F_k^A(b) = G_k^A(b), k = 2, \ldots, N+1$, we get $F_{N+1}^D(x) - G_{N+1}^D(x) = F_{N+1}^A(x) - G_{N+1}^A(x)$. Thus, the assertion holds. $\square$

We note that Theorem 3.3 tell us that the preference of risk averters and risk seekers could be of the same directions sometimes while in other situations, it could be opposite. Moreover, one could also easily obtain the "transitivity" property for MSD as shown in the following theorem.

Theorem 3.4.

1. *If* $\vec{X} \succeq_i \vec{Y}$ *and* $\vec{Y} \succeq_j \vec{Z}$*, then* $\vec{X} \succeq_k \vec{Z}$*, and*
2. *if* $\vec{X} \succeq^i \vec{Y}$ *and* $\vec{Y} \succeq^j \vec{Z}$*, then* $\vec{X} \succeq^k \vec{Z}$*,*

where $k = \max(i, j)$*.*

Proof. One could prove Theorem 3.4 by applying the results in Theorems 3.1 and 3.2. Without loss of generality, we assume $k = \max(i, j) = j$. In the following, we focus on part 1 of Theorem 3.4. According to Theorem 3.2, if $\vec{X} \succeq_i \vec{Y}$, we can obtain that $\vec{X} \succeq_j \vec{Y}$. Applying Theorem 3.1, we get $E\big[u(\vec{X})\big] \geq E\big[u(\vec{Y})\big]$ for any u in U_j^A. We also obtain $E\big[u(\vec{Y})\big] \geq E\big[u(\vec{Z})\big]$ for any u in U_j^A. Thus, we have $E\big[u(\vec{X})\big] \geq E\big[u(\vec{Z})\big]$ for any u in U_j^A. Consequently, we can get $\vec{X} \succeq_k \vec{Z}$. □

Now, we turn to study the properties for convex combinations of random variables. If $X, Y, \ldots$ is the returns of individual assets, convex combinations of $X, Y, \ldots$ are the returns of the portfolios of different assets. Note that for any pair of random variables X and Y, $X \succeq_m Y$, and $F \succeq_m G$ are the same for any integer m. However, for $n > 1$, the convex combinations of random variables, say, $\sum_{i=1}^n \alpha_i X_i \succeq_m \sum_{i=1}^n \alpha_i Y_i$ are different from the convex combinations of distribution functions, say, $\sum_{i=1}^n \alpha_i F_i \succeq_m \sum_{i=1}^n \alpha_i G_i$. Readers may refer to the convex stochastic dominance theorems [35] for the convex combinations of distribution functions while this paper studies the convex combinations of random variables.

Hadar and Russell [13] and others have developed some results of the univariate SD of random variables that are in the same location and scale family [36]. In this paper we extend their work by developing the following theorem for MSD.

Theorem 3.5. *Let* $\vec{X}$ *be a random vector with mean* $\mu_{\vec{X}} = (\mu_{X_1}, \ldots, \mu_{X_n})'$ *and elements* $\{X_i\}$*, each defined in* $[a, b]$*. Suppose the random variable* $\vec{Y} = p + q\vec{X}$ *with mean* $\mu_{\vec{Y}} = (\mu_{Y_1}, \ldots, \mu_{Y_n})'$*.*

1. *If* $p + qy \geq y$ *for all* $y \in [a, b]$*, then* $\vec{Y} \succeq_1 \vec{X}$*, equivalently* $\vec{Y} \succeq^1 \vec{X}$*.*
2. *If* $0 \leq q < 1$ *such that* $p/(1-q) \geq \mu_{X_i}$*; that is,* $\mu_{Y_i} \geq \mu_{X_i}$*, for each* i*, then* $\vec{Y} \succeq_2 \vec{X}$*.*
3. *If* $0 \leq q < 1$ *such that* $p/(1-q) \leq \mu_{X_i}$*; that is,* $\mu_{Y_i} \leq \mu_{X_i}$*, for each* i*, then* $\vec{X} \succeq^2 \vec{Y}$*.*

Readers could apply Theorems 3.1 and 8 of [13] to obtain the proof of Theorem 3.5. Hadar and Russell [13] and others have developed some relationships for linear combinations of random variables for univariate SD. In this paper, we extend their work to obtain the following theorem for linear combinations of random vectors in MSD.

Theorem 3.6. *Let* $\{\vec{X}_{n,1}, \ldots, \vec{X}_{n,m}\}$ *and* $\{\vec{Y}_{n,1}, \ldots, \vec{Y}_{n,m}\}$ *be two sets of independent vectors for random variables. For* $j = 1, 2, \ldots, N$*, we have:*

1. $\vec{X}_{n,i} \succeq_j \vec{Y}_{n,i}$ *for* $i = 1, \ldots, m$ *if and only if* $\sum_{i=1}^m \alpha_i \vec{X}_{n,i} \succeq_j \sum_{i=1}^m \alpha_i \vec{Y}_{n,i}$ *for any* $\alpha_i \geq 0, i = 1, \ldots, m$*; and*
2. $\vec{X}_{n,i} \succeq^j \vec{Y}_{n,i}$ *for* $i = 1, \ldots, m$ *if and only if* $\sum_{i=1}^m \alpha_i \vec{X}_{n,i} \succeq^j \sum_{i=1}^m \alpha_i \vec{Y}_{n,i}$ *for any* $\alpha_i \geq 0, i = 1, \ldots, m$*.*

Proof. To prove the above theorem, we only need to show that $X_{n,il} \succeq_j Y_{n,il}$ for $i = 1, \ldots, m$ if and only if $\sum_{i=1}^m \alpha_i X_{n,il} \succeq_j \sum_{i=1}^m \alpha_i Y_{n,il}$ for any $\alpha_i \geq 0, i = 1, \ldots, m$ and any $l = 1, \ldots, n$.

The proofs for the necessary parts are obvious. For the sufficient part, it suffices to prove the following two lemmas:

Lemma 3.7. X *and* Y *are random variables. For any integer* j *and for* $\alpha > 0, X \succeq_j (\succ_j) Y$ *implies* $\alpha X \succeq_j (\succ_j) \alpha Y$*.*

Lemma 3.8. *Suppose* X_i *and* Y_i $(i = 1, 2)$ *are random variables such that* X_1 *and* X_2 *are independent, and* Y_1 *and* Y_2 *are independent. For any integer* j *if* $X_i \succeq_j (\succ_j) Y_i$ *for* $i = 1$ *and* 2*, then* $X_1 + X_2 \succeq_j (\succ_j) Y_1 + Y_2$*.*

The proof of Lemma 3.7 is obvious. For Lemma 3.8, we suppose without loss of generality, we assume X_i and Y_i $(i = 1, 2)$ to be defined in $[a, b]$. Li and Wong [24] have proved the result for the second order. In the following, we give the proof for the general j-order. Let $\tilde{X}_1 = X_1 + k$ and $\tilde{Y}_1 = Y_1 + k$. Let the probability distribution functions of X_1, $\tilde{X}_1$, Y_1 and $\tilde{Y}_1$ be F_1, $\tilde{F}_1$, G_1 and $\tilde{G}_1$, respectively.

We can know that $\tilde{F}_1(x) = F_1(x - k)$ and $\tilde{G}_1(x) = G_1(x - k)$.

From Proposition 1 in Ogryczak and Ruszczynski (2001), we know that

$$F_j^A(\eta) = \frac{1}{(j-1)!}\int_{-\infty}^{\eta}(\eta - x)^{j-1}\mathrm{d}F(x) = \frac{1}{(j-1)!}E(\eta - X)_+^{j-1}.$$

Here, the function $t \mapsto (t)_+ = \max(0, t)$ and $j \geq 3$. Thus, we can have

$$\begin{aligned}
\tilde{F}_j^A(\eta) &= \frac{1}{(j-1)!}\int_{-\infty}^{\eta}(\eta - x)^{j-1}d\tilde{F}(x) \\
&= \frac{1}{(j-1)!}\int_{-\infty}^{\eta}(\eta - x)^{j-1}\mathrm{d}F_1(x-k) \\
&= \frac{1}{(j-1)!}\int_{-\infty}^{\eta-k}(\eta - k - t)^{j-1}\mathrm{d}F_1(t) = \frac{1}{(j-1)!}E(\eta - k - X)_+^{j-1}.
\end{aligned}$$

Consequently, if $X_1 \succeq_j (\succ_j) Y_1$ then $X_1 + k \succeq_j (\succ_j) Y_1 + k$. As a result, we can have $E\big[u(X_1+k)\big] \geq E\big[u(Y_1+k)\big]$ for any $u \in U_j^A$. Now we turn to compare $X_1 + X_2$ with $Y_1 + Y_2$. Note that

$$E\big[u(X_1 + X_2)\big] = E\big[E(u(X_1 + X_2)|X_2)\big] \geq E\big[E(u(Y_1 + X_2)|X_2)\big] = E\big[u(Y_1 + X_2)\big].$$

Thus, we conclude that $X_1 + X_2 \succeq_j (\succ_j) Y_1 + X_2$. Similarly, we can prove that $Y_1 + X_2 \succeq_j (\succ_j) Y_1 + Y_2$. Based on these results, we conclude that the assertion $X_1 + X_2 \succeq_j (\succ_j) Y_1 + Y_2$ holds. □

On the other hand, Hadar and Russell [13] and others have developed some results for risk averters to compare their preference on the individual assets with partial and completed diversified portfolios for univariate SD while Li and Wong [24] extend the result to the preference of risk seekers. In this paper, we extend their results for the convex combinations of random variables for MSD as shown in the following theorem:

Theorem 3.9. *Let $m \geq 2$. If $\vec{X}_{n,1}, \ldots, \vec{X}_{n,m}$ are i.i.d., then*

$$\text{(a)} \qquad \frac{1}{m}\sum_{i=1}^{m}\vec{X}_{n,i} \succeq_2 \sum_{i=1}^{m}\lambda_i\vec{X}_{n,i} \succeq_2 \vec{X}_{n,i} \qquad \textit{for any} \quad (\lambda_1, \ldots, \lambda_m) \in S_m^0\,, \qquad \textit{and}$$

$$\text{(b)} \qquad \vec{X}_{n,i} \succeq^2 \sum_{i=1}^{m}\lambda_i\vec{X}_{n,i} \succeq^2 \frac{1}{m}\sum_{i=1}^{m}\vec{X}_{n,i} \qquad \textit{for any} \quad (\lambda_1, \ldots, \lambda_m) \in S_m^0\,,$$

where S_m^0 is defined in (2.1).

Readers could apply Theorem 12 of Li and Wong (1999) and Theorem 3.1 to prove Theorem 3.9.

3.2. Stochastic dominance and majorization

Now, we turn to develop the MSD results by applying the theory of majorization and Dalton transfer. To do so, we first state the following proposition [10] that could be used to develop some stochastic dominance relationships.

Proposition 3.10. *Let $\vec{\alpha}_n$, $\vec{\beta}_n \in S_n$, $\vec{\beta}_n \succeq_M \vec{\alpha}_n$ if and only if $\vec{\alpha}_n$ can be obtained from $\vec{\beta}_n$ by applying a finite number of Dalton transfers, denoted by $\vec{\beta}_n \xrightarrow{D} \vec{\alpha}_n$.*

Theorem 3.9 enables investors to compare the preferences among any specialized asset, any partially diversified portfolio, with the completely diversified portfolio. However, it cannot be used to compare any two different partially diversified portfolios. We now generalize Theorem 7 in [10] to obtain the following theorem to make such comparison become possible.

Theorem 3.11. *For $n > 1$, let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$*[8] *and $\vec{X}_n = (X_1, \ldots, X_n)'$ where $X_1, \ldots, X_n$ are i.i.d. If $\vec{\beta}_n \succeq_M \vec{\alpha}_n$, then $\vec{\alpha}'_n \vec{X}_n \succeq_2 \vec{\beta}'_n \vec{X}_n$ and $\vec{\beta}'_n \vec{X}_n \succeq^2 \vec{\alpha}'_n \vec{X}_n$.*

In addition, one could easily applying Proposition 3.10 and Theorem 3.11 to obtain the following corollary:

Corollary 3.12. *For $n > 1$, let $\vec{\alpha}_n, \vec{\beta}_n \in S_n$ and $\vec{X}_n = (X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ are i.i.d. If $\overrightarrow{\beta}_n \stackrel{D}{\rightarrow} \overrightarrow{\alpha}_n$, then $\vec{\alpha}'_n \vec{X}_n \succeq_2 \vec{\beta}'_n \vec{X}_n$ and $\vec{\beta}'_n \vec{X}_n \succeq^2 \vec{\alpha}'_n \vec{X}_n$.*

Between any pair of two partially diversified portfolios $\vec{\alpha}'_n \vec{X}_n$ and $\vec{\beta}'_n \vec{X}_n$, if the conditions in Theorem 3.11 or Corollary 3.12 are satistified, Theorem 3.11 and Corollary 3.12 tell us that risk averters will prefer $\vec{\alpha}'_n \vec{X}_n$ whereas risk seekers will prefer $\vec{\beta}'_n \vec{X}_n$.

Can the i.i.d. assumption be dropped in the diversification problem and the completely diversified portfolio still be optimal? Samuelson [30] tells us that the answer is no in general. He further establishes some results to relax the i.i.d. assumption. In this paper, we complement Samuelson's work by applying Proposition 3.10 and Theorem 3.11 to obtain the following corollaries:

Corollary 3.13. *For $n > 1$, let $\vec{X}_n = (X_1, \ldots, X_n)'$ be a series of random variables that could be dependent. For any $\vec{\alpha}_n$ and $\vec{\beta}_n$, if there exist $\vec{Y}_n$ and A_{nn} with $\vec{Y}_n = (Y_1, \ldots, Y_n)'$ in which $\{Y_1, \ldots, Y_n\}$ are i.i.d., $\vec{X}_n = A_{nn} \vec{Y}_n$ such that*

$$\vec{\beta}'_n A_{nn} \succeq_M \vec{\alpha}'_n A_{nn} \quad \text{or} \quad \vec{\beta}'_n A_{nn} \stackrel{D}{\rightarrow} \vec{\alpha}'_n A_{nn} \,,$$

with $\vec{\alpha}'_n A_{nn}$, $\vec{\beta}'_n A_{nn} \in S_n$, then

$$\vec{\alpha}'_n \vec{X}_n \succeq_2 \vec{\beta}'_n \vec{X}_n \quad \text{and} \quad \vec{\beta}'_n \vec{X}_n \succeq^2 \vec{\alpha}'_n \vec{X}_n$$

Corollary 3.14. *For $n > 1$, let $\vec{X}_n = (X_1, \ldots, X_n)'$ and $\vec{Y}_n = (Y_1, \ldots, Y_n)'$ be two series of random variables that could be dependent. For any $\vec{\alpha}_n$ and $\vec{\beta}_n$, if there exist $\vec{U}_n = (U_1, \ldots, U_n)'$, $\vec{V}_n = (V_1, \ldots, V_n)'$, A_{nn}, and B_{nn} in which $\{U_1, \ldots, U_n\}$ and $\{V_1, \ldots, V_n\}$ are two series of i.i.d. random variables with $\vec{X}_n = A_{nn} \vec{U}_n$, $\vec{Y}_n = B_{nn} \vec{V}_n$, $U_i \succeq_2 V_i$ for all $i = 1, 2, ..., n$ such that*

$$\vec{\beta}'_n B_{nn} \succeq_M \vec{\alpha}'_n A_{nn} \quad \text{or} \quad \vec{\beta}'_n B_{nn} \stackrel{D}{\rightarrow} \vec{\alpha}'_n A_{nn} \,,$$

where $\vec{\alpha}'_n A_{nn}$, $\vec{\beta}'_n B_{nn} \in S_n$, then

$$\vec{\alpha}'_n \vec{X}_n \succeq_2 \vec{\beta}'_n \vec{Y}_n \quad \text{and} \quad \vec{\beta}'_n \vec{Y}_n \succeq^2 \vec{\alpha}'_n \vec{X}_n$$

4. Discussions

In this section we will discuss briefly the importance and the use of the results developed in this paper. Theorem 3.1 states that ranking multivariate prospects in the sense of multivariate ascending and descending MSD is equivalent to the expected-utility maximization for risk averters and risk seekers, respectively. With this establishment, if one would like to compare the preferences on different prospects for different types of investors, it is not necessary to measure the utilities for different types of investors and analyze their expected utilities. One only needs to find out the orders and the types of SD for different prospects. This information could then enable us to draw conclusion on the preferences for different types of investors on the prospects. This is the

[8] We keep the condition $\sum_{i=1}^{n} s_i = 1$ in S_n for convenience. One could exclude this condition and relax it to be $\vec{1}'_n \vec{\alpha}_n = \vec{1}'_n \vec{\beta}_n$.

basic principle academics apply the SD theory to many areas like economics and finance. For example, Qiao *et al.* [28] find that stocks SASD dominates futures and futures SDSD dominates stocks and conclude that risk averters prefer to buy stocks, whereas risk seekers prefer long index futures. Recently, Davidson and Duclos [4] and others develop test statistics for ASD while Bai *et al.* [1] extend their work by developing test statistics for both ASD and DSD. The tests could be used to apply the theory of ASD and DSD to empirical issues.

Theorem 3.2 establishes the hierarchical relationship in ASD and DSD. This relationship is important because, with the establishment of this theorem, only the lowest dominance order of SD is needed to be reported in practice. On the other hand, Theorem 3.3 is important because it enables one to realize the DSD preferences of prospects if their ASD preferences are known and vice versa. On the other hand, Theorem 3.4 tells us that the "transitivity" property holds for MSD, and thus, investors only need to evaluate the set of the most efficient ones and ignore the inefficient assets.

We call a person a ASD risk averter if his/her utility function belongs to U_2^A, and a SDSD risk seeker if his/her utility function belongs to U_2^D. The other orders of ASD risk averter and DSD risk seeker can be defined similarly. Applying Theorem 3.5 to Corollary 3.14, we could obtain several interesting properties. We state the following two properties in this paper:

Property 4.1. *For the portfolio of n vectors of i.i.d. prospects with $n \geq 2$,*

1. *SASD risk averters will prefer to invest in the completely diversified portfolio than any partially diversified portfolio, which, in turn, is preferred to any specialized asset; and*
2. *SDSD risk seekers will prefer to invest in any specialized asset than any partially diversified portfolio, which, in turn, is preferred to the completely diversified portfolio.*

Property 4.2. *Between the two partially diversified portfolios $\vec{\alpha}_n'\vec{X}_n$ and $\vec{\beta}_n'\vec{Y}_n$, if $\vec{\beta}_n$ majorizes $\vec{\alpha}_n$, or if the conditions in Corollaries 3.13 and 3.14 are satisfied, then SASD risk averters prefer to invest in $\vec{\alpha}_n'\vec{X}_n$ while SDSD risk seekers prefer to invest in $\vec{\beta}_n'\vec{Y}_n$.*

5. Concluding remarks

This paper develops some properties of the multivariate ASD and DSD for risk averters and risk seekers, respectively, and discuss how to apply the results to investment decision-making. We remark that though we have developed some MSD relationships when attributes are dependent but the dependent situations are restricted to some special situations. Further research could extend the results to include more general situations. In addition, it would be interesting to extend the theory developed in this paper for utility that is not additively or separable. Further study could extend univariate SD theory to multivariate SD theory for other types of investors, for example, S-shaped and reverse S-shaped investors. Readers may refer to, for example, [34], Egozcue, *et al.* (2011), Clark, *et al.* (2015), and the references therein for more information on S-shaped and reverse S-shaped investors.

At last, we note that the SD theory could be used to explain many financial anomalies. For example, Jegadeesh and Titman [20] document a financial anomaly on momentum profit in stock markets that extreme movements in stock prices will be followed by subsequent price movements in the same direction. In another words, former winners are still winners and former losers are still losers. If investors know that past winners are still be winners and past losers are still be losers, they would buy winners and sell losers. This will drive up the price of winners relative to losers until the market price of winners relative to losers is high enough to make the momentum profit disappear. However, after many years, many studies still find momentum profits empirically. We note that the SD theory could explain this financial anomaly well. For example, extending the work of Fong [12], Sriboonchita *et al.* [31] conclude that winners dominate losers in the sense of the second order ASD while losers dominate winners in the sense of the second order DSD, inferring that risk averters will prefer to invest in winners whereas risk seekers will prefer to invest in losers. This finding could explain why the momentum profit could still exist after discovery. If the market do consists of both risk averters and risk seekers, risk averters prefer to invest in winners while risk seekers prefer to invest in losers. Thus, both risk averters and risk seekers would get what they want in the market and will not drive up the price of winners or drive down the price of losers and thus the momentum profit could still exist after discovery.

Dentcheva and Ruszczynski [6] consider stochastic optimization problem with multivariate stochastic dominance constraints. They introduce the concept of positive linear multivariate stochastic dominance. They consider linear scalarization with positive coefficients and apply a univariate SSD constraint to all nonnegative weighted combinations of random outcomes. To be precise, two random vectors, X is said to dominate Y in positive linear second order, written $X \succeq_{(2)}^{Plin} Y$, if $c^\tau X \succeq_{(2)} c^\tau Y$ for all $c \in \mathrm{R}_+^m$. Following this linear scalarization idea, Homem-de-Mello and Mehrotra [17] further propose the polyhedral second order dominance by allowing the set of scalarization coefficients to be an arbitrary polyhedron. Hu *et al.* [18] develop an even more general concept of dominance by allowing arbitrary convex scalarization sets. These several interesting papers all focus on optimization problem with multivariate stochastic dominance constraints, refining and extending earlier results for optimization under univariate stochastic dominance constraints, see, for example, [5, 8] for more information.

Acknowledgements. The authors are grateful to A. Ridha Mahjoub, the Editor-in-Chief for substantive comments that have significantly improved this manuscript. The authors would like to thank Martin Egozcue for his valuable comments that have significantly improved this manuscript. The second author would also like to thank Robert B. Miller and Howard E. Thompson for their continuous guidance and encouragement. The research described here is supported by the Natural Science Foundation of Jiangsu Province, China (grant No. BK20150732), Hong Kong Baptist University, and the Research Grants Council (RGC) of Hong Kong (projects Nos. 12502814 and 12500915), and Chiang Mai University.

References

[1] Z.D. Bai, H.X. LiU and W.K. Wong, Enhancement of the Applicability of Markowitz's Portfolio Optimization by Utilizing Random Matrix Theory. *Math. Finance* **19** (2009) 639–667.

[2] Z.D. Bai, H. Li, M. McAleer and W.K. Wong, Stochastic Dominance Statistics for Risk Averters and Risk Seekers: An Analysis of Stock Preferences for USA and China. *Quant. Finance* **15** (2015) 889–900.

[3] J.C. Cox, A Theorem on Additively-Separable Quasi-Concave Functions. *J. Econ. Theory* **6** (1973) 210–212.

[4] R. Davidson and J.Y. Duclos, Statistical inference for stochastic dominance and for the measurement of poverty and inequality. *Econometrica* **68** (2000) 1435–1464.

[5] D. Dentcheva and A. Ruszczynski, Optimization with stochastic dominance constraints. *SIAM J. Optim.* **14** (2003) 548–566.

[6] D. Dentcheva and A. Ruszczynski, Optimality and duality theory for stochastic optimization problems with nonlinear dominance constraints. *Math. Program.* **99** (2004) 329–350.

[7] D. Dentcheva and A. Ruszczynski, Optimization with multivariate stochastic dominance constraints, *Math. Program.* **117** (2009) 111–127.

[8] M. Denuit and L. Eeckhoudt, Bivariate stochastic dominance and substitute risk (in)dependent utilities. *Decis. Anal.* **7** (2010) 302–312.

[9] M. Denuit, L. Eeckhoudt and B. Rey, Some consequences of correlation aversion in decision science. *Ann. Oper. Res.* **176** (2010) 259–269.

[10] M. Egozcue and W.K. Wong, Gains from Diversification: A Majorization and Stochastic Dominance Approach. *Eur. J. Oper. Res.* **200** (2010) 893–900.

[11] R. Eisner and R. Strotz, Flight insurance and the theory of choice. *J. Political Economy* **69** (1961) 355–368.

[12] W.M. Fong, W.K. Wong and H.H. Lean, International Momentum Strategies: A Stochastic Dominance Approach. *J. Financ. Mark.* **8** (2005) 89–109.

[13] J. Hadar and W.R. Russell, Stochastic Dominance and Diversification. *J. Econ. Theory* **3** (1971) 288–305.

[14] J.S. Hammond, Simplifying the Choice between Uncertain Prospects where Preference is Nonlinear. *Manage. Sci.* **20** (1974) 1047–1072.

[15] G.H. Hardy, J.E. Littlewood and G. Pólya, Inequalities, Cambridge University Press. Cambridge, MA (1934).

[16] G.B. Hazen, Partial information, dominance, and potential optimality in multiattribute utility theory. *Oper. Res.* **34** (1986) 296–310.

[17] T. Homem-de-Mello and S. Mehrotra, A cutting surface method for uncertain linear programs with linear stochastic dominance constraints. *SIAM J. Optim.* **20** (2009) 1250–1273.

[18] J. Hu, T. Homem-de-Mello and S. Mehrotra, Sample average approximation of stochastic dominance constrained programs. *Math. Program.* **133** (2012) 171–201.

[19] W.H. Jean, The geometric mean and stochastic dominance. *J. Finance* **35** (1980) 151–158.

[20] N. Jegadeesh and S. Titman, Returns to Buying Winners and Selling Losers: Implications for Stock Market Efficiency. *J. Finance* **48** 1993 65–91.

[21] R.L. Keeney and H. Raiffa, Decisions with Multiple Objectives: Preferences and Value Tradeoffs. Wiley, New York (1976).

[22] H. Levy, Stochastic Dominance, Efficiency Criteria, and Efficient Portfolios: The Multi-Period Case. *Am. Econ. Rev.* **43** (1973) 986-994.

[23] H. Levy, Stochastic dominance and expected utility: Survey and analysis. Management Science **38** (1992) 555–593.

[24] C.K. Li and W.K. Wong, Extension of Stochastic Dominance Theory to Random Variables. *RAIRO: OR* **33** (1999) 509–524.

[25] H.M. Markowitz, The Utility of Wealth. *J. Political Economy* **60** (1952a) 151–156.

[26] H.M. Markowitz, Portfolio Selection. *J. Finance* **7** (1952b) 77–91.

[27] E. Ok and L. Kranich, The Measurement of Opportunity Inequality: a Cardinality-Based Approach. *Social Choice Welfare* **15** (1998) 263–287.

[28] Z. Qiao, E. Clark and W.K., Wong, Investors' Preference towards Risk: Evidence from the Taiwan Stock and Stock Index Futures Markets. *Accounting & Finance* **54** (2014) 251–274.

[29] J.P. Quirk and R. Saposnik, Admissibility and Measurable Utility Functions. *Rev. Econ. Stud.* **29** (1962) 140–146.

[30] P.A. Samuelson, General Proof that Diversification Pays. *J. Finance Quant. Anal.* **2** (1967) 1–13.

[31] S. Sriboonchita, W.K. Wong, D. Dhompongsa and H.T. Nguyen, Stochastic Dominance and Applications to Finance, Risk and Economics. Chapman and Hall/CRC, Boca Raton, Florida (2009).

[32] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior. Princeton University Press, Princeton N.J. (1944).

[33] W.K. Wong, Stochastic Dominance and Mean-Variance Measures of Profit and Loss for Business Planning and Investment. *Eur. J. Oper. Res.* **182** (2007) 829–843.

[34] W.K. Wong and R. Chan, Markowitz and Prospect Stochastic Dominances. *Ann. Finance* **4** (2008) 105–129.

[35] W.K. Wong and C.K. Li, A Note on Convex Stochastic Dominance Theory. *Econ. Lett.* **62** (1999) 293–300.

[36] W.K. Wong and C. Ma, Preferences over Location-Scale Family. *Econom. Theory* **37** (2008) 119–146.