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CHANCE-CONSTRAINED DATA ENVELOPMENT ANALYSIS MODELING WITH RANDOM-ROUGH DATA *

Rashed Khanjani Shiraz[1], Madjid Tavana[2,3,a] and Debora Di Caprio[4,5]

Abstract. Data envelopment analysis (DEA) is a useful management tool for measuring the relative efficiency of decision making units (DMUs) which consumes multiple inputs to produce multiple outputs. Although precise input and output data are fundamentally indispensable in classical DEA models, real-world problems often involve random and/or rough input and output data. We present a chance-constrained DEA model with random and rough (random-rough) input and output data and propose a deterministic equivalent model with quadratic constraints to solve the model. The main contributions of this paper are fourfold: (3.1) we propose a DEA model for problems characterized by random-rough variables; (3.2) we transform the proposed chance-constrained model with random-rough variables into a deterministic equivalent non-linear form that could be simplified as a deterministic model with quadratic constraints; (3.3) we perform sensitivity analysis to investigate the stability and robustness of the proposed model; and (3.4) we use a numerical example to demonstrate the feasibility and richness of the obtained solutions.

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1. Introduction

Data envelopment analysis (DEA), initially developed by Charnes *et al.* [3], is a non-parametric method to identify efficient production frontiers and evaluate the relative efficiency of decision making units (DMUs) where each unit is responsible for converting multiple inputs into multiple outputs. The conventional DEA models require precise and known values for the inputs and outputs. However, this assumption may not be satisfied in many real-world problems characterized by imprecise and unknown data. As a consequence, a wide

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* *Business Systems and Analytics Department, Distinguished Chair of Business Analytics, La Salle University, Philadelphia, PA 19141, United States.*

[1] School of Mathematics Science, University of Tabriz, Tabriz, Iran.
Rashed.shiraz@gmail.com

[2] Business Systems and Analytics Department, Distinguished Chair of Business Analytics, La Salle University, Philadelphia, PA 19141, USA.
[a] Corresponding author: tavana@lasalle.edu; http://tavana.us/

[3] Business Information Systems Department, Faculty of Business Administration and Economics, University of Paderborn, 33098 Paderborn, Germany

[4] Department of Mathematics and Statistics, York University, Toronto, M3J 1P3, Canada. dicaper@mathstat.yorku.ca

[5] Polo Tecnologico IISS G. Galilei, Via Cadorna 14, 39100, Bolzano, Italy. debora.dicaprio@istruzione.it

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range of DEA models have been proposed to evaluate DMUs with uncertain data. Fuzzy, random, and rough sets are commonly used to formalize the uncertainties inherent to real-world performance evaluation problems.

Land *et al.* [13] extended the chance-constrained DEA model proposed by Charnes and Cooper 1959 to compute the efficiency of DMUs facing the uncertainty deriving from the inputs being deterministic and the outputs jointly normally distributed. Olesen and Petersen [22] developed a chance-constrained DEA model by imposing chance-constraints on the multiplier model. Olesen [21] presented a comparison of two different models (Land *et al.* [13] and Olesen and Petersen [22]), both designed to extend DEA to the case of stochastic inputs and outputs. Morita and Seiford [19] studied robustness of the efficiency results when input and output data are subject to stochastic measurement error. Huang and Li [10] developed stochastic models in DEA by taking into account the possibility of random variations in the input and output data. They proposed dominance structures on the DEA and removed the Pareto efficient DMUs from the DEA envelopment side. Huang and Li [11] developed the chance-constrained DEA model with separate chance-constraints. They defined the efficiency dominance of a DMU using joint probabilistic comparisons of inputs and outputs with other DMUs. Talluri *et al.* [32] applied chance-constrained DEA model to vendor evaluation.

Cooper *et al.* [4] presented a joint chance-constrained programming in the multiplier DEA model. Cooper *et al.* [7], Li [14], and Bruni *et al.* [1] utilized joint chance- constraints to extend the concept of "Stochastic efficiency" to a measure called "α-stochastic efficiency". Cooper *et al.* [6] used chance-constrained programming to extend congestion in DEA models. Cooper *et al.* [5] provided chance-constrained programming models for identifying technical efficiencies and inefficiencies. Sueyoshi [30] proposed a stochastic DEA model to plan the restructuring strategy in the petroleum industry by incorporating futuristic data in their stochastic model. Tsionas and Papadakis [36] developed Bayesian inference techniques in stochastic DEA models. Udhayakumar *et al.* [37] used a genetic algorithm to solve the chance-constrained DEA models involving the concept of satisficing. Wu *et al.* [38] proposed a stochastic DEA model by considering undesirable outputs with weak disposability. This model not only deals with the existence of random errors in the collected data, but also depicts the production rules uncovered by the weak disposability of undesirable outputs.

Tavana *et al.* [33] developed three imprecise DEA models in the presence of probability-possibility, probability-necessity and probability-credibility constraints where fuzziness and randomness simultaneously exist in an evaluation problem. Tavana *et al.* [34] introduced random fuzzy variables in DEA when randomness and vagueness coexist in the same problem. The authors propose three DEA models for measuring the radial efficiency of DMUs when the input and output data are random fuzzy variables with Poisson, uniform and normal distributions. Hence, they extend the formulation of the possibility-probability and the necessity-probability DEA models with random fuzzy parameters to a production possibility set where the random fuzzy inputs and outputs have normal distributions with fuzzy means and variances. Tavana *et al.* [35] proposed a chance-constrained DEA model with birandom input and output data. They formulate a super-efficiency model with birandom constraints and obtain a non-linear deterministic equivalent model to solve the super-efficiency model.

Since the pioneering work of stochastic theory and rough set theory, several approaches have been well-developed and applied to a wide variety of real-world problems (Lempel and Moran, [15]). These approaches include expectation models Liu, [17], chance-constrained programming (Charnes and Cooper, 1959; Liu, [17]), and dependent-chance programming (Liu, [16]; Liu, [18]). However, randomness and roughness are always treated separately in all of the models in the literature.

Random phenomenon is one class of uncertainty that has been studied by many scholars in connection with mathematical programming problems. In real-world problems, it is not unusual to have to deal with two or more concurrent uncertainty factors. However, many researchers believe that the classical one-fold uncertain variables (random, fuzzy, and rough variables) cannot always be used to clearly represent complicated and involved real-world problems where randomness and roughness coexist at the same time. In such cases, the concept of random-rough variable turns out to be a useful tool in dealing with these two types of uncertainty simultaneously. Recently Khanjani *et al.* [12], Tavana *et al.* ([33, 34], 2014) and Paryab *et al.* [23] presented DEA models with two-fold uncertain data. Khanjani *et al.* [12] proposed fuzzy rough DEA models based on the expected value and possibility approaches. Paryab *et al.* [23] proposed DEA models using a bi-fuzzy data based possibility approach. However, there has been no attempt to study randomness and roughness simultaneously in DEA problems.

An input or output variable in a DEA problem can be a normally distributed random variable, and its mean values can still be a rough variable. In this study, we consider random variables with rough parameters (random-rough variables) in DEA problems. Liu [17] introduced the concept of random-rough variable by combining randomness and roughness. Liu [18] extended the concept of random-rough variable and proposed a random-rough expected value model (EVM). Xu and Yao ([40, 41, 42, 43]) discussed the basic definitions and properties of random-rough variables, and introduced EVM, chance-constrained, dependent-chance, and bi-level models. Xu and Yao [40] showed that random-rough variables can represent many real-world problems including seasonal products selling (*e.g.*, ice cream, Christmas trees, and woolen materials) where demand may vary from one year to another. They further argue that the historical data in these seasonal problems are subject to stochastic variations and that the expected value of the stochastic distribution is imprecise and varies from year to year. Finally, Xu and Yao ([40,41,42,43]) proposed several crisp equivalent models and applied them to many real-world queuing, inventory, and production planning problems.

The main contributions of this paper are fourfold: (3.1) we propose a DEA model for problems characterized by random-rough variables; (3.2) we transform the proposed chance-constrained model with random-rough variables into a deterministic equivalent non-linear form that could be simplified as a deterministic model with quadratic constraints; (3.3) we perform sensitivity analysis to investigate the stability and robustness of the proposed model; and (3.4) we use a numerical example to demonstrate the feasibility and richness of the obtained solutions. The remainder of this paper is organized as follows. In Section 2, we provide some preliminaries and definitions about rough and random-rough variables. In Section 3, we introduce the random-rough CCR model proposed in this study. In Section 4, we present the mathematical details of the expected value operator and in Section 5, we introduce a sensitivity analysis framework for this model. In Section 6, we present a numerical example to demonstrate the applicability of the proposed random-rough CCR model. Finally, in Section 7, we present our conclusions and future research directions.

2. Preliminaries and definitions

Rough set theory (Pawlak, [24, 25]) is an efficient mathematical tool to deal with imprecise, inconsistent, and incomplete data. Pawlak and Skowron [26, 27, 28] have extensively studied rough sets and their applications. Dubois and Prade [8] extended rough set theory into the fuzzy direction. Tao and Xu [31] presented a rough multiple objective programming model for a solid transportation problem. Other researchers have successfully applied rough set theory to feature selection, attribute reduction, and rule learning problems (Nguyen, [20]; Qian *et al.*, [29]).

Trust theory, introduced by Liu [17], is a branch of mathematics that studies the behavior of rough events. Trust theory is the foundation for rough programming like probability theory is for stochastic programming and possibility theory is for fuzzy programming. In particular, in order to describe two-fold uncertain events, such as random-rough variables, Liu [17] mixed trust measures with probability measures. Random-rough variables turn out to be useful tools to deal with two types of uncertainty (namely, randomness and roughness) simultaneously. In this section we present a series of definitions, axioms, and theorems which provide the basis of theory of random-rough variables.

Definition 2.1 (Liu, [18])**.**

The structure $(\Lambda, \Delta, \mathcal{A}, \pi)$ is called a *rough space* if Λ is a nonempty set, $\mathcal{A}$ is a σ-algebra of subsets of Λ, Δ is an element in $\mathcal{A}$, and π is a set function on $\mathcal{A}$ satisfying the following axioms:
Axiom 1. $\pi\{\Lambda\} < \infty$;
Axiom 2. $\pi\{\Delta\} > 0$;
Axiom 3. $\pi\{A\} \geqslant 0$, for any $A \in \mathcal{A}$;

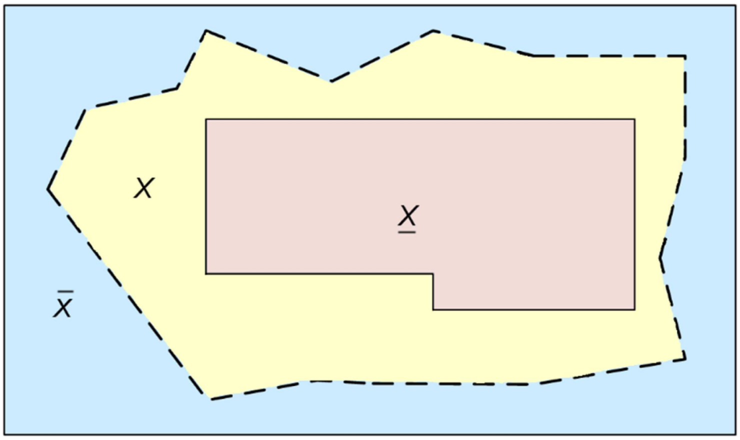

FIGURE 1. Rough approximation.

Axiom 4. For every countable sequence of mutually disjoint events $\{A_i\}_{i=1}^{\infty}$ in $\mathcal{A}$, we have:

$$\pi\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \pi\{A_i\}.$$

If, in particular, $\pi\{\emptyset\} = 0$, then π is a measure on $(\Lambda, \mathcal{A})$ and the tuple $(\Lambda, \mathcal{A}, \pi)$ is also a measure space.

Definition 2.2 (Liu, [18]).
A rough variable ξ is a measurable function from a rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$ to the set of real numbers.

The concept of rough set (Powlak, 1982) is based on the existence of a *universe of objects* U (a non-empty finite set of objects) and an *indiscernibility relation* R on U, such as an equivalence relation, representing the lack of knowledge about the objects in U. Given $X \subseteq U$one can approximate X by constructing the lower and upper approximations of X. The lower approximation of X, denoted by $\underline{X}$, is a subset of X containing all the objects *surely* belonging to X with respect to R. The upper approximation of X, denoted by $\overline{X}$, is a superset of X containing all the objects *possibly* belonging to X with respect to R.

Definition 2.3. Let X be a subset of a finite universe of objects Uendowed with an indiscernibility relation R. The pair $(\underline{X}, \overline{X})$ represents the collection of all subsets of U having the same lower and upper approximations as X and it is called a *rough set*

Figure 1 provides a visual representation of a rough set and its approximations. The dotted curve together with its internal points represents X. The two thick curves together with their inner points represent the upper $(\overline{X})$ and lower $(\underline{X})$ approximation, respectively.

Definition 2.4 Liu, [18].
Let $(\Lambda, \Delta, \mathcal{A}, \pi)$ be a rough space. The upper and lower trust of an event $K \in A$ are defined by:

$$(Tr)^{\text{Upper}}\{K\} = \frac{\pi\{K\}}{\pi\{\mathit{\Lambda}\}} \quad \text{and} \quad (Tr)^{\text{Lower}}\{K\} = \frac{\pi\{K \cap \mathit{\Delta}\}}{\pi\{\mathit{\Delta}\}}.$$

Finally, the trust of $K$is defined by:

$$\text{Tr}\{K\} = \frac{1}{2}\left[(\text{Tr})^{\text{Upper}}\{K\} + (\text{Tr})^{\text{Lower}}\{K\}\right].$$

The trust also defines a measure on $\mathcal{A}$. More precisely: (a) $\mathrm{Tr}\{\Lambda\}=1$; (b) $\mathrm{Tr}\{\emptyset\}=0$; (c)

$$\mathrm{Tr}\{A\} \leqslant \mathrm{Tr}\{B\}, \text{ for every } A, B \in \mathcal{A} \text{with } A \subseteq B; \text{and}(d) \mathrm{Tr}\{A\} + \mathrm{Tr}\{A^c\} = 1, \text{for every} A \in \mathcal{A}.$$

Assume that $\Lambda = \{\lambda : c \leqslant \lambda \leqslant d\}$, $\Delta = \{\lambda : a \leqslant \lambda \leqslant b\}$, where $c \leqslant a < b \leqslant d$, $\mathcal{A}$ coincides with the Borel algebra on Λ and π is the Lebesgue measure. Then, the identity function $\xi(\lambda) = \lambda$ is a rough variable. Such a rough variable is usually denoted by $\bar{\xi} = ([a, b], [c, d])$.

Definition 2.5 (Rough Arithmetic) (Liu, [17])**.**
Let $\bar{\xi} = ([a_1, a_2], [a_3, a_4])$, $a_3 \leqslant a_1 < a_2 \leqslant a_4$, and $\bar{\eta} = ([b_1, b_2], [b_3, b_4])$, $b_3 \leqslant b_1 < b_2 \leqslant b_4$. The rough arithmetic of $\bar{\xi}$ and $\bar{\eta}$is defined as follows:

$$\bar{\xi} + \bar{\eta} = ([a_1 + b_1, a_2 + b_2], [a_3 + b_3, a_4 + b_4]) \quad \text{and} \quad k\bar{\xi} = \begin{cases} ([ka_1, ka_2], [ka_3, ka_4]), & \text{if} \quad k > 0; \\ ([ka_2, ka_1], [ka_4, ka_3]), & \text{if} \quad k < 0. \end{cases}$$

where k is a non-zero real value.

Definition 2.6 Liu, [18]**.**
Let $\bar{\xi}$ be a rough variable defined on a rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$. The expected value of $\bar{\xi}$ is defined by:

$$\mathrm{E}(\bar{\xi}) = \int_0^{+\infty} \mathrm{Tr}\{\xi \geqslant r\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\{\xi \leqslant r\} \mathrm{d}r,$$

provided that at least one of the two integrals is finite. In particular, if $\bar{\xi} = ([a, b], [c, d])$ with $c \leqslant a < b \leqslant d$, then $E\left[\bar{\xi}\right] = \frac{1}{4}(a + b + c + d)$

Definition 2.7 Liu, [18]**.** Let $\bar{\xi}$ be a rough variable and $\alpha \in (0, 1]$.
Then:

$$\xi_{\sup}(\alpha) = \sup\left\{\mathrm{r} : \mathrm{Tr}\left\{\bar{\xi} \geqslant \mathrm{r}\right\} \geqslant \alpha\right\} \text{ is called the } \alpha - \text{optimistic value of } \bar{\xi}, \text{and}$$
$$\xi_{\inf}(\alpha) = \inf\left\{\mathrm{r} : \mathrm{Tr}\left\{\bar{\xi} \leqslant \mathrm{r}\right\} \geqslant \alpha\right\} \text{ is called } \alpha - \text{pessimistic value of } \bar{\xi}.$$

Definition 2.8. A random-rough variable is a function $\tilde{\bar{\xi}}$ from a rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$ to a collection of random variables $\tilde{\xi}(\lambda)$ such that for every Borel set B of $\Re$, $\tilde{\xi}(\lambda)(B) = \Pr\left\{\tilde{\xi}(\lambda) \in B\right\}$ is a measurable function of $\lambda \in \Lambda$.

Definition 2.9 Liu, [18]**.** A random-rough variable $\tilde{\bar{\xi}}$ is said to have a normal distribution, if the random variable $\tilde{\xi}(\lambda)$ has a normal distribution whose expected value $\bar{\mu}$ is approximated by a rough set $(\underline{X}, \overline{X})$, or its standard deviation $\bar{\sigma}$ is approximated by a rough set $(\underline{X}, \overline{X})$, or both of them.

Henceforth, we will write $\tilde{\bar{\xi}} \sim N\left(\bar{\mu}, \sigma^2\right)$ to denote a random-rough variable whose expected value $\bar{\mu}$ is approximated by a rough variable of the form $([a, b], [c, d])$, with $c \leqslant a < b \leqslant d$.

Definition 2.10 Liu, [18]**.** Let $\tilde{\bar{\xi}}$ be a random-rough variable defined on the rough space $(\Lambda, \Delta, \mathcal{A}, \pi)$. Then, its expected value is defined by:

$$\mathrm{E}(\tilde{\bar{\xi}}) = \int_0^{+\infty} \mathrm{Tr}\left\{\lambda \in \Lambda \middle| E\left[\tilde{\xi}(\lambda)\right] \geqslant r\right\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\left\{\lambda \in \Lambda \middle| E\left[\tilde{\xi}(\lambda)\right] \leqslant r\right\} \mathrm{d}r$$

Definition 2.11 Convex function. Bazaraa *et al.*, 1990**.**
Let S be a nonempty set of $\Re^n$. A function $f : S \to \Re$ is called a convex function if, for every x_1, $x_2 \in \Re^n$, $f(x_1) \neq f(x_2)$, and every $\lambda \in (0, 1)$, we have:

$$f(\lambda x_1 + (1 - \lambda) x_2) \leqslant \lambda f(x_1) + (1 - \lambda) f(x_2).$$

Definition 2.12 Bazaraa *et al.*, 1990**.** A programming problem is called convex if it has a convex feasible set and a convex objective function.

Theorem 2.13.

(a) $\xi_{\inf}(\alpha)$ *is an increasing function of* $\alpha \in [0,\ 1]$ *and* $\xi_{\sup}(\alpha)$ *is a decreasing function of* $\alpha \in [0,\ 1]$.
(b) $\xi_{\inf}(\alpha) = \xi_{\sup}(1-\alpha)$ *and* $\xi_{\sup}(\alpha) = \xi_{\inf}(1-\alpha)$ *for* $0 \leqslant \alpha \leqslant 1$
(c) $\xi_{\inf}(\alpha) \leqslant \xi_{\sup}(\alpha)$ *for* $0 \leqslant \alpha \leqslant 0.5$
(d) $\xi_{\inf}(\alpha) \geqslant \xi_{\sup}(\alpha)$ *for* $0.5 \leqslant \alpha \leqslant 1$.

3. The CCR model with random-rough data

Stochastic programming problems are optimization problems with random parameters. In stochastic programming, the parameters or the coefficients are usually characterized by a probability distribution. After the introduction of rough sets by Pawlak [24], some scholars developed the concept of two-fold uncertain variables and combined rough variables with fuzzy and random variables. An explicit use of the classical deterministic DEA models for measuring the true relative efficiency is not possible due to the lack of complete knowledge and information in complex real-life problems. In this section, we present the mathematical details of the approach proposed in this study for solving CCR models in which the input and output data are assumed to be random-rough variables.

As in Section 2, we will use an over-lining bar to indicate rough data and an over-lining tilde to denote random data. Moreover, we will add the superscripts I or O to show that a variable refers to the inputs or outputs, respectively.

Let $\tilde{\bar{X}}_j = (\tilde{\bar{x}}_{1j}, \ldots, \tilde{\bar{x}}_{mj})^t \in \Re^m$ and $\tilde{\bar{Y}}_j = (\tilde{\bar{y}}_{1j}, \ldots, \tilde{\bar{y}}_{sj})^t \in \Re^s$ be the random-rough input and output vectors for the jth DMU, DMU_j, with $j = 1, \ldots, n$, each of them endowed with a normal distribution. For every $j = 1, \ldots, n$ and $i = 1, \ldots, m$, let $\bar{x}_{ij}$ and σ_{ij} denote the expected value and the variance of the random variable $\tilde{\bar{x}}_{ij}$, respectively, with the expected value $\bar{x}_{ij}$ being represented by a rough variable. Similarly, for every $j = 1, \ldots, n$ and $r = 1, \ldots, s$, let $\bar{y}_{rj}$ and σ_{rj} denote the expected value and the standard deviation of the random variable $\tilde{\bar{y}}_{rj}$, respectively, with the expected value $\bar{y}_{rj}$ being represented by a rough variable. In summary, we have:

$$\begin{aligned}
&\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, \sigma_{ij}^2\right), \bar{x}_{ij} = ([x_{ij}^a, x_{ij}^b], [x_{ij}^c, x_{ij}^d]), x_{ij}^c \leqslant x_{ij}^a < x_{ij}^b \leqslant x_{ij}^d \\
&\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, \sigma_{rj}^2\right), \bar{y}_{rj} = ([y_{rj}^a, y_{rj}^b], [y_{rj}^c, y_{rj}^d]), y_{rj}^c \leqslant y_{rj}^a < y_{rj}^b \leqslant y_{rj}^d,
\end{aligned} \tag{3.1}$$

Finally, let DMU_p be the generic but fixed DMU under assessment. Thus, the sub-index p will be added to indicate that a quantity refers to DMU_p.

The chance-constrained CCR model with the random-rough data can be formulated as follows:

$$\begin{aligned}
&\min \quad \theta_p \\
&\text{s.t.} \\
&\operatorname{Tr}\left[P\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p \leqslant 0\right) \geqslant 1-\beta\right] \geqslant \alpha, \quad i = 1, \ldots, m, \\
&\operatorname{Tr}\left[P\left(\sum_{j=1}^{n} \tilde{\bar{y}}_{rj}\lambda_j - \tilde{\bar{y}}_{rp} \geqslant 0\right) \geqslant 1-\beta\right] \geqslant \alpha, \quad r = 1, \ldots, s, \\
&\lambda_j \geqslant 0, \qquad j = 1, \ldots, n.
\end{aligned} \tag{3.2}$$

where $\beta \in (0,1]$ and $\alpha \in (0,1]$ are the pre-determined thresholds defined by the decision maker for identifying an allowable chance of failing to satisfy the constraints. P($\bullet$) denotes the probability of the event ($\bullet$) and Tr[$\bullet$] stands for the trust measure of the event [$\bullet$].

Assume $\tilde{\bar{h}}_i = \sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p$. Due to the normal distribution of $\tilde{\bar{x}}_{ij}$, $\tilde{\bar{h}}_i$ also has a normal distribution with the following mean:

$$\bar{h}_i = E(\tilde{\bar{h}}_i) = \sum_{j=1}^{n} \bar{x}_{ij}\lambda_j - \bar{x}_{ip}\theta_p$$

For the inner part of the first constraints of model (3.2), we have:

$$\begin{aligned} P\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p \leqslant 0\right) \geqslant 1-\beta \Leftrightarrow P\left(\frac{\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p\right) - E\left(\tilde{\bar{h}}_i\right)}{\sqrt{\operatorname{var}\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p\right)}} \leqslant -\frac{E\left(\tilde{\bar{h}}_i\right)}{\sqrt{\operatorname{var}\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p\right)}}\right) \\ \geqslant 1-\beta \end{aligned}$$

Thus, we have:

$$\sum_{j=1}^{n} \bar{x}_{ij}\lambda_j - \bar{x}_{ip}\theta_p - \sigma^I_{\tilde{\bar{h}}_i}\Phi^{-1}(\beta) \leqslant 0$$

where Φ is the cumulative distribution function of the standard normal distribution and:

$$\begin{aligned} \sigma^I_{\tilde{\bar{h}}_i}(\lambda_j,\theta_p) &= \sqrt{\operatorname{var}\left(\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p\right)} \\ &= \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ik}) + \theta_p^2 \operatorname{var}(\tilde{\bar{x}}_{ip}) - 2\theta_p \sum_{j=1}^{n} \lambda_j \operatorname{cov}(\tilde{\bar{x}}_{ij},\tilde{\bar{x}}_{ip})} \end{aligned}$$

Therefore, it results in the following constraint:

$$\operatorname{Tr}\left[\sum_{j=1}^{n} \bar{x}_{ij}\lambda_j - \bar{x}_{ip}\theta_p - \sigma^I_{\tilde{\bar{h}}_i}\Phi^{-1}(\beta) \leqslant 0\right] \geqslant \alpha$$

A similar method can be applied to the second constraint in (3.2) and obtain:

$$\operatorname{Tr}\left(\sum_{j=1}^{n} \bar{y}_{rj}\lambda_j - \bar{y}_{rp} + \Phi^{-1}(\beta)\sigma^O_{\tilde{\bar{h}}_r} \geqslant 0\right) \geqslant \alpha$$

where $\tilde{\bar{h}}_r = \sum_{j=1}^{n} \tilde{\bar{y}}_{rj}\lambda_j - \tilde{\bar{y}}_{rp}$, $\bar{h}_r = E(\tilde{\bar{h}}_r) = \sum_{j=1}^{n} \bar{y}_{rj}\lambda_j - \bar{y}_{rp}$ and

$$\sigma^O_{\tilde{\bar{h}}_r}(\lambda_j) = \sqrt{\operatorname{var}\left(\sum_{j=1}^{n} \tilde{\bar{y}}_{rj}\lambda_j - \tilde{\bar{y}}_{rp}\right)} = \sqrt{\sum_{j=1}^{n}\sum_{k=1}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{ik}) + \operatorname{var}(\tilde{\bar{y}}_{rp}) - 2\sum_{j=1}^{n} \lambda_j \operatorname{cov}(\tilde{\bar{y}}_{rj},\tilde{\bar{y}}_{rp})}.$$

As a result, model (3.2) gives rise to the following chance-constrained programming model:

$$\begin{aligned}&\min \quad \theta_p\\&\text{s.t.}\\&\operatorname{Tr}\left[\sum_{j=1}^{n}\bar{x}_{ij}\lambda_j-\bar{x}_{ip}\theta_p-\sigma^{I}_{\tilde{\bar{h}}_i}(\lambda_j,\theta_p)\Phi^{-1}(\beta)\leqslant 0\right]\geqslant\alpha,\qquad i=1,\ldots,m,\\&\operatorname{Tr}\left[\sum_{j=1}^{n}\bar{y}_{rj}\lambda_j-\bar{y}_{rp}+\Phi^{-1}(\beta)\sigma^{O}_{\tilde{\bar{h}}_r}(\lambda_j)\geqslant 0\right]\geqslant\alpha,\qquad r=1,\ldots,s,\\&\lambda_j\geqslant 0,\qquad j=1,\ldots,n.\end{aligned}\tag{3.3}$$

Liu [18] has proposed three methods for converting rough programming model into a deterministic programming model. These three methods are known as the rough EVMs, rough chance-constrained programming, and the α-optimistic and α-pessimistic value operators for rough variables.

Following Xu *et al.* [44], we use the α-optimistic and α-pessimistic value operators for rough variables to transform the rough programming model (3.3) into a deterministic model, that is, into maximum programming and minimum programming under the α trust level of a rough variable ξ. The α-optimistic and α-pessimistic values of the rough variables are as follows.

Based on Theorem 1, if $\alpha\in(0.5,1]$, then, $\xi_{\inf}(\alpha)\geqslant\xi_{\sup}(\alpha)$. Thus, the α-pessimistic and α-optimistic values for the rough variables are represented by the interval $\left[\xi_{\sup}(\alpha),\xi_{\inf}(\alpha)\right]$. It follows that for $\alpha\in(0.5,1]$, the rough variables $\bar{x}_{ij}$ and $\bar{y}_{rj}$ can be transformed into the intervals $\left[x_{ij}^{\sup(\alpha)},x_{ij}^{\inf(\alpha)}\right]$ and $\left[y_{rj}^{\sup(\alpha)},y_{rj}^{\inf(\alpha)}\right]$, respectively. Hence, the implementation of the αoptimistic and αpessimistic operators to (3.3) yields the following interval non-linear program:

$$\begin{aligned}&\min\ \theta_p\\&\text{s.t.}\\&\sum_{j=1}^{n}\lambda_j\left[x_{ij}^{\sup(\alpha)},x_{ij}^{\inf(\alpha)}\right]-\sigma^{I}_{\tilde{\bar{h}}_i}(\lambda_j,\theta_p)\,\Phi^{-1}(\beta)\leqslant\theta_p\left[x_{ip}^{\sup(\alpha)},x_{ip}^{\inf(\alpha)}\right],\qquad i=1,\ldots,m,\\&\sum_{j=1}^{n}\lambda_j\left[y_{rj}^{\sup(\alpha)},y_{rj}^{\inf(\alpha)}\right]+\Phi^{-1}(\beta)\,\sigma^{O}_{\tilde{\bar{h}}_r}(\lambda_j)\geqslant\left[y_{rp}^{\sup(\alpha)},y_{rp}^{\inf(\alpha)}\right],\qquad r=1,\ldots,s,\\&\lambda_j\geqslant 0,\ j=1,\ldots,n.\end{aligned}\tag{3.4}$$

The minimum efficiency score of model (3.4) for DMU_p is attained if its observations consist of minimum inputs and maximum outputs. If$\alpha\in(0.5,1]$, then the α-optimistic value (and hence the upper bound) is obtained by solving the following non-linear programming model:

$$\begin{aligned}&\left(\theta_p^*\right)^{\inf(\alpha)}_{RRDEA}=\min\ \theta_p^{\inf(\alpha)}\\&\text{s.t.}\\&\sum_{\substack{j=1\\j\neq p}}^{n}x_{ij}^{\inf(\alpha)}\lambda_j+\lambda_p x_{ip}^{\sup(\alpha)}-\sigma^{I}_{\tilde{\bar{h}}_i}\left(\lambda_j,\theta_p^{\inf(\alpha)}\right)\Phi^{-1}(\beta)\leqslant x_{ip}^{\sup(\alpha)}\theta_p^{\inf(\alpha)},\qquad i=1,\ldots,m,\\&\sum_{\substack{j=1\\j\neq p}}^{n}y_{rj}^{\sup(\alpha)}\lambda_j+\lambda_p y_{rp}^{\inf(\alpha)}+\Phi^{-1}(\beta)\,\sigma^{O}_{\tilde{\bar{h}}_r}(\lambda_j)\geqslant y_{rp}^{\inf(\alpha)},\qquad r=1,\ldots,s,\\&\lambda_j\geqslant 0,\qquad j=1,\ldots,n.\end{aligned}\tag{3.5}$$

The α-pessimistic value (lower bound) is obtained by solving the following non-linear programming model:

$$\begin{aligned}
&\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)} = \min\ \theta_p^{\sup(\alpha)}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\sup(\alpha)}\lambda_j + \lambda_p x_{ip}^{\inf(\alpha)} - \sigma^I_{\tilde{\tilde{h}}_i}\left(\lambda_j, \theta_p^{\sup(\alpha)}\right)\Phi^{-1}(\beta) \leqslant x_{ip}^{\inf(\alpha)}\theta_p^{\sup(\alpha)}, \qquad i=1,\ldots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\inf(\alpha)}\lambda_j + \lambda_p y_{rp}^{\sup(\alpha)} + \Phi^{-1}(\beta)\,\sigma^O_{\tilde{\tilde{h}}_r}(\lambda_j) \geqslant y_{rp}^{\sup(\alpha)}, \qquad r=1,\ldots,s,\\
&\lambda_j \geqslant 0, \qquad j=1,\ldots,n.
\end{aligned} \tag{3.6}$$

Given the functional forms of $\sigma^I_{\tilde{\tilde{h}}_i}(\lambda_j, \theta_p^{\cdot})$ and $\sigma^O_{\tilde{\tilde{h}}_r}(\lambda_j)$, it is obvious that models (3.5) and (3.6) are non-linear programming problems. These two non-linear programming models can be transformed into programming models with quadratic constraints. Suppose that $v_i$and u_r are the nonnegative variables substituting $\sigma^I_{\tilde{\tilde{h}}_i}(\lambda_j, \theta_p^*)$ and $\sigma^O_{\tilde{\tilde{h}}_r}(\lambda_j)$, respectively. If $\alpha \in (0.5, 1]$, then the α-optimistic value (and hence the upper bound) is obtained by solving the following model:

$$\begin{aligned}
&\left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha)} = \min\ \theta_p^{\inf(\alpha)}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\lambda_j + \lambda_p x_{ip}^{\sup(\alpha)} - v_i\Phi^{-1}(\beta) \leqslant x_{ip}^{\sup(\alpha)}\theta_p^{\inf(\alpha)},\ i=1,\ldots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + \lambda_p y_{rp}^{\inf(\alpha)} + \Phi^{-1}(\beta)\,u_r \geqslant y_{rp}^{\inf(\alpha)},\ r=1,\ldots,s,\\
&v_i^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ik}) + \left(\theta_p^{\inf(\alpha)}\right)^2 \operatorname{var}(\tilde{\tilde{x}}_{ip}) - 2\theta_p^{\inf(\alpha)} \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ip}), \quad i=1,\ldots,m,\\
&u_r^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{ik}) + \operatorname{var}(\tilde{\tilde{y}}_{rp}) - 2\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{rp}), \quad r=1,\ldots,s,\\
&v_i,\ u_r,\ \lambda_j \geqslant 0, \quad i=1,\ldots,m,\ r=1,\ldots,s,\ j=1,\ldots,n.
\end{aligned} \tag{3.7}$$

And the α-pessimistic value (lower-bound) is obtained by solving the following model:

$$
\begin{aligned}
&\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)} = \min\ \theta_p^{\sup(\alpha)}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\sup(\alpha)}\lambda_j + \lambda_p x_{ip}^{\inf(\alpha)} - v_i\Phi^{-1}(\beta) \leqslant x_{ip}^{\inf(\alpha)}\theta_p^{\sup(\alpha)}, \qquad i=1,\ldots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\inf(\alpha)}\lambda_j + \lambda_p y_{rp}^{\sup(\alpha)} + \Phi^{-1}(\beta)\, u_r \geqslant y_{rp}^{\sup(\alpha)}, \qquad r=1,\ldots,s,\\
&v_i^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\tilde{x}}_{ij},\tilde{\tilde{x}}_{ik}) + \left(\theta_p^{\sup(\alpha)}\right)^2 \operatorname{var}(\tilde{\tilde{x}}_{ip}) - 2\theta_p^{\sup(\alpha)} \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}(\tilde{\tilde{x}}_{ij},\tilde{\tilde{x}}_{ip}),\ i=1,\ldots,m,\\
&u_r^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}(\tilde{\tilde{y}}_{rj},\tilde{\tilde{y}}_{ik}) + \operatorname{var}(\tilde{\tilde{y}}_{rp}) - 2\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}(\tilde{\tilde{y}}_{rj},\tilde{\tilde{y}}_{rp}),\ r=1,\ldots,s,\\
&v_i,\, u_r,\, \lambda_j \geqslant 0, \qquad i=1,\ldots,m, \qquad r=1,\ldots,s, \qquad j=1,\ldots,n.
\end{aligned}
\tag{3.8}
$$

Proposition 3.1. *Both model* (3.7) *and model* (3.8) *are feasible for every level of α and β*

Proof. Let $\lambda_j = \begin{cases} 1 & j=p\\ 0 & j\neq p\end{cases}$, $j=1,\ldots,n$. Also, let $\theta_p^{\inf(\alpha)} = 1$ for model (3.7) and $\theta_p^{\sup(\alpha)} = 1$ for model (3.8).

It is trivial to check that $v_i \geqslant 0$ and $u_r \geqslant 0$. Therefore, this solution is feasible for both model (3.7) and model (3.8). $\square$

Proposition 3.2. *Let $\beta \in (0, 0.5]$ and $\alpha \in (0.5, 1]$. Then, for DMU$_p$, $0 < \left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha)} \leqslant 1$ and $0 < \left(\theta_p^*\right)_{RRDEA}^{sup(\alpha)} \leqslant 1$.*

Proof. Let $\lambda_p = 1$, $\lambda_j = 0$ for all $j \neq p$, and $\theta_p^{\sup(\alpha)} = 1$. Then, $v_i \geqslant 0$, $u_r \geqslant 0$ and all constraints of model (3.8) are satisfied. Thus, this is a solution. Due to the minimization of model (3.8), the lower bound of $\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)}$ must be less than or equal to unity. Then, $-\Phi^{-1}(\beta) \geqslant 0$ and $v_i \geqslant 0$ with respect to $\beta \leqslant 0.5$ and, as a result, the first constraint in model (3.8) is converted into the following inequality:

$$\theta_p^{\sup(\alpha)} \geqslant \frac{\sum_{j=1}^{n} x_{ij}^{\sup(\alpha)}\lambda_j + \lambda_p x_{ip}^{\inf(\alpha)}}{x_{ip}^{\inf(\alpha)}} \geqslant 0,\ i=1,\ldots,m,$$

We claim that $\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)} > 0$. Indeed, assume that $\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)} \leqslant 0$. Then, $\lambda_j = 0$. From the second constraint in model (3.8) it follows that $y_{rp}^{\sup(\alpha)} \leqslant 0$ which contradicts $y_{rp}^{\sup(\alpha)} \geqslant 0$. Therefore, the lower bound of $\theta_p^{\sup(\alpha)}$ must be greater than zero. As a consequence, $0 < \left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha)} \leqslant 1$ holds. A similar reasoning shows that $0 < \left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha)} \leqslant 1$ also holds. $\square$

Proposition 3.3. *Let $\beta \in (0, 0.5]$. Then, there exists at least one efficient DMU_p, that is, there exists $p \in \{1, \ldots, n\}$ such that $\left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha)} = 1$.*

Proof. Suppose that $\forall p \in \{1, \ldots, n\}$, $\left(\theta_p^*\right)^{\inf(\alpha)} < 1$ with λ_p^* and $\left(\theta_p^*\right)^{\inf(\alpha)}$ representing the optimal solution to model (3.7). By the first constraint of model (3.7), we have:

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} x_{ij}^{\inf(\alpha)} \lambda_j - v_i \Phi^{-1}(\beta) \leqslant x_{ip}^{\sup(\alpha)} \left(\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*\right), \quad \forall i = 1, \ldots, m.$$

We claim that $\lambda_p^* < \left(\theta_p^*\right)^{\inf(\alpha)}$. Indeed, if $\lambda_p^* \geqslant \left(\theta_p^*\right)^{\inf(\alpha)}$, then $\left(\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*\right) \leqslant 0$, which implies that the right hand-side of the inequality above is non-positive; a contradiction to the fact that $-\Phi^{-1}(\beta)$ is positive for $\beta \leqslant 0.5$. It follows that $\lambda_p^* < \left(\theta_p^*\right)^{\inf(\alpha)} < 1$.

Now, divide both sides of all the constraints of model (3.7) by $1 - \lambda_p^*$. The constraints become:

$$\textbf{Const. 1}: \quad \sum_{\substack{j=1 \\ j \neq p}}^{n} \left(\frac{\lambda_j}{1-\lambda_p^*}\right) x_{ij}^{\inf(\alpha)} - \frac{v_i}{1-\lambda_p^*} \Phi^{-1}(\beta) \leqslant x_{ip}^{\sup(\alpha)} \frac{\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*}{1-\lambda_p^*}, \ i = 1, \ldots, m,$$

$$\textbf{Const. 2}: \quad \sum_{\substack{j=1 \\ j \neq p}}^{n} \left(\frac{\lambda_j}{1-\lambda_p^*}\right) y_{rj}^{\sup(\alpha)} + \frac{u_r}{1-\lambda_p^*} \Phi^{-1}(\beta) \geqslant y_{rp}^{\inf(\alpha)}, \ r = 1, \ldots, s,$$

$$\textbf{Const. 3}: \quad \left(\frac{v_i}{1-\lambda_p^*}\right)^2 = \sum_{\substack{j=1 \\ j \neq p}}^{n} \sum_{\substack{k=1 \\ k \neq p}}^{n} \frac{\lambda_j}{1-\lambda_p^*} \frac{\lambda_k}{1-\lambda_p^*} \operatorname{cov}\left(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}\right) + \left(\frac{\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*}{1-\lambda_p^*}\right)^2 \operatorname{var}\left(\tilde{\bar{x}}_{ip}\right)$$
$$-2 \frac{\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*}{1-\lambda_p^*} \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{\lambda_j}{1-\lambda_p^*} \operatorname{cov}\left(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ip}\right),$$

$$\textbf{Const. 4}: \quad \left(\frac{u_r}{1-\lambda_p^*}\right)^2 = \sum_{\substack{j=1 \\ j \neq p}}^{n} \sum_{\substack{k=1 \\ k \neq p}}^{n} \frac{\lambda_j}{1-\lambda_p^*} \frac{\lambda_k}{1-\lambda_p^*} \operatorname{cov}\left(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{ik}\right) + \left(\frac{1-\lambda_p^*}{1-\lambda_p^*}\right)^2 \operatorname{var}\left(\tilde{\bar{y}}_{rp}\right) - 2 \sum_{\substack{j=1 \\ j \neq p}}^{n} \frac{\lambda_j}{1-\lambda_p^*} \operatorname{cov}\left(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rp}\right)$$

Therefore, we have

$$\textbf{Const. 1}: \quad \sum_{\substack{j=1 \\ j \neq p}}^{n} \bar{\lambda}_j x_{ij}^{\inf(\alpha)} - \bar{v}_i \Phi^{-1}(\beta) \leqslant x_{ip}^{\sup(\alpha)} \left(\bar{\theta}_p\right)^{\inf(\alpha)}, \ i = 1, \ldots, m,$$

$$\textbf{Const. 2}: \quad \sum_{\substack{j=1 \\ j \neq p}}^{n} \bar{\lambda}_j y_{rj}^{\sup(\alpha)} + \bar{u}_r \Phi^{-1}(\beta) \geqslant y_{rp}^{\inf(\alpha)}, \ r = 1, \ldots, s,$$

$$\textbf{Const. 3}: \quad \left(\bar{v}_i^2\right)^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \bar{\lambda}_j\bar{\lambda}_k \mathrm{cov}\left(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ik}\right) + \left(\left(\bar{\theta}_p\right)^{\inf(\alpha)}\right)^2 \mathrm{var}\left(\tilde{\tilde{x}}_{ip}\right) - 2\left(\bar{\theta}_p\right)^{\inf(\alpha)} \sum_{\substack{j=1\\ j\neq p}}^{n} \bar{\lambda}_j \mathrm{cov}\left(\tilde{\tilde{x}}_{ij}, \tilde{\tilde{x}}_{ip}\right),$$

$$\textbf{Const. 4}: \quad \left(\bar{u}_r\right)^2 = \sum_{\substack{j=1\\ j\neq p}}^{n}\sum_{\substack{k=1\\ k\neq p}}^{n} \bar{\lambda}_j\bar{\lambda}_k \mathrm{cov}\left(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{ik}\right) + \mathrm{var}\left(\tilde{\tilde{y}}_{rp}\right) - 2\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{\lambda}_j \mathrm{cov}\left(\tilde{\tilde{y}}_{rj}, \tilde{\tilde{y}}_{rp}\right)$$

where $\left(\bar{\theta}_p\right)^{\inf(\alpha)} = \frac{\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^*}{1-\lambda_p^*}$, $\bar{\lambda}_j = \frac{\lambda_j}{1-\lambda_p^*}$, $j \neq p, \bar{v}_i = \frac{v_i}{1-\lambda_p^*}$, $\bar{u}_r = \frac{u_r}{1-\lambda_p^*}$, and $\bar{\lambda}_p = 0$. These are feasible solutions to model (3.7) and $\left(\bar{\theta}_p\right)^{\inf(\alpha)} < \left(\theta_p^*\right)^{\inf(\alpha)}$ which contradicts the optimality of $\left(\theta_p^*\right)^{\inf(\alpha)}$ in the minimization problem. Thus, it can be deduced that there exists $p \in \{1, \ldots, n\}$ such that $\left(\theta_p^*\right)^{\inf(\alpha)} = 1$. □

Proposition 3.4. *Let $\beta \in (0, 0.5]$ and $\alpha_1, \alpha_2 \in (0.5, 1]$ with $\alpha_1 \geqslant \alpha_2$. Then, for DMU_P, we have*:

(a) $\left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha_1)} \geqslant \left(\theta_p^*\right)_{RRDEA}^{\inf(\alpha_2)}$.

(b) $\left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha_2)} \geqslant \left(\theta_p^*\right)_{RRDEA}^{\sup(\alpha_1)}$.

Proof. Assume that $\alpha_1 \geqslant \alpha_2$ Theorem 1 yields:

$$\begin{cases} y_{rj}^{\inf(\alpha_1)} \geqslant y_{rj}^{\inf(\alpha_2)} \\ y_{rj}^{\sup(\alpha_1)} \leqslant y_{rj}^{\sup(\alpha_2)} \end{cases} \quad \text{and} \quad \begin{cases} x_{ij}^{\inf(\alpha_2)} \leqslant x_{ij}^{\inf(\alpha_1)} \\ x_{ij}^{\sup(\alpha_2)} \geqslant x_{ij}^{\sup(\alpha_1)} \end{cases} \tag{3.9}$$

The constraints in (3.5) can be written as:

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j x_{ij}^{\inf(\alpha)} - \sigma_{\tilde{\tilde{h}}_i}^{I}\left(\lambda_j, \theta\right) \Phi^{-1}(\beta) \leqslant \left(\theta_p^{\inf(\alpha)} - \lambda_p\right) x_{ip}^{\sup(\alpha)}, \quad i = 1, \ldots, m \tag{3.10}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j y_{rj}^{\sup(\alpha)} + \Phi^{-1}(\beta)\, \sigma_{\tilde{\tilde{h}}_r}^{O}\left(\lambda_j\right) \geqslant \left(1-\lambda_p\right) y_{rp}^{\inf(\alpha)}, \quad r = 1, \ldots, s \tag{3.11}$$

If $\beta \leqslant 0.5$ and $\Phi^{-1}(\beta) \leqslant 0$, the left-hand-side of (3.10) is non-negative for positive data and we have $\theta_p^{\inf(\alpha)} \geqslant \lambda_p$. Furthermore, $\theta_p^{\inf(\alpha)} = \lambda_p = 1$ is a feasible solution in (3.5) and hence we have $1 \geqslant \theta_p^{\inf(\alpha)}$. It follows that $1 \geqslant \lambda_p$. The relationships in (3.9), (3.10) and (3.11) indicate that the feasible region of the constraints in (3.5) for α_1 is not greater than the one for α_2. Consequently, (a) holds. Reasoning in a similar way, we can shows that (b) also holds □

Proposition 3.5. *Let $\beta_1,\ \beta_2 \in (0, 1]$ with $-\Phi^{-1}(\beta_1) \geqslant -\Phi^{-1}(\beta_2)$ and $\alpha \in (0.5, 1]$. Then, for DMU_p, we have*:

(a) $\left(\theta^*\right)_{\beta_1}^{\inf(\alpha)} \geqslant \left(\theta^*\right)_{\beta_2}^{\inf(\alpha)}$;

(b) $\left(\theta^*\right)_{\beta_1}^{sup(\alpha)} \geqslant \left(\theta^*\right)_{\beta_2}^{sup(\alpha)}$.

Proof. Let $\bar{\lambda}_j$, $\bar{v}_i$, and $\bar{\theta}_p^{\inf(\alpha)}$ be a feasible solution to model (3.7) at the level (α, β_1). Since $-\Phi^{-1}(\beta_1) \geqslant -\Phi^{-1}(\beta_2)$, we must have:

$$\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\bar{\lambda}_j + \bar{\lambda}_p x_{ip}^{\sup(\alpha)} - \bar{v}_i\Phi^{-1}(\beta_2) \leqslant \sum_{j=1}^{n} x_{ij}^{\inf(\alpha)}\bar{\lambda}_j + \bar{\lambda}_p x_{ip}^{\sup(\alpha)} - \bar{v}_i\Phi^{-1}(\beta_1) \leqslant x_{ip}^{\sup(\alpha)}\bar{\theta}_p^{\inf(\alpha)},\ i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} y_{rj}^{\sup(\alpha)}\bar{\lambda}_j + \bar{\lambda}_p y_{rp}^{\inf(\alpha)} + \Phi^{-1}(\beta_2)\bar{u}_r \geqslant \sum_{j=1}^{n} y_{rj}^{\sup(\alpha)}\bar{\lambda}_j + \bar{\lambda}_p y_{rp}^{\inf(\alpha)} + \Phi^{-1}(\beta_1)\bar{u}_r \geqslant y_{rp}^{\inf(\alpha)},\ r = 1, \ldots, s,$$

Thus, $\bar{\lambda}_j$, $\bar{v}_i$and $\bar{\theta}_p^{\inf(\alpha)}$ still is a feasible solution to model (3.7) at the level (α, β_2). Hence, the efficiency at β_1 is greater than or equal to the efficiency at β_2.

The proof of (b) is similar. □

Proposition 3.6. *Let $\beta_1,\ \beta_2 \in (0, 1]$, with $-\Phi^{-1}(\beta_1) \geqslant -\Phi^{-1}(\beta_2)$, and $\alpha_1, \alpha_2 \in (0.5, 1]$, with $\alpha_1 \geqslant \alpha_2$. Then, for DMU$_p$, we have*:

(a) $(\theta^*)_{\beta_1}^{\inf(\alpha_1)} \geqslant (\theta^*)_{\beta_2}^{\inf(\alpha_2)}$;

(b) $(\theta^*)_{\beta_1}^{\sup(\alpha_2)} \leqslant (\theta^*)_{\beta_2}^{\sup(\alpha_1)}$.

Proof. Let $\bar{\lambda}_j$, $\bar{v}_i$ and $\bar{\theta}_p^{\inf(\alpha)}$ be a feasible solution to model (3.7) at the level (α_1, β_1). The constraints of (3.5) can be rewritten as:

$$\sum_{j=1}^{n} x_{ij}^{\inf(\alpha)}\lambda_j - v_i\Phi^{-1}(\beta) \leqslant x_{ip}^{\sup(\alpha)}\left(\theta_p^{\inf(\alpha)} - \lambda_p\right),\ i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + \Phi^{-1}(\beta)u_r \geqslant y_{rp}^{\inf(\alpha)}(1 - \lambda_p),\ r = 1, \ldots, s,$$

According to Theorem 2.13, we know that:

$$\begin{cases} y_{rj}^{\inf(\alpha_1)} \geqslant y_{rj}^{\inf(\alpha_2)} \\ y_{rj}^{\sup(\alpha_2)} \geqslant y_{rj}^{\sup(\alpha_1)} \end{cases} \quad \text{and} \quad \begin{cases} x_{ij}^{\inf(\alpha_1)} \geqslant x_{ij}^{\inf(\alpha_2)} \\ x_{ij}^{\sup(\alpha_2)} \geqslant x_{ij}^{\sup(\alpha_1)} \end{cases}$$

For $-\Phi^{-1}(\beta_1) \geqslant -\Phi^{-1}(\beta_2)$ and $\alpha_1 \geqslant \alpha_2$ we must have the following:

$$\sum_{j=1}^{n} x_{ij}^{\inf(\alpha_2)}\bar{\lambda}_j - \bar{v}_i\Phi^{-1}(\beta_2) \leqslant \sum_{j=1}^{n} x_{ij}^{\inf(\alpha_1)}\bar{\lambda}_j - \bar{v}_i\Phi^{-1}(\beta_1) \leqslant x_{ip}^{\sup(\alpha_1)}\left(\bar{\theta}_p^{\inf(\alpha_1)} - \bar{\lambda}_p\right) \leqslant$$
$$x_{ip}^{\sup(\alpha_2)}\left(\bar{\theta}_p^{\inf(\alpha_2)} - \bar{\lambda}_p\right), i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} y_{rj}^{\sup(\alpha_2)}\bar{\lambda}_j + \Phi^{-1}(\beta_2)\bar{u}_r \geqslant \sum_{j=1}^{n} y_{rj}^{\sup(\alpha_1)}\bar{\lambda}_j + \Phi^{-1}(\beta_1)\bar{u}_r \geqslant y_{rp}^{\inf(\alpha_1)}\left(1 - \bar{\lambda}_p\right) \geqslant y_{rp}^{\inf(\alpha_2)}\left(1 - \bar{\lambda}_p\right),\ r = 1, \ldots, s,$$

Thus, $\bar{\lambda}_j$, $\bar{v}_i$ and $\bar{\theta}_p^{\inf(\alpha)}$ still is a feasible solution to model (3.7) at probability level (α_2, β_2). Hence, it can be concluded that the efficiency at (α_1, β_1) is greater or equal to the efficiency at (α_2, β_2). Similarly, we can establish (b). □

Proposition 3.7. *If* $\beta \in (0, 0.5]$*, then, both model* (3.5) *and model* (3.6) *represent convex programming problems and have global optimal solutions.*

Proof. First, we prove that $\sigma^I_{\tilde{\bar{h}}_i}(\lambda_j, \theta)$ and $\sigma^O_{\tilde{\bar{h}}_r}(\lambda_j)$ are convex functions. $\sigma^I_{\tilde{\bar{h}}_i}(\lambda_j, \theta)$ and $\sigma^O_{\tilde{\bar{h}}_r}(\lambda_j)$ can be redefined as follows:

$$\sigma^I_{\tilde{\bar{h}}_i}(\lambda_j, \theta) = \sqrt{(\theta, -\lambda_1, \ldots, -\lambda_n)\, V\, (\theta, -\lambda_1, \ldots, -\lambda_n)^t}$$

$$\sigma^O_{\tilde{\bar{h}}_r}(\lambda_j) = \sqrt{(1, -\lambda_1, \ldots, -\lambda_n)\, \bar{V}\, (1, -\lambda_1, \ldots, -\lambda_n)^t}$$

where V and $\bar{V}$are the variance-covariance matrices for corresponding set of constraints.

To simplify notations, we let $X = (\theta, -\lambda_1, \ldots, -\lambda_n)^t$ and $\Psi(X) = \sqrt{X^t V X}$. It is easy to prove that $\Psi(X)$ is a convex function. Indeed, given $\Psi(X) = \sqrt{X^t V X}$ and $\eta \in (0, 1)$, we have:

$$\begin{aligned} \Psi(\eta X_1 + (1-\eta) X_2) &= \sqrt{\eta^2 X_1^t V X_1 + (1-\eta)^2 X_2^t V X_2 + 2\eta(1-\eta) X_1^t V X_2} \\ &\leqslant \eta\sqrt{X_1^t V X_1} + (1-\eta)\sqrt{X_2^t V X_2} = \eta \Psi(X_1) + (1-\eta)\Psi(X_2) \end{aligned}$$

Thus, $\Psi(X)$ is a convex function.

Consider now the following functions:

$$H_i(\lambda) = \sum_{\substack{j=1 \\ j \neq p}}^{n} x_{ij}^{\inf(\alpha)} \lambda_j + \lambda_p x_{ip}^{\sup(\alpha)} - \sigma^I_{\tilde{\bar{h}}_i}\left(\lambda_j, \theta_p^{\inf(\alpha)}\right) \Phi^{-1}(\beta) - x_{ip}^{\sup(\alpha)} \theta_p^{\inf(\alpha)} \leqslant 0,$$

$$g_r(\lambda) = -\sum_{\substack{j=1 \\ j \neq p}}^{n} y_{rj}^{\sup(\alpha)} \lambda_j - \lambda_p y_{rp}^{\inf(\alpha)} - \Phi^{-1}(\beta)\, \sigma^O_{\tilde{\bar{h}}_r}(\lambda_j) + y_{rp}^{\inf(\alpha)} \leqslant 0.$$

Since $\beta \leqslant 0.5$, we have $-\Phi^{-1}(\beta) \geqslant 0$ and, hence, $H_i(\lambda)$ and $g_r(\lambda)$ are convex functions. Therefore, models (3.5) and (3.6) represent convex programming problems and have global optimal solutions. □

4. Deterministic random-rough CCR model with expected value operator

In order to solve an uncertain model with random-rough parameters, the model should be first converted into a deterministic model. The technique for computing the expected value is straight-forward and efficient. Assume that n DMUs ($j = 1, \ldots, n$) are to be assessed, each using amounts $\tilde{\bar{x}}_{ij}$ of m random-rough inputs ($i = 1, \ldots, m$) to produce amounts $\tilde{\bar{y}}_{rj}$ of s random-rough outputs ($r = 1, \ldots, s$). The following random-rough CCR model results from considering the random-rough inputs and outputs for DMU_p:

$$\begin{aligned} &\min \quad \theta_p \\ &\text{s.t.} \\ &\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p \leqslant 0, \quad i = 1, \ldots, m, \\ &\sum_{j=1}^{n} \tilde{\bar{y}}_{rj}\lambda_j - \tilde{\bar{y}}_{rp} \geqslant 0, \quad r = 1, \ldots, s, \\ &\lambda_j \geqslant 0, \qquad j = 1, \ldots, n. \end{aligned} \tag{4.1}$$

The random-rough expected value form of the above model is given by:

$$\begin{aligned}
&\min \quad \theta_p \\
&\text{s.t.} \\
&E\left[\sum_{j=1}^{n} \tilde{\bar{x}}_{ij}\lambda_j - \tilde{\bar{x}}_{ip}\theta_p\right] \leqslant 0, \qquad i = 1, \ldots, m, \\
&E\left[\sum_{j=1}^{n} \tilde{\bar{y}}_{rj}\lambda_j - \tilde{\bar{y}}_{rp}\right] \geqslant 0, \qquad r = 1, \ldots, s, \\
&\lambda_j \geqslant 0, \qquad j = 1, \ldots, n.
\end{aligned} \tag{4.2}$$

Proposition 4.1. *Let $\tilde{\bar{\xi}}$ be a normally distributed random-rough variable, $\tilde{\bar{\xi}} \sim N\left(\bar{\mu}, \sigma^2\right)$, whose density function, denoted by $f(x)$, is defined as follows*:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(x-\bar{\mu})^2}{2\sigma^2}}, \quad -\infty < x < +\infty$$

where $\bar{\mu} = ([a, b], [c, d])$is a rough variable. Then, the expected value of $\tilde{\bar{\xi}}$ is obtained as follows:

$$E[\tilde{\bar{\xi}}] = \frac{1}{4}[a + b + c + d].$$

Proof. By the definition of the expected value operator (see Def. 10), we have:

$$\begin{aligned}
\mathrm{E}\left(\tilde{\bar{\xi}}\right) &= \int_0^{+\infty} \mathrm{Tr}\left\{\lambda \in \Lambda \middle| E\left[\bar{\xi}(\lambda)\right] \geqslant r\right\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\left\{\lambda \in \Lambda \middle| E\left[\bar{\xi}(\lambda)\right] \leqslant r\right\} \mathrm{d}r \\
&= \int_0^{+\infty} \mathrm{Tr}\left\{\int_{x\in\Lambda} x f(x)\,\mathrm{d}x \geqslant r\right\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\left\{\int_{x\in\Lambda} x f(x)\,\mathrm{d}x \leqslant r\right\} \mathrm{d}r \\
&= \int_0^{+\infty} \mathrm{Tr}\left\{\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} x \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\bar{\mu}}{\sigma}\right)^2} \mathrm{d}x \geqslant r\right\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\left\{\int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} x \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\bar{\mu}}{\sigma}\right)^2} \mathrm{d}x \leqslant r\right\} \mathrm{d}r \\
&= \int_0^{+\infty} \mathrm{Tr}\{\bar{\mu} \geqslant r\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Tr}\{\bar{\mu} \leqslant r\} \mathrm{d}r = \frac{1}{4}(a + b + c + d).
\end{aligned}$$

Employing the expected value of the normally-distributed random-rough variables, we are able to construct a deterministic linear programming model to evaluate the efficiency of each DMU in the uncertain environment. Subsequently, applying Proposition 4.1, we can represent the deterministic CCR model based on the expected value approach as follows:

$$\begin{aligned}
&\theta^*_{EVM} = \min \quad \theta_p \\
&\text{s.t.} \\
&\sum_{j=1}^{n} \left(x_{ij}^a + x_{ij}^b + x_{ij}^c + x_{ij}^d\right) \lambda_j \leqslant \left(x_{ip}^a + x_{ip}^b + x_{ip}^c + x_{ip}^d\right) \theta_p, \quad i = 1, \ldots, m, \\
&\sum_{j=1}^{n} \left(y_{rj}^a + y_{rj}^b + y_{rj}^c + y_{rj}^d\right) \lambda_j \geqslant \left(y_{rp}^a + y_{rp}^b + y_{rp}^c + y_{rp}^d\right), \quad r = 1, \ldots, s, \\
&\lambda_j \geqslant 0, \qquad j = 1, \ldots, n.
\end{aligned}$$

5. Sensitivity analysis

Sensitivity analysis is one of the most interesting and promising research areas in linear programming. In this section, sensitivity analysis is proposed to adjust the input and output data.

Let σ_{ij}^I and σ_{rj}^O be the standard deviations of $\tilde{\bar{x}}_{ij}$ and $\tilde{\bar{y}}_{rj}$, respectively, and assume that all inputs and outputs are statistically independent. The independence assumption implies both $\operatorname{cov}(\bar{x}_{ij}, \bar{x}_{ik}) = 0$ and $\operatorname{cov}(\bar{y}_{rj}, \bar{y}_{rk}) = 0$ and allows for the following definition of adjusted inputs and outputs.

Definition 5.1. Let all inputs and outputs be statistically independent. The adjusted inputs and outputs can be defined as follows:

$$\bar{y}_{rp}^{\inf(\alpha)} = y_{rp}^{\inf(\alpha)} - \sigma_{rp}^{o}\Phi^{-1}(\beta), \bar{y}_{rp}^{\sup(\alpha)} = y_{rp}^{\sup(\alpha)} - \sigma_{rp}^{o}\Phi^{-1}(\beta), \qquad r = 1, \ldots, s \tag{5.1}$$

$$\bar{x}_{ip}^{\inf(\alpha)} = x_{ip}^{\inf(\alpha)} + \sigma_{ip}^{I}\Phi^{-1}(\beta), \quad \bar{x}_{ip}^{\sup(\alpha)} = x_{ip}^{\sup(\alpha)} + \sigma_{ip}^{I}\Phi^{-1}(\beta), \qquad i = 1, \ldots, m \tag{5.2}$$

$$\bar{y}_{rj}^{\sup(\alpha)} = y_{rj}^{\sup(\alpha)} + \sigma_{rj}^{o}\Phi^{-1}(\beta), \bar{y}_{rj}^{\inf(\alpha)} = y_{rj}^{\inf(\alpha)} + \sigma_{rj}^{o}\Phi^{-1}(\beta), \qquad j \neq p, r = 1, \ldots, s \tag{5.3}$$

$$\bar{x}_{ij}^{\sup(\alpha)} = x_{ij}^{\sup(\alpha)} - \sigma_{ij}^{I}\Phi^{-1}(\beta), \bar{x}_{ij}^{\inf(\alpha)} = x_{ij}^{\inf(\alpha)} - \sigma_{ij}^{I}\Phi^{-1}(\beta), \qquad j \neq p, i = 1, \ldots, m \tag{5.4}$$

Proposition 5.2. *If $\beta \in (0, 0.5]$, then*

$$(a) \quad -\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + (\lambda_p - 1)\, y_{rp}^{\inf(\alpha)} - \Phi^{-1}(\beta)\,\sigma_r^o(\lambda) \leqslant (\lambda_p - 1)\,\bar{y}_{rp}^{\inf(\alpha)} - \sum_{\substack{j=1\\ j\neq p}}^{n} \bar{y}_{rj}^{\sup(\alpha)}\lambda_j, r = 1, \ldots, s$$

$$(b) \quad \sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\lambda_j + \left(\lambda_p - \theta_p^{\inf(\alpha)}\right) x_{ip}^{\sup(\alpha)} - \Phi^{-1}(\beta)\,\sigma_i^I(\lambda, \theta) \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \bar{x}_{ij}^{\inf(\alpha)}\lambda_j - \bar{x}_{ip}^{\sup(\alpha)},\ i = 1, .., m$$

$$(c) \quad -\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\inf(\alpha)}\lambda_j + (\lambda_p - 1)\, y_{rp}^{\sup(\alpha)} - \Phi^{-1}(\beta)\,\sigma_r^o(\lambda) \leqslant (\lambda_p - 1)\,\bar{y}_{rp}^{\sup(\alpha)} - \sum_{\substack{j=1\\ j\neq p}}^{n} \bar{y}_{rj}^{\inf(\alpha)}\lambda_j, r = 1, \ldots, s$$

$$(d) \quad \sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\sup(\alpha)}\lambda_j + \left(\lambda_p - \theta_p^{inf(\alpha)}\right) x_{ip}^{\inf(\alpha)} - \Phi^{-1}(\beta)\,\sigma_i^I(\lambda, \theta) \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \bar{x}_{ij}^{\sup(\alpha)}\lambda_j - \bar{x}_{ip}^{\inf(\alpha)},\ i = 1, .., m$$

where

$$\sigma_i^I(\lambda, \theta) = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2\left(\sigma_{ij}^I\right)^2 + \left(\theta_p^{\inf(\alpha)} - \lambda_p\right)^2\left(\sigma_{ip}^I\right)^2} \text{ and } \sigma_r^O(\lambda) = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2\left(\sigma_{rj}^O\right)^2 + (1 - \lambda_p)^2\left(\sigma_{rp}^O\right)^2}.$$

Proof. Reasoning as in the proof of Proposition 3.3, we have $\lambda_p^* < \left(\theta_p^*\right)^{\inf(\alpha)} < 1$. Since it is an optimal solution, we have $\left(\theta_p^*\right)^{\inf(\alpha)} \leqslant 1$, $\lambda_p^* \leqslant 1$, and $\left(\theta_p^*\right)^{\inf(\alpha)} - \lambda_p^* \geqslant 0$. Therefore:

$$\sigma_i^I(\lambda, \theta) = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2\left(\sigma_{ij}^I\right)^2 + \left(\theta_p^{\inf(\alpha)} - \lambda_p\right)^2\left(\sigma_{ip}^I\right)^2} \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j\sigma_{ij}^I + \left(\theta_p^{\inf(\alpha)} - \lambda_p\right)\sigma_{ip}^I \quad i = 1, \ldots, m,$$

Similarly,

$$\sigma_r^o(\lambda) = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2 \left(\sigma_{rj}^o\right)^2 + (1-\lambda_p)^2 \left(\sigma_{rp}^o\right)^2} \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \sigma_{rj}^o + (1-\lambda_p)\,\sigma_{rp}^o, \quad r = 1,\dots,s.$$

Finally, letting $v_i = \sigma_i^I(\lambda,\theta)$ and $u_r = \sigma_r^O(\lambda)$, we obtain (b) and (a), respectively. Reasoning in a similar way, statements (c) and (d) can be easily obtained. □

Now consider the following linear programming problems:

Deterministic upper-bound CCR model

$$\begin{aligned}
&\tilde{\theta}^*_{\inf} = \min\ \theta_p^{\inf(\alpha)}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{x}_{ij}^{\inf(\alpha)}\lambda_j + \lambda_p \bar{x}_{ip}^{\sup(\alpha)} \leqslant \bar{x}_{ip}^{\sup(\alpha)}\theta_p^{\inf(\alpha)}, \qquad i = 1,\dots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{y}_{rj}^{\sup(\alpha)}\lambda_j + \lambda_p \bar{y}_{rp}^{\inf(\alpha)} \geqslant \bar{y}_{rp}^{\inf(\alpha)}, \qquad r = 1,\dots,s,
\end{aligned} \tag{5.5}$$

$\bar{x}_{ij}^{\inf(\alpha)}, \bar{x}_{ip}^{\sup(\alpha)}, \bar{y}_{rp}^{\inf(\alpha)}, \bar{y}_{rj}^{\sup(\alpha)}$ as in equations (5.1)−(5.4)
$v_i = \sigma_i^I(\lambda,\theta)$ as defined in Proposition 5.2
$u_r = \sigma_r^O(\lambda)$ as defined in Proposition 5.2
$v_i,\ u_r,\ \lambda_j \geqslant 0, j = 1,\dots,n.$

Deterministic lower-bound CCR model

$$\begin{aligned}
&\tilde{\theta}^*_{\sup} = \min\ \theta_p^{\sup(\alpha)}\\
&\text{s.t.}\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{x}_{ij}^{\sup(\alpha)}\lambda_j + \lambda_p \bar{x}_{ip}^{\inf(\alpha)} \leqslant \bar{x}_{ip}^{\inf(\alpha)}\theta_p^{\sup(\alpha)}, \qquad i = 1,\dots,m,\\
&\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{y}_{rj}^{\inf(\alpha)}\lambda_j + \lambda_p \bar{y}_{rp}^{\sup(\alpha)} \geqslant \bar{y}_{rp}^{\sup(\alpha)}, \qquad r = 1,\dots,s,
\end{aligned}$$

$\bar{x}_{ip}^{\inf(\alpha)}, \bar{x}_{ij}^{\sup(\alpha)}, \bar{y}_{rj}^{\inf(\alpha)}, \bar{y}_{rp}^{\sup(\alpha)}$ as in equations (5.1)−(5.4)
$v_i = \sigma_i^I(\lambda,\theta)$ as defined in Proposition 5.2
$u_r = \sigma_r^O(\lambda)$ as defined in Proposition 5.2
$v_i,\ u_r,\ \lambda_j \geqslant 0\,, j = 1,\dots,n.$

Model (5.5) is a deterministic linear programming model that allows us to extend the CCR model with rough variables presented by Xu *et al.* [44] to a CCR model with the adjusted input and output values $(x_{ij}^{\inf(\alpha)}, y_{rj}^{\sup(\alpha)})$ and $(x_{ip}^{\sup(\alpha)}, y_{rp}^{\inf(\alpha)})$ defined as in relations (5.1)–(5.4) for DMU$_j$, $j = 1,\dots,n$.

Proposition 5.3. *Let $0 < \beta < 0.5$. The following implication holds.*

(a) If $(\theta^*)^{\inf(\alpha)} = 1$for DMU_p in model (3.7), then $\tilde{\theta}^*_{\inf} = 1$ in model (5.5).
 Equivalently:
(b) If $\tilde{\theta}^*_{\inf} < 1$ for DMU_p in model (5.5), then $(\theta^*)^{\inf(\alpha)} < 1$ in model (3.7).

Proof. We show (b). Suppose that $\tilde{\theta}^*_{\inf} < 1$ in model (5.5). Due to the fact that $0 < \beta < 0.5$, we have $\Phi^{-1}(\beta) < 0$. At the same time, by the definition of adjusted inputs and outputs (Def. 5.1, Eqs. (5.1)−(5.4)), we have $\bar{y}_{rp}^{\inf(\alpha)} > y_{rp}^{\inf(\alpha)}$, $\bar{y}_{rp}^{\sup(\alpha)} > y_{rp}^{\sup(\alpha)}$ and $\bar{x}_{ij}^{\inf(\alpha)} < x_{ij}^{\inf(\alpha)}, \bar{x}_{ip}^{\sup(\alpha)} < x_{ip}^{\sup(\alpha)}$.

Since a solution of model (5.5) is also a solution of model (3.7), there exists a solution with $(\theta^*)^{\inf(\alpha)} = \tilde{\theta}^*_{\inf} < 1$ for model (3.7). □

The counterpart of Proposition 5.3 for the lower bound efficiency (θ^*) can be also proved. We leave it to the reader to work out the details.

Corollary 5.4. *Let $0 < \beta < 0.5$. The following implication holds.*

(a) If DMU_p is stochastic rough efficient, *i.e.* $\tilde{\theta}^*_{\inf} = 1$, then DMU_p is efficient for the adjusted inputs and outputs in the deterministic model (5.5).
 Equivalently:
(b) If DMU_p is inefficient for the adjusted inputs and outputs in model (5.5), then DMU_p is stochastic rough inefficient.

5.1. Sensitivity analysis for the BCC output-oriented model

In this section, we extend the sensitivity analysis to the BCC output-oriented model with adjusted input-output. Let us first consider the following output-oriented BCC model:

$$
\begin{aligned}
&\max \quad \varphi_p \\
&\text{s.t.} \\
&\sum_{j=1}^{n} x_{ij}\lambda_j \leqslant x_{ip}, \qquad i = 1,\ldots,m, \\
&\sum_{j=1}^{n} y_{rj}\lambda_j \geqslant \varphi_p y_{rp}, \qquad r = 1,\ldots,s, \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \geqslant 0, \qquad j = 1,\ldots,n.
\end{aligned}
\tag{5.6}
$$

Definition 5.5. *DMU_p* is said to be efficient if the optimal value of φ_p is equal to one ($\varphi_p^* = 1$).

Similarly to the CCR model with rough variables introduced by Xu *et al.* [44], we can formulate the deterministic output-oriented BCC model with rough variables as follows:

Deterministic upper-bound BCC model

$$
\begin{aligned}
&\max \quad \varphi_p^{\inf(\alpha)} \\
&\text{s.t.} \\
&\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\lambda_j + \lambda_p x_{ip}^{\sup(\alpha)} \leqslant x_{ip}^{\sup(\alpha)}, \qquad i = 1,\ldots,m,
\end{aligned}
$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + \lambda_p y_{rp}^{\inf(\alpha)} \geqslant \varphi_p^{\inf(\alpha)} y_{rp}^{\inf(\alpha)},\ r=1,\ldots,s,$$

$$\sum_{j=1}^{n}\lambda_j = 1,$$

$$\lambda_j \geqslant 0, \qquad j=1,\ldots,n.$$

Deterministic lower-bound BCC model

$$\begin{aligned} \max \quad & \varphi_p^{\sup(\alpha)}\\ \text{s.t.} \quad & \end{aligned}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\sup(\alpha)}\lambda_j + \lambda_p x_{ip}^{\inf(\alpha)} \leqslant x_{ip}^{\inf(\alpha)},\ i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\inf(\alpha)}\lambda_j + \lambda_p y_{rp}^{\sup(\alpha)} \geqslant \varphi_p^{\sup(\alpha)} y_{rp}^{\sup(\alpha)},\ r=1,\ldots,s, \tag{5.7}$$

$$\sum_{j=1}^{n}\lambda_j = 1,$$

$$\lambda_j \geqslant 0, \quad j=1,\ldots,n.$$

In addition to the input-oriented CCR model with random-rough data, we can now formulate the deterministic upper bound output-oriented BCC model as follows:

$$\begin{aligned} \max \quad & \varphi_p^{\inf(\alpha)}\\ \text{s.t.} \quad & \end{aligned}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\lambda_j + (\lambda_p - 1)\, x_{ip}^{\sup(\alpha)} - \Phi^{-1}(\beta)\,\sigma_i^I(\lambda) \leqslant 0, i=1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + \left(\lambda_p - \varphi_p^{\inf(\alpha)}\right) y_{rp}^{\inf(\alpha)} + \Phi^{-1}(\beta)\,\sigma_r^o(\varphi,\lambda) \geqslant 0, r=1,\ldots,s, \tag{5.8}$$

$$\sum_{j=1}^{n}\lambda_j = 1,$$

$$\lambda_j \geqslant 0, \quad j=1,\ldots,n.$$

The deterministic lower bound output-oriented BCC can be obtained in a similar way. Moreover, as in Proposition 9, for an optimal solution, we have $\varphi_p^{\inf(\alpha)} \geqslant 1$ and $\lambda_j^* \leqslant 1$, and, hence, also $\varphi_p^{\inf(\alpha)} - \lambda_p \geqslant 0$. Thus

$$\sigma_r^o(\varphi,\lambda) = u_r = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2 \left(\sigma_{rj}^o\right)^2 + \left(\varphi_p^{\inf(\alpha)} - \lambda_p\right)^2 \left(\sigma_{rp}^o\right)^2} \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \sigma_{rj}^o + \left(\varphi_p^{\inf(\alpha)} - \lambda_p\right)\sigma_{rp}^o$$

Similarly, we have:

$$\sigma_i^I(\lambda) = v_i = \sqrt{\sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j^2 \left(\sigma_{ij}^I\right)^2 + (1-\lambda_p)^2 \left(\sigma_{ip}^I\right)^2} \leqslant \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \sigma_{ij}^I + (1-\lambda_p)\,\sigma_{ip}^I$$

Therefore, non-linear programming model (5.8) can be then converted into the following model:

$$\begin{aligned}
&\max \quad \varphi_p^{\inf(\alpha)}\\
&\text{s.t.}
\end{aligned}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} x_{ij}^{\inf(\alpha)}\lambda_j + (\lambda_p - 1)\, x_{ip}^{\sup(\alpha)} - v_i \Phi^{-1}(\beta) \leqslant 0, \quad i = 1,\ldots,m, \tag{5.9}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} y_{rj}^{\sup(\alpha)}\lambda_j + \left(\lambda_p - \varphi_p^{\inf(\alpha)}\right) y_{rp}^{\inf(\alpha)} + \Phi^{-1}(\beta)\, u_r \geqslant 0, \quad r = 1, \ldots, s,$$

$$\sum_{j=1}^{n}\lambda_j = 1,$$

$$v_i^2 = \sum_{\substack{j=1\\ j\neq p}}^{n} \sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}\left(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}\right) + (\lambda_p - 1)^2\, var\left(\tilde{\bar{x}}_{ip}\right) + 2(\lambda_p - 1) \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}\left(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ip}\right), \quad i = 1,\ldots,m,$$

$$u_r^2 = \sum_{\substack{j=1\\ j\neq p}}^{n} \sum_{\substack{k=1\\ k\neq p}}^{n} \lambda_j\lambda_k \operatorname{cov}\left(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rk}\right) + \left(\varphi_p^{\inf(\alpha)} - \lambda_p\right)^2 var\left(\tilde{\bar{y}}_{rp}\right) - 2\varphi^{\inf(\alpha)} \sum_{\substack{j=1\\ j\neq p}}^{n} \lambda_j \operatorname{cov}\left(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rp}\right), r = 1,\ldots,s,$$

$$\lambda_j, v_i, u_r \geqslant 0, \quad j = 1,\ldots,n;\ i = 1, \quad \ldots, m; \quad r = 1,\ldots,s.$$

Now consider the following linear programming problems for the output-oriented BCC model with adjusted inputs and outputs defined as in (5.1)−(5.4):

Upper-bound BCC model with adjusted data

$$\begin{aligned}
&\tilde{\varphi}^* = \max \quad \varphi_p^{\inf(\alpha)}\\
&\text{s.t.}
\end{aligned}$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{x}_{ij}^{\inf(\alpha)}\lambda_j + \lambda_p \bar{x}_{ip}^{\sup(\alpha)} \leqslant \bar{x}_{ip}^{\sup(\alpha)},\ i = 1,\ldots,m,$$

$$\sum_{\substack{j=1\\ j\neq p}}^{n} \bar{y}_{rj}^{\sup(\alpha)}\lambda_j + \lambda_p \bar{y}_{rp}^{\inf(\alpha)} \geqslant \varphi_p^{\inf(\alpha)} \bar{y}_{rp}^{\inf(\alpha)},\ r = 1,\ldots,s,$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geqslant 0, \quad j = 1, \ldots, n.$$

Lower-bound BCC model with adjusted data

$$\begin{array}{ll} \max & \varphi_p^{\sup(\alpha)} \\ \text{s.t.} & \end{array}$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \bar{x}_{ij}^{\sup(\alpha)} \lambda_j + \lambda_p \bar{x}_{ip}^{\inf(\alpha)} \leqslant \bar{x}_{ip}^{\inf(\alpha)}, \qquad i = 1, \ldots, m,$$

$$\sum_{\substack{j=1 \\ j \neq p}}^{n} \bar{y}_{rj}^{\inf(\alpha)} \lambda_j + \lambda_p \bar{y}_{rp}^{\sup(\alpha)} \geqslant \varphi_p^{\sup(\alpha)} \bar{y}_{rp}^{\sup(\alpha)}, \qquad r = 1, \ldots, s, \tag{5.10}$$

$$\sum_{j=1}^{n} \lambda_j = 1,$$

$$\lambda_j \geqslant 0, \qquad j = 1, \ldots, n.$$

This simply is the deterministic BCC model described by model (3.2) for DMU_p with the adjusted input and output values defined as in relations (5.1)–(5.4).

The discussion about the lower bound model is omitted in Propositions 5.6 and 5.7.

Proposition 5.6. *Let* $0 < \beta < 0.5$. *The following implication holds.*

(a) If $(\varphi_p^{\inf(\alpha)})^* = 1$ for DMU_p in model (5.9), then $\tilde{\varphi}^* = 1$ in model (5.10).
    Equivalently:
(b) If $\tilde{\varphi}^* > 1$ for DMU_p in model (5.10), then $(\varphi_p^{\inf(\alpha)})^* > 1$ in model (5.9).

Proof. We show (b). Suppose that $\tilde{\varphi}^* > 1$ in model (5.10). Due to the fact that $0 < \beta < 0.5$, we have $\Phi^{-1}(\beta) < 0$. At the same time, by the definition of adjusted inputs and outputs (Def. 5.1, Eqs. (5.1)−(5.4)), we have $\bar{y}_{rp}^{\inf(\alpha)} > y_{rp}^{\inf(\alpha)}$, $\bar{y}_{rp}^{\sup(\alpha)} > y_{rp}^{\sup(\alpha)}$ and $\bar{x}_{ij}^{\inf(\alpha)} < x_{ij}^{\inf(\alpha)}, \bar{x}_{ip}^{\sup(\alpha)} < x_{ip}^{\sup(\alpha)}$.

Since a solution of model (5.10) is also a solution of model (5.9), there exists a solution with $(\varphi_p^{\inf(\alpha)})^* = \tilde{\varphi}^* > 1$ for model (5.9). □

Corollary 5.7. *Let* $0 < \beta < 0.5$. *The following implication holds.*

(a) If DMU_p is stochastic rough efficient, *i.e.* $(\varphi_p^{\inf(\alpha)})^* = 1$, then DMU_p is efficient for the adjusted inputs and outputs in the deterministic model (5.10).

Table 1. Random-rough inputs and outputs.

DMU	Input 1	Input 2	Output 1	Output 2
1	N(([15,26], [13,38]), 1)	N(([38,50], [26,66]), 1)	N(([13,16], [11,19]), 1)	N(([13,17], [9,25]), 1)
2	N(([8,12], [6,14]), 1)	N(([15,18], [12,20]), 1)	N(([16,20], [13,23]), 1)	N(([15,19], [13,23]), 1)
3	N(([30,35], [18,49]), 1)	N(([35,40], [23,46]), 1)	N(([12,18], [9,22]), 1)	N(([16,20], [8,22]), 1)
4	N(([11,19], [9,20]), 1)	N(([9,15], [6,20]), 1)	N(([50,61], [39,74]), 1)	N(([49,63], [35,76]), 1)
5	N(([30,46], [20,52]), 1)	N(([41,55], [28,66]), 1)	N(([60,65], [50,75]), 1)	N(([45,50], [40,65]), 1)

Equivalently:

(b) If DMU_p is inefficient for the adjusted inputs and outputs in model (5.10), then DMU_p is stochastic rough inefficient.

Proposition 5.8. *Let* $0.5 < \beta < 1$. *The following implication holds.*

(a) If $\tilde{\varphi}^* = 1$ for DMU_p in model (5.10), then, $\left(\varphi_p^{\inf(\alpha)}\right)^* = 1$ in model (5.9).

Equivalently:

(b) If $\left(\varphi_p^{\inf(\alpha)}\right)^* > 1$ for DMU_p in model (5.9), then $\tilde{\varphi}^* > 1$ in model (5.10).

Proof. We show (b). Since $0.5 < \beta < 1$, we have $\Phi^{-1}(\beta) > 0$. Moreover, by Definition 5.1, we have $\bar{y}_{rj}^{\sup(\alpha)} < y_{rj}^{\sup(\alpha)}$, $\bar{y}_{rp}^{\inf(\alpha)} < y_{rp}^{\inf(\alpha)}$ and $\bar{x}_{ij}^{\inf(\alpha)} < x_{ij}^{\inf(\alpha)}$, $\bar{x}_{ip}^{\sup(\alpha)} < x_{ip}^{\sup(\alpha)}$. Thus, assuming that $(\varphi_p^{\inf(\alpha)})^* > 1$ in model (5.9) implies that there exists a solution with $\tilde{\varphi}^* = (\varphi_p^{\inf(\alpha)})^* > 1$ for model (5.10), when evaluating DMU_p. □

Corollary 5.9. *Let* $0.5 < \beta < 1$. *The following implication holds.*

(a) If DMU_p is efficient for the adjusted inputs and outputs in model (5.10), then DMU_p is stochastic rough efficient.

Equivalently:

(b) If DMU_p is stochastic rough inefficient, then DMU_p is inefficient for the adjusted inputs and outputs in model (5.10).

6. Numerical example

In this section, we present a numerical example to demonstrate the applicability of the proposed framework and exhibit the efficacy of the described procedures and algorithms. In this example, we consider five DMUs with two random-rough inputs and two random-rough outputs as shown in Table 1. The random-rough inputs, $\tilde{\bar{x}}_{ij}$, and the random-rough outputs,$\tilde{\bar{y}}_{rj}$, are normally distributed with the following rough means and known variances:

$$\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, 1\right)$$

where $\bar{x}_{ij} = \left(\left[x_{ij}^a, x_{ij}^b\right], \left[x_{ij}^c, x_{ij}^d\right]\right)$ with $x_{ij}^c \leqslant x_{ij}^a < x_{ij}^b \leqslant x_{ij}^d$ and $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, 1\right)$. Similarly, $\bar{y}_{rj} = \left(\left[y_{rj}^a, y_{rj}^b\right], \left[y_{rj}^c, y_{rj}^d\right]\right)$ with $y_{rj}^c \leqslant y_{rj}^a < y_{rj}^b \leqslant y_{rj}^d$.

The data in Table 1 can be summarized as follows:

$\tilde{\bar{x}}_{ij} \sim N\left(\bar{x}_{ij}, 1\right)$ and $\tilde{\bar{y}}_{rj} \sim N\left(\bar{y}_{rj}, 1\right)$ where $\bar{x}_{ij} = \left(\left[x_{ij}^a, x_{ij}^b\right], \left[x_{ij}^c, x_{ij}^d\right]\right)$, $\bar{y}_{rj} = \left(\left[y_{rj}^a, y_{rj}^b\right], \left[y_{rj}^c, y_{rj}^d\right]\right)$

It is also assumed that inputs and outputs of two different DMUs are independent from each other. This independence assumption implies that $\operatorname{cov}\left(\tilde{\bar{x}}_{ij}, \tilde{\bar{x}}_{ik}\right) = 0$ and $\operatorname{cov}\left(\tilde{\bar{y}}_{rj}, \tilde{\bar{y}}_{rk}\right) = 0$.

The upper and lower bounds corresponding to models (3.7) and (3.8), respectively, have been estimated by using GAMS software. Recall that the lower bound provides the α-optimistic value, while the upper bound

TABLE 2. Stochastic efficiency of model (3.7) and model (3.8).

DMU	Stochastic Efficiency					
	$\alpha = 0.6,\ \beta = 0.1$	$\alpha = 0.6,\ \beta = 0.0$	$\alpha = 0.6,\ \beta = 0.0$	$\alpha = 0.7,\ \beta = 0.1$	$\alpha = 0.7,\ \beta = 0.0$	$\alpha = 0.7,\ \beta = 0.0$
1	[0.1814, 0.2743]	[0.1910, 0.2883]	[0.1999, 0.3013]	[0.1478, 0.3388]	[0.1559, 0.3561]	[0.1633, 0.3721]
2	[0.5033, 0.7217]	[0.5377, 0.7730]	[0.5705, 0.8225]	[0.4203, 0.8657]	[0.4487, 0.9291]	[0.4758, 0.9908]
3	[0.1420, 0.1940]	[0.1488, 0.2028]	[0.1550, 0.2107]	[0.1209, 0.2262]	[0.1270, 0.2361]	[0.1325, 0.2452]
4	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]	[1.0000, 1.0000]
5	[0.4275, 0.5885]	[0.4422, 0.6078]	[0.4556, 0.6254]	[0.3645, 0.6920]	[0.3776, 0.7145]	[0.3894, 0.7349]

TABLE 3. EVM efficiency model.

DMU	EVM
1	0.1841
2	0.4741
3	0.1323
4	1.0000
5	0.4449

provides the α-pessimistic value, respectively. Six different α, β) threshold levels have been considered to evaluate the performance of DMUs using models (3.7) and (3.8), that is:

($\alpha = 0.6$, $\beta = 0.04$), ($\alpha = 0.6$, $\beta = 0.05$), ($\alpha = 0.6$, $\beta = 0.1$), ($\alpha = 0.7$, $\beta = 0.04$), ($\alpha = 0.7$, $\beta = 0.05$), and ($\alpha = 0.7$, $\beta = 0.1$).

Table 2 presents the lower and upper bound efficiency values associated with the six above specified threshold levels. As shown in this table, DMU_4 turns out to be stochastic rough efficient at all the six given levels, whereas DMUs 1, 2, 3 and 5 results inefficient at all the levels.

When β was kept unchanged and α was increasedfrom 0.6 to 0.7, the lower bound efficiency of DMUs reduced while the upper bound efficiency increased. Consider for instance the ($\alpha = 0.6$, $\beta = 0.1$) and ($\alpha = 0.7$, $\beta = 0.1$) levels, as an index for the behavioral analysis of the changes in the efficiency scores induced by α and β. The lower bound efficiency score of DMU_2 at ($\alpha = 0.6$, $\beta = 0.1$) and ($\alpha 0.7 =$, $\beta = 0.1$) levels are 0.5033 and 0.4203, respectively. Also, the upper bound efficiency score of DM_2 at ($\alpha = 0.6$, $\beta = 0.1$) and ($\alpha = 0.7$, $\beta = 0.1$) levels are 0.7217 and 0.8657, respectively. Thus, Proposition 4 has been confirmed.

On the other hand, when α was kept unchanged and $-\Phi^{-1}(\beta)$ was increased, the corresponding upper and lower bounds of the efficiency scores of the DMUs increased or remained unchanged. See Table 2. For example, the upper bound efficiency score of DMU_3 at the ($\alpha = 0.6, \beta = 0.1$), ($\alpha = 0.6, \beta = 0.04$), and ($\alpha = 0.6, \beta = 0.05$) levels are 0.1940, 0.2028 and 0.2107, respectively. Also the lower bound efficiency score of DMU_3 for ($\alpha = 0.6$, $\beta = 0.1$), ($\alpha = 0.6$, $\beta = 0.04$), and ($\alpha = 0.6$, $\beta = 0.05$) levels are 0.1420, 0.1488, and 0.1550, respectively. Thus, Proposition (3.5) has been confirmed.

Finally, it deserves to comment on the computational results obtained for the upper and lower bound efficiency models (3.7) and (3.8) by changing α and β. When α and $-\Phi^{-1}(\beta)$ were increased simultaneously, the corresponding upper bound efficiency value increases, whereas the corresponding lower bound decreases slightly. For instance, as is shown in Table 2, the upper bound scores of DMU_2 at the ($\alpha = 0.7$, $\beta = 0.05$) and ($\alpha = 0.6$, $\beta = 0.04$) levels are 0.9908 and 0.7730, respectively. Also the lower bound efficiency score of DMU_2 at the ($\alpha = 0.7$, $\beta = 0.05$) and ($\alpha = 0.6$, $\beta = 0.04$) are 0.4758 and 0.5377, respectively. Thus, Proposition 6 has been confirmed.

Next, we applied the random-rough EVM model (4.1) to calculate the efficiency of the DMUs. The results are reported in Table 3. DMU_4 was identified as the efficient unit with a score of one (Tab. 3). Compared to the existing chance-constrained DEA models or approaches, the random-rough EVM was much easier to solve

TABLE 4. Geometric efficiency and ranking of the DMUs.

DMU	Stochastic Efficiency					
	$\alpha = 0.6,\, \beta = 0.1$	$\alpha = 0.6,\, \beta = 0.0$	$\alpha = 0.6,\, \beta = 0.0$	$\alpha = 0.7, \beta = 0.1$	$\alpha = 0.7, \beta = 0.0$	$\alpha = 0.7, \beta = 0.0$
1	0.2231 (4)	0.2347 (4)	0.2454 (4)	0.2238 (4)	0.2356 (4)	0.2465 (4)
2	0.6027 (2)	0.6447 (2)	0.6850 (2)	0.6032 (2)	0.6457 (2)	0.6866 (2)
3	0.1660 (5)	0.1737 (5)	0.1807 (5)	0.1654 (5)	0.1732 (5)	0.1802 (5)
4	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0000 (1)	1.0000 (1)
5	0.5016 (3)	0.5184 (3)	0.5338 (3)	0.5022 (3)	0.5194 (3)	0.5349 (3)

and implement. Using the random-rough EVM, to determine the efficiencies, only one pair of LP models was solved for each DMU without any need to solve non-linear programming models. In particular, the random-rough inputs and outputs made it easy to identify the best performing DMU.

However, the fi nal efficiency score for each DMU was characterized by an interval $\left[(\theta^*)^{\sup(\alpha)}, (\theta^*)^{inf(\alpha)}\right]$ with respect to the (α, β) trust level and the ranking approach was used to rank the efficiencies of different DMUs. The geometric average efficiency scores and the final rankings of the five DMUs are presented in the Table 4.

Wang *et al.* [39] suggested a geometric average efficiency index for DMU_j, namely:

$$\theta_j^{\text{Geometric}(\alpha)} = \sqrt{\theta_j^{\inf(\alpha)} \times \theta_j^{\sup(\alpha)}}$$

The geometric average efficiency measures the overall performance of each DMU and is more comprehensive than either of the lower bound and upper bound efficiencies. This efficiency can be seen as an overall performance measure for each DMU and is much easier to compute and rank. Table 4 also shows the geometric average efficiencies and ranking of the five DMUs as $DMU_4 \succ DMU_2 \succ DMU_5 \succ DMU_1 \succ DMU_3$. Based on this ranking, DMU_4 is the best-performing DMU and DMU_3 is the worst-performing DMU among the five DMUs under consideration.

7. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

The conventional DEA is a well-established methodology for measuring the relative efficiency of DMUs which consumes crisp inputs and produce crisp outputs. Due to the uncertainties inherent in the real-world performance assessment problems, precise input-output data values are often unavailable in the production process. The variables in the real-world problems are often characterized as random-rough data.

In this study, we have considered a chance-constrained DEA model for solving DEA models with random-rough inputs and outputs. By assuming that the DMUs operate in a random-rough environment, we have proposed a deterministic non-linear model equivalent to the non-deterministic chance-constrained model. We have also investigated the stability and robustness of the proposed random-rough DEA model through sensitivity analysis. Finally, we have presented a numerical example to demonstrate the applicability of the proposed framework and exhibit the feasibility and richness of the obtained solutions. Although, in this study, we have considered a CCR DEA model, the proposed approach can provide insights for future research to address uncertainty in other types of DEA models.

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