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TRANSIENT BEHAVIOUR OF THREE-HETEROGENEOUS SERVERS QUEUE WITH SYSTEM DISASTER AND SERVER REPAIR

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Abstract. Three-heterogeneous servers queue with system disaster, server failure and repair is investigated. The arrival of customers follows Poisson process and service time is exponentially distributed. Explicit expressions are derived for the transient-state probabilities using generating function, modified Bessel function and Laplace transform. Further, the steady-state system size probabilities are deduced and certain important performance measures are acquired. Finally, numerical interpretations are presented to depict the system behaviour.

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1. Introduction

Heterogeneity of service is a major aspect of multiprocessor queueing environment such as banks, hospitals, telecommunication networks, manufacturing systems and several business organisations. Morse [13] introduced the concept of heterogeneity in service. The heterogeneous servers have the characteristic of containing distinct levels and speed of service. The Internet itself is an example of a heterogeneous network.

The investigation of the time-dependent behaviour of queueing models is efficient in acquiring optimal solution which helps in the development of more efficient congestion control of the system. The time-dependent studies have really proved significant and are largely adopted by business organizations.

Trivedi [19] analyzed the two-server heterogeneous system in steady-state. For the same model, Dharmaraja [5] obtained the exact transient-state probabilities using generating function. Two heterogeneous servers queueing system with balking was discussed by Singh [16]. Ammar [1] derived the transient solution of an $M/M/2$ heterogeneous servers queue subject to balking and reneging.

Krishna Kumar and Pavai Madheswari [10] analyzed two heterogeneous servers queue with multiple vacations using matrix geometric method. Recently the authors [21] examined two heterogeneous servers perishable inventory system and calculated the system performance measures in steady-state.

Singh [15] performed steady-state analysis for three heterogeneous servers queueing system. Vijayalakshmi and Jyothsna [20] investigated a renewal input multiple working vacation queue with balking, reneging and

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heterogeneous servers. In [9] Ke *et al.* considered a machine repair problem with queue-dependent heterogeneous repairmen.

Tien Van Do [6] examined a tollbooth queueing model with two heterogeneous servers. He and Chao [8] considered a toll booth tandem queueing system with K heterogeneous servers using matrix-analytic method and obtained the explicit results for various performance measures. Li and Stanford [12] developed a multi-server queueing model with heterogeneous servers and established the waiting time distribution.

Moreover, the adoption of the First Come First Served (FCFS) discipline may not be realistic in modelling service systems with heterogeneous structures. Ramasamy *et al.* [14] considered the $M/G/2$ heterogeneous servers queueing model with a minimum violation of FCFS queue discipline.

There is a growing interest in the study of queueing models in the event of system disaster and repairable server. A disaster is also known as catastrophe, mass exodus, or queue flushing [3]. A disaster can be treated as a server reset which breaks down the server and causes all the customers in the system to get lost. The loss of customers due to disastrous breakdowns (also referred to as a kind of negative arrivals) was first introduced by Gelenbe [7]. Krishna Kumar *et al.* [11] studied two heterogeneous servers queue with catastrophe. Sudhesh *et al.* [18] analyzed an $M/M/1$ queue with $N-$ policy and system disaster.

Most of the works in the previous literature are carried out based on the assumption that disaster (catastrophe) occurs only when the system is not empty. However, this situation is predominantly unsuitable in many real-life systems. In various practical situations disaster may occur even when the system is empty. This motivates us to consider a system with disastrous breakdowns when the system is both idle and busy.

Ammar [2] expands the model described in Dharmaraja [5] subject to disastrous breakdowns and obtained an exact time-dependent solution. Dharmaraja and Rakesh Kumar [4] obtained the transient solution of the $c-$heterogeneous servers queue with system disaster. In recent times Sudhesh *et al.* [17] inferred two heterogeneous servers queue with system disaster, server repair and customers' impatience and obtained the transient-state and steady-state probabilities in closed form.

However, no work has been developed in the previous literature which analyzes queueing systems with more than two heterogeneous servers taking together the effect of system disaster and server repair. Based on this survey, we have evaluated the transient-state system size probabilities for the three-heterogeneous server queueing system subject to system disaster and server repair using generating function, Bessel function and Laplace transform.

The framework of this article is categorized as follows: in Section 2, the mathematical model and the system of difference equations determining this model are presented. In Section 3, the transient-state probabilities using generating function, Bessel function and Laplace Transform were obtained. In Section 4, the steady-state probabilities of the system size are deduced. In Section 5, certain performance measures of the system are given. In Section 6, numerical illustrations are presented to get more insight of the system behaviour and finally the concluding remarks are summarized in Section 7.

2. Model description

Observe a three-processor heterogeneous system with system disaster and server repair. Customer arrival process is Poisson with rate λ and the system has one waiting line. Service time is exponentially distributed where the three servers provide heterogeneous service with different service rates μ_1, μ_2 and μ_3 such that $\mu_1 > \mu_2 > \mu_3$. Each customer needs only one server for service and chooses on Fastest Server First (FSF) basis.

An accomplishing customer identifying more than one idle server prefers the faster among the free servers. If the customer observes exactly one idle server , customer go for it. If the customer finds all the three servers busy, customer joins the queue and wait to take service from either faster server or slower server based on who completed the service first. This schedule is reiterated whenever there are four or more customers in the system.

When the system is idle or busy, disaster occurs according to a Poisson procedure of rate γ. Whenever disaster happens, the entire customers (both waiting and served) are flushed out from the system and all the servers are subjected to failure. A repair process then begins instantly and the repair time of the system is exponentially

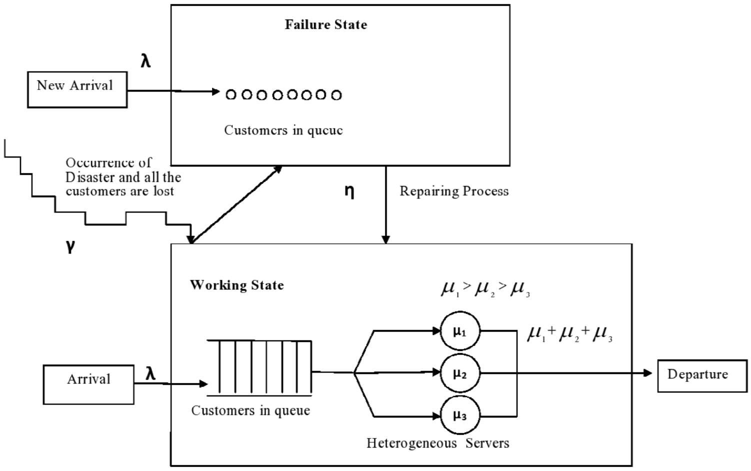

FIGURE 1. Pictorial Representation of the Model.

distributed with mean η^{-1}. After repairing, the servers become ready for service when a new arrival occurs. The pictorial depiction of the system is demonstrated in Figure 1.

Let $\{(X(t), Y(t)), t \geq 0\}$ be a two-dimensional continuous-time Markov chain, where $X(t)$ represents the amount of customers in the system at time t and $Y(t)$ denotes the state of the system at time t. Let $Q_{n,j}(t)$ denote the time dependent system size probabilities where there are n customers in the system at time t and j takes values $0, 1, 2, 3, 12, 13, 23, 4$ and 5. Mathematically,

$$Q_{n,j}(t) = Q[X(t) = n, Y(t) = j], n \geq 0; j = 0, 1, 2, 3, 12, 13, 23, 4, 5.$$

Let $Q_{0,0}(t)$ be the probability that there are no customers in the system and the servers are ready to serve customers.

For $n = 1$, let $Q_{1,j}(t)$ be the probability that there are one customer in the system and only the jth server is on the working state where j varies from 1 to 3.

For $n = 2$, let $Q_{2,ij}(t) = \sum_{i<j}^{3} Q_{2,ij}(t)$ be the probability that there are two customers in the system where ith and jth servers are on the working state such that i takes values 1 and 2 and j takes values 2 and 3.

For $j = 4$, let $Q_{n,4}(t)$ be the probability that there are $n \geq 3$ customers in the system and all the three servers are on the working state and

For $j = 5$, let $Q_{n,5}(t)$ be the probability that there are $n \geq 0$ customers in the system and all the three servers are on the failure state.

The transition-rate diagram of the proposed model is depicted in Figure 2.

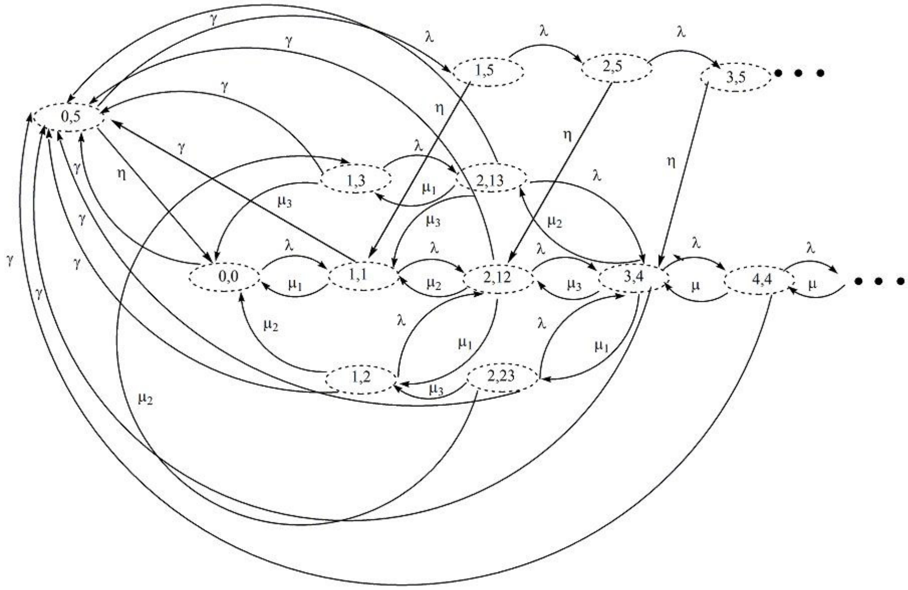

FIGURE 2. Transition-rate diagram of the system.

2.1. Governing equations

In view of the aforementioned assumptions, the behaviour of resulting system is described by a set of Chapman−Kolmogorov forward equations which can be written as :

$$Q_{0,0}^{'}(t) = -(\lambda+\gamma)Q_{0,0}(t) + \mu_1 Q_{1,1}(t) + \mu_2 Q_{1,2}(t) + \mu_3 Q_{1,3}(t) + \eta Q_{0,5}(t), \tag{2.1}$$

$$Q_{1,1}^{'}(t) = -(\lambda+\mu_1+\gamma)Q_{1,1}(t) + \lambda Q_{0,0}(t) + \mu_2 Q_{2,12}(t) + \mu_3 Q_{2,13}(t) + \eta Q_{1,5}(t), \tag{2.2}$$

$$Q_{1,2}^{'}(t) = -(\lambda+\mu_2+\gamma)Q_{1,2}(t) + \mu_1 Q_{2,12}(t) + \mu_3 Q_{2,23}(t), \tag{2.3}$$

$$Q_{1,3}^{'}(t) = -(\lambda+\mu_3+\gamma)Q_{1,3}(t) + \mu_1 Q_{2,13}(t) + \mu_2 Q_{2,23}(t), \tag{2.4}$$

$$Q_{2,12}^{'}(t) = -(\lambda+\mu_1+\mu_2+\gamma)Q_{2,12}(t) + \lambda Q_{1,1}(t) + \lambda Q_{1,2}(t) + \mu_3 Q_{3,4}(t) + \eta Q_{2,5}(t), \tag{2.5}$$

$$Q_{2,13}^{'}(t) = -(\lambda+\mu_1+\mu_3+\gamma)Q_{2,13}(t) + \lambda Q_{1,3}(t) + \mu_2 Q_{3,4}(t), \tag{2.6}$$

$$Q_{2,23}^{'}(t) = -(\lambda+\mu_2+\mu_3+\gamma)Q_{2,23}(t) + \mu_1 Q_{3,4}(t), \tag{2.7}$$

$$Q_{3,4}^{'}(t) = -(\lambda+\gamma+\mu)Q_{3,4}(t) + \lambda Q_{2,13}(t) + \lambda Q_{2,12}(t) + \lambda Q_{2,23}(t) + \mu Q_{4,4}(t) + \eta Q_{3,5}(t), \tag{2.8}$$

$$Q_{n,4}^{'}(t) = -(\lambda+\gamma+\mu)Q_{n,4}(t) + \lambda Q_{(n-1),4}(t) + \mu Q_{(n+1),4}(t) + \eta Q_{n,5}(t), \quad n \geq 4, \tag{2.9}$$

$$Q_{0,5}^{'}(t) = -(\eta+\lambda)Q_{0,5}(t) + \gamma\left[1 - \sum_{n=0}^{\infty} Q_{n,5}(t)\right], \tag{2.10}$$

$$Q_{n,5}^{'}(t) = -(\eta+\lambda)Q_{n,5}(t) + \lambda Q_{n-1,5}(t), \quad n \geq 1, \tag{2.11}$$

where $\mu = \mu_1 + \mu_2 + \mu_3$.

We assume that the amount of customers available at first is random with the probability $p_r, r \geq 0$ where r denotes the amount of customers in the system at the beginning.

3. Transient behaviour

Define the probability generating function

$$G(t,z) = R(t) + \sum_{n=0}^{\infty} Q_{(n+4),4}(t) z^{n+1}, \tag{3.1}$$

where

$$R(t) = Q_{0,0}(t) + \sum_{i=1}^{3} Q_{1,i}(t) + Q_{2,12}(t) + Q_{2,13}(t) + Q_{2,23}(t) + Q_{3,4}(t),$$

with initial condition $G(0,z) = \sum_{r=0}^{\infty} p_r z^r$.

The system of equations (2.1)–(2.9) yield the following partial differential equation

$$\begin{aligned}\frac{\partial G}{\partial t} =& -\gamma R(t) + \eta\left[\sum_{i=0}^{3} Q_{i,5}(t) + \sum_{n=1}^{\infty} Q_{(n+3),5}(t) z^n\right]\\ &+ \lambda(z-1)Q_{3,4}(t) + (\lambda z - (\lambda+\gamma+\mu) + \frac{\mu}{z})\Big[G(t,z) - R(t)\Big].\end{aligned} \tag{3.2}$$

On integration, we get

$$\begin{aligned}G(t,z) =& \left[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\} \exp\left\{\left(\lambda z + \frac{\mu}{z}\right)(t-y)\right\} \mathrm{d}y\right]\\ &\times\left[\lambda(z-1)Q_{3,4}(y) + \left[(\lambda+\mu) - \left(\lambda z + \frac{\mu}{z}\right)\right] R(y) + \eta\left(\sum_{i=0}^{3} Q_{i,5}(y) + \sum_{n=1}^{\infty} Q_{(n+3),5}(y) z^n\right)\right]\\ &+ \sum_{r=0}^{\infty} p_r z^r \exp\{-(\lambda+\gamma+\mu)(t)\} \exp\left\{\left(\lambda z + \frac{\mu}{z}\right)(t)\right\}.\end{aligned} \tag{3.3}$$

If $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$ then

$$\exp\left\{\left(\lambda z + \frac{\mu}{z}\right)(t-y)\right\} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n[\alpha(t-y)],$$

where $I_n(.) = I_n[\alpha(t-y)]$ is the modified Bessel function of first kind.

3.1. Evaluation of $Q_{(n+3),4}(t), n \geq 1$

Comparing the coefficients of z^n on both sides of equation (3.3), for $n \geq 1$, we have

$$\begin{aligned}\beta^{-n} Q_{(n+3),4}(t) =& \left[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\}\mathrm{d}y\right] \times \left[\left[\lambda\beta^{-1} I_{n-1}(.) - \lambda I_n(.)\right] Q_{3,4}(y)\right.\\ &+ \left[(\lambda+\mu) I_n(.) - \lambda\beta^{-1} I_{n-1}(.) - \mu\beta I_{n+1}(.)\right] R(y)\\ &\left.+ \eta[\sum_{i=0}^{3} Q_{i,5}(y) I_n(.) + \sum_{i=1}^{\infty} Q_{(i+3),5}(y) \beta^{-i} I_{n-i}(.)]\right]\\ &+ \exp\{-(\lambda+\gamma+\mu)(t)\} \sum_{r=0}^{\infty} p_r \beta^{-r} I_{n-r}(\alpha t).\end{aligned} \tag{3.4}$$

The above equation (3.4) holds for negative powers of n with left hand side replaced by zero. Using Bessel property $I_{-n}(.) = I_n(.)$ for $n \geq 1$, we get

$$\begin{aligned}0 = &\left[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\}\mathrm{d}y\right] \times \Bigg[\left[\lambda\beta^{-1}I_{n+1}(.) - \lambda I_n(.)\right] Q_{3,4}(y) \\ &+ \left[(\lambda+\mu)I_n(.) - \lambda\beta^{-1}I_{n+1}(.) - \mu\beta I_{n-1}(.)\right] R(y) \\ &+ \eta\left[\sum_{i=0}^{3} Q_{i,5}(y)I_n(.) + \sum_{i=1}^{\infty} Q_{(i+3),5}(y)\beta^{-i}I_{n+i}(.)\right]\Bigg] \\ &+ \exp\{-(\lambda+\gamma+\mu)(t)\}\sum_{r=0}^{\infty} p_r\beta^{-r}I_{n+r}(\alpha t). \end{aligned} \tag{3.5}$$

Difference of equation (3.5) from equation (3.4), for $n \geq 1$ we obtain

$$\begin{aligned}Q_{(n+3),4}(t) = &\left[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\}\mathrm{d}y\right] \\ &\times \left[\lambda\beta^{n-1}[I_{n-1}(.) - I_{(n+1)}(.)]Q_{3,4}(y) + \eta\sum_{i=1}^{\infty}\beta^{n-i}Q_{(i+3),5}(y)[I_{n-i}(.) - I_{(n+i)}(.)]\right] \\ &+ \exp\{-(\lambda+\gamma+\mu)(t)\}\sum_{r=0}^{\infty} p_r\beta^{n-r}[I_{n-r}(\alpha t) - I_{n+r}(\alpha t)]. \end{aligned} \tag{3.6}$$

3.2. Evaluation of $Q_{3,4}(t)$

We rewrite the system of equation (2.1)−(2.7) in the following matrix form:

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = BH(t) + \eta Q_{0,5}(t)e_1 + \eta Q_{1,5}(t)e_2 + \left(\mu_3 Q_{3,4}(t) + \eta Q_{2,5}(t)\right)e_3 + \mu_2 Q_{3,4}(t)e_4 + \mu_1 Q_{3,4}(t)e_5, \tag{3.7}$$

where

$$H(t) = \left(Q_{0,0}(t), Q_{1,1}(t), Q_{1,2}(t), Q_{1,3}(t), Q_{2,12}(t), Q_{2,13}(t), Q_{2,23}(t)\right)^T,$$

$$B = \begin{bmatrix} a & \mu_1 & \mu_2 & \mu_3 & 0 & 0 & 0 \\ \lambda & b & 0 & 0 & \mu_2 & \mu_3 & 0 \\ 0 & 0 & c & 0 & \mu_1 & 0 & \mu_3 \\ 0 & 0 & 0 & d & 0 & \mu_1 & \mu_2 \\ 0 & \lambda & \lambda & 0 & f & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h \end{bmatrix},$$

$a = -(\lambda+\gamma), b = -(\lambda+\mu_1+\gamma), c = -(\lambda+\mu_2+\gamma), d = -(\lambda+\mu_3+\gamma), f = -(\lambda+\mu_1+\mu_2+\gamma), g = -(\lambda+\mu_1+\mu_3+\gamma),$ $h = -(\lambda+\mu_2+\mu_3+\gamma), e_1 = (1,0,0,0,0,0,0)^T, e_2 = (0,1,0,0,0,0,0)^T, e_3 = (0,0,0,0,1,0,0)^T,$ $e_4 = (0,0,0,0,0,1,0)^T$ and $e_5 = (0,0,0,0,0,0,1)^T$.

Let $\tilde{f}(s)$ denotes the Laplace transform of $f(t)$. Applying Laplace transform on equation (3.7), we get

$$\tilde{H}(s) = (sI-B)^{-1}\left[H(0) + \eta\tilde{Q}_{0,5}(s)e_1 + \eta\tilde{Q}_{1,5}(s)e_2 + \left(\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)\right)e_3 + \mu_2\tilde{Q}_{3,4}(s)e_4 + \mu_1\tilde{Q}_{3,4}(s)e_5\right], \tag{3.8}$$

with

$$\begin{aligned}H(0) &= (Q_{0,0}(0), Q_{1,1}(0), Q_{1,2}(0), Q_{1,3}(0), Q_{2,12}(0), Q_{2,13}(0), Q_{2,23}(0))^T\\ &= (Q_1(0), Q_2(0), Q_3(0), Q_4(0), Q_5(0), Q_6(0), Q_7(0))^T \text{ (say).}\end{aligned}$$

To find $\tilde{Q}_{3,4}(s)$, we have

$$\tilde{R}(s) = \mathrm{e}^T \tilde{H}(s) + \tilde{Q}_{3,4}(s), \tag{3.9}$$

where $e = (1, 1, 1, 1, 1, 1, 1)^T$. Comparing the constant term in either side of equation (3.3) and using Bessel property, we obtain

$$\begin{aligned}R(t) = &\left[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\}\mathrm{d}y\right] \times \Bigg[\left[\lambda\beta^{-1}I_1(.) - \lambda I_0(.)\right] Q_{3,4}(y)\\ &+ \left[(\lambda+\mu)I_0(.) - \lambda\beta^{-1}I_1(.) - \mu\beta I_1(.)\right] R(y)\\ &+ \eta\left[\sum_{i=0}^{3} Q_{i,5}(y)I_0(.) + \sum_{i=1}^{\infty} Q_{(i+3),5}(y)\beta^{-i}I_i(.)\right]\Bigg]\\ &+ \exp\{-(\lambda+\gamma+\mu)(t)\}\sum_{r=0}^{\infty} p_r\beta^{-r}I_r(\alpha t).\end{aligned} \tag{3.10}$$

Taking Laplace transform and after some simplifications, the above equation yield

$$\begin{aligned}\tilde{R}(s)(s+\gamma) = &\frac{1}{2}\tilde{Q}_{3,4}(s)\left[w - \sqrt{w^2-\alpha^2} - 2\lambda\right] + \eta\sum_{i=0}^{3}\tilde{Q}_{i,5}(s)\\ &+ \eta\sum_{i=1}^{\infty}\tilde{Q}_{(i+3),5}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^i + \sum_{r=0}^{\infty} p_r\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r,\end{aligned} \tag{3.11}$$

where $w = s + \lambda + \gamma + \mu$.

Using equations (3.8) and (3.9) in equation (3.11) and simplifying we get,

$$\begin{aligned}\tilde{Q}_{3,4}(s) = &\left[s+\gamma - \frac{1}{2}[w - \sqrt{w^2-\alpha^2} - 2\lambda] + (s+\gamma)\mathrm{e}^T(sI-B)^{-1}(\mu_3 e_3 + \mu_2 e_4 + \mu_1 e_5)\right]^{-1}\\ &\times\Bigg[\sum_{r=0}^{\infty} p_r\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r + \eta\sum_{i=0}^{3}\tilde{Q}_{i,5}(s) + \eta\sum_{i=1}^{\infty}\tilde{Q}_{(i+3),5}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^i\\ &- \mathrm{e}^T(sI-B)^{-1}(s+\gamma)\left[H(0) + \eta\sum_{i=0}^{2}\tilde{Q}_{i,5}(s)e_{i+1}\right]\Bigg].\end{aligned} \tag{3.12}$$

Let

$$(sI-B)^{-1} = (b^*_{ij}(s))_{7\times 7}, \tag{3.13}$$

which can be found using matlab and $s_k, (k = 1, 2, \ldots, 7)$ be the eigen values of equation (3.13). Using matlab, it is known that the eigen values of B are all different. Hence, the inverse transform $b_{ij}(t)$ of $b^*_{ij}(s)$ can be obtained by partial fraction decomposition. Now using equation (3.13), we get

$$\mathrm{e}^T(sI-B)^{-1}H(0)=\sum_{i=1}^{7}\sum_{j=1}^{7}b_{ij}^*(s)Q_j(0), \tag{3.14}$$

$$\mathrm{e}^T(sI-B)^{-1}e_1=\sum_{j=1}^{7}b_{j1}^*(s), \tag{3.15}$$

$$\mathrm{e}^T(sI-B)^{-1}e_2=\sum_{j=1}^{7}b_{j2}^*(s), \tag{3.16}$$

$$\mathrm{e}^T(sI-B)^{-1}e_3=\sum_{j=1}^{7}b_{j5}^*(s), \tag{3.17}$$

and

$$\mathrm{e}^T(sI-B)^{-1}[\mu_3e_3+\mu_2e_4+\mu_1e_5]=\left[\mu_3\sum_{j=1}^{7}b_{j5}^*(s)+\mu_2\sum_{j=1}^{7}b_{j6}^*(s)+\mu_1\sum_{j=1}^{7}b_{j7}^*(s)\right]. \tag{3.18}$$

Using (3.14)–(3.18) in (3.12), we get

$$\begin{aligned}\tilde{Q}_{3,4}(s)=&\left[s+\gamma-\frac{1}{2}\left[w-\sqrt{w^2-\alpha^2}-2\lambda\right]+c_4^*(s)\right]^{-1}\\ &\times\left[\sum_{r=0}^{\infty}p_r\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r+\eta\sum_{i=0}^{3}\tilde{Q}_{i,5}(s)\right.\\ &\left.+\eta\sum_{i=1}^{\infty}\tilde{Q}_{(i+3),5}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^i-c_0^*(s)-\eta\sum_{i=1}^{3}c_i^*(s)\tilde{Q}_{(i-1),5}(s)\right],\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}c_0^*(s)&=(s+\gamma)\sum_{i=1}^{7}\sum_{j=1}^{7}b_{ij}^*(s)Q_j(0),\\ c_1^*(s)&=(s+\gamma)\sum_{j=1}^{7}b_{j1}^*(s),\\ c_2^*(s)&=(s+\gamma)\sum_{j=1}^{7}b_{j2}^*(s),\\ c_3^*(s)&=(s+\gamma)\sum_{j=1}^{7}b_{j5}^*(s),\end{aligned}$$

and

$$c_4^*(s)=(s+\gamma)\left[\mu_3\sum_{j=1}^{7}b_{j5}^*(s)+\mu_2\sum_{j=1}^{7}b_{j6}^*(s)+\mu_1\sum_{j=1}^{7}b_{j7}^*(s)\right].$$

After some simple algebraic manipulations, equation (3.19) reduces to

$$\begin{aligned}\tilde{Q}_{3,4}(s) = &\left[\frac{2}{\alpha}\left(\frac{w-\sqrt{w^2-\alpha^2}}{\alpha}\right)\left(\sum_{r=0}^{\infty} p_r\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r + \eta\sum_{i=0}^{3}\tilde{Q}_{i,5}(s)\right.\right.\\ &\left.\left.+\eta\sum_{i=1}^{\infty}\tilde{Q}_{(i+3),5}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^i - c_0^*(s) - \eta\sum_{i=1}^{3}c_i^*(s)\tilde{Q}_{(i-1),5}(s)\right)\right]\\ &\times\left[\sum_{m=0}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{m}{2}}\left(\frac{w-\sqrt{w^2-\alpha^2}}{\alpha}\right)^m\right]\times\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\frac{c_4^*(s)}{\mu}\right)^k\right].\end{aligned}$$

On Laplace inversion, we get an explicit expression for $Q_{3,4}(t)$ as

$$\begin{aligned}Q_{3,4}(t) = &\left(\sum_{m=0}^{\infty}\left(\frac{\mu}{\lambda}\right)^{\frac{m}{2}}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\frac{1}{\mu}\right)^k\right)\\ &\times\left(\int_0^t c_4^{*^k}(t-u)\exp\{-(\lambda+\gamma+\mu)u\}\sum_{r=0}^{\infty}p_r\beta^{-r}[I_{m+r}(\alpha u)-I_{m+r+2}(\alpha u)]\mathrm{d}u\right.\\ &+\eta\int_0^t c_4^{*^k}(t-u)\left[\int_0^u\sum_{i=0}^{3}Q_{i,5}(u-v)\exp\{-(\lambda+\gamma+\mu)v\}[I_m(\alpha v)-I_{m+2}(\alpha v)]\mathrm{d}v\right]\mathrm{d}u\\ &+\eta\int_0^t c_4^{*^k}(t-u)\left[\int_0^u\sum_{i=1}^{\infty}Q_{(i+3),5}(u-v)\beta^{-i}\exp\{-(\lambda+\gamma+\mu)v\}[I_{m+i}(\alpha v)-I_{m+i+2}(\alpha v)]\mathrm{d}v\right]\mathrm{d}u\\ &\left.-\int_0^t c_4^{*^k}(t-u)\left[\int_0^u\left[c_0(u-v)+\eta\left[\sum_{i=0}^{2}Q_{i,5}(t-u)c_{i+1}(u-v)\right]\right]\exp\{-(\lambda+\gamma+\mu)v\}[I_m(\alpha v)-I_{m+2}(\alpha v)]\mathrm{d}v\right]\mathrm{d}u\right),\end{aligned} \tag{3.20}$$

where $c_4^{*^k}$ is the k-fold convolution of $c_4(t)$ with itself and $c_4^{*^0}(t)=\delta(t)$.

3.3. Evaluation of $Q_{0,0}(t), Q_{1,1}(t), Q_{1,2}(t), Q_{1,3}(t), Q_{2,12}(t), Q_{2,13}(t), Q_{2,23}(t)$

Using equation (3.13) in equation (3.8), we have

$$\begin{aligned}\tilde{Q}_{0,0}(s) = &\left[\sum_{j=1}^{7}b_{1j}^*(s)Q_j(0)+\eta\tilde{Q}_{0,5}(s)\right]b_{11}^*(s)\\ &+\eta\tilde{Q}_{1,5}(s)b_{12}^*(s)+[\mu_3\tilde{Q}_{3,4}(s)+\eta\tilde{Q}_{2,5}(s)]b_{15}^*(s)\\ &+\mu_2\tilde{Q}_{3,4}(s)b_{16}^*(s)+\mu_1\tilde{Q}_{3,4}(s)b_{17}^*(s),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\tilde{Q}_{1,1}(s) = &\left[\sum_{j=1}^{7}b_{2j}^*(s)Q_j(0)+\eta\tilde{Q}_{0,5}(s)\right]b_{21}^*(s)\\ &+\eta\tilde{Q}_{1,5}(s)b_{22}^*(s)+[\mu_3\tilde{Q}_{3,4}(s)+\eta\tilde{Q}_{2,5}(s)]b_{25}^*(s)\\ &+\mu_2\tilde{Q}_{3,4}(s)b_{26}^*(s)+\mu_1\tilde{Q}_{3,4}(s)b_{27}^*(s),\end{aligned} \tag{3.22}$$

$$\tilde{Q}_{1,2}(s) = \left[\sum_{j=1}^{7} b_{3j}^*(s)Q_j(0) + \eta\tilde{Q}_{0,5}(s)\right] b_{31}^*(s)$$
$$+ \eta\tilde{Q}_{1,5}(s)b_{32}^*(s) + [\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)]b_{35}^*(s)$$
$$+ \mu_2\tilde{Q}_{3,4}(s)b_{36}^*(s) + \mu_1\tilde{Q}_{3,4}(s)b_{37}^*(s), \tag{3.23}$$

$$\tilde{Q}_{1,3}(s) = \left[\sum_{j=1}^{7} b_{4j}^*(s)Q_j(0) + \eta\tilde{Q}_{0,5}(s)\right] b_{41}^*(s)$$
$$+ \eta\tilde{Q}_{1,5}(s)b_{42}^*(s) + [\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)]b_{45}^*(s)$$
$$+ \mu_2\tilde{Q}_{3,4}(s)b_{46}^*(s) + \mu_1\tilde{Q}_{3,4}(s)b_{47}^*(s), \tag{3.24}$$

$$\tilde{Q}_{2,12}(s) = \left[\sum_{j=1}^{7} b_{5j}^*(s)Q_j(0) + \eta\tilde{Q}_{0,5}(s)\right] b_{51}^*(s)$$
$$+ \eta\tilde{Q}_{1,5}(s)b_{52}^*(s) + [\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)]b_{55}^*(s)$$
$$+ \mu_2\tilde{Q}_{3,4}(s)b_{56}^*(s) + \mu_1\tilde{Q}_{3,4}(s)b_{57}^*(s), \tag{3.25}$$

$$\tilde{Q}_{2,13}(s) = \left[\sum_{j=1}^{7} b_{6j}^*(s)Q_j(0) + \eta\tilde{Q}_{0,5}(s)\right] b_{61}^*(s)$$
$$+ \eta\tilde{Q}_{1,5}(s)b_{62}^*(s) + [\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)]b_{65}^*(s)$$
$$+ \mu_2\tilde{Q}_{3,4}(s)b_{66}^*(s) + \mu_1\tilde{Q}_{3,4}(s)b_{67}^*(s), \tag{3.26}$$

and

$$\tilde{Q}_{2,23}(s) = \left[\sum_{j=1}^{7} b_{7j}^*(s)Q_j(0) + \eta\tilde{Q}_{0,5}(s)\right] b_{71}^*(s)$$
$$+ \eta\tilde{Q}_{1,5}(s)b_{72}^*(s) + [\mu_3\tilde{Q}_{3,4}(s) + \eta\tilde{Q}_{2,5}(s)]b_{75}^*(s)$$
$$+ \mu_2\tilde{Q}_{3,4}(s)b_{76}^*(s) + \mu_1\tilde{Q}_{3,4}(s)b_{77}^*(s). \tag{3.27}$$

Using equations (3.21)−(3.27) and inverting, we obtain

$$Q_{0,0}(t) = \int_0^t \sum_{j=1}^{7} b_{1j}(u)Q_j(0)b_{11}(t-u)\mathrm{d}u$$
$$+ \int_0^t \eta Q_{0,5}(u)b_{11}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{15}(t-u)\mathrm{d}u$$
$$+ \int_0^t \mu_2 Q_{3,4}(u)b_{16}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{17}(t-u)\mathrm{d}u, \tag{3.28}$$

$$Q_{1,1}(t) = \int_0^t \sum_{j=1}^{7} b_{2j}(u)Q_j(0)b_{21}(t-u)\mathrm{d}u$$
$$+ \int_0^t \eta Q_{0,5}(u)b_{21}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{25}(t-u)\mathrm{d}u$$
$$+ \int_0^t \mu_2 Q_{3,4}(u)b_{26}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{27}(t-u)\mathrm{d}u, \tag{3.29}$$

$$\begin{aligned} Q_{1,2}(t) = &\int_0^t \sum_{j=1}^{7} b_{3j}(u)Q_j(0)b_{31}(t-u)\mathrm{d}u \\ &+ \int_0^t \eta Q_{0,5}(u)b_{31}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{35}(t-u)\mathrm{d}u \\ &+ \int_0^t \mu_2 Q_{3,4}(u)b_{36}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{37}(t-u)\mathrm{d}u, \end{aligned} \tag{3.30}$$

$$\begin{aligned} Q_{1,3}(t) = &\int_0^t \sum_{j=1}^{7} b_{4j}(u)Q_j(0)b_{41}(t-u)\mathrm{d}u \\ &+ \int_0^t \eta Q_{0,5}(u)b_{41}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{45}(t-u)\mathrm{d}u \\ &+ \int_0^t \mu_2 Q_{3,4}(u)b_{46}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{47}(t-u)\mathrm{d}u, \end{aligned} \tag{3.31}$$

$$\begin{aligned} Q_{2,12}(t) = &\int_0^t \sum_{j=1}^{7} b_{5j}(u)Q_j(0)b_{51}(t-u)\mathrm{d}u \\ &+ \int_0^t \eta Q_{0,5}(u)b_{51}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{55}(t-u)\mathrm{d}u \\ &+ \int_0^t \mu_2 Q_{3,4}(u)b_{56}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{57}(t-u)\mathrm{d}u, \end{aligned} \tag{3.32}$$

$$\begin{aligned} Q_{2,13}(t) = &\int_0^t \sum_{j=1}^{7} b_{6j}(u)Q_j(0)b_{61}(t-u)\mathrm{d}u \\ &+ \int_0^t \eta Q_{0,5}(u)b_{61}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{65}(t-u)\mathrm{d}u \\ &+ \int_0^t \mu_2 Q_{3,4}(u)b_{66}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{67}(t-u)\mathrm{d}u, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} Q_{2,23}(t) = &\int_0^t \sum_{j=1}^{7} b_{7j}(t)Q_j(0)b_{71}(t-u)\mathrm{d}u \\ &+ \int_0^t \eta Q_{0,5}(u)b_{71}(t-u)\mathrm{d}u + \int_0^t (\mu_3 Q_{3,4}(u) + \eta Q_{2,5}(u))b_{75}(t-u)\mathrm{d}u \\ &+ \int_0^t \mu_2 Q_{3,4}(u)b_{76}(t-u)\mathrm{d}u + \int_0^t \mu_1 Q_{3,4}(u)b_{77}(t-u)\mathrm{d}u. \end{aligned} \tag{3.34}$$

3.4. Evaluation of $Q_{n,5}(t), n \geq 0$

The transient system size probabilities $Q_{n,5}(t), n \geq 0$ are obtained by using Laplace transform. Applying Laplace transform in equation (2.11), we have

$$\tilde{Q}_{n,5}(s) = \left(\frac{\lambda}{s+\eta+\lambda}\right)^n \tilde{Q}_{0,5}(s) \quad n \geq 1. \tag{3.35}$$

On taking Laplace transform of equation (2.10) and using equation (3.35), we obtain

$$\tilde{Q}_{0,5}(s) = \left(\frac{\gamma}{s(s+\eta+\lambda)}\right)\left(\frac{s+\eta}{s+\gamma+\eta}\right) \quad \text{provided} \quad \left|\frac{\lambda}{s+\eta+\lambda}\right| < 1, \tag{3.36}$$

with the assumption that $Q_{0,5}(0) = 0$.

Taking inverse Laplace transform, we get

$$Q_{0,5}(t) = \left[\frac{\gamma}{\eta+\lambda}\Big[1-\exp\{-(\eta+\lambda)t\}\Big]\right] \times \left[1-\frac{\gamma}{(\gamma+\eta)}\right] + \frac{\gamma^2}{(\gamma+\eta)(\lambda-\gamma)}\left[\exp\{-(\gamma+\eta)t\} - \exp\{-(\eta+\lambda)t\}\right]. \tag{3.37}$$

Using (3.36) in (3.35), we yield

$$\tilde{Q}_{n,5}(s) = \frac{\lambda^n\gamma}{(s+\eta+\lambda)^{n+1}}\left[\frac{s+\eta}{s(s+\gamma+\eta)}\right], \tag{3.38}$$

with the assumption that $Q_{n,5}(0) = 0$.

On Laplace inversion, we have for $n > 0$,

$$\begin{aligned} Q_{n,5}(t) =& \left[\frac{\lambda^n\gamma}{n!(\gamma+\eta)}\left[\sum_{r=0}^{n} nC_r t^{n-r} r!(-1)^r \exp\{-(\eta+\lambda)t\}\right]\right] \\ &\times \left[\frac{\eta}{\big[-(\eta+\lambda)\big]^{r+1}} + \frac{\gamma}{\big[-(\lambda-\gamma)\big]^{r+1}}\right] \\ &+ \frac{\lambda^n\eta\gamma}{(\gamma+\eta)(\eta+\lambda)^{n+1}} + \frac{\lambda^n\gamma^2\exp\{-(\gamma+\eta)t\}}{(\gamma+\eta)(\lambda-\gamma)^{n+1}}. \end{aligned} \tag{3.39}$$

Thus equations (3.6), (3.20), (3.28)–(3.34), (3.37) and (3.39) completely determined all the transient-state probabilities.

4. Steady-state probabilities

The steady-state system size probabilities for the three-heterogeneous servers queue with system disaster and server repair are derived. Let

$$Q_{nj} = \lim_{t\to\infty} Q\{X(t) = n, Y(t) = j\}, n \geq 0; j = 0, 1, 2, 3, 12, 13, 23, 4, 5.$$

From equation (3.19), for $\gamma, \eta > 0$, we get

$$\begin{aligned} \tilde{Q}_{3,4}(s) =& \left[s+\gamma-\frac{1}{2}\left[w-\sqrt{w^2-\alpha^2}-2\lambda\right]+c_4^*(s)\right]^{-1} \\ &\times \left[\sum_{r=0}^{\infty} p_r \left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r + \eta\sum_{i=0}^{3}\tilde{Q}_{i,5}(s)\right. \\ &\left. + \eta\sum_{i=1}^{\infty}\tilde{Q}_{(i+3),5}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^i - c_0^*(s) - \eta\sum_{i=1}^{3} c_i^*(s)\tilde{Q}_{(i-1),5}(s)]\right]. \end{aligned}$$

By using Tauberian theorem, we get

$$\begin{aligned}Q_{3,4} =& \left[\gamma - \frac{1}{2}\left[(\lambda+\gamma+\mu) - \sqrt{(\lambda+\gamma+\mu)^2-\alpha^2} - 2\lambda\right] + \lim_{s\to 0} sc_4^*(s)\right]^{-1}\\ &\times\left[\eta\sum_{i=0}^{3} Q_{i,5} + \eta\sum_{i=1}^{\infty} Q_{(i+3),5}\left(\frac{(\lambda+\gamma+\mu) - \sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}}{2\lambda}\right)^i\right.\\ &\left.-\lim_{s\to 0} sc_0^*(s) - \eta\lim_{s\to 0} s\sum_{i=1}^{3} c_i^*(s)Q_{(i-1),5}\right]. \qquad (4.1)\end{aligned}$$

Taking Laplace transform of (3.6) and after considerable simplifications, we have for $n \geq 1$

$$\begin{aligned}\tilde{Q}_{(n+3),4}(s) =& \left[\sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\left[(s+\lambda+\gamma+\mu) + \sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right]^n\right]^{-1}\\ &\times\left[\left[(2\lambda)^n\sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right]\tilde{Q}_{3,4}(s)\right.\\ &+\left[\eta\sum_{i=1}^{\infty}(2\lambda)^{n-i}\left((s+\lambda+\gamma+\mu) + \sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right)^i\right.\\ &\left.\times\left(1-\alpha^{2i}\left((s+\lambda+\gamma+\mu) + \sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right)^{-2i}\right)\right]\tilde{Q}_{(i+3),5}(s)\\ &+\sum_{r=0}^{\infty} p_r(2\lambda)^{n-r}\left((s+\lambda+\gamma+\mu) + \sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right)^r\\ &\left.\times\left(1-\alpha^{2r}\left((s+\lambda+\gamma+\mu) + \sqrt{(s+\lambda+\gamma+\mu)^2-\alpha^2}\right)^{-2r}\right)\right].\end{aligned}$$

Then, again by using Tauberian theorem, we obtain

$$\begin{aligned}Q_{(n+3),4} =& \left[\sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}\left[(\lambda+\gamma+\mu) + \sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}\right]^n\right]^{-1}\\ &\times\left[\left[(2\lambda)^n\sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}\right]Q_{3,4} + \left[\eta\sum_{i=1}^{\infty}(2\lambda)^{n-i}\left((\lambda+\gamma+\mu) + \sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}\right)^i\right.\right.\\ &\left.\left.\times\left(1-\alpha^{2i}\left((\lambda+\gamma+\mu) + \sqrt{(\lambda+\gamma+\mu)^2-\alpha^2}\right)^{-2i}\right)\right]Q_{(i+3),5}\right]. \qquad (4.2)\end{aligned}$$

In a similar way, equation (3.21) gives

$$\begin{aligned}Q_{0,0} =& \left[\sum_{j=1}^{7}\lim_{s\to 0} sb_{1j}^*(s)Q_j(0) + \eta\lim_{s\to 0} sQ_{0,5}\right]b_{11}^*(s)\\ &+\eta\lim_{s\to 0} sQ_{1,5}b_{12}^*(s) + [\mu_3\lim_{s\to 0} sQ_{3,4} + \eta\lim_{s\to 0} sQ_{2,5}]b_{15}^*(s)\\ &+\mu_2\lim_{s\to 0} sQ_{3,4}b_{16}^*(s) + \mu_1\lim_{s\to 0} sQ_{3,4}b_{17}^*(s). \qquad (4.3)\end{aligned}$$

From (3.22), we have

$$Q_{1,1} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{2j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{21}^*(s) + \eta \lim_{s\to 0} s Q_{1,5} b_{22}^*(s)$$
$$+ [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{25}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{26}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{27}^*(s). \tag{4.4}$$

From (3.23), we get

$$Q_{1,2} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{3j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{31}^*(s) + \eta \lim_{s\to 0} s Q_{1,5} b_{32}^*(s)$$
$$+ [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{35}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{36}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{37}^*(s). \tag{4.5}$$

From (3.24), we obtain

$$Q_{1,3} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{4j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{41}^*(s) + \eta \lim_{s\to 0} s Q_{1,5} b_{42}^*(s)$$
$$+ [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{45}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{46}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{47}^*(s). \tag{4.6}$$

From (3.25), we yield

$$Q_{2,12} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{5j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{51}^*(s)$$
$$+ \eta \lim_{s\to 0} s Q_{1,5} b_{52}^*(s) + [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{55}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{56}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{57}^*(s). \tag{4.7}$$

From (3.26), we have

$$Q_{2,13} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{6j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{61}^*(s) + \eta \lim_{s\to 0} s Q_{1,5} b_{62}^*(s)$$
$$+ [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{65}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{66}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{67}^*(s) \tag{4.8}$$

and from (3.27), we get

$$Q_{2,23} = \left[\sum_{j=1}^{7} \lim_{s\to 0} s b_{7j}^*(s) Q_j(0) + \eta \lim_{s\to 0} s Q_{0,5}\right] b_{71}^*(s) + \eta \lim_{s\to 0} s Q_{1,5} b_{72}^*(s)$$
$$+ [\mu_3 \lim_{s\to 0} s Q_{3,4} + \eta \lim_{s\to 0} s Q_{2,5}] b_{75}^*(s)$$
$$+ \mu_2 \lim_{s\to 0} s Q_{3,4} b_{76}^*(s) + \mu_1 \lim_{s\to 0} s Q_{3,4} b_{77}^*(s). \tag{4.9}$$

Using (3.38), we obtain for $n \geq 1$

$$Q_{n,5} = \frac{\lambda^n \eta\gamma}{(\gamma+\eta)(\eta+\lambda)^{n+1}}\cdot \tag{4.10}$$

From (3.36), we get

$$Q_{0,5} = \frac{\eta\gamma}{(\gamma+\eta)(\eta+\lambda)}\cdot \tag{4.11}$$

Remark 4.1. The aforementioned model ensures the convergence of steady-state probabilities due to the presence of system disaster. As the disaster rate γ is positive, the disaster event will eventually happen with probability one and hence the proposed system would never be unstable.

5. Performance measures

In this section, we obtain some important performance measures of the system.

5.1. Probability of arriving customers joining the queue

The probability that an accomplishing customer is mandatory to wait in line at time t is accounted as

$$\begin{aligned} Q(X(t)\geq 3) = Q_{3,4}(t) + \sum_{n=1}^{\infty}\Bigg(\Bigg[\int_0^t \exp\{-(\lambda+\gamma+\mu)(t-y)\}\mathrm{d}y\Bigg] \\ \times\Bigg[\lambda\beta^{n-1}[I_{n-1}(.) - I_{(n+1)}(.)]Q_{3,4}(y) \\ + \eta\sum_{i=1}^{\infty}\beta^{n-i}Q_{(i+3),5}(y)[I_{n-i}(.) - I_{(n+i)}(.)]\Bigg] \\ + \exp\{-(\lambda+\gamma+\mu)(t)\}\sum_{r=0}^{\infty}p_r\beta^{n-r}[I_{n-r}(\alpha t) - I_{n+r}(\alpha t)]\Bigg). \end{aligned} \tag{5.1}$$

5.2. Probability that the system is in breakdown

The probability that the system is in breakdown at time t is alloted by

$$\begin{aligned} \sum_{n=0}^{\infty}Q_{n,5}(t), = \Bigg[\frac{\gamma}{\eta+\lambda}\Big[1-\exp\{-(\eta+\lambda)t\}\Big]\Bigg]\times\Bigg[1-\frac{\gamma}{(\gamma+\eta)}\Bigg] \\ + \frac{\gamma^2}{(\gamma+\eta)(\lambda-\gamma)}\Big[\exp\{-(\gamma+\eta)t\} - \exp\{-(\eta+\lambda)t\}\Big] \\ + \Bigg[\frac{\lambda^n\gamma}{n!(\gamma+\eta)}\Bigg[\sum_{r=0}^{n} nC_r t^{n-r} r!(-1)^r \exp\{-(\eta+\lambda)t\}\Bigg]\Bigg] \\ \times\Bigg[\frac{\eta}{\big[-(\eta+\lambda)\big]^{r+1}} + \frac{\gamma}{\big[-(\lambda-\gamma)\big]^{r+1}}\Bigg] \\ + \frac{\lambda^n\eta\gamma}{(\gamma+\eta)(\eta+\lambda)^{n+1}} + \frac{\lambda^n\gamma^2\exp\{-(\gamma+\eta)t\}}{(\gamma+\eta)(\lambda-\gamma)^{n+1}}\cdot \end{aligned} \tag{5.2}$$

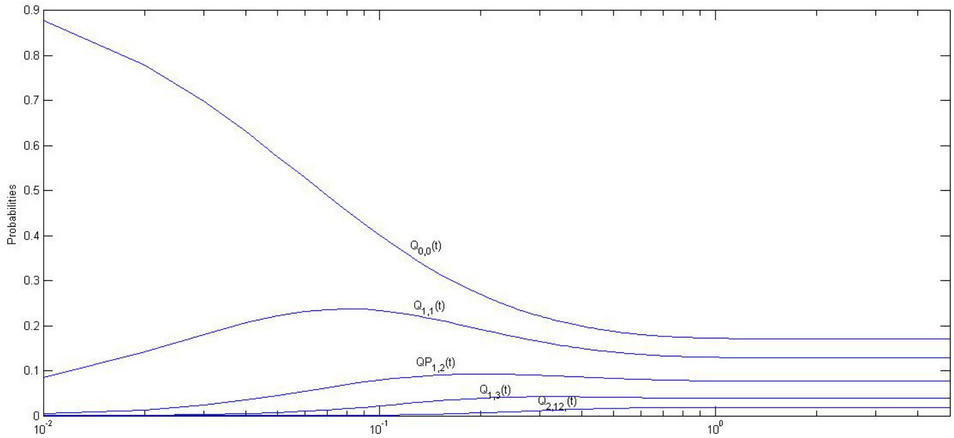

FIGURE 3. Transient-state probabilities.

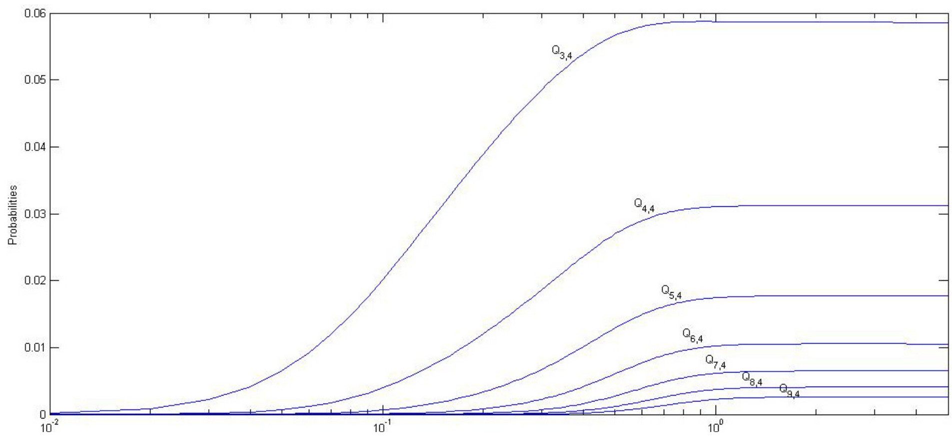

FIGURE 4. Transient-state probabilities in working state.

6. Numerical interpretations

In order to gain more perception of the model behaviour, some numerical experiments are performed in this section.

In Figures 3–5, the transient-state probabilities $Q_{0,0}(t)$, $Q_{1,1}(t)$, $Q_{1,2}(t)$, $Q_{1,3}(t)$, $Q_{2,12}(t)$, $Q_{n,4}(t)(n = 3, \ldots, 9)$, $Q_{n,5}(t)(n = 0, 1, \ldots, 6)$ are plotted against time with the arrival rate $\lambda = 10$, service rates $\mu_1 = 9$, $\mu_2 = 7.5$, $\mu_3 = 5.5$, system disaster rate $\gamma = 3.5$ and repair rate $\eta = 5.5$. We assume that initially the system is empty. We see that initially all the probability curves except $Q_{0,0}(t)$ increases and then decreases gradually up to some time interval. In addition, the probability distribution attains steady-state as time increases.

Figures 6 and 7 indicates the expected system size $E(X(t))$ for various values of system disaster rate γ and repair rate η respectively with the arrival rate $\lambda = 5$ and service rates $\mu_1 = 4$, $\mu_2 = 2$ and $\mu_3 = 1$. The value of $E(X(t))$ grows gradually with the increase in time t until it takes its maximum value and consequently the system reaches the steady-state.

Figures 8 and 9 depicts the values $\mathrm{Var}(X(t))$ for various values of γ and η respectively with $\lambda = 5$, $\mu_1 = 4$, $\mu_2 = 2$ and $\mu_3 = 1$. The variance $\mathrm{Var}(X(t))$ increases with time t until it takes its maximum value and consequently the system transits to the steady-state.

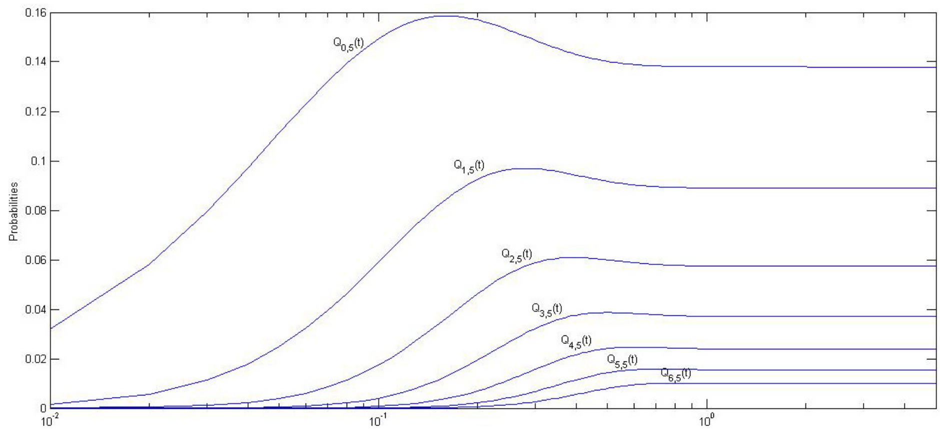

FIGURE 5. Transient-state probabilities in failure state.

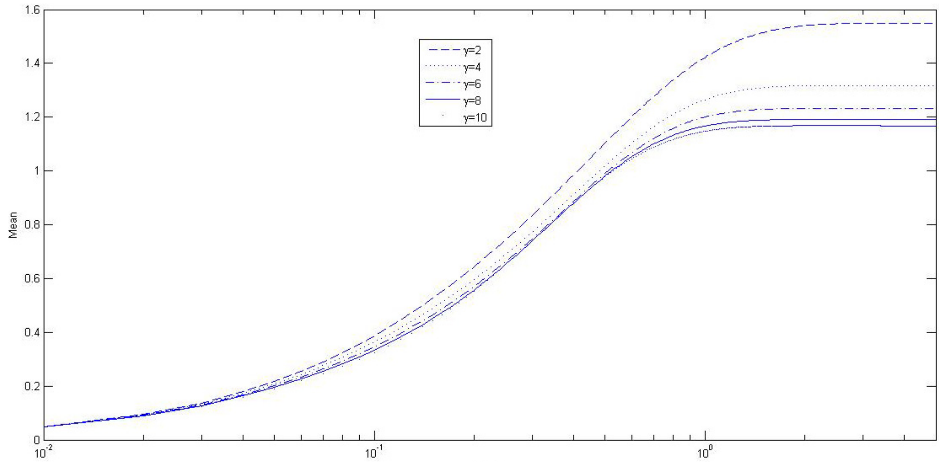

FIGURE 6. Expected system size for various disaster rates γ and t corresponding to $\eta = 4.5$.

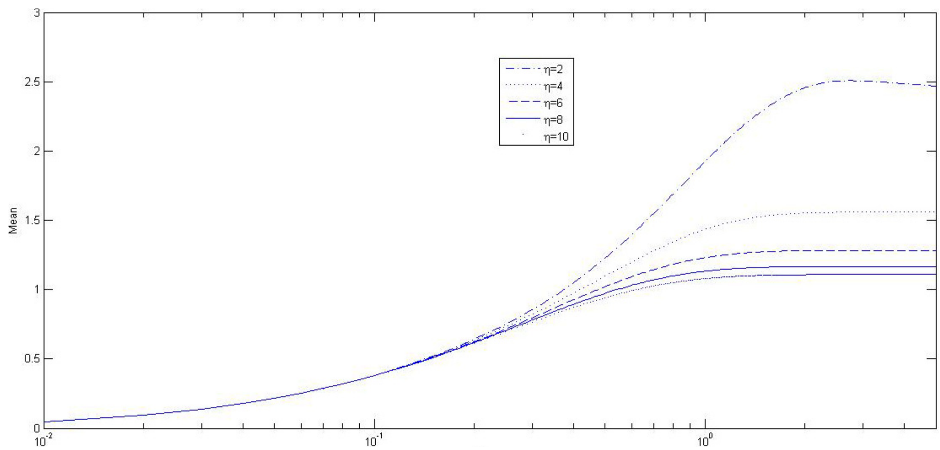

FIGURE 7. Expected system size for various repair rates η and t corresponding to $\gamma = 2.5$.

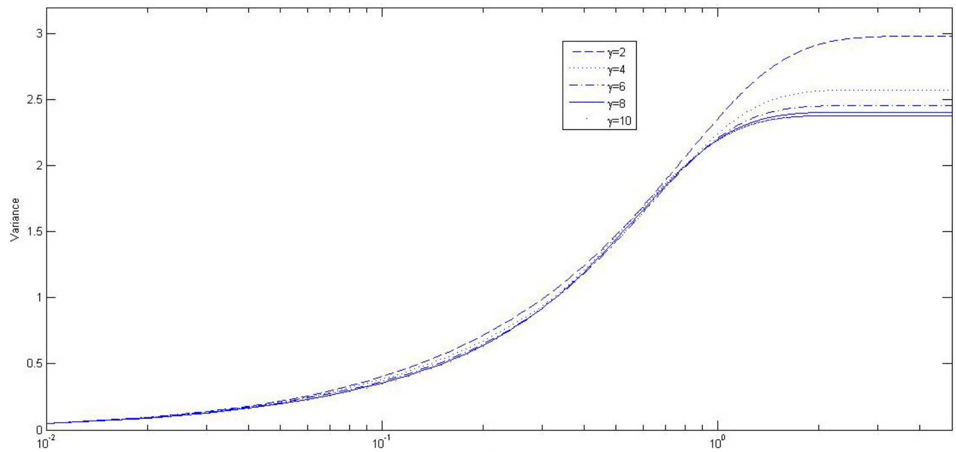

FIGURE 8. The variation of the variance with evolution of system disaster γ and t with $\eta = 4.5$.

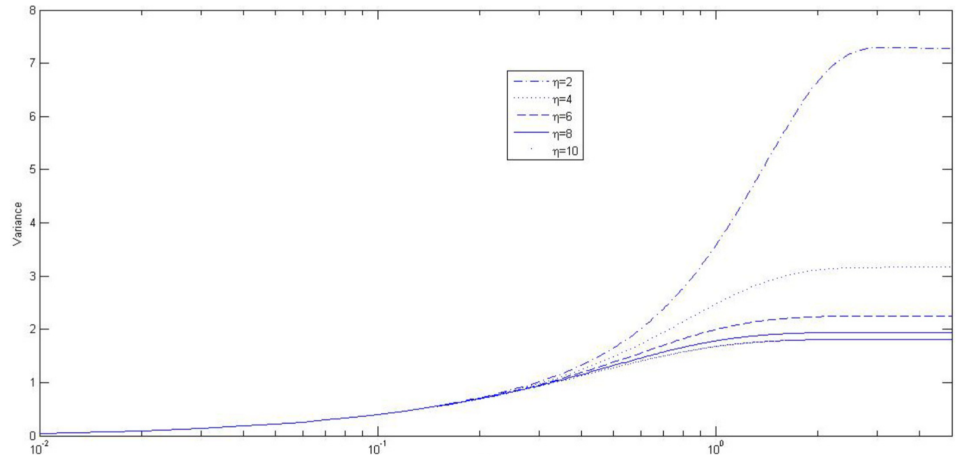

FIGURE 9. The variation of the variance with the effect of repair rate η and t with $\gamma = 2.5$.

7. Conclusion and future scope

The three-server heterogeneous queueing system with system disaster and server repair is discussed and the explicit solution is derived in closed form for the time-dependent system size probabilities using generating function technique. These probabilities and performance measures are useful to know the transient behaviour of the system. Further, we wish to extend this model for $c-$heterogeneous servers with customers' impatience.

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