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# ECONOMIES OF SCOPE IN TWO-STAGE PRODUCTION SYSTEMS: A DATA ENVELOPMENT ANALYSIS APPROACH

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**Abstract.** Traditional data envelopment analysis (DEA) models consider a production system as a black-box without taking into consideration its internal linked activities. In recent years, a number of DEA studies have been presented to estimate efficiency score of two-stage network production systems in which all outputs of the first stage (intermediate products) are used as inputs of the second stage to produce final outputs. This paper aims to develop a two-stage network DEA model to study economic notion of economies of scope (ES) between two products. It intends to determine profitability of joint production of two products by one firm. Numerical illustrations are presented to show applicability of proposed methods.

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## 1. Introduction

Data envelopment analysis (DEA) is a non-parametric approach introduced by Charnes *et al.* [4] which is used for evaluating relative performance of decision making units (DMUs). DMUs use multiple inputs to produce multiple outputs. Conventional DEA models consider production systems as a black-box. In other words, DMUs under assessment perform as single aggregated process that transforms inputs to outputs. Using these models, internal linked activities of systems have been neglected. However, in real world, there are many systems such as banks and hospitals that have some sub-processes.

Recently, some researchers have focused on evaluating efficiency score of systems with some sub-processes which have interrelationship in producing final outputs. For example, Kazemi Matin and Azizi [14] proposed a new DEA model for measuring efficiency of network production processes with arbitrary internal structures. Kao [12] reviewed network DEA models and found out that a large number of studies are devoted to two-stage production systems. In typical two-stage production systems, outputs of the first stage are used as inputs of second stage (intermediate products). Fare and Grosskopf [10] are the first scholars to deal with efficiency assessments in such processes. Seiford and Zhu [21] presented a two-stage system to measure both profitability and marketability of US commercial banks. They measured profitability using labor and assets as inputs and profits and revenue as outputs. For marketability in the second stage outputs are market value, returns, and earnings per share. Liang *et al.* [16] examined and extended DEA models for two-stage systems using game theory concepts. Kao and Hwang [13] developed a different approach where the overall efficiency of the system could

*Keywords.* Network data envelopment analysis, Economies of scope, Intermediate product, Two-stage production system.

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be decomposed into product of efficiencies of two-stage structures. This multiplicative approach to efficiency decomposition is restricted to constant returns to scale (CRS) situations. Chen *et al.* [5] presented a model similar to Kao and Hwang [13], but in an additive format. The proposed method of Chen *et al.* [5] can be applied to both constant and variable returns to scale (VRS) situations. Chen *et al.* [6] developed an approach for determining frontier points (projections) for inefficient DMUs within framework of two-stage DEA. Zhu[23] used a two-stage process to measure efficiency of an airline industry in 2007 and 2008. In the first stage, cost per available seat mile (ASM), salaries and benefits, and fuel are considered as inputs to keep load factor and fleet size, while in the second stage, outputs are revenue passenger miles and passenger revenue. Li *et al.* [15] developed approach of Liang *et al.* [16] to analyze performance of two-stage network structures in which the second stage had its own inputs in addition to outputs from the first stage.

Akther *et al.* [1] studied performance of 21 banks (19 private commercial banks and 2 government-owned banks) in Bangladesh and used a two-stage network approach for maximizing desirable outputs and minimizing bad outputs. Lozano *et al.* [17] proposed a directional distance approach that deal with network problems that had undesirable outputs. They applied their model for modeling and benchmarking airport operations in Spain. Yang *et al.* [22] adopted two-stage DEA method extended by Chen *et al.* [5] to evaluate efficiency of National Basketball Association (NBA) teams. They broke down overall team efficiency into first-stage wage efficiency and second-stage on court efficiency and discovered individual endogenous weights for each stage. Despotis and Koronakos [9] introduced a new approach to assess both individual and overall efficiencies of two-stage systems, which effectively overcomes shortcomings of multiplicative and additive decomposition methods. Their modeling approach is based on selection of an output orientation for the first stage and an input orientation for the second stage. The above studies, among the many other papers that could be mentioned, show that two-stage production systems play basic and important role in analysing new modelling ideas incorporated in network production systems.

Economies of scope (ES) are also one of the important and interesting discussions in production theory. ES exists when joint production is more efficient than separate production. Scope economies (reflecting benefit of a multi output firm) have been formally defined by Panzar and Willig [19] and Baumol *et al.* [2] within context of cost functions. Duality relationships between input distance function and cost function are presented in Hajargasht *et al.* [11]. They proposed a method to obtain a measure of ES without need to estimate cost function. To evaluate ES, Sahoo and Tone [20] discussed two DEA models. They used these models to estimate a cost frontier exhibiting ES in production, due to process indivisibilities that arise from task-specific multi-stage production processes. Nemoto and Furumatsu [18] proposed a duality approach based on input distance function. They estimated an input distance function for analyzing cost structure of Japanese private universities without requiring input prices. De Witte *et al.* [8] proposed a non-parametric method to study presence of ES between teaching and research.

All previous studies on analyzing ES and estimating degree of economies of scope (DES) have dealt with production systems as single stage production process (black-box) while we extend these concepts in two-stage production systems (network structure) and study effect of internal linked activities (intermediate products) of systems of ES and DES using non-parametric method. Moreover, we determine whether joint production of products is more efficient in two-stage network structures. Main contribution of this paper is to present new and novel DEA models for evaluating degree of ES in two-stage production systems. The objective of this paper is to determine profitability of producing two products by one firm in comparison with producing them separately.

The rest of this paper unfolds as follows. Section 2 is devoted to provide the required background. Analyzing ES in two-stage production systems is given in Section 3. An illustrative example in Taiwanese non-life insurance companies is provided in Section 4. Conclusions are given in Section 5.

## 2. Background

This section is devoted to a brief review of Kao and Hwang's model [13] and concepts and models used to determine ES. Suppose there are $n$ DMUs ($j = 1, \dots, n$), each using $m$ inputs to produce $s$ outputs.

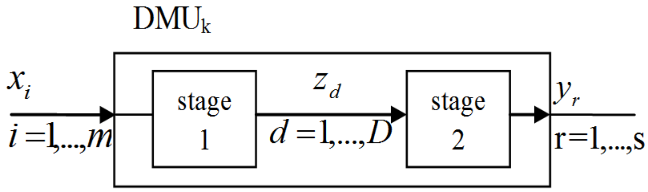


FIGURE 1. Two stage system.

Let the observed input and output vectors of $DMU_j$ are denoted by $x_{ij} \geqslant 0, (x_{ij} \neq 0, i = 1, \ldots, m)$ and $y_{rj} \geqslant 0, (y_{rj} \neq 0, r = 1, \ldots, s)$, respectively. Conventional DEA model which measures efficiency of $DMU_k$ under CRS assumption, is CCR (Charnes–Cooper–Rhodes) model that was presented by Charnes *et al.* [4]:

$$
\begin{aligned}
E_k = \max \quad & \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} & & \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leqslant 1 & & j = 1, \ldots, n \\
& u_r, v_i \geqslant 0, & & r = 1, \ldots s;\ i = 1, \ldots, m.
\end{aligned} \tag{2.1}
$$

Model (2.1) is a nonlinear and fractional program which can be easily transformed to an equivalent linear form (Charnes and Cooper [3]). Envelopment (dual) model of the above CCR model is presented as follows:

$$
\begin{aligned}
\min \quad & \theta_k & & \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta_k x_{ik} & & i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \geqslant y_{rk} & & r = 1, \ldots, s \\
& \lambda_j \geqslant 0 & & j = 1, \ldots, n.
\end{aligned} \tag{2.2}
$$

The CCR model is applicable for systems performing as black boxes where some inputs are used to produce some outputs. Therefore, it is not useful to evaluate performance of two-stage production systems due to overestimating efficiency scores [13].

## 2.1. Two-stage production systems

In many problems, for example in production process of non-life insurance industry or banking industry, production systems may have two-stage network structures which outputs of the first stage are considered as inputs of the second stage as shown in Figure 1. Production process of non-life insurance industry is divided into two stages: marketability and profitability. This production process consists of two inputs, two intermediate products, and two final outputs (see Kao and Hwang [13]). Also, in banking industry, production process can be illustrated as a two-stage system. In stage 1, banks collect deposits using number of personnel and physical capital. In stage 2, banks utilize their managerial and marketing skills to transform the deposits into loans and investment.

In this system, $z_{dj}$ is denoted as $d^{th}$ intermediate product,$d = 1, \ldots, D$, of $DMU_j$ which are introduced as outputs of the first stage as well as inputs of the second stage. Kao and Hwang [13] presented following model to evaluate efficiency score of two-stage systems more accurately than the other classical models such as CCR model.

$$
\begin{aligned}
E_K = \max \quad & \sum_{r=1}^{s} u_r Y_{rk} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i x_{ik} = 1 \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leqslant 0 && j = 1, \ldots, n, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leqslant 0 && j = 1, \ldots, n, \\
& u_r, v_i, w_d \geqslant 0, && r = 1, \ldots, s;\ i = 1, \ldots, m;\ d = 1, \ldots, D.
\end{aligned}
\tag{2.3}
$$

Envelopment (dual) model of model (2.3) is as follows:

$$
\begin{aligned}
\min \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta x_{ik} && i = 1, \ldots, m \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geqslant y_{rk} && r = 1, \ldots, s \\
& \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} \geqslant 0 && d = 1, \ldots, D \\
& \lambda_j, \mu_j \geqslant 0 && j = 1, \ldots, n.
\end{aligned}
\tag{2.4}
$$

Here, we have two types of intensity weights $\lambda_j$ and $\mu_j$ which are used for intermediate products $z_{dj}$. They play two roles, *i.e.* when appear as outputs of the first stage as well as inputs of the second stage. Note that Kao and Hwang [13] examined internal structure of DMUs and introduced radial efficiency measure to investigate performance of two-stage network systems. This measure is more accurate than traditional measures.

We will benefit this formulation for introducing DES in DEA framework to decide about possibility of lowering average cost by producing more types of products in two-stage production systems.

### 2.2. ES in DEA framework

The notion 'economies of scope' is used to study effect of diversity of products on production costs. Economies of scope exist if joint production of several products by a single diversified firm has less cost than their separate production in specialized firms. For example, cost of joint production of hamburgers, French fries, and salads in McDonald is less than cost of producing any of these products in separate restaurants.

To define economies and diseconomies of scope between two products, Baumol *et al.* [2] introduced diversified and specialized firms. Diversified firms are the firms that jointly produce two products whereas specialized firms produce only one of them.

In DEA framework and for diversified firm, outputs and inputs (cost) are denoted by $\mathbf{y}_j = (y_{1j}, y_{2j})$ and $\mathbf{c}_j (j = 1, \ldots, n)$, respectively. Production possibility set (PPS) of diversified firms under the CRS assumption are presented as follows:

$$P_{1,2} = \left\{ (\mathbf{c},\mathbf{y}) | \mathbf{c} = \sum_{j=1}^{n} \mathbf{c}_j \lambda_j, \quad \mathbf{y} \leqslant \sum_{j=1}^{n} \mathbf{y}_j \lambda_j, \forall j : \lambda_j \geqslant 0 \right\}$$

Furthermore, it is also supposed that there exists $(p+q)$, specialized firms that produce only one of two products. Let $S_1$ to be set of $p$ specialized firms which produce output $y_{1j}$ using input (cost) $v_{1j}(j = 1, \ldots, p)$. There are also q specialized firms which are denoted by $S_2$, producing $y_{2j}$, using input $v_{2j}(j = 1, \ldots, q)$. Therefore, PPS of these firms can be defined by $P_1$ and $P_2$, respectively.

$$P_1 = \left\{ (v, y_1) | v = \sum_{j=1}^{p} v_{1j} \lambda_j, \quad y_1 \leqslant \sum_{j=1}^{p} y_{1j} \lambda_j, \quad \forall j : \lambda_j \geqslant 0 \right\}$$

$$P_2 = \left\{ (v, y_2) | v = \sum_{j=1}^{q} v_{2j} \lambda_j, \quad y_2 \leqslant \sum_{j=1}^{q} y_{2j} \lambda_j, \quad \forall j : \lambda_j \geqslant 0 \right\}$$

Baumol *et al.* [2] expressed that ES exists if cost of producing two products by a single diversified firm is less than aggregated cost of producing two products separately by two specialized firms. Therefore, in DEA framework, ES between two products exists if

$$C(y_1, y_2) < C(y_1, 0) + C(0, y_2)$$

where $C(y_1, 0)$and $C(0, y_2)$ are cost of producing products 1 and 2 separately by two specialized firms. $C(y_1, y_2)$ shows cost of producing both products by one diversified firm. Furthermore, degree of economies of scope for firm j is defined as follows:

$$DES_j = \frac{C(y_1, 0) + C(0, y_2) - C(y_1, y_2)}{C(y_1, y_2)} \tag{2.5}$$

DES is a unit-free measure which reflects cost saving percentage that occurs when products are produced jointly in a single diversified firm. Thus, $DES > 0$ means that firm $j$ exhibits ES, $DES < 0$ means diseconomies of scope, and $DES = 0$ implies costs are additive in nature.

## 3. DEA evaluation of ES in two-stage production systems

It is possible to evaluate profitability of a virtual merger which is combination of two specialized firms (Cooper *et al.* [7]). For two-stage production units, to evaluate ES of virtually merged firms, we create a set of virtually diversified firms $(M)$ by combining p specialized firms of group $S_1$ with q specialized firms of group $S_2$. In a more general setting, it is supposed that specialized firms in both groups have two-stage structure which their inputs are identical, but their final outputs and intermediate products are different. Note that primary inputs are identical in both specialized groups (both of them are cost type) but there is difference in final outputs and intermediate productions. Consequently, definition for integration of both firms to create virtual merged firms is $v_{kh} = v_k^* + v_h^*$. However, this merged firm consists of two final output $y'_{ks} = \begin{pmatrix} y'_k \\ y'_s \end{pmatrix}$ and two intermediate productions $w_{ks} = \begin{pmatrix} w_k \\ w_s \end{pmatrix}$. Virtual firms consist of one input $v = v_{1k}^* + v_{2h}^*$ and two outputs $\begin{pmatrix} y'_1 = y_{1k} \\ y'_2 = y_{2h} \end{pmatrix}$, where $v_{1k}^*$ and $v_{2h}^*$ are optimal cost of group 1 and 2, respectively. Then, we define $M$ and PPS corresponding to $M$ for virtual diversified firms as follows:

$$M = \{(v_j, y'_{1j}, y'_{2j}) | j = 1, \ldots, pq\}$$

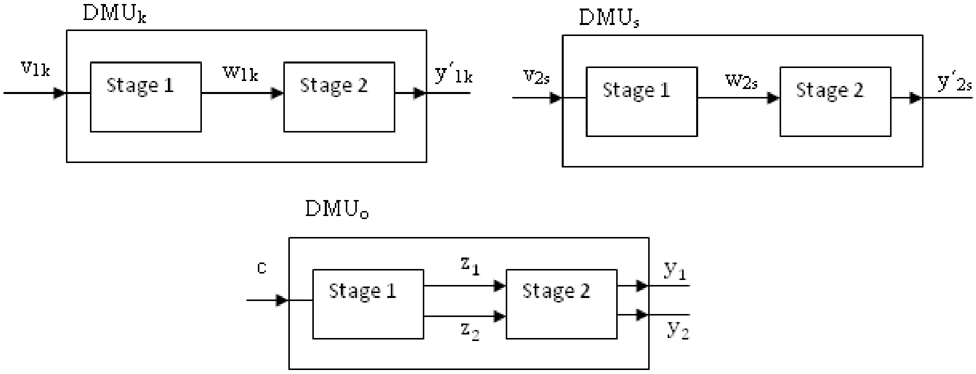


FIGURE 2. $\mathrm{DMU}_k$ in group 1, $\mathrm{DMU}_s$ in group 2, divershified $\mathrm{DMU}_o$.

$$P_M = \left\{ (v, y_1', y_2') \mid v \geqslant \sum_{j=1}^{n} c_j \lambda_j, \quad y' \leqslant \sum_{j=1}^{n} y_j \lambda_j, \quad \forall j : \lambda_j \geqslant 0 \right\}.$$

where $v_j = v_{1j}^* + v_{2j}^*$. Also, $y$ represents outputs (*i.e.*, products 1 and 2). The DMUs of groups and diversified set are shown in Figure 2. Now, to check the ES of virtual two-stage systems in PPS of diversified firms under the CRS assumption, we suggest following model:

$$\begin{aligned} \min \quad & \theta \\ s.t. \quad & v_k \theta \geqslant \sum_{j=1}^{n} \lambda_j c_j \\ & y'_{rk} \leqslant \sum_{j=1}^{n} \mu_j y_{rj} && r = 1, 2 \\ & 0 \leqslant \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{dj} && d = 1, 2 \\ & \lambda_j, \mu_j \geqslant 0 && j = 1, \ldots, n \end{aligned} \tag{3.1}$$

where $\lambda_j$ and $\mu_j$ are intensity variables of each stage and $v_k$ and $y_k^{'}$ denote inputs and outputs of the merged virtual two-stage system, respectively. Right hand side of each constrain in model (3.1) indicates input, intermediate products, and outputs of diversified firms. Also, in this model $v_{1k}^*$ and $v_{2h}^*$ are optimal cost obtained by model (2.4) for each DMU which are located in $S_1$ and $S_2$ groups, respectively ($i.e.$, $v_{1k}^* = \theta_k^* x_k; DMU_k \in S_1$ and $v_{2h}^* = \theta_h^* x_h; DMU_h \in S_2$).$y_1'$ and $y_2'$ are outputs of $S_1$ and $S_2$, respectively.

Each merged firm in model (3.1) is integration of two specialized firms in two diverse groups which specialized firms' intermediate products and final outputs are different but their inputs are same. Consequently, the created merged firm includes two outputs, two intermediate products, and one input similar to diversified firms as shown in Figure 3. Moreover, **(c, z, y)** shows the input, intermediate product, and outputs of diversified firms. Note that although the virtual merged firms have intermediate products, they are eliminated from the third constraint due to fact that the third one is an aggregation of $w_{dk} \leqslant \sum_{j=1}^{n} \lambda_j z_{dj}$ and $w_{dk} \geqslant \sum_{j=1}^{n} \mu_j z_d j$. $w_{dk}$ is $d$th intermediate product of merged virtual two-stage system. Therefore, we can form merged firm as is depicted in Figure 3. Note that $(v_{ks}, w_{ks}, y_{ks}^{'})$ shows input, intermediate product, and output of virtual merged $\mathrm{DMU}_{ks}$, where $v_{ks} = v_k^* + v_s^*, w_{ks} = \begin{pmatrix} w_{1k} \\ w_{2s} \end{pmatrix}$, and $y_{ks}^{'} = \begin{pmatrix} y_{1k}^{'} \\ y_{2s}^{'} \end{pmatrix}$.

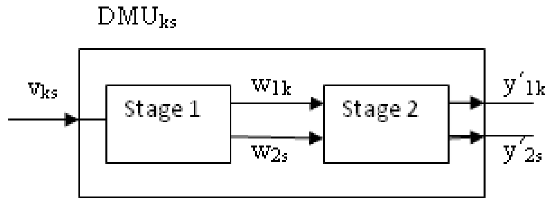


FIGURE 3. Merged $\mathrm{DMU}_{ks}$.

Suppose that optimal $\theta$ for model (3.1) is $\theta^*$. We define optimal cost of producing two products by virtual merged firm as follows:

$$c(y_1, y_2) = \theta^*(v_{1k}^* + v_{2h}^*)$$

Since $v_{1k}^* = c(y_1, 0)$ and $v_{2h}^* = c(0, y_2)$, therefore

$$c(y_1, y_2) = \theta^*(c(y_1, 0) + c(0, y_2)) \tag{3.2}$$

Considering recent relations, we can check desirability of merger by following conditions:

$\theta^* > 1$ : Merger is unfavorable,

$\theta^* = 1$ : Indifference,

$\theta^* < 1$ : Merger is favorable.

Also, DES is computed using formulas (2.5) and (3.2) as follows:

$$DES = \frac{1}{\theta^*} - 1$$

To study existence of ES in two-stage systems which use multiple inputs to produce multiple outputs, assuming availability of input prices, it is sufficient to transform such systems to a single input case and use presented method as follows:

$$\bar{c_j} = c_1 x_{1j} + c_2 x_{2j} + \ldots + c_t x_{tj}$$

where $C = (c_1, c_2, \ldots, c_t)$ is a vector of input prices.

## 4. NUMERICAL EXAMPLES

**Example 4.1.** Assume that there are two kinds of products: $y_1$ and $y_2$. There exist three specialized firms that produce only product $y_1$, three other specialized firms which produce only product $y_2$, and five diversified firms which produce both $y_1$ and $y_2$. Inputs, intermediate products, and outputs are shown in Table 1.

We create nine virtual diversified firms given the method introduced in Section 3. Note that the virtual firms consist of one input $(v_{1k}^* + v_{2h}^*)$, two intermediate products $\begin{pmatrix} z_1 = z_{1k} \\ z_2 = z_{2h} \end{pmatrix}$, and two outputs $\begin{pmatrix} y_1 = y_{1k} \\ y_2 = y_{2h} \end{pmatrix}$. To find out $v_{1k}^*$ and $v_{2k}^*$, we first solve model (2.4) for the specialized firms producing only $y_1$, and then for the specialized firms producing output $y_2$. Results are presented in Table 2.

Now, by applying model (3.1), we estimate ES of each of these virtual firms and their DES using formula (2.5). Results are reported in Table 3 for two different positions (two-stage and black box). Note that in black box case, intermediate products are ignored. Therefore, we treat the DMUs as single stage production process that transforms input to outputs.

In Table 3, $S_{1,k}$-$S_{2,h}$ denotes the virtual firm which is created by combining kth firm of specialized group $S_1$ with the hth firm of specialized group $S_2$. Comparison of computed DES and ES from the two different cases,

TABLE 1. Specialized and diversified firms.

| | DMUs | Input (cost) | Intermediate product ($z_1$) | Intermediate product ($z_2$) | Output ($y_1$) | Output ($y_2$) |
|---|---|---|---|---|---|---|
| DMUs of group 1 | 1 | 4 | 5 | – | 4.5 | – |
| | 2 | 6 | 5 | – | 7 | – |
| | 3 | 1.5 | 2 | – | 3 | – |
| DMUs of group 2 | 1 | 2 | – | 1.5 | – | 1.5 |
| | 2 | 5 | – | 4 | – | 5.5 |
| | 3 | 3 | – | 2 | – | 3 |
| Diversified DMUs | 1 | 7 | 4 | 4 | 5 | 6 |
| | 2 | 6 | 4 | 2 | 4 | 3 |
| | 3 | 13 | 5 | 5.5 | 7 | 6.5 |
| | 4 | 16 | 9 | 7.5 | 9.5 | 10.5 |
| | 5 | 9 | 1 | 5.5 | 1.5 | 7.5 |

TABLE 2. Optimal cost of DMUs, groups 1 and 2.

| DMUs | $v_{1k}^*$ | $v_{2h}^*$ |
|---|---|---|
| 1 | $0.5625(4) = 2.25$ | $0.625(2) = 1.25$ |
| 2 | $0.5833333(6) = 3.4999998$ | $0.9166667(5) = 4.5833335$ |
| 3 | $1(1.5) = 1.5$ | $0.8333333(3) = 2.4999999$ |

TABLE 3. Comparative estimates on ES in two cases: Black box and two stage.

| Virtual firm | $\theta^*$ | | DES | | ES | |
|---|---|---|---|---|---|---|
| | Black box | Two stage | Black box | Two stage | Black box | Two stage |
| $S_{1,1}-S_{2,1}$ | 0.9712879 | 0.7108844 | 0.02956 | 0.40670 | Yes | Yes |
| $S_{1,1}-S_{2,2}$ | 0.8850575 | 0.9076284 | 0.12987 | 0.10177 | Yes | Yes |
| $S_{1,1}-S_{2,3}$ | 0.9207862 | 0.7719299 | 0.08603 | 0.29545 | Yes | Yes |
| $S_{1,2}-S_{2,1}$ | 1.000000 | 0.7368421 | 0 | 0.35714 | **Indifference** | **Yes** |
| $S_{1,2}-S_{2,2}$ | 0.9088584 | 0.8006873 | 0.10028 | 0.24893 | Yes | Yes |
| $S_{1,2}-S_{2,3}$ | 0.9533482 | 0.6944445 | 0.04893 | 0.44000 | Yes | Yes |
| $S_{1,3}-S_{2,1}$ | 0.9420290 | 0.7212121 | 0.06154 | 0.38655 | Yes | Yes |
| $S_{1,3}-S_{2,2}$ | 0.9871795 | 1.019528 | 0.01299 | -0.01915 | **Yes** | **No** |
| $S_{1,3}-S_{2,3}$ | 0.8920056 | 0.8513594 | 0.12107 | 0.17459 | Yes | Yes |

reveals interesting findings. The presence or absence of ES for a special firm is not necessarily the same in both cases. For example, consider firms $S_{1,2}-S_{2,1}$ and $S_{1,3}-S_{2,2}$. ES is exhibited for $S_{1,3}-S_{2,2}$ when it is analyzed as a black box, but there is no ES for $S_{1,3}-S_{2,2}$ when it is analyzed as a two-stage system. Also, analyzing $S_{1,2}-S_{2,1}$ as a black box shows that there is no difference for merging the firms, but there is ES for $S_{1,2}-S_{2,1}$ when it is analyzed as a two-stage unit. ES is existed for other firms in both cases. Furthermore, amount of DES is not necessarily similar to both cases due to considering intermediate products.

Given Table 3, taking into account or not taking into account of the intermediate products affects our decisions. For example, if intermediate products are ignored in firm $S_{1,3}-S_{2,2}$, because of $DES^{\text{Black box}}_{S_{1,3}-S_{2,2}} > 0$, then joint cost production will be lower. Therefore, our decision-making in joint production of products will be for diversified firm. If this firm has two-stage system, it will be diseconomies of scope as $DES^{\text{Two stage}}_{S_{1,3}-S_{2,2}} < 0$. In other words, there is saving cost in separate production in specialized firms. Since separate production

TABLE 4. Optimal changed cost of DMUs, groups 1 and 2.

| DMUs | $v^*_{1k}$ | $v^*_{2h}$ |
|---|---|---|
| 1 | $0.5731875(4) = 2.2928$ | $0.6368750(2) = 1.2738$ |
| 2 | $0.5944167(6) = 3.5665$ | $0.9166667(5.095) = 4.6704$ |
| 3 | $1(1.5285) = 1.5285$ | $0.8491667(3) = 2.5475$ |

TABLE 5. Comparison of estimation of ES: Before and after cost increase.

| DMUs | $\theta^*_{\text{two-stage}}$ | $\theta^{*\text{new}}_{\text{two-stage}}$ | DES | $DES_{\text{new}}$ | ES | $ES_{\text{new}}$ |
|---|---|---|---|---|---|---|
| $DMU_{1-1,2-1}$ | 0.7108844 | 0.7108811 | 0.40670 | 0.40670 | Yes | Yes |
| $DMU_{1-1,2-2}$ | 0.9076284 | 0.9037958 | 0.10177 | 0.10644 | Yes | Yes |
| $DMU_{1-1,2-3}$ | 0.7719299 | 0.7719301 | 0.29545 | 0.29545 | Yes | Yes |
| $DMU_{1-2,2-1}$ | 0.7368421 | 0.7368552 | 0.35714 | 0.35711 | Yes | Yes |
| $DMU_{1-2,2-2}$ | 0.8006873 | 0.8006950 | 0.24893 | 0.24892 | Yes | Yes |
| $DMU_{1-2,2-3}$ | 0.6944445 | 0.6944586 | 0.44000 | 0.43997 | Yes | Yes |
| $DMU_{1-3,2-1}$ | 0.7212121 | 0.7212114 | 0.38655 | 0.38656 | Yes | Yes |
| $DMU_{1-3,2-2}$ | 1.019528 | 1.0000003 | −0.0191 | 0 | **No** | **Indifference** |
| $DMU_{1-3,2-3}$ | 0.8513594 | 0.8520543 | 0.17459 | 0.17363 | Yes | Yes |

is cost-effective, we choose specialized firms which produce outputs while this decision is contrary to decision-making in black box status. Also, given $\theta^*$ and DES in two-stage system for firm $S_{1,3}-S_{2,2}$, it is clear that cost of producing outputs in this virtual diversified firm is 1.9% more than cost of producing them separately in specialized firms (firm 3 of group $S_1$ and firm 2 of group $S_2$). Therefore, selection of these specialized firms for production is profitable. As a result, diversified firms can be used for production on condition that specialized firms' costs are more than 6.20212884. In other words, cost of specialized firms in outputs production should be increased to 1.9%.

$$C_1(2,0,3,0) + C_2(0,4,0,5.5) = v^*_{1,3} + v^*_{2,2} = 1.5 + 4.5833335 = 6.083335$$
$$C(2,4,3,5.5) = \theta^*_{1,3-2,2} \times [C_1(2,0,3,0) + C_2(0,4,0,5.5)] = 1.019528(6.08335) = 6.20212884$$

Following tables report changes in calculation of $v^*_{1k}, v^*_{2h}, \theta^*$ and DES after increase in cost of specialized firms (firm 3 of group $S_1$ and firm 2 of group $S_2$).

Increasing input of specialized firm 3 in group $S_1$ and specialized firm 2 in group $S_2$ is equal to:

$$c_{\text{new}} = \theta^* c_{\text{old}}, \quad \theta^*_{1-3,2-2} = 1.019528 \cong 1.019$$
$$c^{\text{new}}_{1-3} = \theta^* c^{\text{old}}_{1-3} = 1.019(1.5) = 1.5285, \quad c^{\text{new}}_{2-2} = \theta^* c^{\text{old}}_{2-2} = 1.019(5) = 5.095$$

Note that if increase is more than $\theta^*$, then DES$>0$. In this case, we prefer to use diversified firms for production

**Example 4.2.** In order to illustrate results of new proposed method in a real application, we take dataset from Kao and Hwang [13]. There are two specialized groups and one diversified one. All the specialized firms have one input, one intermediate product, and one final output. Primary input is identical in both specialized groups but there is difference in intermediate products and final outputs. In first group, input, intermediate product, and output of specialized firms are operation expenses, direct written premiums, and underwriting profit, respectively. In second group, input, intermediate product, and output of specialized firms are insurance

Table 6. Specialized and diversified firms.

| | DMUs | Input (cost) | Intermediate product ($z_1$) | Intermediate product ($z_2$) | Output ($y_1$) | Output ($y_2$) |
|---|---|---|---|---|---|---|
| DMUs of group 1 | 1 | 1,178,744 | 7,451,757 | – | 984,143 | – |
| | 2 | 1,381,822 | 10,020,274 | – | 1,228,502 | – |
| | 3 | 1,177,494 | 4,776,548 | – | 293,613 | – |
| | 4 | 601,320 | 3,174,851 | – | 248,709 | – |
| | 5 | 6,699,063 | 37,392,862 | – | 7,851,229 | – |
| | 6 | 2,627,707 | 9,747,908 | – | 1,713,598 | – |
| | 7 | 1,942,833 | 10,658,457 | – | 2,239,593 | – |
| | 8 | 3,789,001 | 17,267,266 | – | 3,899,530 | – |
| | 9 | 1,567,746 | 11,473,162 | – | 1,043,778 | – |
| | 10 | 1,303,249 | 8,210,389 | – | 1,697,941 | – |
| | 11 | 1,962,448 | 7,222,378 | – | 1,486,014 | – |
| | 12 | 2,592,790 | 9,434,406 | – | 1,574,191 | – |
| DMUs of group 2 | 1 | 673,512 | – | 856,735 | – | 681,687 |
| | 2 | 1,352,755 | – | 1,812,894 | – | 834,754 |
| | 3 | 592,790 | – | 560,244 | – | 658,428 |
| | 4 | 594,259 | – | 371,863 | – | 177,331 |
| | 5 | 3,531,614 | – | 1,753,794 | – | 3,925,272 |
| | 6 | 668,363 | – | 952,326 | – | 415,058 |
| | 7 | 1,443,100 | – | 643,412 | – | 439,039 |
| | 8 | 1,873,530 | – | 1,134,600 | – | 622,868 |
| | 9 | 950,432 | – | 546,337 | – | 264,098 |
| | 10 | 1,298,470 | – | 504,528 | – | 554,806 |
| | 11 | 672,414 | – | 643,178 | – | 18,259 |
| | 12 | 650,952 | – | 1,118,489 | – | 909,295 |
| Diversified DMUs | 1 | 3,978,743 | 13,921,464 | 811,343 | 3,609,236 | 223,047 |
| | 2 | 2,384,890 | 7,396,396 | 465,509 | 1,401,200 | 332,283 |
| | 3 | 2,836,007 | 10,422,297 | 749,893 | 3,355,197 | 555,482 |
| | 4 | 1,626,787 | 5,606,013 | 402,881 | 854,054 | 197,947 |
| | 5 | 2,538,816 | 7,695,461 | 342,489 | 3,144,484 | 371,984 |
| | 6 | 1,305,512 | 3,631,484 | 995,620 | 692,731 | 163,927 |
| | 7 | 341,760 | 1,141,950 | 483,291 | 519,121 | 46,857 |
| | 8 | 198,960 | 316,829 | 131,920 | 355,624 | 26,537 |
| | 9 | 110,395 | 225,888 | 40,542 | 51,950 | 6491 |
| | 10 | 26,495 | 52,063 | 14,574 | 82,141 | 4181 |
| | 11 | 83,101 | 245,910 | 49,864 | 0.1 | 18,980 |
| | 12 | 398,391 | 476,419 | 644,816 | 142,370 | 16,976 |

expenses, reinsurance premiums, and investment profit, respectively. Input of diversified group is aggregation of operational expenses and insurance expenses. Its intermediate products are direct written premiums and reinsurance premiums and its outputs are underwriting profit and investment profit. Dataset is reported in Table 6.

First, model (2.4) is applied for the specialized firms $S_1$ and $S_2$ to determine their optimal input (cost). Table 7 reports computed optimal costs.

The first and second group of optimal costs are denoted by $v_{1k}^*$ and $v_{2s}^*$, respectively. The virtual firms' costs are determined by $v_{1k}^* + v_{2s}^* = v_{ks}$. Now, to evaluate ES in two-stage production systems, we use model (3.1). Results are reported in Table 8.

TABLE 7. Optimal cost of DMUs, groups 1 and 2.

| DMUs | $v_{1k}^*$ | $v_{2s}^*$ |
|---|---|---|
| 1 | 595473.1 | 177260.1 |
| 2 | 743326.7 | 217062.4 |
| 3 | 177655.7 | 171212 |
| 4 | 150485.7 | 46111.66 |
| 5 | 4750524 | 1943819 |
| 6 | 1036843 | 107928.2 |
| 7 | 1355105 | 114164 |
| 8 | 2537791 | 161965.3 |
| 9 | 637370.4 | 68673.82 |
| 10 | 1027369 | 144267 |
| 11 | 899138.9 | 4747.917 |
| 12 | 952492 | 285128.4 |

Here, two different cases are considered: In the first one, internal operations of systems are considered and virtual merged two-stage firms are created. Therefore, using model (3.1), ES of each virtual merged two-stage firm is analyzed. In second case, intermediate products are not considered and virtual merged systems are created as black boxes. Therefore, ES of virtual merged black box firms are determined when intermediate products are ignored. In Table 8, $S_{1,k}-S_{2,l}$ shows virtual merged firm which is created by merging $k$th firm of specialized group $S_1$ with the lth firm of specialized group $S_2$. Both $\theta^*$ and DES measures provide information on ES on virtual firms. Table 9 shows results more clearly. According to Table 9, both cases yield similar results on ES. Some of virtually diversified firms, by combining $S_1$ and $S_2$, have ES. For example, combination of firm 11 of group $S_2$ with any firm in group $S_1$ have ES. It can also be seen that ES exists for combination of firm 5 of group $S_1$ with any firm of group $S_2$, except for fifth one. To check these results, we evaluate merger of firm 8 of group $S_1$ with firm 8 of group $S_2$. This merged firm shows costs of $v_8 = 2537791 + 161965.3 = 2699756.3$ that produce intermediate products $w_{18} = 17267266, w_{28} = 1134600$, and final outputs $y_{18} = 3899530, y_{28} = 622868$. Using model (3.1), optimal cost of mentioned virtual merged firm can be equal to:

$$C(17267266, 1134600, 3899530, 622868) = 0.823 \times 2699756.3 = 2221899.43$$

Also, the aggregation cost of firm 8 in group 1 and firm 8 in group 2 is equal to:

$$C_1(17267266, 0, 3899530, 0) + C_2(0, 1134600, 0, 622868) = 2699756.3.$$

It is clear that $2221899.43 < 2699756.3$. We evaluate this dataset using model (3.1) and find out that cost of virtually merged firms used for producing two outputs is lower than aggregation cost that exists for the firms in two distinct groups when producing same outputs. In other words, joint production of two outputs is less costly than their separate production. Hence, merged is recommended. Now, using formula (2.5), we compute DES of this firm as follows:

$$\begin{aligned} DES_{S_{1,8}-S_{2,8}} &= \frac{C_1(w_{18}, 0, y_{18}, 0) + C_2(0, w_{28}, 0, y_{28}) - C(w_{18}, w_{28}, y_{18}, y_{28})}{C(w_{18}, w_{28}, y_{18}, y_{28})} \\ &= \frac{2699756.3 - 2221899.43}{2221899.43} = 0.21506683 \cong 0.215 \end{aligned}$$

or

$$DES_{S_{1,8}-S_{2,8}} = \frac{1}{\theta^*} - 1 = \frac{1}{0.823} - 1 = 0.21506683.$$

TABLE 8. ES evaluation of virtual firms in two different cases.

| Virtual firm | $\theta^*$ Black box | $\theta^*$ Two stage | DES Black box | DES Two stage | Virtual firm | $\theta^*$ Black box | $\theta^*$ Two stage | DES Black box | DES Two stage |
|---|---|---|---|---|---|---|---|---|---|
| $S_{1,1}-S_{2,1}$ | 2.479 | 3.146 | −0.597 | −0.682 | $S_{1,7}-S_{2,1}$ | 1.454 | 1.587 | −0.312 | −0.37 |
| $S_{1,1}-S_{2,2}$ | 2.774 | 3.664 | −0.64 | −0.727 | $S_{1,7}-S_{2,2}$ | 1.674 | 1.894 | −0.403 | −0.472 |
| $S_{1,1}-S_{2,3}$ | 2.430 | 3.063 | −0.588 | −0.674 | $S_{1,7}-S_{2,3}$ | 1.418 | 1.539 | −0.295 | −0.35 |
| $S_{1,1}-S_{2,4}$ | 0.991 | 0.986 | 0.009 | 0.014 | $S_{1,7}-S_{2,4}$ | 0.542 | 0.451 | 0.845 | 1.217 |
| $S_{1,1}-S_{2,5}$ | 4.848 | 5.514 | −0.794 | −0.819 | $S_{1,7}-S_{2,5}$ | 3.844 | 4.244 | −0.74 | −0.764 |
| $S_{1,1}-S_{2,6}$ | 1.820 | 2.105 | −0.451 | −0.525 | $S_{1,7}-S_{2,6}$ | 1.012 | 1.012 | −0.012 | −0.012 |
| $S_{1,1}-S_{2,7}$ | 1.889 | 2.207 | −0.471 | −0.547 | $S_{1,7}-S_{2,7}$ | 1.055 | 1.066 | −0.052 | −0.062 |
| $S_{1,1}-S_{2,8}$ | 2.352 | 2.933 | −0.575 | −0.659 | $S_{1,7}-S_{2,8}$ | 1.363 | 1.464 | −0.266 | −0.317 |
| $S_{1,1}-S_{2,9}$ | 1.328 | 1.418 | −0.247 | −0.295 | $S_{1,7}-S_{2,9}$ | 0.723 | 0.662 | 0.383 | 0.511 |
| $S_{1,1}-S_{2,10}$ | 2.193 | 2.675 | −0.544 | −0.626 | $S_{1,7}-S_{2,10}$ | 1.253 | 1.320 | −0.202 | −0.242 |
| $S_{1,1}-S_{2,11}$ | 0.413 | 0.299 | 1.421 | 2.344 | $S_{1,7}-S_{2,11}$ | 0.417 | 0.301 | 1.398 | 2.322 |
| $S_{1,1}-S_{2,12}$ | 2.901 | 3.683 | −0.655 | −0.728 | $S_{1,7}-S_{2,12}$ | 1.774 | 1.977 | −0.436 | −0.494 |
| $S_{1,2}-S_{2,1}$ | 2.171 | 2.641 | −0.539 | −0.621 | $S_{1,8}-S_{2,1}$ | 0.969 | 0.895 | 0.032 | 0.117 |
| $S_{1,2}-S_{2,2}$ | 2.452 | 3.100 | −0.592 | −0.677 | $S_{1,8}-S_{2,2}$ | 1.126 | 1.081 | −0.112 | −0.075 |
| $S_{1,2}-S_{2,3}$ | 2.125 | 2.568 | −0.529 | −0.611 | $S_{1,8}-S_{2,3}$ | 0.944 | 0.867 | 0.059 | 0.153 |
| $S_{1,2}-S_{2,4}$ | 0.840 | 0.801 | 0.19 | 0.248 | $S_{1,8}-S_{2,4}$ | 0.403 | 0.276 | 1.481 | 2.623 |
| $S_{1,2}-S_{2,5}$ | 4.612 | 5.210 | −0.783 | −0.808 | $S_{1,8}-S_{2,5}$ | 3.029 | 3.124 | −0.670 | −0.68 |
| $S_{1,2}-S_{2,6}$ | 1.564 | 1.739 | −0.361 | −0.425 | $S_{1,8}-S_{2,6}$ | 0.670 | 0.560 | 0.493 | 0.786 |
| $S_{1,2}-S_{2,7}$ | 1.626 | 1.826 | −0.385 | −0.452 | $S_{1,8}-S_{2,7}$ | 0.699 | 0.590 | 0.431 | 0.695 |
| $S_{1,2}-S_{2,8}$ | 2.052 | 2.454 | −0.513 | −0.593 | $S_{1,8}-S_{2,8}$ | 0.906 | 0.823 | 0.104 | 0.215 |
| $S_{1,2}-S_{2,9}$ | 1.130 | 1.160 | −0.115 | −0.138 | $S_{1,8}-S_{2,9}$ | 0.486 | 0.362 | 1.058 | 1.762 |
| $S_{1,2}-S_{2,10}$ | 1.904 | 2.229 | −0.475 | −0.551 | $S_{1,8}-S_{2,10}$ | 0.831 | 0.738 | 0.203 | 0.355 |
| $S_{1,2}-S_{2,11}$ | 0.414 | 0.300 | 1.415 | 2.333 | $S_{1,7}-S_{2,11}$ | 0.418 | 0.280 | 1.392 | 2.571 |
| $S_{1,2}-S_{2,12}$ | 2.575 | 3.153 | −0.612 | −0.683 | $S_{1,8}-S_{2,12}$ | 1.199 | 1.149 | −0.166 | −0.13 |
| $S_{1,3}-S_{2,1}$ | 4.225 | 6.851 | −0.763 | −0.854 | $S_{1,9}-S_{2,1}$ | 2.396 | 2.985 | −0.583 | −0.665 |
| $S_{1,3}-S_{2,2}$ | 4.477 | 7.543 | −0.777 | −0.867 | $S_{1,9}-S_{2,2}$ | 2.687 | 3.485 | −0.628 | −0.713 |
| $S_{1,3}-S_{2,3}$ | 4.180 | 6.732 | −0.761 | −0.851 | $S_{1,9}-S_{2,3}$ | 2.347 | 2.904 | −0.574 | −0.656 |
| $S_{1,3}-S_{2,4}$ | 2.287 | 2.827 | −0.563 | −0.646 | $S_{1,9}-S_{2,4}$ | 0.949 | 0.925 | 0.054 | 0.081 |
| $S_{1,3}-S_{2,5}$ | 5.672 | 6.600 | −0.824 | −0.848 | $S_{1,9}-S_{2,5}$ | 4.788 | 5.424 | −0.791 | −0.816 |
| $S_{1,3}-S_{2,6}$ | 3.534 | 5.184 | −0.717 | −0.807 | $S_{1,9}-S_{2,6}$ | 1.749 | 1.986 | −0.428 | −0.496 |
| $S_{1,3}-S_{2,7}$ | 3.616 | 5.366 | −0.723 | −0.814 | $S_{1,9}-S_{2,7}$ | 1.817 | 2.084 | −0.450 | −0.52 |
| $S_{1,3}-S_{2,8}$ | 4.106 | 6.542 | −0.756 | −0.847 | $S_{1,9}-S_{2,8}$ | 2.270 | 2.779 | −0.559 | −0.64 |
| $S_{1,3}-S_{2,9}$ | 2.861 | 3.824 | −0.65 | −0.738 | $S_{1,9}-S_{2,9}$ | 1.273 | 1.334 | −0.214 | −0.25 |
| $S_{1,3}-S_{2,10}$ | 3.949 | 6.147 | −0.747 | −0.837 | $S_{1,9}-S_{2,10}$ | 2.114 | 2.532 | −0.527 | −0.605 |
| $S_{1,3}-S_{2,11}$ | 0.458 | 0.358 | 1.183 | 1.793 | $S_{1,9}-S_{2,11}$ | 0.413 | 0.297 | 1.421 | 2.367 |
| $S_{1,3}-S_{2,12}$ | 4.577 | 7.008 | −0.782 | −0.857 | $S_{1,9}-S_{2,12}$ | 2.813 | 3.516 | −0.645 | −0.716 |
| $S_{1,4}-S_{2,1}$ | 4.433 | 7.419 | −0.774 | −0.865 | $S_{1,10}-S_{2,1}$ | 1.761 | 2.018 | −0.432 | −0.504 |
| $S_{1,4}-S_{2,2}$ | 4.667 | 8.101 | −0.786 | −0.877 | $S_{1,10}-S_{2,2}$ | 2.012 | 2.393 | −0.503 | −0.582 |
| $S_{1,4}-S_{2,3}$ | 4.390 | 7.300 | −0.772 | −0.863 | $S_{1,10}-S_{2,3}$ | 1.720 | 1.959 | −0.419 | −0.49 |
| $S_{1,4}-S_{2,4}$ | 2.521 | 3.217 | −0.603 | −0.689 | $S_{1,10}-S_{2,4}$ | 0.661 | 0.589 | 0.513 | 0.698 |
| $S_{1,4}-S_{2,5}$ | 5.735 | 6.685 | −0.826 | −0.85 | $S_{1,10}-S_{2,5}$ | 4.219 | 4.712 | −0.763 | −0.788 |
| $S_{1,4}-S_{2,6}$ | 3.774 | 5.729 | −0.735 | −0.825 | $S_{1,10}-S_{2,6}$ | 1.241 | 1.304 | −0.194 | −0.233 |
| $S_{1,4}-S_{2,7}$ | 3.854 | 5.917 | −0.741 | −0.831 | $S_{1,10}-S_{2,7}$ | 1.293 | 1.372 | −0.227 | −0.271 |
| $S_{1,4}-S_{2,8}$ | 4.322 | 7.110 | −0.769 | −0.859 | $S_{1,10}-S_{2,8}$ | 1.656 | 1.868 | −0.396 | −0.465 |
| $S_{1,4}-S_{2,9}$ | 3.109 | 4.298 | −0.678 | −0.767 | $S_{1,10}-S_{2,9}$ | 0.888 | 0.859 | 0.126 | 0.164 |
| $S_{1,4}-S_{2,10}$ | 4.173 | 6.714 | −0.76 | −0.851 | $S_{1,10}-S_{2,10}$ | 1.528 | 1.689 | −0.346 | −0.408 |
| $S_{1,4}-S_{2,11}$ | 0.514 | 0.420 | 0.946 | 1.381 | $S_{1,10}-S_{2,11}$ | 0.416 | 0.300 | 1.404 | 2.333 |

TABLE 8. Continued.

| Virtual firm | $\theta^*$ Black box | $\theta^*$ Two stage | DES Black box | DES Two stage | Virtual firm | $\theta^*$ Black box | $\theta^*$ Two stage | DES Black box | DES Two stage |
|---|---|---|---|---|---|---|---|---|---|
| $S_{1,4}-S_{2,12}$ | 4.759 | 7.445 | −0.79 | −0.866 | $S_{1,10}-S_{2,12}$ | 2.124 | 2.471 | −0.529 | −0.595 |
| $S_{1,5}-S_{2,1}$ | 0.578 | 0.493 | 0.73 | 1.028 | $S_{1,11}-S_{2,1}$ | 1.924 | 2.259 | −0.480 | −0.557 |
| $S_{1,5}-S_{2,2}$ | 0.670 | 0.599 | 0.493 | 0.669 | $S_{1,11}-S_{2,2}$ | 2.188 | 2.667 | −0.543 | −0.625 |
| $S_{1,5}-S_{2,3}$ | 0.564 | 0.477 | 0.773 | 1.096 | $S_{1,11}-S_{2,3}$ | 1.880 | 2.194 | −0.468 | −0.544 |
| $S_{1,5}-S_{2,4}$ | 0.412 | 0.299 | 1.427 | 2.344 | $S_{1,11}-S_{2,4}$ | 0.729 | 0.669 | 0.372 | 0.495 |
| $S_{1,5}-S_{2,5}$ | 2.034 | 2.091 | −0.508 | −0.522 | $S_{1,11}-S_{2,5}$ | 4.388 | 4.925 | −0.772 | −0.797 |
| $S_{1,5}-S_{2,6}$ | 0.411 | 0.306 | 1.433 | 2.268 | $S_{1,11}-S_{2,6}$ | 1.367 | 1.470 | −0.268 | −0.32 |
| $S_{1,5}-S_{2,7}$ | 0.427 | 0.323 | 1.342 | 2.096 | $S_{1,11}-S_{2,7}$ | 1.423 | 1.545 | −0.297 | −0.353 |
| $S_{1,5}-S_{2,8}$ | 0.542 | 0.452 | 0.845 | 1.212 | $S_{1,11}-S_{2,8}$ | 1.812 | 2.094 | −0.448 | −0.522 |
| $S_{1,5}-S_{2,9}$ | 0.407 | 0.298 | 1.457 | 2.355 | $S_{1,11}-S_{2,9}$ | 0.981 | 0.973 | 0.019 | 0.028 |
| $S_{1,5}-S_{2,10}$ | 0.500 | 0.405 | 1 | 1.469 | $S_{1,11}-S_{2,10}$ | 1.676 | 1.897 | −0.403 | −0.473 |
| $S_{1,5}-S_{2,11}$ | 0.419 | 0.302 | 1.387 | 2.311 | $S_{1,11}-S_{2,11}$ | 0.415 | 0.300 | 1.410 | 2.333 |
| $S_{1,5}-S_{2,12}$ | 0.713 | 0.644 | 0.403 | 0.553 | $S_{1,11}-S_{2,12}$ | 2.305 | 2.739 | −0.566 | −0.635 |
| $S_{1,6}-S_{2,1}$ | 1.750 | 2.003 | −0.429 | −0.501 | $S_{1,12}-S_{2,1}$ | 1.8527 | 2.152 | −0.460 | −0.535 |
| $S_{1,6}-S_{2,2}$ | 2.000 | 2.374 | −0.5 | −0.579 | $S_{1,12}-S_{2,2}$ | 2.111 | 2.546 | −0.526 | −0.607 |
| $S_{1,6}-S_{2,3}$ | 1.709 | 1.944 | −0.415 | −0.486 | $S_{1,12}-S_{2,3}$ | 1.810 | 2.090 | −0.448 | −0.522 |
| $S_{1,6}-S_{2,4}$ | 0.657 | 0.584 | 0.522 | 0.712 | $S_{1,12}-S_{2,4}$ | 0.699 | 0.633 | 0.431 | 0.58 |
| $S_{1,6}-S_{2,5}$ | 4.207 | 4.697 | −0.762 | −0.787 | $S_{1,12}-S_{2,5}$ | 4.316 | 4.834 | −0.768 | −0.793 |
| $S_{1,6}-S_{2,6}$ | 1.233 | 1.293 | −0.189 | −0.227 | $S_{1,12}-S_{2,6}$ | 1.311 | 1.396 | −0.237 | −0.284 |
| $S_{1,6}-S_{2,7}$ | 1.284 | 1.360 | −0.221 | −0.265 | $S_{1,12}-S_{2,7}$ | 1.366 | 1.468 | −0.268 | −0.319 |
| $S_{1,6}-S_{2,8}$ | 1.645 | 1.853 | −0.392 | −0.46 | $S_{1,12}-S_{2,8}$ | 1.744 | 1.993 | −0.427 | −0.498 |
| $S_{1,6}-S_{2,9}$ | 0.882 | 0.852 | 0.134 | 0.174 | $S_{1,12}-S_{2,9}$ | 0.940 | 0.922 | 0.064 | 0.085 |
| $S_{1,6}-S_{2,10}$ | 1.518 | 1.675 | −0.341 | −0.403 | $S_{1,12}-S_{2,10}$ | 1.611 | 1.804 | −0.379 | −0.446 |
| $S_{1,6}-S_{2,11}$ | 0.416 | 0.300 | 1.404 | 2.333 | $S_{1,12}-S_{2,11}$ | 0.416 | 0.300 | 1.404 | 2.333 |
| $S_{1,6}-S_{2,12}$ | 2.112 | 2.453 | −0.527 | −0.592 | $S_{1,12}-S_{2,12}$ | 2.226 | 2.621 | −0.551 | −0.618 |

Thus, cost of joint production of two products in diversified firm is 21.5% less than cost of producing them separately. As a result, saving costs through joint production of all products in diversified firm is possible. Since $DES_{S_{1,8}-S_{2,8}} > 0$, it is better to produce all outputs as a group, *i.e.* this virtual firm exhibits ES. Note that desirability of merging virtual firm which is created by combination of firm 1 of group $S_1$ with firm 1 of group $S_2$, cannot be concluded as

Input: $v_1 = 595473.1 + 177260.1 = 772733.2$

Intermediate products: $w_{11} = 7451757$ and $w_{21} = 856735$

Final outputs: $y_{11} = 984143$ and $y_{21} = 681687$

Therefore, $C_1(7451757, 0, 984143, 0) + C_2(0, 856735, 0, 681687) = 772733.2$

$$C(7451757, 856735, 984143, 681687) = 3.146 \times 772733.2 = 2431018.6472 > 772733.2$$

This means that the diversified group can produce the same outputs at higher cost than the specialized firms. Therefore, separate production of products is cost-effective and profitable.

$$DES_{S_{1,1}-S_{2,1}} = \frac{C_1(w_{11}, 0, y_{11}, 0) + C_2(0, w_{21}, 0, y_{21} - C(w_{11}, w_{21}, y_{11}, y_{21})}{C(w_{11}, w_{21}, y_{11}, y_{21})}$$
$$= \frac{772733.2 - 2431018.6472}{2431018.6472} = -0.68213604 \cong -0.682.$$

$DES_{S_{1,1}-S_{2,1}} < 0$. Therefore, it is profitable to produce all outputs separately (ES).

TABLE 9. Presence or absence of ES in two different cases.

| Virtual firm | ES | | Virtual firm | ES | | Virtual firm | ES | | Virtual firm | ES | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Black box | Two stage | | Black box | Two stage | | Black box | Two stage | | Black box | Two stage |
| $S_{1,1}{-}S_{2,1}$ | No | No | $S_{1,4}{-}S_{2,1}$ | No | No | $S_{1,7}{-}S_{2,1}$ | No | No | $S_{1,10}{-}S_{2,1}$ | No | No |
| $S_{1,1}{-}S_{2,2}$ | No | No | $S_{1,4}{-}S_{2,2}$ | No | No | $S_{1,7}{-}S_{2,2}$ | No | No | $S_{1,10}{-}S_{2,2}$ | No | No |
| $S_{1,1}{-}S_{2,3}$ | No | No | $S_{1,4}{-}S_{2,3}$ | No | No | $S_{1,7}{-}S_{2,3}$ | No | No | $S_{1,10}{-}S_{2,3}$ | No | No |
| $S_{1,1}{-}S_{2,4}$ | Yes | Yes | $S_{1,4}{-}S_{2,4}$ | No | No | $S_{1,7}{-}S_{2,4}$ | Yes | Yes | $S_{1,10}{-}S_{2,4}$ | Yes | Yes |
| $S_{1,1}{-}S_{2,5}$ | No | No | $S_{1,4}{-}S_{2,5}$ | No | No | $S_{1,7}{-}S_{2,5}$ | No | No | $S_{1,10}{-}S_{2,5}$ | No | No |
| $S_{1,1}{-}S_{2,6}$ | No | No | $S_{1,4}{-}S_{2,6}$ | No | No | $S_{1,7}{-}S_{2,6}$ | No | No | $S_{1,10}{-}S_{2,6}$ | No | No |
| $S_{1,1}{-}S_{2,7}$ | No | No | $S_{1,4}{-}S_{2,7}$ | No | No | $S_{1,7}{-}S_{2,7}$ | No | No | $S_{1,10}{-}S_{2,7}$ | No | No |
| $S_{1,1}{-}S_{2,8}$ | No | No | $S_{1,4}{-}S_{2,8}$ | No | No | $S_{1,7}{-}S_{2,8}$ | No | No | $S_{1,10}{-}S_{2,8}$ | No | No |
| $S_{1,1}{-}S_{2,9}$ | No | No | $S_{1,4}{-}S_{2,9}$ | No | No | $S_{1,7}{-}S_{2,9}$ | Yes | Yes | $S_{1,10}{-}S_{2,9}$ | Yes | Yes |
| $S_{1,1}{-}S_{2,10}$ | No | No | $S_{1,4}{-}S_{2,10}$ | No | No | $S_{1,7}{-}S_{2,10}$ | No | No | $S_{1,10}{-}S_{2,10}$ | No | No |
| $S_{1,1}{-}S_{2,11}$ | Yes | Yes | $S_{1,4}{-}S_{2,11}$ | Yes | Yes | $S_{1,7}{-}S_{2,11}$ | Yes | Yes | $S_{1,10}{-}S_{2,11}$ | Yes | Yes |
| $S_{1,1}{-}S_{2,12}$ | No | No | $S_{1,4}{-}S_{2,12}$ | No | No | $S_{1,7}{-}S_{2,12}$ | No | No | $S_{1,10}{-}S_{2,12}$ | No | No |
| $S_{1,2}{-}S_{2,1}$ | No | No | $S_{1,5}{-}S_{2,1}$ | Yes | Yes | $S_{1,8}{-}S_{2,1}$ | Yes | Yes | $S_{1,11}{-}S_{2,1}$ | No | No |
| $S_{1,2}{-}S_{2,2}$ | No | No | $S_{1,5}{-}S_{2,2}$ | Yes | Yes | $S_{1,8}{-}S_{2,2}$ | No | No | $S_{1,11}{-}S_{2,2}$ | No | No |
| $S_{1,2}{-}S_{2,3}$ | No | No | $S_{1,5}{-}S_{2,3}$ | Yes | Yes | $S_{1,8}{-}S_{2,3}$ | Yes | Yes | $S_{1,11}{-}S_{2,3}$ | No | No |
| $S_{1,2}{-}S_{2,4}$ | Yes | Yes | $S_{1,5}{-}S_{2,4}$ | Yes | Yes | $S_{1,8}{-}S_{2,4}$ | Yes | Yes | $S_{1,11}{-}S_{2,4}$ | Yes | Yes |
| $S_{1,2}{-}S_{2,5}$ | No | No | $S_{1,5}{-}S_{2,5}$ | No | No | $S_{1,8}{-}S_{2,5}$ | No | No | $S_{1,11}{-}S_{2,5}$ | No | No |
| $S_{1,2}{-}S_{2,6}$ | No | No | $S_{1,5}{-}S_{2,6}$ | Yes | Yes | $S_{1,8}{-}S_{2,6}$ | Yes | Yes | $S_{1,11}{-}S_{2,6}$ | No | No |
| $S_{1,2}{-}S_{2,7}$ | No | No | $S_{1,5}{-}S_{2,7}$ | Yes | Yes | $S_{1,8}{-}S_{2,7}$ | Yes | Yes | $S_{1,11}{-}S_{2,7}$ | No | No |
| $S_{1,2}{-}S_{2,8}$ | No | No | $S_{1,5}{-}S_{2,8}$ | Yes | Yes | $S_{1,8}{-}S_{2,8}$ | Yes | Yes | $S_{1,11}{-}S_{2,8}$ | No | No |
| $S_{1,2}{-}S_{2,9}$ | No | No | $S_{1,5}{-}S_{2,9}$ | Yes | Yes | $S_{1,8}{-}S_{2,9}$ | Yes | Yes | $S_{1,11}{-}S_{2,9}$ | Yes | Yes |
| $S_{1,2}{-}S_{2,10}$ | No | No | $S_{1,5}{-}S_{2,10}$ | Yes | Yes | $S_{1,8}{-}S_{2,10}$ | Yes | Yes | $S_{1,11}{-}S_{2,10}$ | No | No |
| $S_{1,2}{-}S_{2,11}$ | Yes | Yes | $S_{1,5}{-}S_{2,11}$ | Yes | Yes | $S_{1,8}{-}S_{2,11}$ | Yes | Yes | $S_{1,11}{-}S_{2,11}$ | Yes | Yes |
| $S_{1,2}{-}S_{2,12}$ | No | No | $S_{1,5}{-}S_{2,12}$ | Yes | Yes | $S_{1,8}{-}S_{2,12}$ | No | No | $S_{1,11}{-}S_{2,12}$ | No | No |
| $S_{1,3}{-}S_{2,1}$ | No | No | $S_{1,6}{-}S_{2,1}$ | No | No | $S_{1,9}{-}S_{2,1}$ | No | No | $S_{1,12}{-}S_{2,1}$ | No | No |
| $S_{1,3}{-}S_{2,2}$ | No | No | $S_{1,6}{-}S_{2,2}$ | No | No | $S_{1,9}{-}S_{2,2}$ | No | No | $S_{1,12}{-}S_{2,2}$ | No | No |
| $S_{1,3}{-}S_{2,3}$ | No | No | $S_{1,6}{-}S_{2,3}$ | No | No | $S_{1,9}{-}S_{2,3}$ | No | No | $S_{1,12}{-}S_{2,3}$ | No | No |
| $S_{1,3}{-}S_{2,4}$ | No | No | $S_{1,6}{-}S_{2,4}$ | Yes | Yes | $S_{1,9}{-}S_{2,4}$ | Yes | Yes | $S_{1,12}{-}S_{2,4}$ | Yes | Yes |
| $S_{1,3}{-}S_{2,5}$ | No | No | $S_{1,6}{-}S_{2,5}$ | No | No | $S_{1,9}{-}S_{2,5}$ | No | No | $S_{1,12}{-}S_{2,5}$ | No | No |
| $S_{1,3}{-}S_{2,6}$ | No | No | $S_{1,6}{-}S_{2,6}$ | No | No | $S_{1,9}{-}S_{2,6}$ | No | No | $S_{1,12}{-}S_{2,6}$ | No | No |
| $S_{1,3}{-}S_{2,7}$ | No | No | $S_{1,6}{-}S_{2,7}$ | No | No | $S_{1,9}{-}S_{2,7}$ | No | No | $S_{1,12}{-}S_{2,7}$ | No | No |
| $S_{1,3}{-}S_{2,8}$ | No | No | $S_{1,6}{-}S_{2,8}$ | No | No | $S_{1,9}{-}S_{2,8}$ | No | No | $S_{1,12}{-}S_{2,8}$ | No | No |
| $S_{1,3}{-}S_{2,9}$ | No | No | $S_{1,6}{-}S_{2,9}$ | Yes | Yes | $S_{1,9}{-}S_{2,9}$ | No | No | $S_{1,12}{-}S_{2,9}$ | Yes | Yes |
| $S_{1,3}{-}S_{2,10}$ | No | No | $S_{1,6}{-}S_{2,10}$ | No | No | $S_{1,9}{-}S_{2,10}$ | No | No | $S_{1,12}{-}S_{2,10}$ | No | No |
| $S_{1,3}{-}S_{2,11}$ | Yes | Yes | $S_{1,6}{-}S_{2,11}$ | Yes | Yes | $S_{1,9}{-}S_{2,11}$ | Yes | Yes | $S_{1,12}{-}S_{2,11}$ | Yes | Yes |
| $S_{1,3}{-}S_{2,12}$ | No | No | $S_{1,6}{-}S_{2,12}$ | No | No | $S_{1,9}{-}S_{2,12}$ | No | No | $S_{1,12}{-}S_{2,12}$ | No | No |

## 5. Conclusions

Traditional DEA models deal with a system as a black-box without taking into consideration its internal activities. Recently, DEA models have been introduced to study internal structure of DMUs and measure efficiency score with interrelationship within stages. This paper studied ES between two products in two-stage production systems. ES exists if costs of joint production of products are lower than producing products separately. To analyze existence of ES and determine DES in two-stage network structures, virtual merged two-stage systems were created by combining specialized firms of two different groups and use of new proposed model. These merged systems undergo evaluation in PPS of diversified firms to determine whether joint production is more efficient in two-stage network structures or separate production. To show effect of intermediate products on ES and DES, we considered numerical examples in black box and two-stage structures and introduced virtual merged systems which were compared with diversified systems. Then, we applied our new proposed model to compare the results with conventional DEA models. Some interesting and notable differences were presented in comparing DES and ES. Decision making based on results of new proposed model is more accurate than conventional DEA models. This case was discussed in Example 4.1. In the results, presence or absence of ES for some of DMUs was not necessarily same in both structures. Also, DES was not necessarily similar to both structures because of considering or ignorance of intermediate products. Furthermore, we illustrated how ES could be used for decision making in joint production of products in one diversified firm or their separate production in specialized firms based on saved costs. We suggest prospective researchers develop fuzzy and stochastic version of our proposed model.

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