RAIRO-Oper. Res. 52 (2018) 935–942
https://doi.org/10.1051/ro/2017024

RAIRO Operations Research
www.rairo-ro.org

A NEW AXIOMATIZATION OF A CLASS OF EQUAL SURPLUS DIVISION VALUES FOR TU GAMES

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Abstract. In this paper, we propose a variation of weak covariance named as non-singleton covariance, requiring that changing the worth of a non-singleton coalition in a TU game affects the payoffs of all players equally. We establish that this covariance is characteristic for the convex combinations of the equal division value and the equal surplus division value, together with efficiency and a one-parameterized axiom treating a particular kind of players specially. As special cases, parallel axiomatizations of the two values are also provided.

Mathematics Subject Classification. 91A12.

Received November 10, 2016. Accepted March 31, 2017.

1. Introduction

The equal division (ED) value and the equal surplus division (ESD) value[3] [6] are two well-known single-value solutions for transferable utility cooperative games (TU games) with some egalitarian flavour. The ED-value associates to every player an equal share of the worth of the grand coalition, while the ESD-value first assigns to every player his individual worth, and then splits the remainder of the worth of the grand coalition equally among all players.

The ED-value assumes every player has zero right on his individual worth, while the ESD-value full right. These two assumptions represent extremes.

Recently, convex combinations of the ED-value and the ESD-value have been studied by Xu *et al.* [14]. This kind of convex combinations assume every player has partial right on his individual worth. It first assigns to every player some uniform share ($\alpha \in [0, 1]$) of his individual worth, and then splits the remainder of the worth of the grand coalition equally among all players. Xu *et al.* [14] first proposed a one-parameterized axiom named as α-individual rationality[4], requiring the payoff of a player to be no less than the α-share of his individual worth. They showed that this axiom is characteristic for the convex combinations together with the well-known efficiency, additivity, and symmetry. Then, they replaced the α-individual rationality in their axiomatization with α-dummifying player property, requiring an α-dummifying player to get the α-share of his individual worth.

Keywords. TU game, equal division value, equal surplus division value, nullifying player, dummifying player.

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[3] The ESD-value is often called as the centre-of-gravity of the imputation-set (CIS) value.

[4] Which is also being used by van den Brink *et al.* [13].

Roughly, an α-dummifying player's participation of a non-empty coalition not only prevents the cooperations in the resulting coalition, but also decreases the production of every player partially. That is, only the α-share of every individual worth is left. Obviously, an α-dummifying player degenerates to a dummifying player of Casajus and Huettner [4] if $\alpha = 1$. Yet, it looks very similar with the nullifying player of van den Brink [10] if $\alpha = 0$. However, Xu *et al.* [14] did not give any further information about the relation between 0-dummifying player[5] and nullifying player.

The contributions of this paper are in two respects. First, we propose a weak version of nullifying player, coinciding with the 0-dummifying player of Xu *et al.* [14]. We show that the nullifying player in van den Brink [10] can be weakened into weak nullifying player. Then, we replace the additivity and symmetry in the second axiomatization of Xu *et al.* [14] with a variation of weak covariance [11] named as non-singleton covariance, stating that changing the worth of a non-singleton coalition affects the payoffs of all players equally. Since both the ED-value and the ESD-value neglect most information contained in the characteristic function, non-singleton covariance may capture the essences of them, just as the strong monotonicity of Young [16] to the Shapley value [9]. As corollaries of our axiomatization, we also provide parallel axiomatizations of the ED-value ($\alpha = 0$) and the ESD-value ($\alpha = 1$).

Other axiomatizations of the convex combinations can also be found in van den Brink *et al.* [13]. First, they characterized it by simplifying the β-consistency of [12] into projection reduced game consistency. Then, they extended the axiomatization of the ESD-value with population solidarity [5] to the convex combinations. Both β-consistency and population solidarity requires a variable population, hence the axiomatizations of van den Brink *et al.* [13]. Meanwhile, all the axioms we use as well as Xu *et al.* [14] are defined on fixed player sets, so does our axiomatizations.

The rest of the paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, the non-singleton covariance is defined, thereupon a new axiomatization of the convex combinations is presented, as well as parallel axiomatizations of the ED-value and the ESD-value.

2. Preliminaries

A *transferable utility cooperative game* (TU game) over a finite player set N can be represented as an ordered pair (N, v), where $v : 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$ is a characteristic function. A subset of N is called a *coalition*. Given a coalition $S \subseteq N$, $v(S)$ is called the *worth* of S, representing the guaranteed award that can be obtained by S without cooperating with $N \setminus S$. Denote the set of all TU games over N by $\mathcal{G}^N$.

A player $i \in N$ is a *nullifying player* of $(N, v) \in \mathcal{G}^N$ if $v(S \cup \{i\}) = 0$ for all $S \subseteq N \setminus \{i\}$, while it is a *dummifying player* if $v(S \cup \{i\}) = \sum_{j \in S \cup \{i\}} v(\{j\})$. Players $i, j \in N$ are *symmetric players* of $(N, v) \in \mathcal{G}^N$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A *value* f over $\mathcal{G}^N$ associates a vector $f(N, v) \in \mathbb{R}^N$ to every $(N, v) \in \mathcal{G}^N$. For every $i \in N$, the component $f_i(N, v)$ represents the *payoff* of i according to f.

The *equal division value* (ED-value) assigns to every player an equal share of the worth of the grand coalition, *i.e.*, for every $(N, v) \in \mathcal{G}^N$ and $i \in N$,

$$\mathrm{ED}_i(N, v) = \frac{v(N)}{|N|}\cdot$$

The *equal surplus division value* (ESD-value) first assigns to every player his individual worth, and then splits the remainder of the worth of the grand coalition equally among all players, *i.e.*, for every $(N, v) \in \mathcal{G}^N$ and $i \in N$,

$$\mathrm{ESD}_i(N, v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}\cdot$$

Consider the following axioms for a value f over $\mathcal{G}^N$.

[5] Note that the 0-dummifying player here is different from the one of Béal *et al.* [1].

- *Efficiency.* For all $(N,v) \in \mathcal{G}^N$, $\sum_{i\in N} f_i(N,v) = v(N)$.
- *Additivity.* For all $(N,u),(N,v) \in \mathcal{G}^N$, $f(N,u+v) = f(N,u) + f(N,v)$, where $(u+v)(S) = u(S)+v(S)$ for all $S \subseteq N$.
- *Symmetry.* For all $(N,v) \in \mathcal{G}^N$ and $i,j \in N$ such that i and j are symmetric players of (N,v), $f_i(N,v) = f_j(N,v)$.
- *Nullifying player property.* For all $(N,v) \in \mathcal{G}^N$ and $i \in N$ such that i is a nullifying player of (N,v), $f_i(N,v) = 0$.
- *Dummifying player property.* For all $(N,v) \in \mathcal{G}^N$ and $i \in N$ such that i is a dummifying player of (N,v), $f_i(N,v) = v(\{i\})$.

Theorem 2.1. (1) [10] *A value f over $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry, and nullifying player property if and only if $f = ED$.*

(2) [4] *A value f over $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry, and dummifying player property if and only if $f = ESD$.*

In this paper, we consider the convex combinations of the ED-value and the ESD-value, which is defined for every $(N,v) \in \mathcal{G}^N$ and $\alpha \in [0,1]$ by

$$\varphi^\alpha(N,v) = \alpha \mathrm{ESD}(N,v) + (1-\alpha)\mathrm{ED}(N,v). \tag{2.1}$$

Obviously, $\varphi^0 = \mathrm{ED}$ and $\varphi^1 = \mathrm{ESD}$. Besides, for every $i \in N$, equation (2.1) can be rewritten as

$$\varphi_i^\alpha(N,v) = \alpha v(\{i\}) + \frac{v(N) - \sum_{j\in N} \alpha v(\{j\})}{|N|}. \tag{2.2}$$

That is, φ^α first assigns to every player his α-share of individual worth, and then splits the remainder of the worth of the grand coalition equally among all players.

3. The axiomatization

Similar to Theorem 2.1, Xu *et al.* [14] characterized φ^α with efficiency, additivity, symmetry, and a one-parameterized axiom named as α-dummifying player property.

Definition 3.1 ([14])**.** Let $\alpha \in [0,1]$. A player $i \in N$ is an *α-dummifying player* of $(N,v) \in \mathcal{G}^N$ if for all $S \subseteq N \setminus \{i\}$ with $S \neq \varnothing$, $v(S \cup \{i\}) = \sum_{j \in S\cup\{i\}} \alpha v(\{j\})$.

An α-dummifying player's participation of a non-empty coalition not only prevents the cooperations in the resulting coalition, but also decreases the production of every player partially. That is, only the α-share of every individual worth is left. It "can be interpreted economically as such a person who has a membership, when he invites any other consumers to form a coalition then all of them can get a discount with fraction α" [14]. Obviously, an α-dummifying player degenerates to a dummifying player if $\alpha = 1$. Yet, it looks quite similar to a nullifying player if $\alpha = 0$.

α-dummifying player property. Let $\alpha \in [0,1]$. A value f over $\mathcal{G}^N$ satisfies α-dummifying player property if for all $(N,v) \in \mathcal{G}^N$ and $i \in N$ such that i is an α-dummifying player of (N,v), $f_i(N,v) = \alpha v(\{i\})$.

α-dummifying player property "means that the α-dummifying player just pays the fraction α of his individual cost when allocating the total cost among all players" [14]. Considered the α-dummifying player is a membership, this requirement seems natural.

α-dummifying player property degenerates to dummifying player property if $\alpha = 1$.

Next we define a weak version of nullifying player, coinciding with the 0-dummifying player.

Definition 3.2. A player $i \in N$ is a *weak nullifying player* of (N,v) if $v(S \cup \{i\}) = 0$ for all $S \subseteq N \setminus \{i\}$ with $S \neq \varnothing$.

A weak nullifying player's participation of a non-empty coalition neutralises the production of the resulting coalition, irrespective of the production of himself. This makes it weaker than a nullifying player, which requires the production of the underlying player to be zero. Nevertheless, we will show that this requirement is redundant for the first part of Theorem 2.1.

Weak nullifying player property. A value f over $\mathcal{G}^N$ satisfies *weak nullifying player property* if for all $(N, v) \in \mathcal{G}^N$ and $i \in N$ such that i is a weak nullifying player of (N, v), $f_i(N, v) = 0$.

Weak nullifying player property requires a weak nullifying player to get zero-payoff, irrespective of his production. This axiom looks natural in the sense that a weak nullifying player destroys the production of other coalitions, thus he should be expelled by the other players and ought to get nothing from the game. Obviously, the ED-value satisfies weak nullifying player property. Meanwhile, weak nullifying player property coincides with 0-dummifying player property.

So as to present our main result, we still need to define the non-singleton covariance mentioned before, which is a variation of the weak covariance of van den Brink and Ju [11].

Weak covariance. A value f over $\mathcal{G}^N$ satisfies *weak covariance* if for all $(N, v) \in \mathcal{G}^N$, $\rho \in \mathbb{R}$, $i \in N$, and $j, k \in N \setminus \{i\}$,

$$f_j(N, v + \rho u_{\{i\}}) - f_j(N, v) = f_k(N, v + \rho u_{\{i\}}) - f_k(N, v),$$

where $u_{\{i\}}$ represents the characteristic function of the $\{i\}$-unanimity game, for all $S \subset N$,

$$u_{\{i\}}(S) = \begin{cases} 1, & \text{if } i \in S; \\ 0, & \text{otherwise.} \end{cases}$$

Weak covariance requires that for any pair of players, changing the worth of all coalitions containing a given third player uniformly affects their payoffs by the same amount. All the Shapley value, the ED-value, and the ESD-value satisfy weak covariance, hence their convex combinations. That is, the egalitarian Shapley value [7] (*i.e.*, convex combinations of the Shapley value and the ED-value), the consensus value [8] (*i.e.*, convex combinations of the Shapley value and the ESD-value), and φ^α. Thereby, weak covariance does not capture the essence of φ^α. However, if the change only happens to the worth of one coalition, then the corresponding covariance is just satisfied by φ^α, at the cost of the underlying coalition to be non-singleton.

Non-singleton covariance. A value f over $\mathcal{G}^N$ satisfies *non-singleton covariance* if for all $(N, v) \in \mathcal{G}^N$, $\rho \in \mathbb{R}$, $T \subseteq N$ with $|T| \geq 2$, and $i, j \in N$,

$$f_i(N, v + \rho e_T) - f_i(N, v) = f_j(N, v + \rho e_T) - f_j(N, v).$$

Here e_T represents the characteristic function of the T-standard game[6], for all $S \subseteq N$,

$$e_T(S) = \begin{cases} 1, & \text{if } S = T; \\ 0, & \text{otherwise.} \end{cases}$$

Non-singleton covariance states that if the worth of a non-singleton coalition changes, then the payoffs of all players should be affected by the same amount. Thus, no player can benefit from changing the worth of a non-singleton coalition. This implies that non-singletons are less responsible for the payoffs of players than singletons.

Remark 3.3. Non-singleton covariance can also be viewed as a variation of the addition invariance on bi-partitions of Béal *et al.* [3], which requires that if the worths a coalition and its complementary coalition change by the same amount, then the payoffs of all players should not be affected. Addition invariance on bi-partitions can also be viewed as a variation of the addition invariance and the transfer invariance of Béal *et al.* [2].

[6] Also known as the Dirac game.

Since both the ED-value and the ESD-value neglect most information contained in the characteristic function, non-singleton covariance may capture the essences of them, just like the strong monotonicity to the Shapley value.

Theorem 3.4. *Let $\alpha \in [0,1]$. Then a value f over $\mathcal{G}^N$ satisfies efficiency, non-singleton covariance, and α-dummifying player property if and only if $f = \varphi^\alpha$.*

Proof. Existence. According to equation (2.2), one can easily verify that φ^α satisfies these three axioms.

Uniqueness. Let $(N,v) \in \mathcal{G}^N$ and $i \in N$. Note that

$$v = \sum_{T\subseteq N:|T|=1} v(T)\bar{u}_T + \sum_{T\subseteq N:|T|\geq 2}\left(v(T) - \sum_{j\in T}\alpha v(\{j\})\right)e_T,$$

where $\bar{u}_T$ is a modification of u_T, for all $S \subseteq N$,

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } S = T; \\ \alpha, & \text{if } T \subsetneq S; \\ 0, & \text{otherwise.} \end{cases}$$

(1) Consider the TU game $(N,v_0) \equiv \left(N, \sum_{T\subseteq N:|T|=1} v(T)\bar{u}_T\right) \in \mathcal{G}^N$. Since every player $i \in N$ is an α-dummifying player, according to α-dummifying player property,

$$f_i(N,v_0) = \varphi_i^\alpha(N,v_0) = \alpha v(\{i\}) \quad \text{for all } i \in N.$$

(2) Let $\mathcal{T} = \{T \subseteq N | 2 \leq |T| \leq |N|-1\}$. Without loss of generality, denote

$$\mathcal{T} = \{T_1, T_2, \ldots, T_{2^{|N|}-|N|-2}\}$$

such that $|T_1| \leq |T_2| \leq \ldots \leq |T_{2^{|N|}-|N|-2}|$. For every $l \in \{1,2,\ldots,2^{|N|}-|N|-2\}$, define

$$(N,v_l) \equiv \left(N, v_0 + \sum_{l'=1}^{l}\left(v(T_{l'}) - \sum_{j\in T_{l'}}\alpha v(\{j\})\right)e_{T_{l'}}\right) \in \mathcal{G}^N.$$

For $l=1$. According to non-singleton covariance, for all $i,j \in N$,

$$f_i(N,v_1) - f_i(N,v_0) = f_j(N,v_1) - f_j(N,v_0).$$

According to case (1),

$$f_i(N,v_1) = f_j(N,v_1) + \alpha v(\{i\}) - \alpha v(\{j\}).$$

Fix i and let j run over N,

$$f_i(N,v_1) = \alpha v(\{i\}) + \frac{\sum_{j\in N} f_j(N,v_1) - \alpha\sum_{j\in N} v(\{j\})}{|N|}.$$

Using efficiency,

$$f_i(N,v_1) = \alpha v(\{i\}) = \varphi_i^\alpha(N,v_1) \quad \text{for all } i \in N.$$

Let $l' \in \{2,3,\ldots,2^{|N|}-|N|-3\}$. Assume that

$$f_i(N,v_{l'}) = \varphi_i^\alpha(N,v_{l'}) = \alpha v(\{i\}) \quad \text{for all } i \in N.$$

For $l = l' + 1$. According to non-singleton covariance,

$$f_i(N, v_l) - f_i(N, v_{l-1}) = f_j(N, v_l) - f_j(N, v_{l-1}).$$

According to inductive hypothesis,

$$f_i(N, v_l) = f_j(N, v_l) + \alpha v(\{i\}) - \alpha v(\{j\}).$$

Fix i and let j run over N,

$$f_i(N, v_l) = \alpha v(\{i\}) + \frac{\sum_{j\in N} f_j(N, v_l) - \alpha \sum_{j\in N} v(\{j\})}{|N|}.$$

Using efficiency,

$$f_i(N, v_l) = \alpha v(\{i\}) = \varphi_i^\alpha(N, v_l) \quad \text{for all } i \in N.$$

From the previous, we finally get

$$f_i(N, v_{2^{|N|}-|N|-2}) = \varphi_i^\alpha(N, v_{2^{|N|}-|N|-2}) = \alpha v(\{i\}) \quad \text{for all } i \in N. \tag{3.1}$$

(3) Note that $v = v_{2^{|N|}-|N|-2} + (v(N) - \sum_{j\in N} \alpha v(\{j\}))e_N$. According to non-singleton covariance, for all $i, j \in N$,

$$f_i(N, v) - f_i(N, v_{2^{|N|}-|N|-2}) = f_j(N, v) - f_j(N, v_{2^{|N|}-|N|-2}).$$

Fix i and let j run over N,

$$f_i(N, v) = f_i(N, v_{2^{|N|}-|N|-2}) + \frac{\sum_{j\in N} f_j(N, v) - \sum_{j\in N} f_j(N, v_{2^{|N|}-|N|-2})}{|N|}.$$

Using efficiency and equation (3.1),

$$f_i(N, v) = \alpha v(\{i\}) + \frac{v(N) - \sum_{j\in N} \alpha v(\{j\})}{|N|} = \varphi_i^\alpha(N, v),$$

the desired expression. □

Remark 3.5. The three axioms used in Theorem 3.4 are independent from each other.

(1) The value f^1 over $\mathcal{G}^N$ defined for every $(N, v) \in \mathcal{G}^N$ and $i \in N$ by

$$f_i^1(N, v) = \alpha v(\{i\}) + \frac{v(N) - \sum_{j\in N} \alpha v(\{j\})}{|N| - 1}$$

satisfies all the axioms except efficiency.

(2) The value f^2 over $\mathcal{G}^N$ defined for every $(N, v) \in \mathcal{G}^N$ and $i \in N$ by

$$f_i^2(N, v) = \begin{cases} \varphi_i^\alpha(N, v), & \text{if } \sum_{j\in N} v(\{j\}) = 0 \\ \alpha v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}\big(v(N) - \sum_{j\in N} \alpha v(\{j\})\big), & \text{otherwise} \end{cases}$$

satisfies all the axioms except non-singleton covariance.

(3) The ESD-value satisfies all the axioms except α-dummifying player property in general.

Remark 3.6. Once we view a TU game $(N, v) \in \mathcal{G}^N$ as an $2^{|N|}-1$ dimensional vector, then $\mathcal{G}^N$ forms a linear space. In this case, $\left\{\{\bar{u}_T\}_{T\subseteq N:|T|=1}, \{e_T\}_{T\subseteq N:|T|\geq 2}\right\}$ is a basis of this space. Another interesting basis can also be found in Yokote *et al.* [15].

Remark 3.7. Note that in the proof of Theorem 3.4, α-dummifying player property is only used to determine a payoff vector of (N, v_0), thus it can be replaced with other axioms suitable for this task. For example, the α-individual rationality of Xu *et al.* [14]. Formally, a value f over $\mathcal{G}^N$ satisfies α-individual rationality if for all $(N, v) \in \mathcal{G}^N$ and $i \in N$ such that (N, v) is α-weakly essential (*i.e.*, $v(N) \geq \sum_{j\in N} \alpha v(j)$), $f_i(N, v) \geq \alpha v(i)$. Since $v_0(N) = \sum_{j\in N} \alpha v_0(j)$, α-individual rationality and efficiency together determine a payoff vector of (N, v_0).

As special cases of Theorem 3.4, we can deduce the following parallel axiomatizations of the ED-value and the ESD-value.

Corollary 3.8. (1) *A value f over $\mathcal{G}^N$ satisfies efficiency, non-singleton covariance, and weak nullifying player property if and only if $f = ED$.*
(2) *A value f over $\mathcal{G}^N$ satisfies efficiency, non-singleton covariance, and dummifying player property if and only if $f = ESD$.*

We end this paper by showing that the nullifying player in the first part of Theorem 2.1 can be weakened into weak nullifying player.

Proposition 3.9. *If a value f over $\mathcal{G}^N$ satisfies efficiency, additivity, symmetry, and weak nullifying player property, then it also satisfies non-singleton covariance.*

Proof. Let $(N, v) \in \mathcal{G}^N$, $\rho \in \mathbb{R}$, $T \subseteq N$ with $|T| \geq 2$, and $i, j \in N$.

According to additivity,

$$f_i(N, v + \rho e_T) - f_i(N, v) = f_i(N, \rho e_T).$$

If $i, j \in T$ or $i, j \in N \setminus T$, then they are symmetric players of $(N, \rho e_T)$, thus according to symmetry, non-singleton covariance holds.

If $i \in T$ and $j \in N \setminus T$, then j is a weak nullifying player of $(N, \rho e_T)$. On one hand, according to weak nullifying player property,

$$f_j(N, \rho e_T) = 0 \text{ for all } j \in N \setminus T.$$

On the other hand, all the players in T are symmetric players of $(N, \rho e_T)$, thus according to efficiency and symmetry,

$$f_i(N, \rho e_T) = \frac{\rho e_T(N) - \sum_{j\in N\setminus T} f_j(N, \rho e_T)}{|T|} = 0,$$

the desired expression. □

According to the first part of Corollary 3.8 and Proposition 3.9, the nullifying player property in the first part of Theorem 2.1 can be weakened into weak nullifying player property. In this sense, the Theorem 4.2 of Xu *et al.* [14] can be viewed as an unification of the two parts of Theorem 2.1.

Acknowledgements. This work was supported by the Key Program of National Natural Science Foundation of China (71231003), the National Natural Science Foundation of China (71572040), and the Social Sciences Planning Project of Fujian (FJ2015C230).

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