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A NOTE ON SECOND-ORDER KARUSH–KUHN–TUCKER NECESSARY OPTIMALITY CONDITIONS FOR SMOOTH VECTOR OPTIMIZATION PROBLEMS

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Abstract. The aim of this note is to present some second-order Karush–Kuhn–Tucker necessary optimality conditions for vector optimization problems, which modify the incorrect result in ([10], Thm. 3.2).

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1. Introduction

In this note, we are interested in second-order Karush–Kuhn–Tucker (KKT) optimality conditions for the following constrained vector optimization problem

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } \ x \in Q_0 := \{x \in \mathbb{R}^n : g(x) \leqq 0\}, \end{aligned} \tag{VP}$$

where $f := (f_i)$, $i \in I := \{1, \ldots, l\}$, and $g := (g_j)$, $j \in J := \{1, \ldots, m\}$ are twice continuously differentiable vector-valued functions.

(KKT) optimality conditions are one of the greatest results in Optimization. In literature, there are two types of (KKT) necessary optimality conditions for vector optimization problems; see [3] for more details. The first one is called by strong (KKT) necessary conditions, *i.e.*, when all the Lagrange multipliers of the objective functions are positive. The second one is weak (KKT) necessary conditions, *i.e.*, when not all the Lagrange multipliers of the objective functions are zero.

It is well known that constraint qualifications and regularity conditions play an important role in establishing (KKT)-type necessary optimality conditions. We recall here that these assumptions are called constraint qualifications when they have to be fulfilled by the constraints of the problem, and they are called regularity conditions when they have to be fulfilled by both the objectives and the constraints of the problem; see [3] for more details.

Keywords. Second-order regularity conditions, second-order Karush–Kuhn–Tucker optimality conditions, efficient solution, geoffrion properly efficient solution.

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In 1994, Maeda [6] was the first to introduce a Generalized Guignard regularity condition and established strong first-order (KKT) necessary optimality conditions for differentiable problems. Later on, Preda and Chiţescu [9] derived results analogous to those obtained by Maeda [6] within the more general framework of the semidifferentiable case. On the line of their work, many authors have derived strong first-order (KKT) necessary conditions for efficiency in vector optimization problems both for smooth and nonsmooth cases; see [3–5].

One of the first investigations to obtain second-order (KKT) optimality conditions for smooth vector optimization problems was carried out by Wang [11]. Then, Bigi and Castellani [1, 2] obtained some weak second-order (KKT) optimality conditions by introducing some types of the second-order regularity conditions. In 2004, Maeda [7] was the first to propose a Abadie regularity condition and established strong second-order (KKT) necessary conditions for $C^{1,1}$ vector optimization problems. Then, using the so-called generalized Abadie regularity condition, Rizvi and Nasser [10] obtained some second-order (KKT) necessary conditions for (VP). However, the main result of Rizvi and Nasser ([10], Thm. 3.2) is not correct; see Example 2.5 in Section 2 below.

The aim of this note is to present some second-order (KKT) necessary optimality conditions for (VP), which modify the incorrect result in ([10], Thm. 3.2). The rest of this note is organized as follows. In Section 2, we give an example which shows that the result in ([10], Thm. 3.2) is not correct. Section 3 presents the main results.

2. A counterexample

In order to present a counterexample of ([10], Thm. 3.2), we need to recall some notations as follows. Let $\mathbb{R}^l$ be the l-dimensional Euclidean space. For $a, b \in \mathbb{R}^l$, by $a \leqq b$, we mean $a_i \leqq b_i$ for all $i = 1, \ldots, l$; by $a \leq b$, we mean $a \leqq b$ and $a \neq b$; and by $a < b$, we mean $a_i < b_i$ for all $i = 1, \ldots, l$. For any two vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\mathbb{R}^2$, we denote the lexicographic order by

$$
\begin{aligned}
a \leqq_{\text{lex}} b, \quad &\text{iff} \quad a_1 < b_1 \ \text{ or } \ a_1 = b_1 \quad \text{and} \quad a_2 \leqq b_2,\\
a <_{\text{lex}} b, \quad &\text{iff} \quad a_1 < b_1 \ \text{ or } \ a_1 = b_1 \quad \text{and} \quad a_2 < b_2.
\end{aligned}
$$

Definition 2.1. Let $x^0 \in Q_0$. We say that:

(i) x^0 is an *efficient solution* to (VP) if there is no $x \in Q_0$ satisfies $f(x) \leq f(x^0)$.
(ii) x^0 is a *Geoffrion properly efficient solution* to (VP) if it is efficient and there exists $K > 0$ such that, for each i,

$$\frac{f_i(x) - f_i(x^0)}{f_j(x^0) - f_j(x)} \leqq K,$$

for some j such that $f_j(x^0) < f_j(x)$ whenever $x \in Q_0$ and $f_i(x^0) > f_i(x)$.

Let C be a nonempty subset of $\mathbb{R}^n$, $x^0 \in C$ and $u \in \mathbb{R}^n$. The *tangent cone* to C at $x^0 \in C$ is the set defined by

$$T(C; x^0) := \{d \in \mathbb{R}^n : \exists t_k \to 0^+, \exists d^k \to d, x^0 + t_k d^k \in C \quad \forall k \in \mathbb{N}\}.$$

The *second-order tangent set* to C at x^0 with respect to the direction u is the set defined by

$$T^2(C; x^0, u) := \left\{ v \in \mathbb{R}^n : \exists t_k \to 0^+, \exists v^k \to v, x^0 + t_k u + \frac{1}{2} t_k^2 v^k \in C \quad \forall k \in \mathbb{N} \right\}.$$

Clearly, $T^2(C; x^0, 0) = T(C; x^0)$.

Fix $x^0 \in Q_0$, the *active index set* at x^0 is defined by

$$J(x^0) := \{j \in J : g_j(x^0) = 0\}.$$

For each $u \in \mathbb{R}^n$, put

$$
\begin{aligned}
I(x^0; u) &:= \{i \in I : \langle \nabla f_i(x^0), u \rangle = 0\},\\
J(x^0; u) &:= \{j \in J(x^0) : \langle \nabla g_j(x^0), u \rangle = 0\}.
\end{aligned}
$$

We say that u is a *critical direction* of problem (VP) at $x^0 \in Q_0$ if

$$\begin{aligned} &\langle \nabla f_i(x^0), u\rangle \leqq 0 \quad \forall i \in I,\\ &\langle \nabla f_i(x^0), u\rangle = 0 \ \text{ at least one } \quad i \in I,\\ &\langle \nabla g_j(x^0), u\rangle \leqq 0 \quad \forall j \in J(x^0). \end{aligned}$$

The set of all critical direction of problem (VP) at x^0 is denoted by $K(x^0)$.

The following sets were introduced by Rizvi and Nasser [10]:

$$M^i := Q_0 \cap \{x \in \mathbb{R}^n \,:\, f_i(x) \leqq f_i(x^0)\}, \quad i \in I,$$
$$Q := \{x \in Q_0 \,:\, f(x) \leqq f(x^0)\} = \bigcap_{i=1}^{l} M^i.$$

The *linearizing cone* to M^i (resp. Q) at $x^0 \in Q_0$ is the set defined by

$$L(M^i; x^0) := \{u \in \mathbb{R}^n \,:\, \langle \nabla f_i(x^0), u\rangle \leqq 0, \langle \nabla g_j(x^0), u\rangle \leqq 0, j \in J(x^0)\}, \quad i \in I.$$

$$(\text{resp. } L(Q; x^0) := \{u \in \mathbb{R}^n : \langle \nabla f_k(x^0), u\rangle \leqq 0, k \in I; \langle \nabla g_j(x^0), u\rangle \leqq 0, j \in J(x^0)\}.)$$

For each $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$, put

$$\begin{aligned} F_i^2(x^0; u, v) &:= \left(\langle \nabla f_i(x^0), u\rangle, \langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u, u\rangle\right), \ i \in I,\\ G_j^2(x^0; u, v) &:= \left(\langle \nabla g_j(x^0), u\rangle, \langle \nabla g_j(x^0), v\rangle + \langle \nabla^2 g_j(x^0)u, u\rangle\right), \ j \in J, \end{aligned}$$

and

$$\begin{aligned} L^2(Q_0; x^0, u) \ &:= \{v \in \mathbb{R}^n \,:\, G_j^2(x^0; u, v) \leqq_{\text{lex}} (0,0),\ j \in J(x^0)\},\\ L^2(Q; x^0, u) \ &:= \{v \in \mathbb{R}^n \,:\, F_i^2(x^0; u, v) \leqq_{\text{lex}} (0,0), i \in I\\ &\qquad \text{and } \ G_j^2(x^0; u, v) \leqq_{\text{lex}} (0,0), j \in J(x^0)\},\\ L^2(M^i; x^0, u) \ &:= \{v \in \mathbb{R}^n : F_i^2(x^0; u, v) \leqq_{\text{lex}} (0,0),\\ &\qquad \text{and } \ G_j^2(x^0; u, v) \leqq_{\text{lex}} (0,0),\ j \in J(x^0)\},\ i \in I. \end{aligned}$$

It is easily seen that $L^2(Q; x^0, 0) = L(Q; x^0)$, $L^2(Q_0; x^0, 0) = L(Q_0; x^0)$, and $L^2(M^i; x^0, 0) = L(M^i; x^0)$ for all $i \in I$.

Definition 2.2. Let $x^0 \in Q_0$ and $u \in \mathbb{R}^n$. We say that:
(i) (see [1], Def. 5.1) *The Abadie second-oder constraint qualification* holds at x^0 for the direction u if

$$L^2(Q_0; x^0, u) \subset T^2(Q_0; x^0, u). \tag{$ASOCQ$}$$

(ii) (see [10], Rem. 3.1) *The generalized Abadie second-order regularity condition* holds at x^0 for the direction u if

$$L^2(Q; x^0, u) \subset \bigcap_{i=1}^{l} T^2(M^i; x^0, u). \tag{$GASORC$}$$

(iii) *The weak Abadie second-order regularity condition* holds at x^0 for the direction u if

$$L^2(Q; x^0, u) \subset T^2(Q_0; x^0, u). \tag{$WASORC$}$$

Remark 2.3. It is easily seen that the following implications hold:

(i) ($GASORC$) $\Rightarrow$ ($WASORC$);
(ii) ($ASOCQ$) $\Rightarrow$ ($WASORC$).

We recall the main result in ([10], Thm. 3.2) as follows.

Theorem 2.4 (see [10]). *Let $x^0 \in Q_0$ be an efficient solution to (VP). Suppose that the ($GASORC$) holds at x^0 for each critical direction. Let u^0 be a critical direction at x^0. Then, there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ such that*

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^0) = 0, \tag{2.1}$$

$$\sum_{i=1}^{l} \lambda_i \langle \nabla^2 f_i(x^0)u^0, u^0\rangle + \sum_{j=1}^{m} \mu_j \langle \nabla^2 g_j(x^0)u^0, u^0\rangle \geqq 0, \tag{2.2}$$

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m) \geqq 0, \mu_j = 0, j \notin J(x^0; u^0), \tag{2.3}$$

$$\lambda_i > 0 \quad \textit{if} \quad i \in I(x^0; u^0); \lambda_i = 0, i \notin I(x^0; u^0). \tag{2.4}$$

The following example shows that the conclusions of Theorem 2.4 are not correct.

Example 2.5. Consider the following problem

$$\begin{aligned} &\min f(x) \\ &\text{s. t.} \quad x \in Q_0 := \mathbb{R}^2, \end{aligned} \tag{VP}$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ is a vector-valued function defined by

$$f(x) := (f_1(x), f_2(x), f_3(x)) = (x_2, x_1 + x_2^2, -x_1 + x_2^2).$$

Let $x^0 := (0, 0) \in Q_0$. We have

$$M^1 = \{(x_1, x_2) : x_2 \leqq 0\}, M^2 = \{(x_1, x_2) : x_1 \leqq -x_2^2\}, M^3 = \{(x_1, x_2) : x_2^2 \leqq x_1\},$$

and x^0 is an efficient solution to (VP) since $Q = \{x^0\}$. It is easy to check that

$$K(x^0) = \{(u_1, u_2) \in \mathbb{R}^2 \,:\, u_1 = 0, u_2 \leqq 0\}.$$

For each critical direction $u = (0, u_2)$, where $u_2 < 0$, it is easy to show that $L^2(Q; x^0, u) = \emptyset$. Thus, the ($GASORC$) holds at x^0 for the direction u. We now check the ($GASORC$) at x^0 for the critical direction $0_{\mathbb{R}^2} := (0, 0)$. An easy computation shows that

$$\begin{aligned} &T(M^1; x^0) = M^1, T(M^2; x^0) = \{(u_1, u_2) \in \mathbb{R}^2 \,:\, u_1 \leqq 0\}, \\ &T(M^3; x^0) = \{(u_1, u_2) \in \mathbb{R}^2 \,:\, u_1 \geqq 0\}, \\ &\bigcap_{i=1}^{3} T(M^i; x^0) = L(Q; x^0) = \{(u_1, u_2) \in \mathbb{R}^2 \,:\, u_1 = 0, u_2 \leqq 0\}. \end{aligned}$$

Since $L^2(Q; x^0, 0_{\mathbb{R}^2}) = L(Q; x^0)$ and $T^2(M^i; x^0, 0_{\mathbb{R}^2}) = T(M^i; x^0)$ for all $i = 1, 2, 3$, we have

$$L^2(Q; x^0, 0_{\mathbb{R}^2}) = L(Q; x^0) = \bigcap_{i=1}^{3} T(M^i; x^0) = \bigcap_{i=1}^{3} T^2(M^i; x^0, 0_{\mathbb{R}^2}).$$

Thus, the ($GASORC$) holds at x^0. Since $I(x^0, 0_{\mathbb{R}^2}) = \{1, 2, 3\}$ and

$$\lambda_1 \nabla f_1(x^0) + \lambda_2 \nabla f_2(x^0) + \lambda_3 \nabla f_3(x^0) = 0 \Leftrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = \lambda_3, \end{cases}$$

it follows that the conclusions of Theorem 2.4 is not correct.

Furthermore, we see that the system

$$\begin{cases} F_i^2(x^0; 0_{\mathbb{R}^2}, v) = (0, v_2) & <_{\text{lex}} (0,0) \\ F_i^2(x^0; 0_{\mathbb{R}^2}, v) = (0, v_1) & \leqq_{\text{lex}} (0,0) \\ F_i^2(x^0; 0_{\mathbb{R}^2}, v) = (0, -v_1) & \leqq_{\text{lex}} (0,0), \quad \text{for all } i = 1, 2, 3 \end{cases}$$

admits a solution $v = (0, v_2)$, where $v_2 < 0$. Thus, ([10], Thm. 3.1) is not correct, too.

We conclude this section with the remark that x^0 is not a Geoffrion properly efficient solution to (VP). Indeed, for each $a > 0$, put $x =: (0, -a)$. Then we have $f_1(x) < f_1(x^0)$, $f_2(x) > f_2(x^0)$ and

$$\lim_{a \downarrow 0} \frac{f_1(x) - f_1(x^0)}{f_2(x^0) - f_2(x)} = \lim_{a \downarrow 0} \frac{1}{a} = +\infty.$$

Therefore, x^0 is not a Geoffrion properly efficient solution.

3. Main results

This section presents our main results, both for weak and strong second-order (KKT) necessary optimality conditions.

Theorem 3.1 (Weak second-order (KKT) necessary optimality conditions). *Let $x^0 \in Q_0$ be an efficient solution to* (VP)*. Suppose that the* ($WASORC$) *holds at x^0 for each critical direction. Let u^0 be a critical direction at x^0. Then, there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ satisfying* (2.1)–(2.3) *and*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \geq 0, \lambda_i = 0, i \notin I(x^0; u^0). \tag{3.1}$$

Proof. We first claim that the following system

$$F_i^2(x^0; u, v) <_{\text{lex}} (0,0), \quad i \in I, \tag{3.2}$$

$$G_i^2(x^0; u, v) \leqq_{\text{lex}} (0,0), \quad j \in J(x^0) \tag{3.3}$$

has no solution $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Arguing by contradiction, assume that there exists $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ satisfying the system (3.2)–(3.3). It follows that $v \in L^2(Q; x^0, u)$ and

$$\begin{aligned} \langle \nabla f_i(x^0), u \rangle &\leqq 0, \quad i \in I, \\ \langle \nabla g_j(x^0), u \rangle &\leqq 0, \quad j \in J(x^0). \end{aligned}$$

Thus $u \in L(Q; x^0)$. We claim that $u \in K(x^0)$. Indeed, if otherwise, then

$$\langle \nabla f_i(x^0), u \rangle < 0, \quad \forall i \in I. \tag{3.4}$$

Since $0 \in K(x^0)$ and the ($WASORC$) holds at x^0 for each critical direction, we have

$$L^2(Q; x^0, 0) = L(Q; x^0) \subset T^2(Q_0; x^0, 0) = T(Q_0; x^0).$$

This implies that $u \in T(Q_0; x^0)$, *i.e.*, there exist sequences $\tau_k \to 0^+$, $u^k \to u$ such that

$$x^0 + \tau_k u^k \in Q_0, \quad \forall k \in \mathbb{N}.$$

Since (3.4) and

$$\lim_{k\to\infty} \frac{f_i(x^0 + \tau_k u^k) - f_i(x^0)}{\tau_k} = \langle \nabla f_i(x^0), u \rangle$$

it follows that there exists $k_0 \in \mathbb{N}$ such that

$$f_i(x^0 + \tau_k u^k) < f_i(x^0), \quad \forall i \in I, k \geqq k_0,$$

contrary to the fact that x^0 is an efficient solution to (VP).

Since $u \in K(x^0)$ and the $(WASORC)$ holds at x^0 for each critical direction, we have $v \in T^2(Q_0; x^0, u)$. Thus, there exist a sequence $\{v^k\}$ converging to v and a positive sequence $\{t_k\}$ converging to 0 such that

$$x^k := x^0 + t_k u + \frac{1}{2} t_k^2 v^k \in Q_0, \quad \forall k \in \mathbb{N}.$$

By Taylor's formula, for each $i \in I$, we have

$$f_i(x^k) - f_i(x^0) - t_k \langle \nabla f_i(x^0), u \rangle = \frac{1}{2} t_k^2 \left[\langle \nabla f_i(x^0), v \rangle + \langle \nabla^2 f_i(x^0) u, u \rangle \right] + o(t_k^2)$$

for all $k \in \mathbb{N}$. Thus

$$\lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0) - t_k \langle \nabla f_i(x^0), u \rangle}{\frac{1}{2} t_k^2} = \langle \nabla f_i(x^0), v \rangle + \langle \nabla^2 f_i(x^0) u, u \rangle.$$

For each $i \in I$, we consider two cases as follows.

Case 1. $i \in I(x^0; u)$. This means that $\langle \nabla f_i(x^0), u \rangle = 0$. By (3.2), we have

$$\langle \nabla f_i(x^0), v \rangle + \langle \nabla^2 f_i(x^0) u, u \rangle < 0.$$

From this and

$$\begin{aligned} \lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0)}{\frac{1}{2} t_k^2} &= \lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0) - t_k \langle \nabla f_i(x^0), u \rangle}{\frac{1}{2} t_k^2} \\ &= \langle \nabla f_i(x^0), v \rangle + \langle \nabla^2 f_i(x^0) u, u \rangle \end{aligned}$$

it follows that there exists k large enough such that $f_i(x^k) < f_i(x^0)$.

Case 2. $i \in I \setminus I(x^0; u)$. This means that $\langle \nabla f_i(x^0), u \rangle < 0$. Since

$$\lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0)}{t_k} = \langle \nabla f_i(x^0), u \rangle < 0,$$

it follows that $f_i(x^k) < f_i(x^0)$ for large enough k. Thus

$$f_i(x^k) < f_i(x^0)$$

for all $i \in I$ and large enough k, contrary to the fact that x^0 is an efficient solution to (VP).

Fix $u^0 \in K(x^0)$. Then the system

$$\begin{aligned} &F_i^2(x^0; u^0, v) <_{\text{lex}} (0,0), \quad i \in I, \\ &G_j^2(x^0; u^0, v) \leqq_{\text{lex}} (0,0), \quad j \in J(x^0) \end{aligned}$$

has no solution $v \in \mathbb{R}^n$. This implies that the system

$$\begin{aligned} \langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u^0, u^0\rangle &< 0, \qquad i \in I(x^0; u^0),\\ \langle \nabla g_j(x^0), v\rangle + \langle \nabla^2 g_j(x^0)u^0, u^0\rangle &\leq 0, \qquad j \in J(x^0; u^0), \end{aligned}$$

has no solution $v \in \mathbb{R}^n$. By the Motzkin theorem of the alternative ([8], p. 28), there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ satisfying (2.1)–(2.3) and (3.1). □

Next we prove that for Geoffrion properly efficient solutions, the (*GASORC*) guarantees strong second-order (KKT) optimality conditions.

Theorem 3.2 (Strong second-order (KKT) necessary optimality conditions)**.**
Let $x^0 \in Q_0$ be a Geoffrion properly efficient solution to (VP)*. Suppose that the* (*GASORC*) *holds at x^0 for each critical direction. Let u^0 be a critical direction at x^0. Then, there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ satisfying* (2.1)*–*(2.3) *and*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) > 0. \tag{3.5}$$

Proof. We first claim that the following system

$$F_i^2(x^0; u, v) \leqq_{\text{lex}} (0,0), \qquad i \in I, \tag{3.6}$$

$$F_i^2(x^0; u, v) <_{\text{lex}} (0,0), \qquad \text{at least one } \ i \in I, \tag{3.7}$$

$$G_j^2(x^0; u, v) \leqq_{\text{lex}} (0,0), \qquad j \in J(x^0) \tag{3.8}$$

has no solution $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$. Arguing by contradiction, assume that there exists a point $(u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ such that (3.6)–(3.8) hold. Without loss of generality one may assume that

$$F_1^2(x^0; u, v) <_{\text{lex}} (0,0). \tag{3.9}$$

From (3.6) and (3.8) it follows that $v \in L^2(Q; x^0, u)$ and

$$\begin{aligned} \langle \nabla f_i(x^0), u\rangle &\leqq 0, \qquad i \in I,\\ \langle \nabla g_j(x^0), u\rangle &\leqq 0, \qquad j \in J(x^0). \end{aligned}$$

Since the (*GASORC*) holds at x^0 for any critical direction at x^0, thanks to Theorem ([3], Thm. 4.3), we have

$$\langle \nabla f_i(x^0), u\rangle = 0$$

for all $i \in I$. Thus, u is a critical direction at x^0. Since the (*GASORC*) at x^0 for the critical direction u, we have

$$v \in \bigcap_{i=1}^{l} T^2(M^i; x^0, u).$$

Consequently, $v \in T^2(M^1; x^0, u)$. This implies that there exist a sequence $\{v^k\}$ converging to v and a positive sequence $\{t_k\}$ converging to 0 such that

$$x^k := x^0 + t_k u + \frac{1}{2} t_k^2 v^k \in M^1 \qquad \forall k \in \mathbb{N}.$$

Clearly, $\{x^k\} \subset Q_0$. By (3.6) and (3.9), we have

$$\langle \nabla f_1(x^0), v\rangle + \langle \nabla^2 f_1(x^0)u, u\rangle < 0, \tag{3.10}$$

$$\langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u, u\rangle \leqq 0 \ \ \forall i \in \{2, \ldots, l\}. \tag{3.11}$$

For each $i \in I$, we have

$$\lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0)}{\frac{1}{2}t_k^2} = \langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u, u\rangle. \tag{3.12}$$

From this and (3.10), we have

$$f_1(x^k) - f_1(x^0) < 0$$

for all large enough k. Without loss of generality we may assume that

$$f_1(x^k) < f_1(x^0) \quad \forall k \in \mathbb{N}.$$

For each $k \in \mathbb{N}$, put

$$I_k := \{i \in I \ : \ i \geq 2 \ \text{ and } \ f_i(x^k) > f_i(x^0)\}.$$

We claim that I_k is nonempty for all $k \in \mathbb{N}$. Indeed, if $I_k = \emptyset$ for some $k \in N$, then we have

$$f_i(x^k) \leqq f_i(x^0) \quad \forall i = 2, \ldots, l.$$

Using also the fact that $f_1(x^k) < f_1(x^0)$, we arrive at a contradiction with the efficiency of x^0.

Since $I_k \subset \{2, \ldots, l\}$ for all $k \in \mathbb{N}$, we may assume without loss of generality that $I_k = \bar{I}$ is constant for all $k \in \mathbb{N}$. By (3.12), for each $i \in \bar{I}$, we have

$$\langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u, u\rangle \geqq 0.$$

This and (3.11) imply that

$$\langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u, u\rangle = 0 \quad \forall i \in \bar{I}. \tag{3.13}$$

By (3.10), we can fix $\delta \in \mathbb{R}$ such that $\langle \nabla f_1(x^0), v\rangle + \langle \nabla^2 f_1(x^0)u, u\rangle < \delta < 0$, *i.e.*,

$$-[\langle \nabla f_1(x^0), v\rangle + \langle \nabla^2 f_1(x^0)u, u\rangle] > -\delta > 0.$$

From this and (3.12) it follows that there exists $k_0 \in \mathbb{N}$ such that

$$f_1(x^0) - f_1(x^k) > -\frac{1}{2}\delta t_k^2 > 0$$

for all $k \geq k_0$. Thus, for any $i \in \bar{I}$ and $k \geq k_0$, we have

$$0 < \frac{f_i(x^k) - f_i(x^0)}{f_1(x^0) - f_1(x^k)} \leqq \frac{f_i(x^k) - f_i(x^0)}{-\frac{1}{2}\delta t_k^2}.$$

From this, (3.12) and (3.13), we have

$$\begin{aligned} 0 \leq \lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0)}{f_1(x^0) - f_1(x^k)} &\leqq \lim_{k\to\infty} \frac{f_i(x^k) - f_i(x^0)}{-\frac{1}{2}\delta t_k^2} \\ &= -\frac{1}{\delta}[\langle \nabla f_i(x^0), v\rangle + \langle \xi^i, u\rangle] = 0. \end{aligned}$$

Thus

$$\lim_{k\to\infty} \frac{f_1(x^k) - f_1(x^0)}{f_i(x^0) - f_i(x^k)} = +\infty,$$

contrary to the fact that x^0 is a Geoffrion properly efficient solution to (VP).

Let u^0 be a critical direction at x^0. By ([3], Thm. 4.3), $\langle \nabla f_i(x^0), u^0\rangle = 0$ for all $i \in I$. Since the system

$$\begin{aligned} &F_i^2(x^0; u^0, v) \leq_{\text{lex}} (0,0), \quad && i \in I,\\ &F_i^2(x^0; u^0, v) <_{\text{lex}} (0,0), \quad && \text{at least one } \; i \in I,\\ &G_j^2(x^0; u^0, v) \leqq_{\text{lex}} (0,0), \quad && j \in J(x^0)\end{aligned}$$

has no solution $v \in \mathbb{R}^n$, it follows that the system

$$\begin{aligned} &\langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u^0, u^0\rangle \leq 0, \quad && i \in I,\\ &\langle \nabla f_i(x^0), v\rangle + \langle \nabla^2 f_i(x^0)u^0, u^0\rangle < 0, \quad && \text{at least one} \quad i \in I,\\ &\langle \nabla g_j(x^0), v\rangle + \langle \nabla^2 g_j(x^0)u^0, u^0\rangle \leq 0, \quad && j \in J(x^0; u^0),\end{aligned}$$

has no solution $v \in \mathbb{R}^n$. By the Motzkin theorem of the alternative, there exist $\lambda \in \mathbb{R}^l$ and $\mu \in \mathbb{R}^m$ satisfying (2.1)–(2.3) and (3.5). □

Remark 3.3.

(i) Thanks to Remark 2.3(i), the conclusions of Theorem 3.1 still hold when the (*WASORC*) is replaced by the (*GASORC*). However, by Example 2.5, even if the (*GASORC*) is satisfied, the conclusions of Theorem 2.4 still does not hold.

(ii) Thanks to Example 2.5, Theorem 3.2 cannot be extended to the efficient solution of (VP).

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